Applied Soft Computing Journal 154 (2024) 111315

Contents lists available at ScienceDirect

Applied Soft Computing

journal homepage: www.elsevier.com/locate/asoc

Three-way multi-attribute decision-making under the double hierarchy hesitant fuzzy linguistic information system

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ARTICLE INFO

Keywords:
Three-way decision
Multi-attribute decision-making
Double hierarchy hesitant fuzzy linguistic term set
Conditional probability

ABSTRACT

Three-way multi-attribute decision-making (3MADM) integrated with double hierarchy hesitant fuzzy linguistic term set (DHHFLTS) can not only effectively express the uncertainty of language, but also help reduce the risk of wrong decision-making. However, the existing compared methods for DHHFLTSs dismiss the variances in psychological reference points, resulting in mismatches in form and connotation for some linguistic terms. Furthermore, it is difficult or even impossible to obtain an accurate degree of difference using the existing DHHFLTS distance method for these linguistic terms. This directly affects the accuracy of obtaining conditional probabilities results in the three-way decision model. Therefore, this paper introduces the concept of the superior gradus for double hierarchy linguistic term set (DHLTS) and double hierarchy hesitant fuzzy linguistic element (DHHFLE), respectively. Then, some novel compared methods are defined that allow the identification of differences between linguistic variables. Subsequently, based on the superior gradus, a novel distance measurement is designed with a risk parameters. Through the adjustment of risk parameters, this method can respectively obtain optimistic and pessimistic results. Also, the relative loss functions designed for DHHFLTSs aim at getting more objective decision-making results. Finally, the paper proposes a novel 3MADM method under the double hierarchy hesitant fuzzy linguistic information system (DHHFLIS) and applies it to service assessment. To verify the effectiveness and rationality, the medical diagnosis data set is used, and the results are compared with other classic MAMD methods.

1. Introduction

In the era of big data, computing with words (CW) has become an important part of data analytics. CW was first proposed by Zadeh [1] in 1975, providing a basis for simulating human processing of qualitative information. Qualitative information with the linguistic variables has been investigated by many scholars [2–6]. These representations are varied from the perspective of syntactic form and semantic form according to different qualitative decision-making problems. Linguistic term set (LTS) as originally designed has not been able to convey complex data. The compound structures linguistic terms for expressing complex data are hesitant fuzzy linguistic term set (HFLTS) [7] and probabilistic linguistic term set (PLTS) [8]. HFLTS is proposed to allow a decision maker(DM) to simultaneously hesitate among several linguistic terms. For example, when a consumer is purchasing an electric car, he/she may be hesitant to judge the battery life, and he/she may use HFLTS — normal, good, very good to express his/her opinion. However, all the linguistic terms included in HFLTS have the same important degrees, which is not always adequate when representing the real thoughts of people [9]. PLTS consists of two parts: one part is to utilize the precise numerical value to represent the important degrees of linguistic terms given by people, and the other part is to show the frequencies of linguistic terms. However, the process of using precise numerical values for evaluation tends to be quite time-intensive. The match between precise numerical values and qualitative assessments is what DMs need to consider. Thus, Gou et al. [10] proposed a double hierarchy linguistic term set (DHLTS), which uses linguistic term to represent the important degrees rather than precise numerical values. DMs are more likely to utilize complex and detailed linguistic expressions, such as "only a little

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https://doi.org/10.1016/j.asoc.2024.111315
Received 13 July 2023; Received in revised form 7 January 2024; Accepted 19 January 2024
Available online 2 February 2024
1568-4946/

good", "entirely low", and "a little bit better". The linguistic variables model of DHHFLTS suitably represents the complex data and effectively captures the uncertainty of DM's hesitation. For DMs, using DHHFLTS only requires an easy selection of the language scale based on their intuition, without the need to think hard and evaluate the appropriate numerical values.

In recent years, DHHFLTS has been extensively studied by scholars, including aggregation operator, score function, decision-making methods, etc., and has been applied to a variety of practical scenarios to solve multi-attribute decision-making (MADM) problems. Gou et al. [10] explored the DHHFLTS and the MULTIMOORA method based on the Euclidean distance measure, and applied these methods to the evaluation of haze control measures. Gou et al. [11] proposed a novel score function and a novel distance measure, which were combined with VIKOR to evaluate smart healthcare. Liu et al. [12] proposed the PROMETHEE combining subjective and objective information acquisition methods to evaluate PPP's advancement. Zhang et al. [13] proposed a DHHFL-DEMATEL method to identify key performance indicators in healthcare management. Liu et al. [14] proposed four aggregation operators to calculate the support degree of network accounting software. As mentioned above, the final result of these methods, such as MULTIMOORA, VIKOR, DEMATEL, score function, aggregation operator, are to rank the alternatives according to some indicator. Although MADM methods [10,15–17] generate ranked results that could assist DMs in identifying the superior alternative(s), decision-making necessitates more than just pinpointing the best choice. It may also require consideration of acceptable alternative(s). For example, investment diversification often involves selecting several suitable funds with the goal of spreading investment risk. It is well known that high returns correspond to high risks. Therefore, instead of choosing a few funds with the highest returns, it is better to obtain more stable and appropriate returns through a combination of different risk funds. Especially when the number of alternatives are large, the results produced by the classic MADM methods can only give rankings, but not semantic explanations of the decision results of the alternatives. Since the classic MADM method has the characteristics of two-way decision. This kind of two-way decision-making limited to "accept" or "reject" may reduce decision-making efficiency or increase the risk of wrong decisions. From a human cognitive perspective, triadic thinking is a much more effective model for problem solving.

Three-way decision (3WD) is about thinking, problem-solving, and computing in threes [18]. From the perspective of worldview, Yao's publication in 2022 discussed the mutually supporting relationship between the 3WD and the three-world conception, concreting the TAO (Triading-Acting-Optimizing) framework in more detail. It is a high-level conceptual model whose structure consists of three-part structure, which is in line with the idea of granular computing, and provides the accepted option, the non-commitment option and the rejected option. As early as 2018, Yao [19] discussed the relationship between three-way decision and granular computing from a new perspective, providing a good theoretical and methodological research paradigm for big data analysis. With the development of 3WD, Yao [20,21] proposed the notion of three-way granular computing (3WGC) in 2020 which can effectively reduce decisions risk and yield results that better reflect human cognition. The above models all focus on the structured thinking, problem-solving methodology and information processing aspects of granular computing. The above poins are usually considered to be 3WD in a wide sense. Corresponding to the wide sense of 3WD is the narrow sense of 3WD [22]. The difference between them is that the former focuses on the interpretation of concept connotation and denotation, while the latter focuses on the semantic explanation of three-way decision in practical problems. Research on the narrow sense of 3WD can be roughly divided into two major categories: research into conditional probabilities and research into loss functions. In this model, the loss functions determine a pair of threshold parameters as the basis for the division of three regions: the rule of the positive domain represents acceptance of something, the rule of the negative domain represents rejection something, the rule of the boundary domain represents not to make a commitment on something for the time being. Liang et al. [23] introduced a new similarity relationship by deriving conditional probability under hesitant fuzzy environment. Subsequently, this method was widely extended to type-2 fuzzy environment [24], hesitant fuzzy set [25], fuzzy multi-granulation rough set [26], and others. Jia and Liu [27] proposed the relative loss functions calculation method based on the objects differences, acknowledging that different attributes will lead to different losses. This relative concept was subsequently applied to various fuzzy information systems, such as intuitionistic fuzzy set [28], triangular fuzzy set [29], and PLTS [6]. It is not difficult to see that the above models all have specific mathematical expressions on specific information sets. They have different 3WD models and are applied to problem solving in different fields, such as risk preference decision-making [30], medical decision-making [31], and housing price prediction [32]. Based on existing studies [25,26,33–35], 3WD offers theoretical and technical support for effectively handling MADM problems. Therefore, how to apply the idea of 3WD to solve the MAMD problem is important for the development of 3WD.

The field of three-way multi-attribute decision-making (3MADM) has seen significant growth in recent years. Different from the traditional MADM problem, it can not only realize the function of optimal ranking, but also produce the best/optimal triad in response to the set of strategies. According to the fuzzy expression can be categorized as quantitative and qualitative, the existing 3MADM research can also be classified into two categories. Research in the area of quantitative expression is concerned with the construction of models in new environments, the definition of dominance relationships, and the character of behavioral psychology. Zhang et al. [36] applied the C-means algorithm to the 3MADM model. Gao et al. [37] used the 3MADM method for evaluating target threats. Wang et al. [38] presented a 3MADM method with integrating risk strategies. Zhang et al. [39] developed a 3MADM approach combined with TOPSIS to improve classification and ranking processes. Wang et al. [40] introduced a novel 3MADM method incorporating probabilistic dominance relations. Wang et al. [41,42] introduced behavior theories into 3MADM and established a novel model based on the prospect and regret theories. Wang et al. [43] combined 3MADM with the Z-number environment to solve the decision problem of collaborative human-machine assignment. Among these studies, research combined with behavioral psychology is particularly active. It is worth noting that Zhan et al. [44] conducted a comprehensive review and analysis of three-way behavioral decision making (TW-BDM) with hesitant fuzzy information systems (HFIS) from the perspective of the past, present, and future for the first time. Similarly, qualitatively expressed research is about the construction of rough set models for decision-making in new environments, the determination of dominance degrees, and so on. Table 1 provides a summary of these qualitative research.

Firstly, most of these methods are researched on MADM problems within the framework of decision-theoretic rough set (DTRS). Zhang et al. [3] and Lei et al. [4] further develop two universes rough set frameworks and multi-granularity DTRS frameworks, respectively. Secondly, these researches focuses on loss functions and conditional probabilities. These methods can be categorized into two different types of calculation methods: subjective acquisition and objective computation. Liang et al. [45] proposed linguistic operators for acquiring conditional probabilities and loss functions, while Li et al. [46–48] used different methods to obtain conditional probabilities. Thirdly, these methods construct models in different LTS environments and vertify the effectiveness and feasibility through experimental analyses in various applications. Furthermore, these methods based on the use of different LTS have the most literatures on PLTS [6,48] and HFLTS [3–5], with a small number of other fuzzy linguistic methods [2,45,49,50]. As analyzed earlier, the expression mode of PLTS involves numerical values as the degree of linguistic variable, which may consume time

Table 1
Some relevant studies on three-way decisions combined with linguistic variables.

Methods	Qualitative expression	Application background	Conditional probabilities	Loss functions
Liang et al. [45]	LTS	The new product idea selection	LTS' operational laws	LTS' operational laws
Sun et al. [2]	LTS	The emergency plans evaluation	Fuzzy linguistic logical operator	Subjectively expressed by LTS
Zhang et al. [3]	HFLTS	Person-job fit problem	Rough set	Subjectively expressed by HFLTS
Lei et al. [4]	HFLTS	Green supplier selection	Objectively expressed by HFLTS	Subjectively expressed by HFLTS
Ma et al. [5]	HFLTS	Green supplier selection	Objectively expressed by HFLTS	Subjectively expressed by HFLTS
Li et al. [46]	DHLTS	Enterprise selection for resumption	Grey relational analysis	Subjectively expressed by DHLTS
Li et al. [47]	DHHFLTS	Enterprise selection for resumption	Entropy	Subjectively expressed by DHHFLTS
Li et al. [48]	PLTS	Medical supplies company selection	Belief degrees	Objectively expressed by PLTS
Wang et al. [49]	q-ROULS	Venture Capital Options	Subjectively expressed by q-ROULS	Subjectively expressed by q-ROULS
Qin et al. [50]	LIVIFN	3D printers selection	Grey relational analysis	Objectively expressed by LIVIFN

during the data collection. HFLTS may not adequately capture complex evaluation content. DHHFLTS not only provide a more comprehensive representation of complex evaluation information but also offer greater convenience in practice. Considering the 3MADM and fuzzy linguistic sets are the two main tools for processing uncertain information systems, and are also recognized as related but different and complementary. Based on the above analysis, the following problems remain for their further study:

(1) Regarding compared methods in DHHFLTS, existing expected variance compared methods cannot fully distinguish different DHHFLTSs with the same expected variance value Since these methods do not adequately consider the differences in psychological reference points.
(2) It is a common to use distance measures to characterize the differences between objects and then obtain the equivalence relationship between each object. However, converting DHHFELE to fuzzy numbers to calculate distances weakens the accuracy of the results because there are cases where the different forms of DHHFLE but the same converted fuzzy numbers cannot distinguish the differences.
(3) The current methods for computing conditional probabilities, such as these Refs. [46,47], are often too subjective to keep the accuracy of decision results. While there are a few studies that have combined DHHFLTS with 3WD, even fewer have focused on employing objective methods in handling conditional probabilities and loss functions.
(4) The character of two-way often leads to results lacking in reference when applied in the DHHFLTS environment. This contrasts with 3WD, where a more significant correlation is observed between results and their respective application backgrounds, offering greater relevance and applicability.

CW in the form of DHHFLTS for evaluating values is an emerging phenomenon. Therefore, it is very necessary to build 3MADM models for different CW decision-making environment. The research work of this paper will make new contributions in the following aspects:

(1) In view of the problem that DHLTS cannot be completely distinguished by current researches, based on the fact that people have different psychological reference points for judging things, a new compared method based on the superior gradus is designed to completely distinguish DHLTS and DHHFLE.
(2) In response to the problem of not being able to characterize the distances between different objects due to the inability to fully distinguish between them, proposing a new distance measurement based on the superior gradus. This measurement can take into account both optimistic and pessimistic risk attitudes. The goal of this new measurement is to make conditional probabilities calculations more objective and to provide better sensitivity to various DHHFLTSs.
(3) Aiming at the problem that the existing loss function under the DHHFLIS is set by the experts' experience which makes the decision-making results not objective enough, the relative loss functions for all objects are developed. This method aligns with common sense and intuition.
(4) Constructing a novel 3MADM method under the DHHFLIS, which goes beyond mere expansion and integrates the strengths of DHHFLTS and 3WD method in a comprehensive manner. It could help to enrich the relevant theory research in the field of CW.

The structure of this paper is as follows. Section 2 provides the preliminaries of DHHFLTS and the 3WD model. Section 3 proposes two novel compared methods. In Section 4 the novel 3MADM model under the DHHFLIS is provided. Section 5 shows the processing and algorithm of the novel method. In Section 6, the novel method is used to handle a practical case of service assessment. Section 7 is comparative analysis. Section 8 summarizes the conclusions and future work.

2. Preliminaries

This section briefly reviews some concepts related to the DHHFLTS, 3WD model, and appropriate examples have been added to explain the linguistic variables model of DHHFLTS.

2.1. Double hierarchy hesitant fuzzy linguistic term sets

The DHLTS concept based on the binary linguistic structure was first proposed by Gou et al. [10]. It consists of two levels of completely independent LTS.

Definition 1 (*[10]*). Let $S=\left\{ s_t \middle| t=-\tau,\dots,-1,0,1,\dots,\tau \right\}$ and $\mathrm{O}=\left\{ o_k \middle| k=-\varsigma,\dots,-1,0,1,\dots,\varsigma \right\}$ be the first and second hierarchies of LTS, respectively, and their impact on each other is entirely negligible. Then the DHLTS is defined as $S_O=\left\{ s_{t\langle o_k\rangle} \middle| t=-\tau,\dots,-1,0,1,\dots, \tau; k=-\varsigma,\dots,-1,0,1,\dots,\varsigma \right\}$, where $s_{t\langle o_k\rangle}$ denotes the DHLTS.

Example 1. $\left\{ s_{1\langle o_{-1}\rangle} \right\}$, $\left\{ s_{0\langle o_1\rangle}, s_1, s_{2\langle o_0\rangle} \right\}$, and $\left\{ s_{3\langle o_0\rangle} \right\}$ in Fig. 1 represent "much slightly good", "between much ordinary and just right good", and "just right perfect", respectively. The second linguistic expression, "between much ordinary and just right good", includes all the LTSs from "much ordinary" to "just right good". As a result, s_1 is used to denote the central linguistic terms without using to the DHLTS form. Obviously, s_1 represents "slightly good" and is also the core linguistic terms.

DHLTS can only express a single semantic, but not for more complex semantics, such as "between much ordinary and just right good". Thus, Gou et al. [10] extended DHLTS to hesitant fuzzy set and defined the DHHFLTS.

Definition 2 (*[10]*). Let $S_O=\left\{ s_{t\langle o_k\rangle} \middle| t=-\tau,\dots,-1,0,1,\dots,\tau; k= -\varsigma,\dots,-1,0,1,\dots,\varsigma \right\}$ be a DHLTS. $H_{S_O}=\left\{ \left\langle x, h_{S_O}(x) \right\rangle \middle| x\in U \right\}$ denotes a DHHFLTS on U. When applied to U, it is represented by a membership function, which yields a subset of S_O. $h_{S_O}(x)$ is a set of specific values in S_O, expressed as $h_{S_O}(x)=\left\{ s_{\phi_l\left\langle o_{\varphi_l}\right\rangle}(x) \middle| s_{\phi_l\left\langle o_{\varphi_l}\right\rangle} \in S_O;\right.$

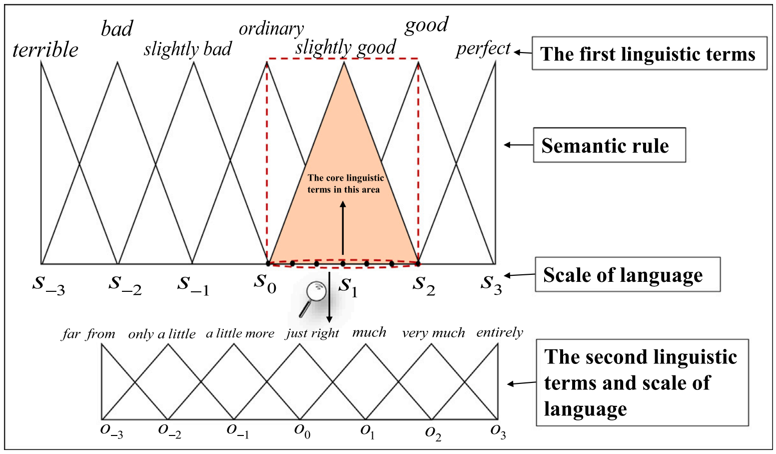


Fig. 1. The second hierarchy LTS O of s_1.

$l = 1,2,\ldots,L; \phi_l = -\tau,\ldots,\tau; \varphi_l = -\varsigma,\ldots,\varsigma\}$,be a continuous DHLTS where L represents the number of DHLTSs in $h_{S_O}(x)$, and the $s_{\phi_l\langle o_{\varphi_l}\rangle}(x)\,(l = 1,2,\ldots,L)$ within each $h_{S_O}(x)$ signifies the potential degree of linguistic variable x with respect to S_O. For convenience, $h_{S_O}(x)$ is called a DHHFLE, and $\Phi \times \Psi$ be the set of all DHHFLEs.

Although the subscripts of DHLTS and DHHFLE are discrete, CW requires that the ambiguity and uncertainty of the information should be preserved as much as possible. In order to realize the operation between the DHLTSs, the discrete subscripts be extended into a continuous form to avoid losing too much information in the calculation processing. For implementing the equivalent transformation of DHLTS and DHHFLE with interval values $[0,1]$, Gou et al. [10] defined two equivalent transform functions for making the mutual transformations between the DHLTS (DHHFLE) and HFE.

Definition 3 (*[10]*). Let $\widehat{S}_O = \left\{ s_{\phi_l\langle o_{\varphi_l}\rangle} \middle| \phi_l \in [-\tau,\tau], \varphi_l \in [-\varsigma,\varsigma] \right\}$ be a continuous DHLTS, $h_{S_O} = \left\{ s_{\phi_l\langle o_{\varphi_l}\rangle} \middle| s_{\phi_l\langle o_{\varphi_l}\rangle} \in \widehat{S}_O; l = 1,2,\ldots,L; \phi_l \in [-\tau,\tau], \varphi_l \in [-\varsigma,\varsigma] \right\}$ be a DHHFLE, and $h_\gamma = \left\{ \gamma_l \middle| \gamma_l \in [0,1]; l = 1,2,\ldots,L \right\}$ be a set of HFEs. There are two transformed functions f and f^{-1} as follows:

$$f : [-\tau,\tau] \times [-\varsigma,\varsigma] \to [0,1], f(\phi_l,\varphi_l) = \frac{\varphi_l + (\tau + \phi_l)\varsigma}{2\varsigma\tau}, \tag{1}$$

$$\begin{aligned} f^{-1} : [0,1] \to [-\tau,\tau] \times [-\varsigma,\varsigma], f^{-1}(\gamma_l) &= [2\tau\gamma_l - \tau]\langle o_{\varsigma(2\tau\gamma_l - \tau - [2\tau\gamma_l - \tau])}\rangle \\ &= [2\tau\gamma_l - \tau] + 1\langle o_{\varsigma((2\tau\gamma_l - \tau - [2\tau\gamma_l - \tau]) - 1)}\rangle. \end{aligned} \tag{2}$$

According to this definition, the transformation functions F and F^{-1} are established as follows:

$$F : \Phi \times \Psi \to \Theta, F\left(h_{S_O}\right) = \left\{ \gamma_l \middle| \gamma_l = f(\phi_l,\varphi_l) \right\} = h_\gamma, \tag{3}$$

$$F^{-1} : \Theta \to \Phi \times \Psi, F^{-1}(h_\gamma) = \left\{ s_{\phi_l\langle o_{\varphi_l}\rangle} \middle| \phi_l\langle\varphi_l\rangle = f^{-1}(\gamma_l) \right\} = h_{S_O}. \tag{4}$$

Then the expect and variance function of h_{S_O} is also defined as below:

$$E\left(h_{S_O}\right) = \frac{1}{L} \sum_{\gamma_l \in F\left(h_{S_O}\right)} \gamma_l, \tag{5}$$

$$v\left(h_{S_O}\right) = \sqrt{\frac{1}{L} \sum_{\gamma_l \in F\left(h_{S_O}\right)} \left(\gamma_l - E\left(h_{S_O}\right)\right)^2}. \tag{6}$$

Based on Eqs. (5) and (6), the compared laws for DHHFLEs are illustrated as follows:

(1) If $E\left(h_{S_O 1}\right) > E\left(h_{S_O 2}\right)$, then $h_{S_O 1}$ is bigger than $h_{S_O 2}$, denoted as $h_{S_O 1} > h_{S_O 2}$,
(2) If $E\left(h_{S_O 1}\right) = E\left(h_{S_O 2}\right)$, then
- If $v\left(h_{S_O 1}\right) < v\left(h_{S_O 2}\right)$, then $h_{S_O 1}$ is bigger than $h_{S_O 2}$, denoted as $h_{S_O 1} > h_{S_O 2}$,
- If $v\left(h_{S_O 1}\right) = v\left(h_{S_O 2}\right)$, then $h_{S_O 1}$ is equivalent with $h_{S_O 2}$, denoted as $h_{S_O 1} = h_{S_O 2}$.

Then a linguistic expected-value of h_{S_O} is obtained as follows:

$$le : \Phi \times \Psi \to \widehat{S}_O, le\left(h_{S_O}\right) = s_{le(\phi_l)\langle o_{le(\varphi_l)}\rangle}, le(\phi_l) = \frac{1}{L}\sum_{l=1}^{L} \phi_l, le(\varphi_l) = \frac{1}{L}\sum_{l=1}^{L} \varphi_l. \tag{7}$$

Example 2 (*Continue with Example 1*). Given that $\left\{ s_{1\langle o_{-1}\rangle} \right\}$ and $\left\{ s_{0\langle o_1\rangle}, s_1, s_{2\langle o_0\rangle} \right\}$ are two DHHFLEs. Let $\tau = 3$, $\varsigma = 3$, $\varphi_l = 1$, and $\phi_l = -1$. According to the transformed functions f, it could get $f(1,-1) = \frac{-1+(3+1)\times 3}{2\times 3\times 3} = \frac{11}{18}$. Then $\gamma_l = \frac{11}{18}$ becomes the independent variable of the inverse function f^{-1}. Two different results can be calculated according to Eq. (2), i.e. $f^{-1}\left(\frac{11}{18}\right) = \left[2\times 3\times\frac{11}{18} - 3\right]\left\langle o_{3\times\left(2\times 3\times\frac{11}{18} - 3 - \left[2\times 3\times\frac{11}{18} - 3\right]\right)}\right\rangle = [0.67]\langle 2\rangle$ or $f^{-1}\left(\frac{11}{18}\right) = \left[2\times 3\times\frac{11}{18} - 3\right] + 1\left\langle o_{3\times\left(\left(2\times 3\times\frac{11}{18} - 3 - \left[2\times 3\times\frac{11}{18} - 3\right]\right) - 1\right)}\right\rangle = [0.67] + 1\langle -1\rangle$. The DHLTS obtained based on these two results are $\left\{ s_{0\langle o_{-2}\rangle} \right\}$ and $\left\{ s_{1\langle o_{-1}\rangle} \right\}$ When the number of DHLTSs in DHHFLE is greater than or equal to 1, the set of γ_l can be obtained by the transformation function F and F^{-1} between the DHHFLE and HFE. Let h_{S_O} be the $\left\{ s_{0\langle o_1\rangle}, s_1, s_{2\langle o_0\rangle} \right\}$, then $F\left(h_{S_O}\right) = \left\{ \frac{5}{9}, \frac{1}{3}, \frac{5}{6} \right\}$. In this way, the expectation and variance of h_{S_O} can be obtained, respectively, as $E\left(h_{S_O}\right) = \frac{1}{3}\times\left(\frac{5}{9} + \frac{1}{3} + \frac{5}{6}\right) = \frac{37}{54} \approx 0.69$ and $v\left(h_{S_O}\right) = \sqrt{\frac{1}{3}\times\left(\left(\frac{5}{9} - \frac{37}{54}\right)^2 + \left(\frac{1}{3} - \frac{37}{54}\right)^2 + \left(\frac{5}{6} - \frac{37}{54}\right)^2\right)} = 0.233$. The linguistic expected-value of h_{S_O} for the subscripts of linguistic variables is $le\left(h_{S_O}\right) = s_{\frac{1}{3}\times(0+1+2)\left\langle o_{\frac{1}{3}\times(1+0+0)}\right\rangle} = s_{1\langle o_{0.33}\rangle}$.

Definition 4 ([10]). Let $\widehat{S}_O$ be a continuous DHLTS, h_{S_O}, $h_{S_{O1}}$, and $h_{S_{O2}}$ be any three DHHFLEs, and μ be a constant. Then some operational laws between DHHFLEs are defined as follows:

$$h_{S_{O1}} \oplus h_{S_{O2}} = F^{-1}\left(\bigcup_{\gamma_1 \in F\left(h_{S_{O1}}\right), \gamma_2 \in F\left(h_{S_{O2}}\right)} \left\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\right\}\right), \tag{8}$$

$$\mu h_{S_O} = F^{-1}\left(\bigcup_{\gamma \in F\left(h_{S_O}\right)} \left\{1 - (1-\gamma)^{\mu}\right\}\right), \tag{9}$$

$$\left(h_{S_O}\right)^{\mu} = F^{-1}\left(\bigcup_{\gamma \in F\left(h_{S_O}\right)} \left\{\gamma^{\mu}\right\}\right), \tag{10}$$

$$\left(h_{S_O}\right)^{C} = F^{-1}\left(\bigcup_{\gamma \in F\left(h_{S_O}\right)} \left\{1 - \gamma\right\}\right). \tag{11}$$

Example 3. Let $h_{S_{O^1}} = \left\{s_{1\langle o_{-1}\rangle}\right\}, h_{S_{O^2}} = \left\{s_{3\langle o_0\rangle}\right\}$, and $\mu = 2$. Then it is possible to get $\gamma_1 = \frac{11}{18}$ and $\gamma_2 = 1$ according to Definition 3. The effect of Eqs. (8)–(11) are as follow: $h_{S_{O1}} \oplus h_{S_{O2}} = F^{-1}\left(\frac{11}{18} + 1 - \frac{11}{18} \times 1\right) = F^{-1}(1)$, $\mu h_{S_{O^1}} = F^{-1}\left(1 - \left(1 - \frac{11}{18}\right)^2\right) = F^{-1}(0.85)$, $\left(h_{S_O}\right)^{\mu} = F^{-1}\left(\left(\frac{11}{18}\right)^2\right) = F^{-1}(0.374)$, and $\left(h_{S_{O^1}}\right)^C = F^{-1}\left(1 - \frac{11}{18}\right) = F^{-1}(0.389)$.

2.2. Three-way decision model

In 3WD model, the domain is divided into three distinct regions. Bayesian minimal risk decision's rules separate the 3WD problem into the sets of states and actions. The state sets indicate $\Xi = \{C, \neg C\}$, indicating whether the object x is in state C or not. Y_{PP}, Y_{BP}, and Y_{NP} represent the incurred losses for executing actions sets $\Lambda = \left\{\dot{a}_P, \dot{a}_B, \dot{a}_N\right\}$, when the object x belongs to C. Similarly, Y_{PN}, Y_{BN}, and Y_{NN} denote the incurred losses for executing the same actions when the object x belongs to $\neg C$. The loss functions values satisfy $0 \le Y_{PP} \le Y_{BP} < Y_{NP}$ and $0 \le Y_{NN} \le Y_{BN} < Y_{PN}$. $[x]$ is the equivalence class of x for all the equivalence relations. The expected loss $\mathfrak{R}\left(\dot{a}_\bullet \middle| [x]\right), \bullet = P, B, N$ can be expressed as follows [51]:

$$\mathfrak{R}\left(\dot{a}_P \middle| [x]\right) = Y_{PP} \Pr\left(C \middle| [x]\right) + Y_{PN} \Pr\left(\neg C \middle| [x]\right), \tag{12}$$

$$\mathfrak{R}\left(\dot{a}_B \middle| [x]\right) = Y_{BP} \Pr\left(C \middle| [x]\right) + Y_{BN} \Pr\left(\neg C \middle| [x]\right), \tag{13}$$

$$\mathfrak{R}\left(\dot{a}_N \middle| [x]\right) = Y_{NP} \Pr\left(C \middle| [x]\right) + Y_{NN} \Pr\left(\neg C \middle| [x]\right), \tag{14}$$

where $\Pr\left(C \middle| [x]\right)$ is the conditional probability of the object x_i belonging to C, and the object x is described by its similarity equivalence class $[x]$. According to the minimum risk decision, the decision rules are obtained as follows:

(P) If $\mathfrak{R}\left(\dot{a}_P \middle| [x]\right) \le \mathfrak{R}\left(\dot{a}_B \middle| [x]\right)$ and $\mathfrak{R}\left(\dot{a}_P \middle| [x]\right) \le \mathfrak{R}\left(\dot{a}_N \middle| [x]\right)$, decide $x \in POS(C)$,

(B) If $\mathfrak{R}\left(\dot{a}_B \middle| [x]\right) \le \mathfrak{R}\left(\dot{a}_P \middle| [x]\right)$ and $\mathfrak{R}\left(\dot{a}_B \middle| [x]\right) \le \mathfrak{R}\left(\dot{a}_N \middle| [x]\right)$, decide $x \in BND(C)$,

(N) If $\mathfrak{R}\left(\dot{a}_N \middle| [x]\right) \le \mathfrak{R}\left(\dot{a}_P \middle| [x]\right)$ and $\mathfrak{R}\left(\dot{a}_N \middle| [x]\right) \le \mathfrak{R}\left(\dot{a}_B \middle| [x]\right)$, decide $x \in NEG(C)$.

Since $\Pr\left(C \middle| [x]\right) + Pr\left(\neg C \middle| [x]\right) = 1$, the decision rules (P)-(N) can be simplified with $u = \frac{Y_{PN} - Y_{BN}}{(Y_{PN} - Y_{BN}) + (Y_{BP} - Y_{PP})}$ and $v = \frac{Y_{BN} - Y_{NN}}{(Y_{BN} - Y_{NN}) + (Y_{NP} - Y_{BP})}$ as (P1)-(N1):

(P1) If $\Pr\left(C \middle| [x]\right) \ge u$, then decide $x \in POS(C)$,

(B1) If $v < \Pr\left(C \middle| [x]\right) < u$, then decide $x \in BND(C)$,

(N1) If $\Pr\left(C \middle| [x]\right) \le v$, then decide $x \in NEG(C)$.

According to (P1)-(N1), for the object x, when its conditional probability is greater than or equal to the threshold u, it will be classified into the positive domain ($POS(C)$). Conversely, the object is classified into the negative domain ($NEG(C)$) with its conditional probability is less than or equal to the threshold v. The conditional probability of object between u and v, it is classified into the boundary domain ($BND(C)$).

3. The novel compared methods for DHLTSs and DHHFLEs

The process of making a choice requires comparison between different objects. Comparing different objects can help DMs identify the advantages and disadvantages of each object. The primary goal of comparison is to optimize the decision-making process by improving its accuracy and efficiency. Therefore, a complete comparison of different objects is essential for making wise and effective decisions. This section will discuss novel compared methods to identify the differences.

3.1. The compared method for DHLTSs based on the superior gradus

People tend not to evaluate things absolutely, but relative to certain standards, expectations, or prior experiences. This relativity is a common phenomenon in human evaluation of things. For example, reference dependence theory states that when people make decisions, they consider losses and gains based on a reference point rather than absolute values. For another example, the anchoring effect points out that during valuation, the starting price often becomes an "anchor", affecting the final valuation decision. Therefore, when evaluating anything, people are always based on certain reference points, whether consciously or unconsciously, these reference points shape people's judgments and decisions. This phenomenon can effectively communicate individual inclinations and decisions in a nuanced manner. Where a description like "far from perfect" is required, the word "perfect" might serve as the core linguistic terms in the compound structure for some individuals. Others might choose different words as core linguistic terms to express the same meaning. These differences of core linguistic terms and their understandings can affect the interpretation of the same object or concept. Unfortunately, these nuances are often overlooked in the existing compared methods. The current methods for comparing DHLTSs, which typically involve transforming DHLTSs s_o into HFEs γ_l using Eq. (1), have limitations in Example 4. The compared method of expected variance based on the calculation process is disregard individual preferences and psychological reference points, resulting in mechanistic calculations that lack sensitivity to the subtleties arising from such subjective factors within a given linguistic scale.

Example 4. Let $s_{1\langle o_3\rangle}$, $s_{2\langle o_0\rangle}$, and $s_{3\langle o_{-3}\rangle}$ be three different DHLTSs. As depicted in Fig. 2, they share the same γ_l, $f\left(s_{1\langle o_3\rangle}\right) = f\left(s_{2\langle o_0\rangle}\right) = f\left(s_{3\langle o_{-3}\rangle}\right) = 0.833$. According to the linguistic scale, $s_{1\langle o_3\rangle}$ signifies "entirely slightly good", $s_{2\langle o_0\rangle}$ conveys the meaning of "just right good", and $s_{3\langle o_{-3}\rangle}$ represents "far from perfect".

It is important to recognize that even though the DHLTSs in Fig. 2 may have the same physical meaning, their interpretative content can be very different. This is due to the non-monotonic nature of Eq. (1), which can result in identical γ_l values within the range $[0, 1]$ for different DHLTSs. As mentioned earlier, there may be differences in how people perceive the same object or concept due to different reference. To better illustrate this concept, Example 5 is going to be considered and explored in the real-life situations.

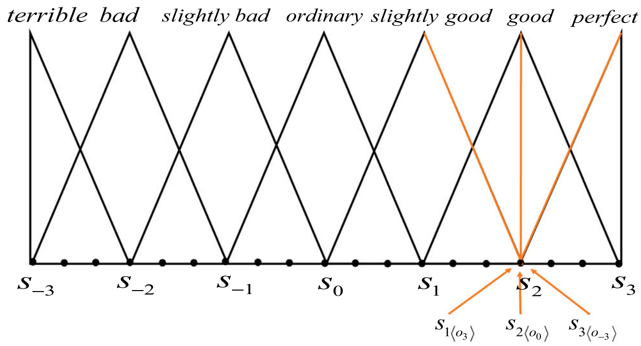


Fig. 2. DHLTS with the same value based on the transformation function f.

Example 5. Suppose two individuals who need to buy an electric car are asked to evaluate a vehicle featuring a driving range of 330 kilometers. It is widely recognized that individuals exhibit varying sensitivities toward electric vehicle mileage, influenced by their different preferences and concerns. If an individual who has only daily commuting needs, would probably rate the kilometers as $s_{2\langle o_0\rangle}$, representing the meaning of "just right good". If another who has the need of short-distance travel, might rate it as $s_{1\langle o_3\rangle}$, represents "entirely slightly good". Obviously, the latter has a little of "mileage anxiety". Although $s_{2\langle o_0\rangle}$ and $s_{1\langle o_3\rangle}$ have the same γ_l, the former gives $s_{2\langle o_0\rangle}$ based on the reference s_2 and gives $s_{1\langle o_3\rangle}$ based on the reference s_1. In other words, the former prefers $s_{2\langle o_0\rangle}$ to $s_{1\langle o_3\rangle}$ for evaluating the mileage of electric vehicles.

To account for variations in linguistic evaluation stemming from psychological preferences, a model is needed to characterize such differences. This model should accurately capture the nuances and differences in Example 5 scenarios. It should preserve the original linguistic evaluation information while incorporating parameters that align with the degree of psychological preference, thereby enhancing the sensitivity of the evaluation. By introducing some parameters from Definition 5, the metric model can effectively highlight the differences observed in Example 5. These parameters can be tailored to reflect the specific psychological preferences and subjective factors relevant to the decision-making process. This is designed to ensure a more precise representation of linguistic evaluation and consider the varying importance and preferences assigned to different linguistic terms. The developed metric model aims to capture the complexity of linguistic evaluation and provide a quantitative framework for comparison. This enables DMs to make more informed evaluations based on their individual psychological preferences and reference points. In summary, the construction of a metric model incorporating parameters that align with psychological preferences offers a comprehensive framework for evaluating linguistic information. Through it, DMs can confidently capture and effectively address the nuanced differences that may arise from a range of psychological perspectives.

Definition 5. Let $\widehat{S}_O = \left\{ s_{\phi_l\left\langle o_{\varphi_l}\right\rangle} \middle| \phi_l \in [-\tau,\tau], \varphi_l \in [-\varsigma,\varsigma]; l = 1,2,\ldots,L \right\}$ be a continuous DHLTSs, and $\phi_{l\langle\varphi_l\rangle}$ be the subscript of $s_{\phi_l\left\langle o_{\varphi_l}\right\rangle}$. Then the superior gradus is defined as follows:

$$sg : [-\tau,\tau]\times[-\varsigma,\varsigma] \to [0,1], sg\left(s_{\phi_l\left\langle o_{\varphi_l}\right\rangle}\right) = \frac{(\varepsilon^{\alpha}+\beta)-1}{\varepsilon-1}, \tag{15}$$

where ε is the base of superior gradus index, $\alpha = \left(\frac{\phi_l}{2\tau}+\frac{1}{2}\right)$, $\beta = \frac{\varphi_l}{2\varsigma\tau}$.

The superior gradus sg is a monotonic function where $sg\left(s_{\phi_l\left\langle o_{\varphi_l}\right\rangle}\right)$ possesses a unique corresponding value, representing the DM's preference degree. The higher the value of superior gradus, the more willing the DM is to accept the DHLTSs expression given by him/her. This quality becomes particularly useful when faced with situations like in Example 4. Even when several DHLTSs share the same value, the superior gradus are capable of ranking them based on the DM's psychological preference with clarity. According to Definition 5, the following properties can be obtained.

Property 1. *Let S_O, S_O^1, and S_O^2 be any DHLTS, they satisfy the following rules:*

(1) $sg\left(S_O\right) = 1$, *iff* $\phi_l = \tau, \varphi_l = 0$,
(2) $sg\left(S_O\right) = 0$, *iff* $\phi_l = -\tau, \varphi_l = 0$,
(3) $sg\left(S_O^1\right) = sg\left(S_O^2\right)$, *iff* $\phi_l^1 = \phi_l^2, \varphi_l^1 = \varphi_l^2$,
(4) If $sg\left(S_O^1\right) \geq sg\left(S_O^2\right)$ *and* $sg\left(S_O^2\right) \geq sg\left(S_O\right)$, *then* $sg\left(S_O^1\right) \geq sg\left(S_O\right)$.

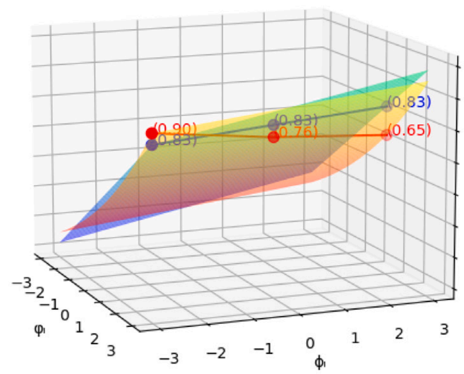
Proof. Given $sg\left(S_O\right) = sg\left(s_{\phi_l\left\langle o_{\varphi_l}\right\rangle}\right) = \frac{(\varepsilon^{\alpha}+\beta)-1}{\varepsilon-1}$, *where* $\alpha = \left(\frac{\phi_l}{2\tau}+\frac{1}{2}\right), \beta = \frac{\varphi_l}{2\varsigma\tau}$.

(1) Substituting $\phi_l = \tau, \varphi_l = 0$ into the above equations, it can be easy to obtain $\alpha = 1, \beta = 0$, and $sg\left(S_O\right) = 1$.
(2) Substituting $\phi_l = -\tau, \varphi_l = 0$ into the above equations, it can be easy to obtain $\alpha = 0, \beta = 0$, and $sg\left(S_O\right) = 0$.
(3) Since $\phi_l^1 = \phi_l^2, \varphi_l^1 = \varphi_l^2$, it is easy to know that $\alpha^1 = \alpha^2$ and $\beta^1 = \beta^2$. $sg\left(S_O^1\right) = sg\left(S_O^2\right)$ is holds.
(4) It is easy to know that $\alpha^1 > \alpha^2 > \alpha$ and $\beta^1 > \beta^2 > \beta$. Since sg is a monotonically increasing function with respect to α and β, it is easy to obtain that $sg\left(S_O^1\right) \geq sg\left(S_O\right)$.

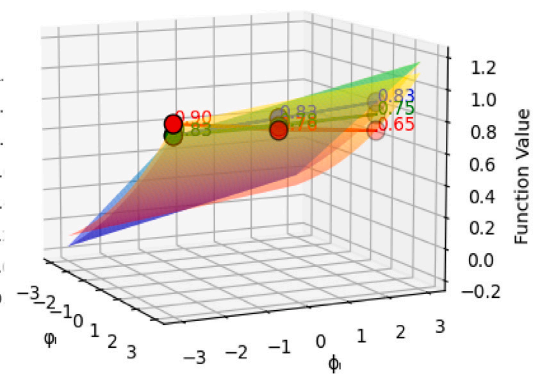
Based on Eq. (15), the compared laws between DHLTS are shown as follows:

(1) If $sg\left(S_O^1\right) > sg\left(S_O^2\right)$, then S_O^1 is superior to S_O^2, denoted as $S_O^1 > S_O^2$,
(2) If $sg\left(S_O^1\right) = sg\left(S_O^2\right)$, then S_O^1 is equivalent with S_O^2, denoted as $S_O^1 = S_O^2$.

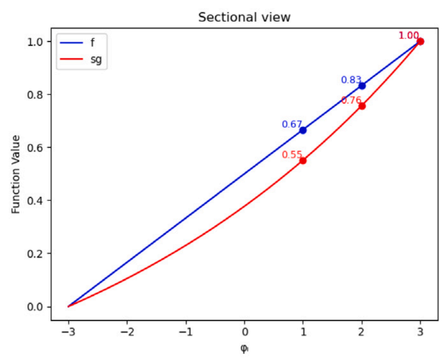
The first hierarchy core linguistic terms in DHLTS is expanded to derive the superior gradus which demonstrates variations in individual preferences when selecting the first hierarchy terms based on their initial intuition. This improvement emphasizes the influence of the first hierarchy terms and could better align with human language habits. The superior gradus resolves the issue of being unable to compare different DHLTS that share the same γ_l. Using the superior gradus, a unique value can be assigned to each DHLTS to reflect the degree of the DM's preference. The superior gradus of DHLTS in Example 4 can be obtained as $sg\left(s_{1\langle o_3\rangle}\right) = 0.65$, $sg\left(s_{2\langle o_0\rangle}\right) = 0.76$, and $sg\left(s_{3\langle o_{-3}\rangle}\right) = 0.90$. Fig. 3 can more clearly see that the sg function can distinguish overlapping points well. Subfigures (a) and (b) present the principle that superior gradus can distinguish partially overlapping points. Subfigures (c) and (d) are cross-sectional views of subfigures (a) and (b) respectively. When the second linguistic terms of DHLTS is fixed, what is shown is the fluctuation of superior gradus as the first linguistic terms changes. The design of superior gradus also conforms to the changing trend of DHLTS itself mapping to the [0,1]. Meanwhile, based on the compared laws, the compered results also can show that $s_{3\langle o_{-3}\rangle} > s_{2\langle o_0\rangle} > s_{1\langle o_3\rangle}$. This method allows for a more detailed evaluation and comparison of DHLTS, while also closely aligning with human linguistic habits, ensuring a more intuitive and accurate reflection of the real feelings and preferences of the DM. Among the compared methods of DHHFLTS, the sg function provides a new perspective.



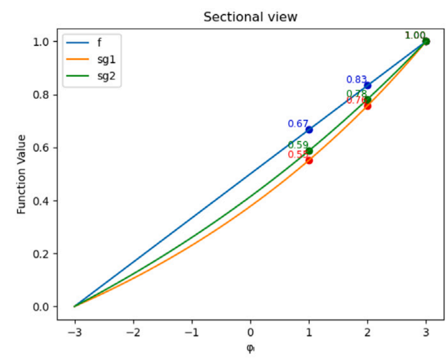
(a) Function f and superior grdus sg.



(b) Function f and superior grdus sg with two ε.



(c) Cross-sections of function f and superior grdus sg.



(d) Cross-sections of function f and superior grdus sg with two ε.

Fig. 3. The differences between function f and superior grdus sg.

Table 2
Two situations of DHHFLEs' compared result.

Situations	DHHFLEs	E	v	$\{sg\}$	F	SG
Same L	$h_{S_O1}=\{s_{1\langle o_{-3}\rangle}, s_{2\langle o_{-3}\rangle}\}$	0.583	0.007	$\{$**0.450,0.660**$\}$	$\{0.500, 0.667\}$	**0.557**
	$h_{S_O2}=\{s_{-1\langle o_3\rangle}, s_{0\langle o_3\rangle}\}$	0.583	0.007	$\{$**0.327,0.475**$\}$	$\{0.500, 0.667\}$	**0.401**
Different L	$h_{S_O3}=\{s_{0\langle o_3\rangle}, s_{1\langle o_0\rangle}, s_{2\langle o_{-3}\rangle}\}$	0.667	0	$\{$**0.475,0.525,0.660**$\}$	$\{0.667, 0.667, 0.667\}$	**0.553**
	$h_{S_O4}=\{s_{1\langle o_0\rangle}, s_{2\langle o_{-3}\rangle}\}$	0.667	0	$\{$**0.552,0.660**$\}$	$\{0.667, 0.667\}$	**0.606**

3.2. The compared method for DHHFLEs based on the superior gradus

The existing methods for comparing any two DHHFLEs are also deficient. There are situations where two DHHFLEs have the same expectation and variance values. DMs may have varying psychological differences. Due to the hesitation characteristics of DHHFLEs, the number of DHLTS contained in DHHFLE often varies. Based on the current compared methods, they are impossible to distinguish the variances among various DHHFLEs. There may be either two or three DHLTS, leading to two situations shown in the Table 2.

Situation 1 The DHHFLEs have the same the number L of DHLTS, but the included DHLTS are not the same. $h_{S_O1}=\left\{s_{1\langle o_{-3}\rangle}, s_{2\langle o_{-3}\rangle}\right\}$ and $h_{S_O2}=\left\{s_{-1\langle o_3\rangle}, s_{0\langle o_3\rangle}\right\}$, they all have the same DHLTS number $L=2$. According to the compared method of expected variances, they also have the same expected values, i.e., $E\left(h_{S_O1}\right)=E\left(h_{S_O2}\right)=0.583$ and $v\left(h_{S_O1}\right)=v\left(h_{S_O2}\right)=0.007$. However, they display differences in both literal interpretation and intuitive understanding. The diversity of core linguistic terms result in the intuitive superiority of h_{S_O1} over h_{S_O2}. Thus, the compared method based on the superior gradus for DHHFLEs is proposed when they have the same DHLTS number L.

Definition 6. Let $h_{S_O}=\left\{s_{\phi_l\langle o_{\varphi_l}\rangle} \middle| s_{\phi_l\langle o_{\varphi_l}\rangle}\in \widehat{S}_O; l=1,2,\ldots,L; \phi_l\in[-\tau,\tau], \varphi_l\in[-\varsigma,\varsigma]\right\}$ be the continuous DHHFLE, and SG be the superior gradus of DHHFLE. Then the SG is defined as follows:

$$SG\left(h_{S_O}\right)=\frac{1}{L}\sum_{l=1}^{L} sg\left(s_{\phi_l\langle o_{\varphi_l}\rangle}\right), \tag{16}$$

where L is the number of DHLTS in h_{S_O}.

Let h_{S_O1} and h_{S_O2} be any DHHFLEs. The number of DHLTS in two DHHFLEs is equivalent. The superior gradus is directly calculated by the Eq. (16), then the two compared laws between DHHFLEs are shown as follows:

(1) If $SG\left(h_{S_O1}\right) > SG\left(h_{S_O2}\right)$, then h_{S_O1} is superior to h_{S_O2}, denoted as $h_{S_O1} > h_{S_O2}$,
(2) If $SG\left(h_{S_O1}\right) = SG\left(h_{S_O2}\right)$, then h_{S_O1} is equivalent with h_{S_O2}, denoted as $h_{S_O1} = h_{S_O2}$.

If the superior gradus of one DHHFLE surpasses the other, the former DHHFLE is determined to be superior in its linguistic representation. If the superior gradus values of the two DHHFLEs are identical, it suggests that their linguistic representations are equivalent in terms of their evaluative impact or meaning.

Situation 2 The DHHFLEs have different number L of DHLTS. $h_{S_O3} = \left\{ s_{0\langle o_3 \rangle}, s_{1\langle o_0 \rangle}, s_{2\langle o_{-3} \rangle} \right\}$ and $h_{S_O4} = \left\{ s_{1\langle o_0 \rangle}, s_{2\langle o_{-3} \rangle} \right\}$ are shown in Table 2. In a hesitant environment, the number of DHLTSs are different due to the diverse references and preferences of core linguistic terms. The core linguistic terms is the semantic corresponding to the scale of the first hierarchy of linguistic terms. The number of DHLTSs in h_{S_O3} is $L_3 = 3$, while the number of DHLTSs in h_{S_O4} is $L_4 = 2$. Since $L_3 > L_4$, there is a higher degree of uncertainty, leading to the intuitive superiority of h_{S_O4} over h_{S_O3}. However, they share the same expectation and variance values, i.e., $E\left(h_{S_O3}\right) = E\left(h_{S_O4}\right) = 0.667$ and $v\left(h_{S_O3}\right) = v\left(h_{S_O4}\right) = 0$. According to Eqs. (5) and (6), this implies they are equal. Obviously, the lower uncertainty h_{S_O4} is superior to h_{S_O3}.

Even so, in order to maintain consistency in the calculation process. Regarding the number of DHLTSs, further thinking can be done. When the number of DHLTSs is the same, the DHHFLEs exhibit equal uncertainty due to the identical number of DHLTSs. But when the numbers differ, how can these uncertainties be effectively measured? Is it possible to extend two DHHFLEs with different numbers of DHLTSs to the same length? Inspired by Xu and Xia [52], they recommended incorporating values according to the decision maker's risk preference to guarantee comparability between shorter and longer DHHFLEs. Consequently, the shorter DHHFLEs are extended to achieve the same length when two different lengths of DHHFLEs are compared. Thus, two types of extension methods are summarized as follows:

- Type I If the decision maker is optimistic, add the maximum DHLTS s_o from the shorter h_{S_O} to complete the same length,
- Type II If the decision maker is pessimistic, add the minimum DHLTS s_o from the shorter h_{S_O} to complete the same length.

Considering the varying risk attitudes of decision makers, two extension methods can be applied to equalize the lengths of differing DHHFLEs. Consequently, different SG values can be computed using Eq. (16). The abbreviation "*opt*" is used to represent the adoption of Type I, while the abbreviation "*pes*" is used to represent the adoption of Type II. To better grasp these two extension methodologies, an illustrative Example 6 is provided below.

Example 6. Comparing h_{S_O3} and h_{S_O4}, h_{S_O4} is shorter than h_{S_O3}, and $s_{2\langle o_{-3} \rangle} > s_{1\langle o_0 \rangle}$. Thus, $h_{S_O4} = \left\{ s_{1\langle o_0 \rangle}, s_{2\langle o_{-3} \rangle} \right\}$ will be extended to $h^{opt}_{S_O4} = \left\{ s_{1\langle o_0 \rangle}, s_{2\langle o_{-3} \rangle}, s_{2\langle o_{-3} \rangle} \right\}$ with the optimistic attitude, $SG\left(h^{opt}_{S_O4}\right) = 0.624$. $h_{S_O4} = \left\{ s_{1\langle o_0 \rangle}, s_{2\langle o_{-3} \rangle} \right\}$ will be extended to $h^{pes}_{S_O4} = \left\{ s_{1\langle o_0 \rangle}, s_{2\langle o_{-3} \rangle}, s_{1\langle o_0 \rangle} \right\}$ with the pessimistic attitude, $SG\left(h^{pes}_{S_O4}\right) = 0.588$. From Table 2, it is shown that $SG\left(h_{S_O3}\right) = 0.553$. In other words, according to the calculation results, regardless of whether it is an/a optimistic or pessimistic attitude, h_{S_O4} is better than h_{S_O3}. This shows that Definition 6 can distinguish them and the results are reasonable.

The idea behind Definition 6 is to combine the superior gradus of each DHLTS into one numerical value for the entire DHHFLE. Any two DHHFLEs can be compared directly using superior gradus. This provides a clear and easily understood compared laws, allowing DHHFLEs with the same number of DHLTS to visually assess their relative superiority. Consider a real application scenario: suppose you are comparing online reviews of two restaurants. The first restaurant has a majority of "very good" reviews, while the second restaurant also has a majority of "mostly very good" reviews. While both sets of reviews may have the same score, it is important to carefully analyze the subtle differences between them. For example, "mostly very good" could mean that there is a small percentage of less positive reviews, which could have an impact on a potential customer's choice. Here, the use of superior gradus model provides the flexibility to deal with these subtle differences, providing consumers with deeper insights and helping them make more informed decisions.

4. 3MADM under the DHHFLIS

In this section, the DHHFLIS of 3MADM is introduced first. Then, the conditional probabilities based on a new distance measurement is proposed. Finally, an objective calculation method of the relative loss functions is presented.

4.1. Conditional probabilities

To increase efficiency when dealing with decision-making involving MADM, the initial step is to translate MADM problems under the DHHFLIS into the 3WD framework.

Definition 7. Let the DHHFLIS be a $IS = \left\{ U, A \cup D, V, h_{S_O} \right\}$, where U is a nonempty finite set of objects, called the universe. A is a nonempty finite set of conditional attributes, $D = \{d\}$ is a singleton set of decision attribute and $A \cap D = \emptyset$, and $V = \cup_{a \in A \cup D} V_a$. Suppose $U/D = \{C, \neg C\}$, $x \in U$, and $V_d = \{0, 1\}$. For any $x \in U$, the value of each conditional attribute is denoted as $h_{S_O}(x)$. Meanwhile, For any $x \in U$ with each conditional attribute has its own loss functions.

Definition 8. A 3MADM framework is $\{IS, LF\}$, where $\{IS\}$ is DHHFLIS, and $\{LF\}$ is loss function. For any $x \in U$, the loss function is denoted as $LF = \left\{ U, h^{\lambda_{PP}}_{S_O} \bigcup h^{\lambda_{BP}}_{S_O} \bigcup h^{\lambda_{NP}}_{S_O} \bigcup h^{\lambda_{PN}}_{S_O} \bigcup h^{\lambda_{BN}}_{S_O} \bigcup h^{\lambda_{NN}}_{S_O} \right\}$, $h^{\gamma_{\bullet\circ}}_{S_O}(x)$ ($\bullet = P, B, N$; $\circ = P, N$).

The 3MADM framework under DHHFLIS is defined. Conditional probabilities in the 3WD model are calculated based on evaluating similarity equivalence classes. Liang et al. [23] discussed why previous similarity equivalence class investigations could not be directly applied, due to data types incompatible with hesitant fuzzy information systems. They proposed a similarity equivalence class for distance and neighborhood models in hesitant fuzzy information systems. Accordingly, this paper defines the similarity equivalence class under DHHFLIS.

Definition 9 (*[23]*). Let $IS = \left\{ U, AT \cup D, V, h_{S_O} \right\}$ be a DHHFLIS, given any $x_i, x_j \in U$, the similarity equivalence class of x_i respect to conditional attributes A is defined as follows:

$$B_{\delta_A} = \left\{ x_j \in U \middle| dist_A\left(x_i, x_j\right) \le \delta \right\}, \tag{17}$$

where B_{δ_A} is the set of the similarity equivalence class of x_i, δ is a constant and $\delta \ge 0$, $dist_A\left(x_i, x_j\right)$ is a distance function.

Equivalent classes $\left[x_i\right]$ of different objects can be clearly obtained by introducing Boolean matrices. In order for the present framework to apply the Eq. (17) to obtain a binary relationship as B_{δ_A}, the distance function must be redefined under the DHHFLIS. However, the existing distance [23] cannot fully distinguish between different DHHFLEs. From $h_{S_O1} = \left\{ s_{1\langle o_{-3} \rangle}, s_{2\langle o_{-3} \rangle} \right\}$ and $h_{S_O2} = \left\{ s_{-1\langle o_3 \rangle}, s_{0\langle o_3 \rangle} \right\}$, it could get $dist\left(h_{S_O1}, h_{S_O2}\right) = \left(\frac{(0.5-0.5)^2+(0.67-0.67)^2}{2} \right)^{1/2} 12$. Obviously, the two DHHFLEs with different semantics should be different, and the distance should not be zero. Based on Section 3, the novel distance under the DHHFLIS will be defined.

Definition 10. A new distance measurement under the DHHFLIS based on the superior gradus can be defined as follows:

$$dist_A\left(x_i, x_j\right) = \left[\frac{1}{|A|} \sum_A \left(\left(SG\left(x_i\right) - SG\left(x_j\right) \right)^2 \right) \right]^{1/2}, \tag{18}$$

Table 3
The relative loss functions.

Actions	$C(P)$	$\neg C(N)$
$\dot{a}_P$	$h^{\Upsilon_{PP}}_{S_O}(x_{ik}) = \{s_{-\tau\langle o_0\rangle}\}$	$h^{\Upsilon_{PN}}_{S_O}(x_{ik}) = \left(h^{\lambda_{NP}}_{S_O}(x_{ik})\right)^C$
$\dot{a}_B$	$h^{\Upsilon_{BP}}_{S_O}(x_{ik}) = \eta \cdot h^{\lambda_{NP}}_{S_O}(x_{ik})$	$h^{\Upsilon_{BN}}_{S_O}(x_{ik}) = \eta \cdot \left(h^{\lambda_{NP}}_{S_O}(x_{ik})\right)^C$
$\dot{a}_N$	$h^{\Upsilon_{NP}}_{S_O}(x_{ik})$	$h^{\Upsilon_{NN}}_{S_O}(x_{ik}) = \{s_{-\tau\langle o_0\rangle}\}$

where $|A|$ denotes the cardinality of set A, $SG\left(h^*_{S_O}(x_i)\right)$ and $SG\left(h^*_{S_O}(x_j)\right)$ are the superior gradus of DHHFLEs after elements' lengths adjusted with different risk attitude type corresponding to $SG(x_i)$ and $SG(x_j)$, $*= pes, opt$.

Property 2. *Let x_1 and x_2 be any two objects, the distance function satisfies the following properties as follows:*

(1) ***Boundary:*** $0 \le dist_A(x_1, x_2) \le 1$,
(2) ***Symmetry:*** $dist_A(x_1, x_2) = dist_A(x_2, x_1)$,
(3) ***Complementarity:*** $dist_A(x_1, x_2) = 1$, *iff* $SG(x_1) = 1, SG(x_2) = 0$ *or* $SG(x_1) = 0, SG(x_2) = 1$,
(4) ***Reflexivity:*** $dist_A(x_1, x_2) = 0$, *iff* $x_1 = x_2$.

Then, combining with the Eq. (18), the conditional probabilities of the object x_i can be obtained as follows:

$$\Pr\left(C\middle|[x_i]\right) = \frac{\left|[x_i] \cap C\right|}{\left|[x_i]\right|}, \Pr\left(\neg C\middle|[x_i]\right) = 1 - \Pr\left(C\middle|[x_i]\right). \tag{19}$$

4.2. Relative loss functions

The determination of the loss function is important in 3WD and selecting a suitable loss function will determine the accuracy of the decision result. However, it becomes evident that the loss values carried by different objects should be different, different selection leads the different loss values. As described in Example 5, when selecting the appropriate electric vehicle, the opportunity cost of selecting one vehicle over another differs for each individual. The loss functions of traditional 3WD is fixed [51], which implies that they have the same loss. This may lead to objects being misclassified. Based on Jia and Liu [27], the relative loss functions calculation method is proposed. To make the decision results more accurate, the concept of relative loss function needs to be extended into DHHFLIS. Suppose $h_{S_O}(x_{ik})$ be the DHHFLEs of the ith object under the kth conditional attribute. Table 3 is the relative loss functions of object x_i respect to the conditional attribute a_k. The relative loss satisfy $h^{\Upsilon_{PP}}_{S_O}(x_{ik}) < h^{\Upsilon_{BP}}_{S_O}(x_{ik}) < h^{\Upsilon_{NP}}_{S_O}(x_{ik})$ and $h^{\Upsilon_{NN}}_{S_O}(x_{ik}) < h^{\Upsilon_{BN}}_{S_O}(x_{ik}) < h^{\Upsilon_{PN}}_{S_O}(x_{ik})$.

A semantic interpretation of the relative loss functions in Table 3 is provided. In state C, when object x_i takes action $\dot{a}_P$, it implies that the gains under the conditional attribute a_k are fully accessible, signifying no loss, which denotes as $h^{\Upsilon_{PP}}_{S_O}(x_{ik}) = \left\{s_{-\tau\langle o_0\rangle}\right\}$. While taking action $\dot{a}_N$ leads to a complete loss of the gain under conditional attribute a_k, it corresponds to the relative loss $h^{\Upsilon_{NP}}_{S_O}(x_{ik})$ with the whole membership degree of x_{ik}. When taking action $\dot{a}_B$, the relative losses are determined based on risk preference and correspond to varying degrees of risk attitude. The parameter η signifies the loss degree arising from adopting non-commitment, acceptance, and rejection. With $\eta \in [0, 1]$, the loss generated by classifying the object into the boundary domain can be measured in terms of the attitude towards the loss. Similarly, in state $\neg C$, $h^{\Upsilon_{PN}}_{S_O}(x_{ik})$ represents the loss that is opposite to the value of object x_i associated with the conditional attribute a_k. It can be considered as the complement. Furthermore, the loss of $h^{\Upsilon_{BN}}_{S_O}(x_{ik})$ represents a partial loss of $h^{\Upsilon_{PN}}_{S_O}(x_{ik})$, and $h^{\Upsilon_{NN}}_{S_O}(x_{ik})$ represents the rejection of taking action results in no loss. If all the conditional attributes are independent, the relative loss functions with different attributes should be consolidated into a comprehensive relative loss function. To preserve the completeness of the information aggregation procedure, the Hamacher aggregation operator [47] of DHLTS is extended to the DHHFLTS. Given the relative loss functions of each object, operators should be used to aggregate them.

Definition 11. Let $h_{S_O}(x_{ik})$ $(i = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$ be the DHHFLEs of the ith object under the kth conditional attribute, and w_k be the conditional attributes' weight vector. The double hierarchy hesitant fuzzy linguistic Hamacher weighted averaging(DHHFLHWA) operator is mapping of $\Omega^m \to \Omega$. The operator is defined as follows:

$$DHHFLHWA\left(h_{S_O}(x_{i1}), h_{S_O}(x_{i2}), \ldots, h_{S_O}(x_{in})\right) = \oplus_{k=1}^n \left(w_k le\left(h_{S_O}(x_{ik})\right)\right), \tag{20}$$

where Ω denotes the set of all DHHFLEs.

In accordance with Definition 4 of operational rules, the result with Hamacher weighted averaging can be deduced.

Theorem 1. *Let $h_{S_O}(x_{ik})$ $(i = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$ be the DHHFLEs, then the Hamacher weighted averaging of DHHFLEs are as below,*

$$\begin{aligned} &DHHFLHWA\left(h_{S_O}(x_{i1}), h_{S_O}(x_{i2}), \ldots, h_{S_O}(x_{in})\right) \\ &= \oplus_{k=1}^n \left(w_k le\left(h_{S_O}(x_{ik})\right)\right) \\ &= F^{-1}\left(\frac{\prod_{k=1}^n \left(1+(\theta-1)\cdot F\left(le\left(h_{S_O}(x_{ik})\right)\right)\right)^{w_k} - \prod_{k=1}^n \left(1-F\left(le\left(h_{S_O}(x_{ik})\right)\right)\right)^{w_k}}{\prod_{k=1}^n \left(1+(\theta-1)\cdot F\left(le\left(h_{S_O}(x_{ik})\right)\right)\right)^{w_k} + (\theta-1)\prod_{k=1}^n \left(1-F\left(le\left(h_{S_O}(x_{ik})\right)\right)\right)^{w_k}}\right) \end{aligned} \tag{21}$$

where θ is the parameter of Hamacher t-norm and t-conorm.

Consequently, the comprehensive relative loss function for the ith object with all conditional attributes, can be obtained as demonstrated in Table 4. According to Eqs. (12)–(14), the expected loss $\Re\left(\dot{a}_\bullet\middle|[x_i]\right)$ ($\bullet$=P, B, N) can be recalculated as follows:

$$\Re\left(\dot{a}_P\middle|[x_i]\right) = SG\left(h^{\Upsilon_{PP}}_{S_O}(x_i)\right)\Pr\left(C\middle|[x_i]\right) + SG\left(h^{\Upsilon_{PN}}_{S_O}(x_i)\right)\Pr\left(\neg C\middle|[x_i]\right), \tag{22}$$

$$\Re\left(\dot{a}_B\middle|[x_i]\right) = SG\left(h^{\Upsilon_{BP}}_{S_O}(x_i)\right)\Pr\left(C\middle|[x_i]\right) + SG\left(h^{\Upsilon_{BN}}_{S_O}(x_i)\right)\Pr\left(\neg C\middle|[x_i]\right), \tag{23}$$

$$\Re\left(\dot{a}_N\middle|[x_i]\right) = SG\left(h^{\Upsilon_{NP}}_{S_O}(x_i)\right)\Pr\left(C\middle|[x_i]\right) + SG\left(h^{\Upsilon_{NN}}_{S_O}(x_i)\right)\Pr\left(\neg C\middle|[x_i]\right). \tag{24}$$

Following the Bayesian decision-making process, the updated decision rules can be derived as follows:

(P3) If $\Re\left(\dot{a}_P\middle|[x_i]\right) \le \Re\left(\dot{a}_B\middle|[x_i]\right)$, and $\Re\left(\dot{a}_P\middle|[x_i]\right) \le \Re\left(\dot{a}_N\middle|[x_i]\right)$, then $x_i \in POS(C)$,
(B3) If $\Re\left(\dot{a}_B\middle|[x_i]\right) \le \Re\left(\dot{a}_P\middle|[x_i]\right)$, and $\Re\left(\dot{a}_B\middle|[x_i]\right) \le \Re\left(\dot{a}_N\middle|[x_i]\right)$, then $x_i \in BND(C)$,
(N3) If $\Re\left(\dot{a}_N\middle|[x_i]\right) \le \Re\left(\dot{a}_P\middle|[x_i]\right)$, and $\Re\left(\dot{a}_N\middle|[x_i]\right) \le \Re\left(\dot{a}_P\middle|[x_i]\right)$, then $x_i \in NEG(C)$.

Table 4
The comprehensive relative loss function of DHHFLIS.

Actions	$C(P)$	$\neg C(N)$
$\dot{a}_P$	$h_{S_O}^{Y_{PP}}(x_i) = \oplus_{k=1}^{n}\left(w_k le\left(h_{S_O}^{Y_{PP}}(x_{ik})\right)\right)$	$h_{S_O}^{Y_{PN}}(x_i) = \oplus_{k=1}^{n}\left(w_k le\left(h_{S_O}^{Y_{PN}}(x_{ik})\right)\right)$
$\dot{a}_B$	$h_{S_O}^{Y_{BP}}(x_i) = \oplus_{k=1}^{n}\left(w_k le\left(h_{S_O}^{Y_{BP}}(x_{ik})\right)\right)$	$h_{S_O}^{Y_{BN}}(x_i) = \oplus_{k=1}^{n}\left(w_k le\left(h_{S_O}^{Y_{BN}}(x_{ik})\right)\right)$
$\dot{a}_N$	$h_{S_O}^{Y_{NP}}(x_i) = \oplus_{k=1}^{n}\left(w_k le\left(h_{S_O}^{Y_{NP}}(x_{ik})\right)\right)$	$h_{S_O}^{Y_{NN}}(x_i) = \oplus_{k=1}^{n}\left(w_k le\left(h_{S_O}^{Y_{NN}}(x_{ik})\right)\right)$

Table 5
The explanation of symbols.

Symbols	Description
S_O	A discrete DHLTS
$\hat{S}_O$	A continuous DHLTS
h_{s_o}	A DHHFLE
τ, ς	The first and second linguistic scale
t, k	The first and second discrete linguistic variables subscripts
ϕ_l, φ_l	The first and second continuous linguistic variables subscripts
$l = 1, 2, \dots, L$	The number of elements in DHHFLE
γ	The DHLT corresponds to the hesitant fuzzy values
$E\left(h_{s_o}\right)/v\left(h_{s_o}\right)$	The expect/variance value of DHHFLE
$le\left(h_{S_O}\right)$	The linguistic expected-value of DHHFLE
sg	The superior gradus of DHLTS
SG	The superior gradus of DHHFLE
θ	The parameter of Hamacher t-norm and t-conorm
$\Xi = \{C, \neg C\}$	The set of states, denoting that an object is in and not in state
$\Lambda = \{\dot{a}_P, \dot{a}_B, \dot{a}_N\}$	The set of actions
$\bullet$	The symbol of action ($\bullet$=P, B, N)
$Y_{PP}, Y_{BP}, Y_{NP}, Y_{PN}, Y_{BN}, Y_{NN}$	The loss function for executing actions when an object is in and not in state
$h_{S_O}^{Y_{PP}}, h_{S_O}^{Y_{BP}}, h_{S_O}^{Y_{NP}}, h_{S_O}^{Y_{PN}}, h_{S_O}^{Y_{BP}}, h_{S_O}^{Y_{NN}}$	The relative loss function for executing actions when an object is in and not in state
B_{δ_A}	The set of objects that satisfy a similarity measure less than a constant δ under conditional attributes A
$dist_A\left(x_i, x_j\right)$	The distance between different objects under conditional attributes A
$\Pr\left(C\middle\|\left[x_i\right]\right), \Pr\left(\neg C\middle\|\left[x_i\right]\right)$	The conditional probability that an object is in and not in state
$\Re\left(\dot{a}_\bullet\middle\|\left[x_i\right]\right)$	The expected loss of action with their equivalence class
u, v	The threshold values

Also, the decision rules (P3)-(N3) can be simplified as (P4)-(N4):

(P4) If $\Pr\left(C\middle|\left[x_i\right]\right) \geq u$, then decide $x_i \in POS(C)$,

(B4) If $v < \Pr\left(C\middle|\left[x_i\right]\right) < u$, then decide $x_i \in BND(C)$,

(N4) If $\Pr\left(C\middle|\left[x_i\right]\right) \leq v$, then decide $x_i \in NEG(C)$.

where the threshold value $u = \frac{SG\left(h_{S_O}^{Y_{PN}}(x_i)\right)-SG\left(h_{S_O}^{Y_{BN}}(x_i)\right)}{\left(SG\left(h_{S_O}^{Y_{PN}}(x_i)\right)-SG\left(h_{S_O}^{Y_{BN}}(x_i)\right)\right)+\left(SG\left(h_{S_O}^{Y_{BP}}(x_i)\right)-SG\left(h_{S_O}^{Y_{PP}}(x_i)\right)\right)}$ and $v = \frac{SG\left(h_{S_O}^{Y_{BN}}(x_i)\right)-SG\left(h_{S_O}^{Y_{NN}}(x_i)\right)}{\left(SG\left(h_{S_O}^{Y_{BN}}(x_i)\right)-SG\left(h_{S_O}^{Y_{NN}}(x_i)\right)\right)+\left(SG\left(h_{S_O}^{Y_{NP}}(x_i)\right)-SG\left(h_{S_O}^{Y_{BP}}(x_i)\right)\right)}$.

According to (P4)-(N4), Objects classified into $POS(C)$, $BND(C)$, and $NEG(C)$. These three regions will have differing semantics depending on the scenario. For example, a disease diagnosis scenario (using benefit attributes) implies a confirmed patient in the positive domain, a further examination in the boundary domain, and an unconfirmed diagnosis in the negative domain. Based on the rules (P3)-(N3), then differently divided regions express the meaning of $POS(C)$ are considered more likely to occur to those in $BND(C)$, and those in $BND(C)$ are considered more likely to occur to those in $NEG(C)$.

This section involves many symbols, most of which are summarized in Table 5 for clarity.

5. The processing of the novel 3MADM method

Section 5.1 provides a brief description of the MADM problem under the DHHFLIS. Section 5.2 shows the processing and the algorithm of the novel 3MAMD method.

5.1. Problem description

In the set $\{IS, LF\}$, there are m objects $U = \{x_1, \dots, x_m\}$, n conditional attributes $AT = \{a_1, \dots, a_n\}$, one decision attribute $D = \{d\}$, and the weight of each conditional attribute a_k is w_k. These attribute weights denote their importance levels, satisfying $\sum_{k=1}^{n} w_k = 1$, with $0 \leq w_k \leq 1$, $k = 1, 2, \dots, n$. The ith object's evaluation value for the kth conditional attribute is $h_{S_O}\left(x_{ik}\right)$, and the values $h_{S_O}\left(x_{ik}\right)$ of all objects under the all conditional attributes constitute a evaluation matrix H_{S_O}. Due to their diverse characteristics, the conditional attributes in DHHFLIS can encompass both cost and benefit attributes with varied dimensions and scales. Negative indicators are employed to normalize measurements and prevent inconsistencies. The method [53] can be employed, as follows:

$$H_{S_O} = \begin{cases} h_{S_O}\left(x_{ik}\right)_{m\times n}, \text{for benefit} - \text{type} \\ \bar{h}_{S_O}\left(x_{ik}\right)_{m\times n}, \text{for cost} - \text{type} \end{cases}, \bar{h}_{S_O} = \bigcup_{s_{\phi_l\left\langle o_{\varphi_l}\right\rangle} \in h_{S_O}} \left\{ s_{-\phi_l\left\langle o_{-\varphi_l}\right\rangle} \right\}. \tag{25}$$

The detailed calculation process is illustrated in Fig. 4.

5.2. Algorithm

The logical connection between each step is vividly demonstrated through the depiction of Algorithm 1.

Remark 1. **Step 1** checks the type of conditional attributes and carries a complexity of $O(1)$. In **Step 2**, the complexity of normalizing for each $a_k \in AT$ is $O(nm)$, n is the number of conditional attributes and m is the number of objects. In **Step 3**, the complexity associated with the computation of distances is $O(m)$. The complexity of determining the conditional probabilities in **Step 4** is $O(m)$. In **Step 5**, the complexity of the relative loss functions for each object is $O\left(nm^2\right)$. In **Step 6**,

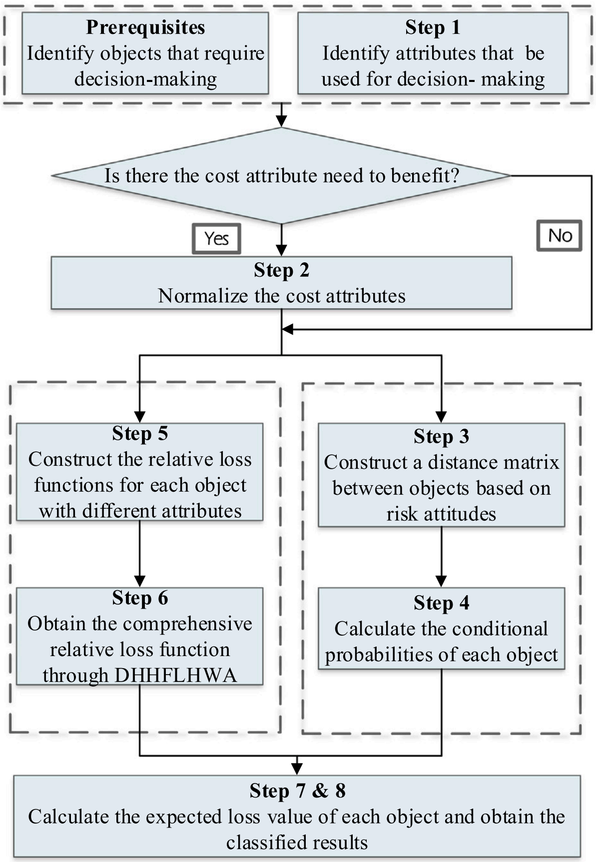


Fig. 4. The specific calculation of processing.

Algorithm 1 The algorithm of the novel 3MADM method

Input: $IS = \{U, AT \cup D, V, h_{S_O}\}$, the first hierarchy linguistic scale τ, the second hierarchy linguistic scale ς, the parameters θ, η, δ.
Output: The decision results of each objects $x_i\,(i = 1, \ldots, m)$.

```
Firstly, check the type of conditional attributes.
for a_k ∈ AT do
    Normalize H_{S_O}(x_{ik}) (i = 1, 2, ..., m; k = 1, 2, ..., n).
end for
for x_i ∈ U do
    Compute the distance by Definition 6 and Eq. (18).
end for
for x_i ∈ U do
    Calculate the conditional probabilities based on the Eqs.(17)(19).
end for
for x_i ∈ U do
    Aggregate the comprehensive relative loss function by the Eqs.(7)(20).
end for
for x_i ∈ U do
    Compute the expect loss ℜ by the Eqs.(22)–(24).
end for
for x_i ∈ U do
    Decision-making through (P4)-(N4).
    if POS(C) then
        x_i ⇒ á_P.
    end if
    if BND(C) then
        x_i ⇒ á_B.
    end if
    if NEG(C) then
        x_i ⇒ á_N.
    end if
end for
return The decision results of all objects.
```

the aggregated of the comprehensive relative loss function complexity is $O\left(m^2\right)$. In **Step 7**, the complexity of computing the expected loss is $O(n)$. Lastly, in **Step 8**, the complexity is $O(m)$. Consequently, the complexity of the novel 3MADM method under the DHHFLIS with is $O\left(nm^2\right)$.

6. Case study

In this section, the novel 3MADM method is applied to a practical case and present its final ranking and classification results in charts and table.

6.1. Case background

As China's population ages and the number of patients with chronic conditions rises, the demand for quality healthcare and rehabilitation services has increased significantly. It is projected that by 2030, the prevalence of chronic diseases in China will reach 65.7% and affect younger individuals. 80% of patients with chronic diseases will require nursing care and rehabilitation to alleviate their symptoms. Traditional Chinese Medicine (TCM) has significant advantages in managing chronic diseases. It plays an important role in maintaining population health through its clinical efficacy, flexible treatment methods, and a focus on preventive care. Community health centers provide essential services in preventing and treating chronic diseases, including disease prevention, health management, medical treatment, rehabilitation, and health education. The number of TCM Rehabilitation Clinics Centers (TCMRCC) in community health services are growing, but there is a lack of standardization in service quality and scientific management. This impedes the clinics' capacity to address the rehabilitation needs of patients with chronic conditions. Enhancing service quality, balancing resource allocation, and strengthening healthcare professionals' rehabilitation and physical therapy skills are crucial for the improvement of clinic services. The quality of clinic services is mainly evaluated based on the conditional attributes listed in Table 6.

Q Community needs to assess some services of TCMRCC. The objects of clinics are $U = \{x_1, x_2, x_3, x_4, x_5\}$. The conditional attributes in Table 7 are noted as $A = \{a_1, a_2, a_3, a_4, a_5\}$. Suppose the weights vector of conditional attribute is $w = \{0.2, 0.2, 0.2, 0.2, 0.2\}$, the DHLTS S_O with the first hierarchy LTS scale is $S = \{s_{-4} = terrible, s_{-3} = very\ bad,\ s_{-2} = bad, s_{-1} = slightly\ bad, s_0 = ordinary, s_1 = slightly\ good, s_2 = good, s_3 = very\ good, s_4 = perfect\}$, and all the second hierarchy LTS scales are the same as $O = \{o_{-4} = far\ from, o_{-3} = scarcely, o_{-2} = only\ a\ little, o_{-1} = a\ little, o_0 = just\ right, o_1 = much, o_2 = very\ much, o_3 = extremely\ much, o_4 = entirely\}$. After several experts in this field evaluated the five TCMRCC, the evaluation information is organized and assembled into Table 7. Decision attribute d indicates the experts' prediction of meeting the service standards of these clinics: "1" is the achievement of standards, "0" is not up to standard, corresponding $\Xi = \{C, \neg C\}$ and $V_d = \{0, 1\}$. In this case, condition state is $C = \{x_1, x_3, x_5\}$. Table 8 is the comprehensive loss functions of object x_i.

6.2. The novel 3MADM method for the case under the DHHFLIS

Under the DHHFLIS, some necessary settings need to be given first. In this case study, all the conditional attributes are benefit-type, so normalizing attribute values is not necessary. Skip the Step 2 and go to the Step 3. According to the explanation of Situation 2 in Section 3.2, the distance results between two different of distance results between objects can be calculated according to the optimistic attitude and the pessimistic attitude by Eq. (18), as shown in Table 9. Also, the process instructions in Steps 5 and 6, the loss degree parameter η is set to 0.5.

In order to get Table 8, according to Eq. (21), it is necessary to assume that the parameter θ is 2. Different risk attitudes are distinguished by assuming the parameter δ of 0.15 for optimistic attitude and 0.1 for pessimistic attitude. As a result, the expected loss values for

Table 6
The explanations of conditional attributes.

Attributes	Explanations
a_1 The Equipments and facilities	Tangibility, including the clinic's facilities,equipments, medical environment, medical procedures, etc.
a_2 The skill of medical staff	Responsibility, including the ability of difficulty help, emergency ability, ability of operation, etc.
a_3 The human resources of health	Guaranteed performance, including the number of doctors and nurses, medical ethics, service attitude, professional training, etc.
a_4 The concept of Chronic disease health management	Professional, including care for the patient, familiar the patient's condition, consider the patient's interests, protect patients privacy, etc.
a_5 The efficiency of service	Reactivity, including waiting time, complaint handling, regular follow-up, etc.

Note: The conditional attributes are set with the model of SERVICE QUALITY [54]. In this case, the conditional attributes are all benefit-type, i.e., the greater the rating, the better the service.

Table 7
The evaluation of TCMRCC.

	c_1	c_2	c_3	c_4	c_5	d
x_1	$\{s_{1\langle o_{-2}\rangle}, s_{2\langle o_2\rangle}, s_{3\langle o_{-2}\rangle}\}$	$\{s_{1\langle o_{-1}\rangle}, s_{3\langle o_{-1}\rangle}, s_{2\langle o_1\rangle}\}$	$\{s_{1\langle o_2\rangle}, s_{0\langle o_{-1}\rangle}, s_{-1\langle o_2\rangle}\}$	$\{s_{2\langle o_2\rangle}, s_{3\langle o_1\rangle}, s_{2\langle o_1\rangle}\}$	$\{s_{-2\langle o_2\rangle}, s_{0\langle o_2\rangle}, s_{-1\langle o_2\rangle}\}$	1
x_2	$\{s_{2\langle o_{-1}\rangle}, s_{1\langle o_2\rangle}, s_{0\langle o_2\rangle}\}$	$\{s_{3\langle o_1\rangle}, s_{3\langle o_0\rangle}, s_{2\langle o_3\rangle}\}$	$\{s_{1\langle o_2\rangle}, s_{0\langle o_{-1}\rangle}, s_{-1\langle o_2\rangle}\}$	$\{s_{2\langle o_2\rangle}, s_{3\langle o_1\rangle}, s_{2\langle o_1\rangle}\}$	$\{s_{-3\langle o_2\rangle}, s_{-2\langle o_1\rangle}, s_{-1\langle o_{-1}\rangle}\}$	0
x_3	$\{s_{2\langle o_{-1}\rangle}, s_{1\langle o_2\rangle}, s_{1\langle o_{-2}\rangle}\}$	$\{s_{3\langle o_2\rangle}, s_{2\langle o_{-2}\rangle}\}$	$\{s_{0\langle o_2\rangle}, s_{-2\langle o_3\rangle}, s_{-2\langle o_2\rangle}\}$	$\{s_{1\langle o_2\rangle}, s_{3\langle o_0\rangle}, s_{2\langle o_3\rangle}\}$	$\{s_{-1\langle o_0\rangle}, s_{-2\langle o_2\rangle}\}$	1
x_4	$\{s_{3\langle o_1\rangle}, s_{2\langle o_{-1}\rangle}, s_{3\langle o_{-2}\rangle}\}$	$\{s_{2\langle o_{-1}\rangle}, s_{1\langle o_1\rangle}\}$	$\{s_{-1\langle o_2\rangle}, s_{-1\langle o_1\rangle}, s_{-2\langle o_3\rangle}\}$	$\{s_{0\langle o_{-2}\rangle}, s_{1\langle o_1\rangle}, s_{0\langle o_3\rangle}\}$	$\{s_{0\langle o_{-1}\rangle}, s_{-1\langle o_2\rangle}, s_{-2\langle o_{-2}\rangle}\}$	0
x_5	$\{s_{2\langle o_0\rangle}, s_{2\langle o_{-1}\rangle}, s_{1\langle o_2\rangle}\}$	$\{s_{2\langle o_{-3}\rangle}, s_{2\langle o_2\rangle}, s_{3\langle o_{-1}\rangle}\}$	$\{s_{2\langle o_{-1}\rangle}, s_{1\langle o_2\rangle}\}$	$\{s_{2\langle o_3\rangle}, s_{3\langle o_{-2}\rangle}, s_{3\langle o_{-3}\rangle}\}$	$\{s_{1\langle o_{-2}\rangle}, s_{0\langle o_{-1}\rangle}, s_{-1\langle o_2\rangle}\}$	1

Table 8
The comprehensive relative loss functions of each objects.

	x_1	x_2	x_3	x_4	x_5
$\dot{a}_P\_C$	0	0	0	0	0
$\dot{a}_B\_C$	0.204	0.183	0.188	0.166	0.222
$\dot{a}_N\_C$	0.679	0.630	0.643	0.594	0.713
$\dot{a}_P\_\neg C$	0.355	0.447	0.417	0.451	0.306
$\dot{a}_B\_\neg C$	0.086	0.114	0.105	0.115	0.071
$\dot{a}_N\_\neg C$	0	0	0	0	0

each clincs with different risk attitudes can be calculated. To provide a clearer representation of the expected losses for these objects, the results are illustrated in Fig. 5. According to the decision rules (P4)-(N4), the classified result for the five clincs can be obtained with an/a optimistic/pessimistic risk attitude as $POS(C) = \{x_1, x_5\}$, $BND(C) = \{x_2, x_3\}$, $NEG(C) = \{x_4\}$. Here the same classification results for optimism and pessimism are caused by the fact that there are only five objects during calculation and the resulting numerical intervals are not very different. However, according to the calculation results, the ranking result of these five clinics can still be obtained with an optimistic risk attitude as $x_5 \succ x_1 \succ x_3 \succ x_2 \succ x_4$ and with a pessimistic risk attitude as $x_5 \approx x_1 \succ x_3 \succ x_2 \succ x_4$. With the optimistic attitude, since the expected loss value of x_5 is relatively the smallest, x_5 has the highest quality of service among all chronic disease rehabilitation clinics in the community. With the pessimistic attitude, cause x_1 and x_5 have the same expected loss value, x_1 and x_5 become the two clinics with high-quality service from the results. For x_2 and x_3, whose expected loss value is at the average level, there are still a lack of some useful evaluation information. That is to say, these clinics need to continue to improve their service capabilities for the clinic. In addition, x_4 does not meet qualified standards requires strengthening supervision.

7. Comparative analysis

In the previous section the novel 3MADM method is used to solve a practical case. In this section, on the one hand, the performance of the novel method is verified through comparison of TCMRCC service assessment and Breast Cancer Coimbra Data Set(BCCD), and on the other hand, the feasibility and effectiveness of the novel method are verified through comparison of ranking and classification. At the end, the similarities and differences between this novel method and other decision-making methods are analyzed, and the advantages of this novel method are summarized.

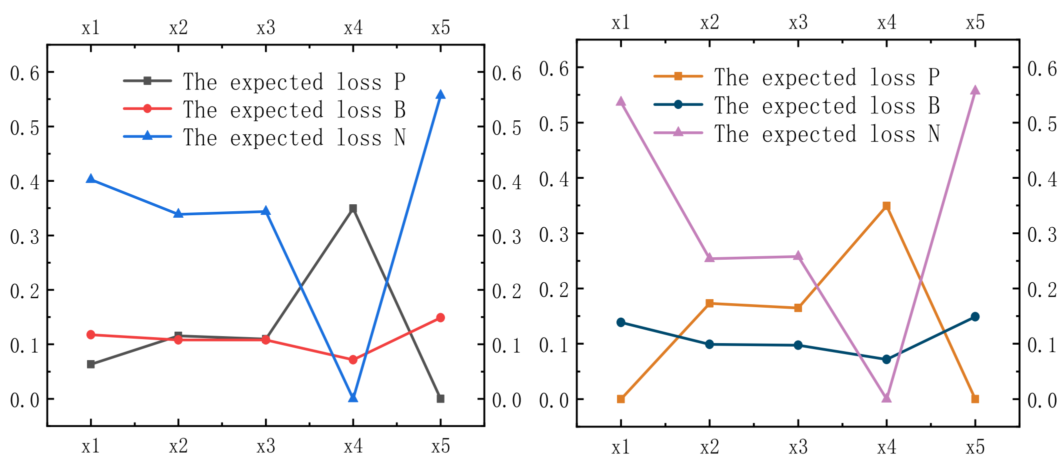
7.1. Comparative analysis of the TCMRCC service assessment

The service assessment results of the TCMRCC are utilized to compare the ranking results with those existed methods, in order to verify the consistency of the results produced by the proposed method. For the purpose of demonstrating the validity of the novel 3MADM method, several classical MADM methods under the DHHFLIS are used to comparing, including Gou et al.'s [10,11,55] and Liu et al.'s [12,14] methods. Furthermore, the reasonableness of the proposed method is justified by comparing the results with Li et al.'s methods [46,47]. The parameters involved in the method of this paper remain the same as Section 6.1. The corresponding parameters mentioned in other literatures also remain unchanged. The ranking results of the service evaluation and the most standardized clinic(s) derived from different methods are shown in Table 10.

Table 10 shows the rankings and selected objects derived from various methods, including the proposed method with optimistic and pessimistic attitudes. A closer examination of these results yields several observations. This shows that the results calculated by the method of this paper are consistent in ranking with the results of the classical method. The proposed method distinguishes itself by considering different decision-making attitudes. With optimistic attitude, x_5 distinctly perform better than other clinics. However, both x_5 and x_1 emerge as front runners with pessimistic attitude, where suggest that these two clinics are deemed preferable when potential risks or unfavorable outcomes are emphasized. Although the rankings for most methods are consistent, Liu et al.'s method [12] presents a distinct ordering for x_2 and x_3. This difference may arise from unique considerations or perspectives incorporated into their approaches. While classical MADM methods effectively rank the clinics, they fail to classify them into different categories or grades. The proposed method fills this gap by not only ranking but also classifying the clinics based on specific decision-making attitudes. Above the analysis, the proposed method offers a richer understanding and nuanced ranking. Specifically, the distinction

Table 9
The distance and conditional probabilities of objects.

Objects	x_1		x_2		x_3		x_4		x_5		Conditional probabilities	
	Opt	Pes	Opt	Pes	Opt	Pes	Opt	Pes	Opt	Pes	Opt	Pes
x_1	0	0	0.137	0.137	0.110	0.125	0.167	0.175	0.116	0.114	0.75	1
x_2	0.137	0.137	0	0	0.072	0.092	0.156	0.164	0.198	0.193	0.67	0.5
x_3	0.110	0.128	0.072	0.002	0	0	0.172	0.174	0.177	0.182	0.67	0.5
x_4	0.167	0.175	0.156	0.164	0.172	0.174	0	0	0.228	0.228	0	0
x_5	0.116	0.114	0.198	0.193	0.177	0.182	0.228	0.228	0	0	1	1



(a) Optimistic risk attitude.

(b) Pessimistic risk attitude.

Fig. 5. The expected loss of $\Re\left(\dot{a}_{\bullet}\middle|[x_i]\right)$ with different risk attitudes.

Table 10
Rankings and the selected object(s) from different methods.

Methods	Ranking results	The selected object(s)
Proposed method(optimistic attitude)	$x_5 \succ x_1 \succ x_3 \succ x_2 \succ x_4$	x_5
Proposed method(pessimistic attitude)	$x_5 \approx x_1 \succ x_3 \succ x_2 \succ x_4$	x_5, x_1
Gou et al.'s method [55]	$x_5 \succ x_1 \succ x_3 \succ x_2 \succ x_4$	x_5
Liu et al.'s method [14]	$x_5 \succ x_1 \succ x_3 \succ x_2 \succ x_4$	x_5
Gou et al.'s method [11]	$x_5 \succ x_1 \succ x_3 \succ x_2 \succ x_4$	x_5
Liu et al.'s method [12]	$x_5 \succ x_1 \succ x_2 \succ x_3 \succ x_4$	x_5
Gou et al.'s method [10]	$x_5 \succ x_1 \succ x_3 \succ x_2 \succ x_4$	x_5

Table 11
Rankings and the selected object(s) of alternatives.

Methods	Regions	Ranking results	The selected object(s)
Proposed method(optimistic attitude)	POS(C) BND(C) NEG(C)	$x_5 \succ x_1$ $x_3 \succ x_2$ x_4	x_5
Proposed method(pessimistic attitude)	POS(C) BND(C) NEG(C)	$x_5 \approx x_1$ $x_3 \succ x_2$ x_4	x_5, x_1
Li et al.'s method [46]	POS(C) BND(C) NEG(C)	$x_5 \succ x_1$ $x_4 \succ x_2 \succ x_3$ $\emptyset$	x_5
Li et al.'s method [47]	POS(C) BND(C) NEG(C)	x_5 $x_4 \succ x_2 \succ x_3 \succ x_1$ $\emptyset$	x_5

between optimistic and pessimistic attitudes allows DMs to understand the impact of their inherent biases or risk preferences on the final ranking. Fig. 6 visually represents these rankings, providing a clearer view of the relative standing of each clinic across the various methods.

The methods compared in Table 11 use the same qualitative linguistic of expression as the methods in this paper. Comparative Analysis with Li et al.'s Methods: Li et al.'s method [46] agrees with optimistic attitude in the POS(C) region by also preferring x_5 over x_1. However, in the BND(C) region, this method exhibits a distinct ranking, placing x_4 ahead of x_2 and x_3. Additionally, the NEG(C) region has no selection for this method. Li et al.'s method [47] offers a more straightforward ranking in the POS(C) region by solely preferring x_5. However, it presents a markedly different ordering in the BND(C) region, with x_4 leading followed by x_2, x_3, and x_1, showing a significant variation from the proposed method. Both Li et al.'s methods have no selection in the NEG(C) region. This absence could suggest that these methods might

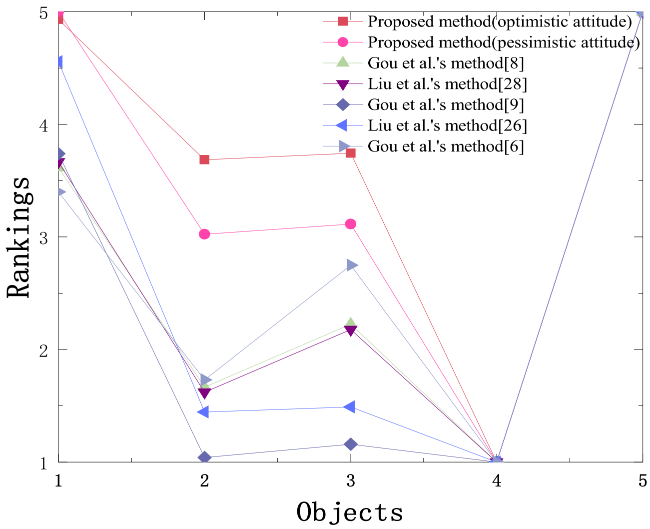


Fig. 6. The ranking results of service assessment with different methods.

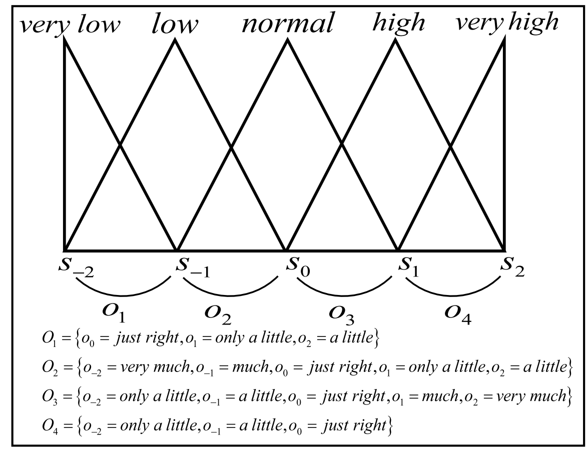


Fig. 7. Distributions of the four parts of the second hierarchy LTSs.

not consider any alternative suitable or might not be equipped to handle this particular region effectively. The proposed method incorporates both optimism and pessimism, providing a range of possibilities for DMs. An optimistic attitude depicts a more risk-taking behavior, while a pessimistic attitude favors caution.

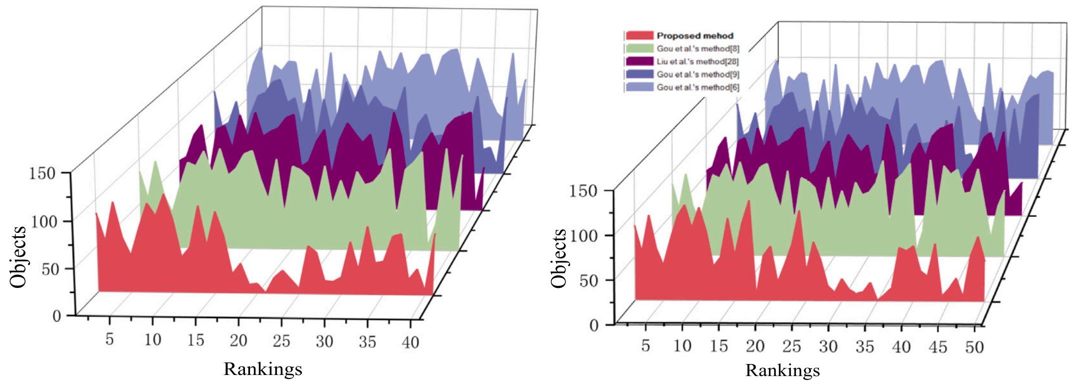
7.2. Comparative analysis of BCCD

To further demonstrate the feasibility and effectiveness of the method, the Breast Cancer Coimbra Data Set (https://archive.ics.uci.edu/datasets) is introduced. The purpose of using this data set is to prove that the novel method has better performance than the classical methods when processing more data. It is also to prove that the classification results of the proposed method can be displayed more intuitively than the ranking results in the case of more data.

This data set does not yet contain estimated values from DHHFLTS and therefore needs to be transformed based on the Eq. (3)(4) before applying the novel 3MADM method. It includes 116 objects and 10 attributes, with nine conditional attributes and 1 decision attribute. Firstly, it is assumed that conditional attributes must have mutually independent properties. Four relevant conditional attributes (Age, HOMA, Resistin, and MCP.1.) are removed. Secondly, several medical experts are invited to convert the crisp numbers of conditional attributes (BMI, Glucose, Insulin, Leptin, and Adiponectin) to DHHFLTS based on the setting of Fig. 7. Thirdly, the DHLTSs are aggregated according to Eq. (20). Suppose all the conditional attributes are of the benefit-type, the decision attributes in the data set include labels “1” represents healthy controls and “2” represents patients. After preprocessing the data, DHHFLIS could be get and be applied to the novel 3MADM method, using Algorithm 1.

Similarly, the parameter settings are the same as above. In addition, since the loss functions in Li et al.’s method is subjectively given and cannot be used for this data set, the relative loss functions proposed in this paper is used to calculate the loss functions during the calculation process. In order to present scientific and reasonable experimental analysis, it is necessary to provide some semantic explanations for the proposed methods and the comparative methods. As can be seen from Fig. 7, the corresponding meaning of the designed linguistic scale is that the higher the value of the linguistic scale, the greater the probability of disease. Take the conditional attribute BMI as an example, the greater the value, the greater the possibility of obesity. It can be seen from public information inquiry, research has found that obesity increases the likelihood of breast cancer. Since obesity may lead to changes in hormone levels, especially increased levels of estrogen, which increases the risk of breast cancer. And obese people have a higher risk of breast cancer than people of normal weight. According to the above explanation, the classic MADM method of obtaining ranking results based on the object’s score can be understood as the higher the object’s score, the higher the possibility of disease. In order to better identify differences, the comparison here focuses on the selected “patients”. Due to the varying number of objects in the positive, boundary, and negative domains between optimistic and pessimistic attitudes, it is crucial to gather an equal number of objects according to the number of negative domains for each risk attitude to facilitate a comparative analysis. According to the optimistic attitude, there are 41 objects categorized to POS(C) and 50 objects categorized to POS(C) with the pessimistic attitude. The following analysis mainly obtains comparison objects based on these two sets of numbers, which are used to compare the correctness of the analysis object selection with the classical MADM methods.

The ranking results of the selected “patients” are displayed in Fig. 8. Although there are some differences in the ranking results, it can still be seen from the figure that most methods show similar ranking trends. However, the proposed method caused significant changes in rankings. Since the computational methods designed are more sensitive to linguistic variables. In other words, when the amount of data is large, the possibility of whether the performance of these methods is still suitable decreases. In contrast, the classical methods applying Eqs. (1)(3) are relatively insensitive to changes in certain linguistic variables, resulting in less obvious fluctuations and interval differences compared with the proposed method. Also the classification results using the 3WD method are presented in Fig. 9. It shows that compared with other methods, the proposed method is more sensitive to the risk attitude of DMs and effectively achieves a clear division of the three classification areas. With the pessimistic attitude, the number of objects included in NEC(C) is larger than the number of objects included in POS(C) with the optimistic attitude. The characteristics presented indicate more cautious and conservative decision-making results. With the optimistic attitude, the number of objects in BND(C) is greater than the number of objects in BND(C) with the pessimistic attitude. Since some conditional attributes are deleted during data processing, resulting in the loss of some relevant important information, and sufficient information needs to be added to obtain more accurate classification results. Although the methods of Li et al. [46,47] are all 3MADM frameworks, they contain fewer patients in the negative domain, making it difficult to make reasonable decisions, resulting in a discount in the efficiency of decision-making.



(a) Optimistic risk attitude. (b) Pessimistic risk attitude.

Fig. 8. The ranking results of the selected "patients".

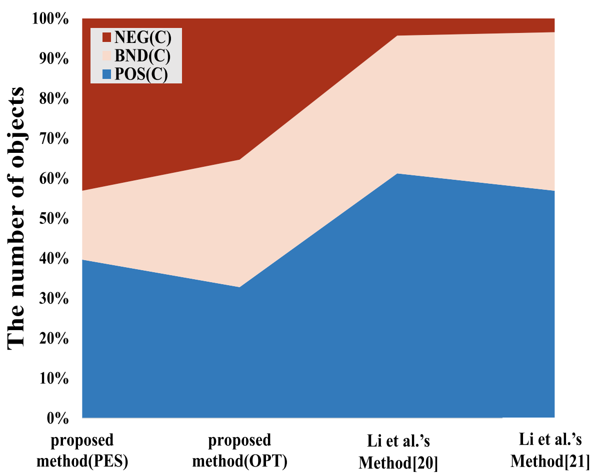


Fig. 9. The comparison of classification with 3WD methods.

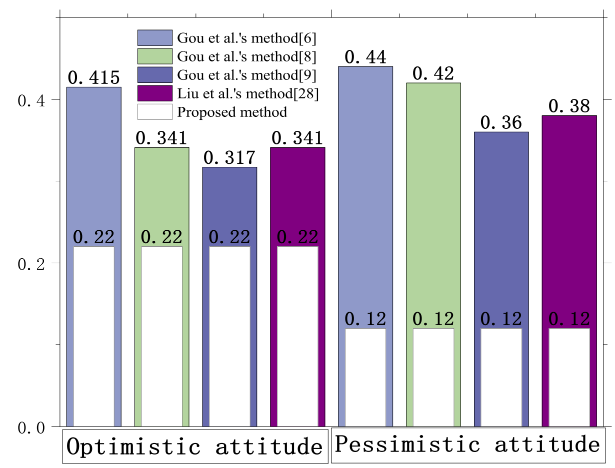


Fig. 10. The ER for different methods.

Moreover, as the objects in this data set are labeled, it is feasible to determine whether the "patients" chosen by these methods are accurate. To verify whether the results selected by these methods are reliable, the ER is the error rate [56] of diagnosis is used to measure the correctness of the "patient" diagnosis. A lower value of ER signifies higher diagnostic accuracy of the method. Fig. 10 reveals that regardless of whether it is optimistic or pessimistic, the method in this paper shows a lower ER. The ER of optimism is significantly higher than that pessimism. This situation reflects the difference of DMs' cautious judgment in the two states, which is consistent with the actual situation. The other four methods are all classical MADM and exhibit relatively high ER. There may be two reasons. One is the performance of the classical MADM decreases when calculating more data. The other one is the methods lack the target concept and do not effectively utilize a priori information. The proposed method surpasses conventional MADM methods in terms of patient diagnostic accuracy. Not only does this method effectively categorize patients, it also minimizes losses during the decision-making process.

7.3. Discussion

In this paper, the novel 3MADM method is applied under the DHHFLIS to solve decision problems with two kinds of data sets and the practicality is demonstrated. Simultaneously, the validity, reasonableness, and effectiveness of the proposed method for ranking and classification have been confirmed. Next, the differences between the novel 3MADM method and other methods will be analyzed, and its advantages will be summarized. From Table 12, these differences can be seen clearly.

The novel 3MADM method proposed in this paper stands out from others in several significant ways:

(1) The proposed 3MADM method uniquely segments objects into positive, boundary, and negative domains, unlike the classical MADM methods.
(2) The proposed method of estimating conditional probabilities under DHHFLIS utilizes a distance measure based on superior gradus. This contrasts with Li et al.'s [46,47] methods through subjective way where the DM directly provides conditional probabilities.
(3) The relative loss functions are designed through objective method. This method tackles a problem that remains unresolved in other methods [46,47], where loss functions are often acquired through a subjective way.
(4) In the comparative analysis, all methods are used to address the MADM problems of DHHFLTS. But there are some differences in the calculation process. Except for the proposed method which uses superior gradus as the compared method, all other methods base their calculations on Eqs. (1)(3).

The advantages of the proposed method are listed as blow:

Table 12
The difference between the proposed method and other methods.

Methods	Conditional probabilities	Loss functions	Rankings	Classifications
Liu et al.'s method [14]	None	None	Yes	No
Gou et al.'s method [11]	None	None	Yes	No
Liu et al.'s method [12]	None	None	Yes	No
Gou et al.'s method [10]	None	None	Yes	No
Gou et al.'s method [55]	None	None	Yes	No
Li et al.'s method [47]	Objective	Subjective	Yes	Yes
Li et al.'s method [46]	Objective	Subjective	Yes	Yes
Proposed method	Objective	Objective	Yes	Yes

(1) It introduces a novel 3MADM method under DHHFLIS that provides two ranking and classification results compared to the method of two-way decision-making.
(2) Distance measurements based on the superior gradus better characterize the differences between the two types of DHHFLEs. This distance which extends Xu and Xia's [52] approach by considering the number of DHLTS for different risk attitudes, allows Eq. (18) to produce different results depending on risk attitude.
(3) This paper adopts the model of the relative loss functions [27], generating the relative loss functions for each object with conditional attributes. The extended Hamacher's weighted averaging operator enhances result reliability and objectivity.
(4) The novel 3MADM method reduces the subjective impact on decision-making results and further diminishes decision-making risk by conditional probabilities-based on superior gradus and relative loss functions while acknowledging differences.

8. Conclusion and future work

This paper introduces a novel 3MADM method under the DHHFLIS. Through comparative analysis of other methods using service assessment and the BCCD with different risk attitudes, the validity, reasonableness, and effectiveness of this novel method are demonstrated. In summary, the proposed method offers the following contributions:

(1) The novel compared methods based on the superior gradus make better distinguish between different DHLTSs or DHHFLEs.
(2) The new way of measuring distance based on the superior gradus concept can characterize the distances between different objects, and make the conditional probabilities calculations more objective.
(3) The relative loss functions calculation method under the DHHFLIS is supplemented, and make the calculation process more convincing.
(4) This paper offers a comprehensive assessment of 3MADM, including the service assessment and the BCCD comparative analyses.

The TAO framework is an effective tool based on the human cognitive process and raised to the philosophical level to deal with uncertain decision-making problems. In the future, due to the uncertainty characteristics of the MADM problem, research on the MADM problem and the TAO framework will also become a trend. Some topics with the TAO-MADM method under the DHHFLIS are worth investigating. Firstly, combining MAGDM [57] using into three-way group conflict analysis model [58] is helpful. Secondly, the weight of the DMs and the corresponding consensus [59] can be studied. Thirdly, the application of sequential 3WD [60] under the DHHFLIS can be considered.

CRediT authorship contribution statement

Nanfang Luo: Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Qinghua Zhang:** Writing – review & editing, Supervision, Resources, Project administration, Methodology, Funding acquisition, Conceptualization. **Longjun Yin:** Writing – review & editing, Resources, Formal analysis. **Qin Xie:** Writing – review & editing. **Chengying Wu:** Writing – review & editing, Resources. **Guoyin Wang:** Supervision, Project administration, Funding acquisition.

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

We confirm that we have given due consideration to the protection of intellectual property associated with this work and that there are no impediments to publication, including the timing of publication, with respect to intellectual property. In so doing we confirm that we have followed the regulations of our institutions concerning intellectual property.

We understand that the Corresponding Author is the sole contact for the Editorial process (including Editorial Manager and direct communications with the office). He is responsible for communicating with the other authors about progress, submissions of revisions and final approval of proofs.

Data availability

Data will be made available on request.

Acknowledgments

This work was supported by the National Key Research and Development Program of China (No. 2021YFF0704101), the National Natural Science Foundation of China (No. 62276038, No. 62221005), the Joint Fund of Chongqing Natural Science Foundation for Innovation and Development under Grant (No. CSTB2023NSCQ-LZX0164), the Chongqing Talent Program (No.CQYC20210202215), the Chongqing Municipal Education Commission (HZ2021008), the Doctoral Talent Training Program of Chongqing University of Posts and Telecommunications (No. BYJS202213), and the Graduate Scientific Research Innovation Project of Chongqing (No. CYB22244).

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