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Incremental updating fuzzy tolerance rough set approach in intuitionistic fuzzy information systems with fuzzy decision

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ABSTRACT

As information technology develops rapidly and data is constantly updated, efficient mining knowledge from dynamic intuitionistic fuzzy information systems with fuzzy decision (IFFD) is a significant topic. In the dynamic data environment, the corresponding IFFD is always changing with time when object sets of datasets may evolve in time. Fuzzy tolerance rough set (FTRS), as one of extended rough set models, has a strong capacity for the expression of information and better representation of uncertainty. Therefore, it is very effective and necessary to use FTRS to acquire valuable knowledge from dynamic IFFD. In this paper, we investigate the dynamic fuzzy tolerance rough set approach for IFFD. Firstly, a new fuzzy tolerance similarity is defined to describe fuzzy tolerance relation between objects in the IFFD. Second, we construct a fuzzy tolerance rough set model in the IFFD, discuss some properties, and propose the corresponding static algorithm. Subsequently, incremental approaches which update fuzzy tolerance rough approximations with the insertion and deletion of objects in the IFFD are investigated and the corresponding dynamic algorithms are also designed. Finally, the feasibility of the FTRS model and the effectiveness and efficiency of the dynamic algorithm in dynamic environments are validated through a series of comparative experiments on nine datasets.

1. Introduction

Rough set theory (RST) proposed by Pawlak [1] is a completely data-driven approach to efficiently handle inconsistent and uncertain information which has been widely applied in information fusion [2], feature selection [3], pattern recognition [4], decision making [5] and other fields. However, the original rough set model cannot directly handle real-valued data. In order to resolve this problem, many researchers proposed a variety of extended rough set models [6–24], where two significant models are fuzzy rough sets [25] and tolerance rough set theory [26,27]. Jensen [28] investigated fuzzy rough sets and tolerance rough sets for avoiding information loss and retaining dataset semantics which can be employed in the task of forensic glass fragment identification. Jrvinen [29] considered that any regular Kleene algebra defined on an algebraic lattice and the rough set Kleene algebra induced by a tolerance associated with irredundant covering are isomorphic. Yan [30] improved the traditional tolerance rough set model and proposed a model for computing sentence similarity associated with the probabilistic tolerance rough set in the view of uncertainty of textual data. Wan [31] defined the constrained tolerance relation based on the matching degree of objects and presented the constrained tolerance rough set under incomplete environment for describing more precise structure of an object class.

As a generalized model of the fuzzy set, intuitionistic fuzzy sets [32] have strong information representation capabilities by considering the membership, non-membership and hesitancy degree of the object simultaneously. Therefore, there exists better ability to control uncertainty and observe the ambiguity of the unstable real-world when data can be tackled by non-membership and hesitancy degree. In view of the properties of intuitionistic fuzzy set and tolerance rough set, a new fusion model can be constructed by Tiwari [33] who investigated intuitionistic fuzzy-rough set based on tolerance degree. By combining the concept of rough set and tolerance associated with knowledge extraction, an intuitionistic fuzzy method [34] was proposed with the aim of choosing more efficient feature for modeling in machine learning. It is worth noting that some existing tolerance-based intuitionistic fuzzy rough models do not consider information system with fuzzy decision attribute. Other models construct similarity relation only by membership of the object and thus ignore the characteristics of intuitionistic fuzzy set. Therefore, one of the motivations of this study is to fully exploit the advantage of intuitionistic fuzzy sets for realizing

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the qualitative description of fuzzy tolerance relation, and construct a novel fuzzy tolerance rough set model in the IFFD.

As society and the economy evolve, high dimensionality of datasets containing a vast number of objects and attributes constrains the process of acquiring knowledge and exhibits dynamic characteristics over time-evolving. The objects, attributes and attribute values of original datasets will be constantly changing due to the insertion of new data and the deletion of obsolete data [35–39]. Meanwhile, original datasets will not be applicable. It is necessary that we formulate some effective strategies to update original knowledge. Computing the entire updated datasets from scratch, which is the core of traditional algorithms for updating knowledge, can lead to inefficiencies in practical applications. Therefore, how to acquire available knowledge from datasets quickly and efficiently becomes an important issue. The incremental updating methods can quickly and efficiently acquire available knowledge in dynamic environment, which has been explored by some scholars [40–47]. In recent years, Huang [48] proposed the matrix-based incremental approaches in composite ordered decision systems with variation of attribute sets over time. Lin [49] established a rough set model in the IIDIS and showed several algorithms for statically/dynamically solving approximate sets. Yang [50] researched dynamic fuzzy neighborhood rough set in IvIS-FD. Above all, researchers have carried out a large number of studies on dynamic updating rough approximation algorithms for different types of information systems. However, these existing methods cannot solve the dynamic intuitionistic fuzzy information systems with fuzzy decision. Subsequently, the realization of dynamic updating rough approximations in dynamic IFFD with time-evolving objects is another motivation of this paper.

The dynamic updating of rough approximations method in IFFD holds practical application value in the context of medical diagnosis. Medical diagnosis often involves dealing with incomplete, fuzzy, and uncertain medical information. Intuitionistic fuzzy sets are capable of handling medical data with missing, fuzzy, or incomplete attributes, thus enabling more comprehensive data analysis. The integration of patients' clinical characteristics, disease severity, and other pertinent factors as input in the form of IFFD, coupled with the employment of theoretical knowledge on approximate sets, contributes to healthcare professionals' enhanced ability to precisely ascertain and evaluate a patient's disease status. Due to the continuous updating of medical data, the dynamic process of incorporating new clinical data into the model enables the ongoing update of approximate sets and adjustment of parameters (essentially involving the dynamic update rough approximations within the IFFD framework). This dynamic adaptation precisely captures trends in patients' changing disease trajectories, enabling healthcare professionals to offer real-time diagnostic assessments and recommendations. The adoption of dynamic updating rough approximations method in IFFD offers a more comprehensive consideration of the fuzziness and uncertainty encompassing various factors in medical diagnoses. By efficiently leveraging real-time data and expert knowledge, this approach provides accurate assessment outcomes and treatment suggestions, thereby optimizing treatment plans, enhancing clinical decision accuracy and personalization, and improving the effectiveness of patient care.

In this paper, an attempt is made to explore incremental updating fuzzy tolerance rough set approach in IFFD with time-evolving objects in light of the aforementioned discussions. Major contributions of this paper are summed up in the following:

- A fuzzy tolerance rough set model based on new fuzzy tolerance similarity is constructed in the context of IFFD and some properties are discussed in detail. Meanwhile, the corresponding static algorithm of calculating approximations in IFFD is investigated.
- The dynamic updating mechanisms of fuzzy tolerance rough set are provided when numerous objects are inserted or deleted from IFFD. Also, the corresponding dynamic algorithms are designed.
- The performance of FTRS model and the proposed dynamic algorithms are evaluated with nine datasets. The feasibility of the proposed FTRS model and the effectiveness and efficiency of the proposed dynamic algorithm are validated through experimental results.

The rest of this paper is organized as follows. Section 2 briefly reviews some theory knowledge of intuitionistic fuzzy set, fuzzy rough set and tolerance-based fuzzy rough set. In Section 3, we define the fuzzy tolerance relation, present a fuzzy tolerance similarity and the corresponding matrix representation in the IFFD. Also, a novel fuzzy tolerance rough set model is constructed and static algorithm for calculating the fuzzy tolerance approximations in IFFD is designed in detail. In Section 4, we investigate the dynamic updating mechanisms of fuzzy tolerance rough set when numerous objects are inserted or deleted from original IFFD and design two dynamic algorithms. In Section 5, the performance of FTRS model and dynamic algorithm are evaluated through a series of experiments. This evaluation verifies their feasibility, effectiveness, and efficiency. In Section 6, we summarize our work and set out further research directions.

2. Preliminaries

This section briefly reviews the concepts of intuitionistic fuzzy set, fuzzy rough set and tolerance-based fuzzy rough set. For more details, please refer to [25,28,32].

Assumed that $\langle m, n\rangle$ is an order pair. We refer to $\langle m, n\rangle$ as an intuitionistic fuzzy value if $0 \leq m, n \leq 1$ and $0 \leq m + n \leq 1$. Denotes $\overline{0} = \langle 0, 1\rangle$ the intuitionistic fuzzy value zero and $\overline{1} = \langle 1, 0\rangle$ the intuitionistic fuzzy value one. For any intuitionistic fuzzy values $\varrho_1 = \langle m_1, n_1\rangle$ and $\varrho_2 = \langle m_2, n_2\rangle$, a partial order on intuitionistic fuzzy values is defined by $\varrho_1 \preceq \varrho_2$, that is, $\langle m_1, n_1\rangle \preceq \langle m_2, n_2\rangle$ if and only if $m_1 \leq m_2$ and $n_1 \geq n_2$. The union, intersection and equivalence operation of ϱ_1 and ϱ_2 are shown as follows:

- $\varrho_1 \cup \varrho_2 = \langle max\{m_1, m_2\}, min\{n_1, n_2\}\rangle$
- $\varrho_1 \cap \varrho_2 = \langle min\{m_1, m_2\}, max\{n_1, n_2\}\rangle$
- $\varrho_1 = \varrho_2 \Longleftrightarrow m_1 = m_2 \wedge n_1 = n_2$

Obviously, the union and intersection of finite intuitionistic fuzzy values $\varrho_i = \langle m_i, n_i\rangle$ can be expressed as

- $\cup_{1\leq i\leq t}\varrho_i = \langle max_{1\leq i\leq t} m_i, min_{1\leq i\leq t} n_i\rangle$
- $\cap_{1\leq i\leq t}\varrho_i = \langle min_{1\leq i\leq t} m_i, max_{1\leq i\leq t} n_i\rangle$

Assumed that U is a universe of discourse. An intuitionistic fuzzy set on U is an object with the form $I = \{\langle u, m_I(u), n_I(u)\rangle \mid u \in U\}$ where $m_I : U \to [0, 1]$ and $n_I : U \to [0, 1]$ such that $0 \leq m_I(u) + n_I(u) \leq 1$ for every $u \in U$. $m_I(u)$ and $n_I(u)$ are referred to as the degree of membership and the degree of non-membership of u with respect to I respectively. $\pi_I(u) = 1 - m_I(u) - n_I(u)$ stands for the hesitancy degree of u with respect to I where $0 \leq \pi_I(u) \leq 1$. If $\pi_I(u) = 0$ and $I = \{\langle u, m_I(u), 1 - m_I(u)\rangle \mid u \in U\}$, intuitionistic fuzzy set I can be degraded as an fuzzy set $I = \{\langle u, m_I(u)\rangle \mid u \in U\}$. Therefore, an intuitionistic fuzzy set can be viewed as an extension of fuzzy set.

An intuitionistic fuzzy information system(IFIS) is usually represented by a quadruple (U, A, V, f), where $U = \{u_1, u_2, \ldots, u_m\}$ is a non-empty finite set of objects, $A = \{a_1, a_2, \ldots, a_n\}$ is a non-empty finite set of attributes, V is the set of all intuitionistic fuzzy values, and V_{a_k} is a intuitionistic fuzzy set of attribute a_k, $f : U \times A \to V$ is an intuitionistic fuzzy information function such that $f(u_i, a_k) = \langle m^{a_k}(u_i), n^{a_k}(u_i)\rangle \in V_{a_k}$ for each $u_i \in U$ and $a_k \in A$. Intuitively, if IFIS is concomitant with fuzzy decision, we call it intuitionistic fuzzy information system with fuzzy decision (IFFD) which is remarked as $(U, A \bigcup \{d\}, V, f)$. In $(U, A \bigcup \{d\}, V, f)$, $d(u_i)$ stands for fuzzy value of fuzzy decision attribute d concerning u_i. For illustrate the concept of IFFD, we provide an illustrate example as follows:

Table 1
An intuitionistic fuzzy information system with fuzzy decision (IFFD).

	a_1	a_2	a_3	a_4	a_5	d
u_1	$\langle 0.4, 0.5\rangle$	$\langle 0.3, 0.5\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.7, 0.1\rangle$	0.3
u_2	$\langle 0.3, 0.5\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.7, 0.3\rangle$	0.4
u_3	$\langle 0.3, 0.5\rangle$	$\langle 0.1, 0.8\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.7, 0.3\rangle$	0.2
u_4	$\langle 0.1, 0.8\rangle$	$\langle 0.1, 0.8\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.1, 0.8\rangle$	$\langle 0.8, 0.2\rangle$	0.2
u_5	$\langle 0.7, 0.3\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.8, 0.1\rangle$	0.4
u_6	$\langle 0.3, 0.6\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.5, 0.5\rangle$	$\langle 0.8, 0.2\rangle$	0.4

Example 1. A medical institution is planning to conduct medical examinations on patients who may potentially have lung cancer. Table 1 illustrates a medical diagnosis IFFD, i.e. $\daleth = (U, A\bigcup\{d\}, V, f)$. In this context, $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ represents the set of patients, $A = \{Breathlessness, Chest\ pain, Weakness, Headache, Cough\ worsening\} = \{a_1, a_2, a_3, a_4, a_5\}$ refers to the conditional attribute set, and $d = \{lung\ cancer\}$ denotes the fuzzy decision attribute.

In 1992, Dubois and Prade [25] introduced fuzzy rough set model on basis of fuzzy similarity relation, which can handle continuous or numerical data. Supposed that U is a universe of discourse. If $R : U \times U \rightarrow [0, 1]$ satisfies

1. $R(u_i, u_i) = 1$, $\forall u_i \in U$
2. $R(u_i, u_j) = R(u_j, u_i)$, $\forall u_i, u_j \in U$

R is a fuzzy similarity relation on U and $[u_i]_R = \{(u_j, R(u_i, u_j)) \mid u_j \in U\}$ is a fuzzy similarity class containing u_i with respect to R.

Definition 1 (*[25]*). Supposed that U is a universe of discourse, R and ξ are fuzzy similarity relation and fuzzy set on U respectively. For every $u_i \in U$, the fuzzy lower and upper approximations of ξ are defined as

$$\underline{R}\xi(u_i) = \inf_{u_j \in U} max\{1 - R(u_i, u_j), \xi(u_j)\} \tag{1}$$

$$\overline{R}\xi(u_i) = \sup_{u_j \in U} min\{R(u_i, u_j), \xi(u_j)\} \tag{2}$$

where $\underline{R}\xi$ and $\overline{R}\xi$ are referred to be the fuzzy lower and upper approximation operators of ξ. The pair $(\underline{R}\xi, \overline{R}\xi)$ is known as fuzzy rough set.

In 2007, Jensen [28] introduced the concept of tolerance-based fuzzy rough set. Assumed that (U, A, P, f) is a fuzzy information system. For any $u_i, u_j \in U$ and $a_k \in A$, the fuzzy similarity relation under a_k is defined as

$$Sim_{a_k}(u_i, u_j) = 1 - \frac{| m^{a_k}(u_i) - m^{a_k}(u_j) |}{| m^{a_{max}} - m^{a_{min}} |} \tag{3}$$

where $m^{a_{max}}$ and $m^{a_{min}}$ are the maximum and the minimum membership grades of u_i with respect to a_k respectively. Let $B \subseteq A$ and λ be a threshold. If $\prod_{a_k \in B} Sim_{a_k}(u_i, u_j) \geq \lambda$, then $(u_i, u_j) \in Sim_B^\lambda$ where λ indicates the level of similarity required to be included in the tolerance classes. Then, tolerance classes based on fuzzy similarity relation are defined by

$$Sim_B^\lambda(u_i) = \{u_j \in U \mid (u_i, u_j) \in Sim_B^\lambda\}$$

For $X \subseteq U$, its lower and upper approximations of X can be represented as

$$\underline{B^\lambda} X = \{u_i \mid Sim_B^\lambda(u_i) \subseteq X\} \tag{4}$$

$$\overline{B^\lambda} X = \{u_i \mid Sim_B^\lambda(u_i) \cap X \neq \phi\} \tag{5}$$

The pair $(\underline{B^\lambda} X, \overline{B^\lambda} X)$ is called tolerance fuzzy rough set.

3. Fuzzy tolerance rough set based on IFFD

In this section, we analyze a similarity between two objects by considering membership and non-membership of objects in the IFIS simultaneously, and define fuzzy tolerance relation and fuzzy tolerance similarity. Accordingly, we give the fuzzy tolerance similarity matrix representation in the IFFD. Meanwhile, we construct a new fuzzy tolerance rough set model in the IFFD and discuss some properties of the model. Afterwards, the fuzzy tolerance approximation accuracy and roughness are provided in the IFFD. Finally, we design a static algorithm for calculating fuzzy tolerance approximations in IFFD and analyze its time complexity.

Definition 2. Let (U, A, V, f) be IFIS. For any $u_i, u_j \in U$ and $a_k \in A$, $f(u_i, a_k) = \langle m^k(u_i), n^k(u_i)\rangle$ and $f(u_j, a_k) = \langle m^k(u_j), n^k(u_j)\rangle$. The intuitionistic similarity between u_i and u_j with respect to a_k in (U, A, V, f) is defined as

$$IS_k(u_i, u_j) = \frac{1}{2}\left(\frac{min(m^k(u_i), m^k(u_j))}{max(m^k(u_i), m^k(u_j))} + \frac{min(1 - n^k(u_i), 1 - n^k(u_j))}{max(1 - n^k(u_i), 1 - n^k(u_j))}\right) \tag{6}$$

where $m^k(u_i)$ and $n^k(u_i)$ are the degree of membership and non-membership of $u_i \in U$ with respect to a_k respectively.

Example 2. For $f(u_1, a_1) = \langle 0.4, 0.5\rangle$ and $f(u_4, a_1) = \langle 0.1, 0.8\rangle$ in Table 1, the intuitionistic similarity between u_1 and u_4 concerning a_1 can be computed by Eq. (6) as follows:

$$IS_1(u_1, u_4) = \frac{1}{2}\left(\frac{min(0.4, 0.1)}{max(0.4, 0.1)} + \frac{min(1 - 0.5, 1 - 0.8)}{max(1 - 0.5, 1 - 0.8)}\right) = \frac{1}{2}\left(\frac{0.1}{0.4} + \frac{0.2}{0.5}\right) = 0.325$$

Property 1. *Let (U, A, V, f) be IFIS. For any $u_i, u_j \in U$ and $a_k \in A$, $IS_k(u_i, u_j)$ defined by Eq. (6) satisfies that*

1. *For any $u_i \in U$ and $a_k \in A$, $IS_k(u_i, u_i) = 1$;*
2. *For any $u_i, u_j \in U$ and $a_k \in A$, $IS_k(u_i, u_j) = IS_k(u_j, u_i)$;*
3. *$IS_k(u_i, u_j) = 0$ if $f(u_i, a_k) = \overline{0}$ and $f(u_j, a_k) = \overline{1}$;*
4. *For any $u_i, u_j, u_l \in U$ and $a_k \in A$, $IS_k(u_i, u_l) \leq min\{IS_k(u_i, u_j), IS_k(u_j, u_l)\}$ whenever $f(u_i, a_k) \leq f(u_j, a_k) \leq f(u_k, a_k)$.*

Proof. (1) For any $u_i \in U$ and $a_k \in A$, we have $IS_k(u_i, u_i) = \frac{1}{2}(\frac{min(m^k(u_i), m^k(u_i))}{max(m^k(u_i), m^k(u_i))} + \frac{min(1-n^k(u_i), 1-n^k(u_i))}{max(1-n^k(u_i), 1-n^k(u_i))}) = 1$ by Eq. (6).

(2) For any $u_i, u_j \in U$ and $a_k \in A$, we have $IS_k(u_i, u_j) = \frac{1}{2}(\frac{min(m^k(u_i), m^k(u_j))}{max(m^k(u_i), m^k(u_j))} + \frac{min(1-n^k(u_i), 1-n^k(u_j))}{max(1-n^k(u_i), 1-n^k(u_j))}) = \frac{1}{2}(\frac{min(m^k(u_j), m^k(u_i))}{max(m^k(u_j), m^k(u_i))} + \frac{min(1-n^k(u_j), 1-n^k(u_i))}{max(1-n^k(u_j), 1-n^k(u_i))}) = IS_k(u_j, u_i)$. Thus, $IS_k(u_i, u_j) = IS_k(u_j, u_i)$.

(3) For any $u_i, u_j \in U$ and $a_k \in A$, if $f(u_i, a_k) = \overline{0} = \langle 0, 1\rangle$ and $f(u_j, a_k) = \overline{1} = \langle 1, 0\rangle$, we have $IS_k(u_i, u_j) = \frac{1}{2}(\frac{min(0,1)}{max(0,1)} + \frac{min(1-1),(1-0)}{max(1-1),(1-0)}) = 0$. So, $IS_k(u_i, u_j) = 0$ if $f(u_i, a_k) = \overline{0}$ and $f(u_j, a_k) = \overline{1}$.

(4) For any $u_i, u_j, u_l \in U$ and $a_k \in A$, if $f(u_i, a_k) \leq f(u_j, a_k) \leq f(u_l, a_k)$, we have $m^k(u_i) \leq m^k(u_j) \leq m^k(u_l)$ and $n^k(u_i) \geq n^k(u_j) \geq n^k(u_l)$. It is obvious that $\frac{min(m^k(u_i), m^k(u_l))}{max(m^k(u_i), m^k(u_l))} \leq \frac{min(m^k(u_i), m^k(u_j))}{max(m^k(u_i), m^k(u_j))}$ and $\frac{min(1-n^k(u_i), 1-n^k(u_l))}{max(1-n^k(u_i), 1-n^k(u_l))} \leq \frac{min(1-n^k(u_i), 1-n^k(u_j))}{max(1-n^k(u_i), 1-n^k(u_j))}$. By Eq. (6), we can obtain $IS_k(u_i, u_l) \leq IS_k(u_i, u_j)$. Similarity, it is easy to prove $IS_k(u_i, u_l) \leq IS_k(u_j, u_l)$. To summarize, $IS_k(u_i, u_l) \leq min\{IS_k(u_i, u_j), IS_k(u_j, u_l)\}$.

Based on the similarity shown in Eq. (6), the fuzzy tolerance relation in the IFFD is defined as follows:

Definition 3. Let $\daleth = (U, A\bigcup\{d\}, V, f)$ be IFFD and $IS_k(u_i, u_j)$ be the similarity between u_i and u_j under a_k. For any $u_i, u_j \in U$, $\lambda \in (0, 1]$ and $B \subseteq A$, the fuzzy tolerance relation between u_i and u_j concerning B in IFFD is defined as

$$ISM_B^\lambda = \{((u_i, u_j), \prod_{a_k \in B} IS_k(u_i, u_j)) : \prod_{a_k \in B} IS_k(u_i, u_j) \geq \lambda, (u_i, u_j) \in U \times U\} \tag{7}$$

Definition 4. Let $\daleth = (U, A\bigcup\{d\}, V, f)$ be IFFD. For any $u_i, u_j \in U$ and $B \subseteq A$. The fuzzy tolerance similarity between u_i and u_j concerning B in IFFD is defined as

$$ISM_B^{\lambda}(u_i,u_j) = \begin{cases} \prod_{a_k\in B} IS_k(u_i,u_j), & \text{if } \prod_{a_k\in B} IS_k(u_i,u_j) \geq \lambda, \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

At the same time, the corresponding fuzzy tolerance similarity matrix can be remarked as $M_U^{ISM_B^{\lambda}}$, i.e., $M_U^{ISM_B^{\lambda}} = [tm_{ij}^{ISM_B^{\lambda}}]_{|U|\times|U|}$ where $tm_{ij}^{ISM_B^{\lambda}} = ISM_B^{\lambda}(u_i,u_j)$. On basis of Definition 3, tolerance classes concerning fuzzy tolerance relation in IFFD are defined as follows:

$$ISM_B^{\lambda}(u_i) = \{u_j \in U \mid (u_i,u_j) \in ISM_B^{\lambda}\} \tag{9}$$

In particular, there exists two special kinds of fuzzy tolerance relations in IFFD as follows:

- If $ISM_B^{\lambda}(u_i) = \{u_i\}$, then ISM_B^{λ} is the finest fuzzy tolerance relation.
- If $ISM_B^{\lambda}(u_i) = U$, then ISM_B^{λ} is the coarsest fuzzy tolerance relation.

Property 2. *Supposed that $\daleth = (U, A\bigcup\{d\}, V, f)$ is IFFD, $\lambda, \lambda_1, \lambda_2 \in (0,1]$, $B, B_1, B_2, \subseteq A$, and $ISM_B^{\lambda}(u_i)$ is tolerance classes based on the fuzzy tolerance relation ISM_B^{λ} concerning B. For any $u_i, u_j \in U$,*

1. *If $\lambda_1 \leq \lambda_2$, then $ISM_B^{\lambda_2}(u_i) \subseteq ISM_B^{\lambda_1}(u_i)$.*
2. *If $B_1 \subseteq B_2$, then $ISM_{B_2}^{\lambda}(u_i) \subseteq ISM_{B_1}^{\lambda}(u_i)$.*

Proof. (1) For any $u_j \in U$, we have $u_j \in ISM_B^{\lambda_2}(u_i)$ and $(u_i,u_j) \in ISM_B^{\lambda_2}$ by the Eq. (9). If $\lambda_1 \leq \lambda_2$, then $\prod_{a_k\in B} IS_k(u_i,u_j) \geq \lambda_2 \geq \lambda_1$. Furthermore, we can obtain $(u_i,u_j) \in ISM_B^{\lambda_1}$ and $u_j \in ISM_B^{\lambda_1}(u_i)$. Thus, $ISM_B^{\lambda_2}(u_i) \subseteq ISM_B^{\lambda_1}(u_i)$.

(2) For any $u_j \in U$, we have $u_j \in ISM_{B_2}^{\lambda}(u_i)$ and $(u_i,u_j) \in ISM_{B_2}^{\lambda}$ by the Eq. (9). If $B_1 \subseteq B_2$, it is obvious that $\prod_{a_k\in B_1} IS_k(u_i,u_j) \geq \prod_{a_k\in B_2} IS_k(u_i,u_j) \geq \lambda$. So, $u_j \in ISM_{B_1}^{\lambda}(u_i)$ and $ISM_B^{\lambda_2}(u_i) \subseteq ISM_B^{\lambda_1}(u_i)$.

Definition 5. Supposed that $\daleth = (U, A\bigcup\{d\}, V, f)$ is IFFD and ISM_B^{λ} is fuzzy tolerance relation concerning $B \subseteq A$. For any $u_i, u_j \in U$ and $\lambda \in (0,1]$, the fuzzy tolerance lower and upper approximation operators (FTAO) of d, which defined on $\daleth = (U, A\bigcup\{d\}, V, f)$ with respect to λ and B, are as follows:

$$\underline{ISM_B^{\lambda}}d(u_i) = \wedge_{u_j\in U}((1 - ISM_B^{\lambda}(u_i,u_j)) \vee d(u_j)) \tag{10}$$

$$\overline{ISM_B^{\lambda}}d(u_i) = \vee_{u_j\in U}(ISM_B^{\lambda}(u_i,u_j) \wedge d(u_j)) \tag{11}$$

The pair $(\underline{ISM_B^{\lambda}}d, \overline{ISM_B^{\lambda}}d)$ is known as fuzzy tolerance rough set (FTRS) of d concerning B in IFFD.

Definition 6. Supposed that $\daleth = (U, A\bigcup\{d\}, V, f)$ is IFFD. For $\forall u_i \in U$, fuzzy decision value matrix is $FM(d) = [fm_{i1}]_{|U|\times 1}$ where $fm_{i1} = d(u_i)$.

Intuitively, supposed that fuzzy tolerance relation ISM_B^{λ} and fuzzy decision d are written as fuzzy tolerance similarity matrix $M_U^{ISM_B^{\lambda}}$ and fuzzy decision value matrix $FM(d)$ in IFFD respectively. The fuzzy tolerance lower and upper approximation matrix are defined by $UI = [\underline{ISM_B^{\lambda}}d_{i1}]_{|U|\times 1}$ and $OI = [\overline{ISM_B^{\lambda}}d_{i1}]_{|U|\times 1}$ where

$$\underline{ISM_B^{\lambda}}d_{i1} = min_{j=1}^{|U|}\{max\{(1 - tm_{ij}^{ISM_B^{\lambda}}), fm_{j1}\}\} \tag{12}$$

$$\overline{ISM_B^{\lambda}}d_{i1} = max_{j=1}^{|U|}\{min\{tm_{ij}^{ISM_B^{\lambda}}, fm_{j1}\}\} \tag{13}$$

Property 3. *Supposed that $\daleth = (U, A\bigcup\{d\}, V, f)$ is IFFD, $\lambda, \lambda_1, \lambda_2 \in (0,1]$, $B, B_1, B_2, \subseteq A$, and $\underline{ISM_B^{\lambda}}d$ and $\overline{ISM_B^{\lambda}}d$ are FTAO of d concerning B in IFFD. For any $u_i, u_j \in U$,*

1. *If $\lambda_1 \leq \lambda_2$, then $\underline{ISM_B^{\lambda_1}}d \subseteq \underline{ISM_B^{\lambda_2}}d$ and $\overline{ISM_B^{\lambda_2}}d \subseteq \overline{ISM_B^{\lambda_1}}d$.*
2. *If $B_1 \subseteq B_2$, then $\underline{ISM_{B_1}^{\lambda}}d \subseteq \underline{ISM_{B_2}^{\lambda}}d$ and $\overline{ISM_{B_2}^{\lambda}}d \subseteq \overline{ISM_{B_1}^{\lambda}}d$.*
3. *$\underline{ISM_B^{\lambda}}d \subseteq \overline{ISM_B^{\lambda}}d$*

Proof. (1) For any $u_i, u_j \in U$, if $\lambda_1 \leq \lambda_2$, we have $ISM_B^{\lambda_2}(u_i,u_j) \leq ISM_B^{\lambda_1}(u_i,u_j)$ by Property 2. Obviously, $\wedge_{u_j\in U}((1 - ISM_B^{\lambda_1}(u_i,u_j)) \vee d(u_j)) \leq \wedge_{u_j\in U}((1 - ISM_B^{\lambda_2}(u_i,u_j)) \vee d(u_j))$. Therefore, $\underline{\underline{ISM_B^{\lambda_1}}}d \subseteq \underline{\underline{ISM_B^{\lambda_2}}}d$ can be proofed. Similarity, it is easy to prove $\underline{\underline{ISM_B^{\lambda_2}}}d \subseteq \underline{\underline{ISM_B^{\lambda_1}}}d$.

(2) For any $u_i, u_j \in U$, if $B_1 \subseteq B_2$, we obtain $ISM_{B_2}^{\lambda}(u_i,u_j) \leq ISM_{B_1}^{\lambda}(u_i,u_j)$. By Eqs. (10) and (11), we have $\wedge_{u_j\in U}((1 - ISM_{B_1}^{\lambda}(u_i,u_j)) \vee d(u_j)) \leq \wedge_{u_j\in U}((1 - ISM_{B_2}^{\lambda}(u_i,u_j)) \vee d(u_j))$ and $\vee_{u_j\in U}(ISM_{B_2}^{\lambda}(u_i,u_j) \wedge d(u_j)) \leq \vee_{u_j\in U}(ISM_{B_1}^{\lambda}(u_i,u_j) \wedge d(u_j))$. To summarize, $\underline{ISM_{B_1}^{\lambda}}d \subseteq \underline{ISM_{B_2}^{\lambda}}d$ and $\overline{ISM_{B_2}^{\lambda}}d \subseteq \overline{ISM_{B_1}^{\lambda}}d$.

(3) For any $u_i, u_j \in U$, according to Definitions 4 and 2, we have $ISM_B^{\lambda}(u_i,u_i) = 1$. Due to $(1 - ISM_B^{\lambda}(u_i,u_i)) \vee d(u_i) = d(u_i)$, we can get $\wedge_{u_j\in U}((1 - ISM_B^{\lambda}(u_i,u_j)) \vee d(u_j)) \leq d(u_j)$. Similarity, we also get $d(u_i) \leq \vee_{u_j\in U}(ISM_B^{\lambda}(u_i,u_j) \wedge d(u_j))$. Thus, $\underline{ISM_B^{\lambda}}d \subseteq \overline{ISM_B^{\lambda}}d$.

Definition 7. Supposed that $\daleth = (U, A\bigcup\{d\}, V, f)$ is IFFD and $\lambda \in (0,1]$. For any $u_i \in U$, the fuzzy tolerance approximation accuracy and roughness of d concerning $B \subseteq A$ in IFFD are respectively defined as

$$TI-acc = \frac{\sum_{u_i\in U} \underline{\underline{ISM_B^{\lambda}}}d(u_i)}{\sum_{u_i\in U} \overline{\overline{ISM_B^{\lambda}}}d(u_i)}, TI-rou = \frac{\sum_{u_i\in U}(1 - \underline{\underline{ISM_B^{\lambda}}}d(u_i))}{\sum_{u_i\in U} \overline{\overline{ISM_B^{\lambda}}}d(u_i)}.$$

Example 3 (*Continuation of Example 1*). Let $B = A$ and $\lambda = 0.44$. Then according to Definitions 2 and 4, the fuzzy tolerance similarity matrix $M_U^{ISM_A^{0.44}}$ can be calculated as follows.

$$M_U^{ISM_A^{0.44}} = \begin{bmatrix} 1 & 0.5577 & 0 & 0 & 0 & 0.4469 \\ 0.5577 & 1 & 0 & 0 & 0 & 0.5569 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0.4469 & 0.5569 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 6}$$

From the above results and Definition 5, we can calculate fuzzy tolerance lower and upper approximation operators of d concerning B in IFFD as follows.

$$\underline{ISM_B^{\lambda}}d = 0.30/u_1 + 0.40/u_2 + 0.20/u_3 + 0.20/u_4 + 0.40/u_5 + 0.40/u_6$$

$$\overline{ISM_B^{\lambda}}d = 0.40/u_1 + 0.40/u_2 + 0.20/u_3 + 0.20/u_4 + 0.40/u_5 + 0.40/u_6$$

According to the above analysis, we design a static algorithm (FTA-IFFD) for calculating fuzzy tolerance approximations in IFFD and discuss its time complexity.

- FTA-IFFD algorithm (See Algorithm 1): In the FTA-IFFD algorithm, steps 1–3 are to generate a fuzzy decision value matrix in IFFD. In steps 4–8, the fuzzy tolerance similarity under attribute set A in IFFD is calculated. Steps 9–12 are to calculate FTAO in IFFD. In step 13, the fuzzy tolerance approximation sets in IFFD are returned.

Table 2
The time complexity of FTA-IFFD algorithm.

Steps	Time complexity
$1-3$	$\mathcal{O}(\varepsilon)$
$4-8$	$\mathcal{O}(\varepsilon^2)$
$9-12$	$\mathcal{O}(\varepsilon)$
Total	$\mathcal{O}(2\varepsilon+\varepsilon^2)$

- Time complexity: The time complexity of FTA-IFFD algorithm is given in Table 2.

Algorithm 1 FTA-IFFD algorithm

Input: An IFFD $\daleth = (U, A\bigcup\{d\}, V, f)$, $\mid U \mid = \varepsilon$ and parameter λ
Output: $\underline{ISM}_A^\lambda d(u_i), \overline{ISM}_A^\lambda d(u_i)$

```
1: for i = 1 to ε do
2:     Calculate FM(d) = [fm_{i1}]_{ε×1} by Definition 6
3: end for
4: for i = 1 to ε do
5:     for j = 1 to ε do
6:         Calculate M_U^{ISM_A^λ} = [tm_{ij}^{ISM_A^λ}]_{ε×ε} via Definition 4
7:     end for
8: end for
9: for i = 1 to ε do
10:    Calculate UI = [\underline{ISM}_A^λ d_{i1}]_{ε×1} via using Eq.(12)
11:    Calculate OI = [\overline{ISM}_A^λ d_{i1}]_{ε×1} via using Eq.(13)
12: end for
13: return \underline{ISM}_A^λ d(u_i), \overline{ISM}_A^λ d(u_i)
```

4. Incremental approaches of fuzzy tolerance rough set with the variation of numerous objects

For dynamic IFFD with time-evolving objects, it is time-consuming to discover knowledge by employing FTA-IFFD algorithm, especially for large-scale IFFD. This algorithm needs to retrain the changed IFFD to the new IFFD and then recalculate the knowledge from scratch. The updating mechanisms of fuzzy tolerance rough set are researched for avoiding double-counting knowledge and improving the efficiency of knowledge discovery in dynamic IFFD.

4.1. The updating mechanisms of FTRS with the insertion of objects

In this subsection, we mainly research updating mechanism of FTRS in dynamic IFFD when inserting numerous objects into IFFD while the attribute set remains constant. It is obviously that the core steps of updating FTRS process in Algorithm 1 are to calculate the corresponding fuzzy decision value matrix and fuzzy tolerance similarity matrix in the incremental manner. Therefore, the corresponding principle of updating fuzzy decision value matrix and fuzzy tolerance similarity matrix are proposed in IFFD.

Proposition 1. *Supposed that $\daleth = (U, A\bigcup\{d\}, V, f)$ is IFFD, $\mid U \mid = \varepsilon$, $U_\Delta = \{u_{\varepsilon+1}, u_{\varepsilon+2}, \ldots, u_{\varepsilon+\tilde{\varepsilon}}\}$ is the inserting object set, $\tilde{U} = U \cup U_\Delta$ is the changed object set. For any $B \subseteq A$, the previous fuzzy decision value matrix is $FM(d) = [fm_{i1}]_{\varepsilon\times 1}$, which is updated to $FM(\tilde{d}) = [\tilde{fm}_{i1}]_{(\varepsilon+\tilde{\varepsilon})\times 1}$ where*

$$\tilde{fm}_{i1} = \begin{cases} fm_{i1}, & 1 \le i \le \varepsilon, \\ d(u_i), & \varepsilon+1 \le i \le \varepsilon+\tilde{\varepsilon}. \end{cases} \tag{14}$$

Proof. After inserting object set $U_\Delta = \{u_{\varepsilon+1}, u_{\varepsilon+2}, \ldots, u_{\varepsilon+\tilde{\varepsilon}}\}$ to $\daleth$, $FM(d)$ is updated to a new fuzzy decision value matrix $FM(\tilde{d})$ by Definition 6. It is obvious that the fuzzy decision of $\daleth$ should not changed when $\daleth$ inserts object set U_Δ. Therefore, for $1 \le i \le \varepsilon$, we have $\tilde{fm}_{i1} = fm_{i1}$. If $u_i \notin U$ and $u_i \in U_\Delta$, the corresponding fuzzy decision value matrix can be computed by Definition 6, namely, $\tilde{fm}_{i1} = d(u_i)$ for $\varepsilon+1 \le i \le \varepsilon+\tilde{\varepsilon}$.

Table 3
A new IFFD after adding objects.

	a_1	a_2	a_3	a_4	a_5	d
u_1	$\langle 0.4, 0.5\rangle$	$\langle 0.3, 0.5\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.7, 0.1\rangle$	0.3
u_2	$\langle 0.3, 0.5\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.7, 0.3\rangle$	0.4
u_3	$\langle 0.3, 0.5\rangle$	$\langle 0.1, 0.8\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.7, 0.3\rangle$	0.2
u_4	$\langle 0.1, 0.8\rangle$	$\langle 0.1, 0.8\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.1, 0.8\rangle$	$\langle 0.8, 0.2\rangle$	0.2
u_5	$\langle 0.7, 0.3\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.8, 0.1\rangle$	0.4
u_6	$\langle 0.3, 0.6\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.5, 0.5\rangle$	$\langle 0.8, 0.2\rangle$	0.4
u_7	$\langle 0.2, 0.4\rangle$	$\langle 0.2, 0.7\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.1, 0.8\rangle$	$\langle 0.4, 0.2\rangle$	0.3
u_8	$\langle 0.5, 0.5\rangle$	$\langle 0.1, 0.8\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.1, 0.8\rangle$	$\langle 0.4, 0.2\rangle$	0.2

Proposition 2. *Supposed that $\daleth = (U, A\bigcup\{d\}, V, f)$ is IFFD, $\mid U \mid = \varepsilon$, $U_\Delta = \{u_{\varepsilon+1}, u_{\varepsilon+2}, \ldots, u_{\varepsilon+\tilde{\varepsilon}}\}$ is the inserting object set, $\tilde{U} = U \cup U_\Delta$ is the changed object set. For any $B \subseteq A$, the previous fuzzy tolerance similarity matrix is $M_U^{ISM_B^\lambda} = [tm_{ij}^{ISM_B^\lambda}]_{\varepsilon\times\varepsilon}$, which is updated to $M_{\tilde{U}}^{ISM_B^\lambda} = [\tilde{m}_{ij}^{ISM_B^\lambda}]_{(\varepsilon+\tilde{\varepsilon})\times(\varepsilon+\tilde{\varepsilon})}$ after inserting objects, where*

$$\tilde{m}_{ij}^{ISM_B^\lambda} = \begin{cases} tm_{ij}^{ISM_B^\lambda}, & 1 \le i, j \le \varepsilon, \\ ISM_B^\lambda(u_i, u_j), & \varepsilon+1 \le i \le \varepsilon+\tilde{\varepsilon} \quad or \quad \varepsilon+1 \le j \le \varepsilon+\tilde{\varepsilon}. \end{cases} \tag{15}$$

Proof. Theoretically, the new fuzzy tolerance similarity matrix $M_{\tilde{U}}^{ISM_B^\lambda} = [\tilde{m}_{ij}^{ISM_B^\lambda}]_{(\varepsilon+\tilde{\varepsilon})\times(\varepsilon+\tilde{\varepsilon})}$ can be viewed as a partitioned matrix which includes 4 submatrix, that is, $M_{\tilde{U}}^{ISM_B^\lambda} = \begin{bmatrix} C_{\varepsilon\times\varepsilon} & D_{\varepsilon\times\tilde{\varepsilon}} \\ E_{\tilde{\varepsilon}\times\varepsilon} & G_{\tilde{\varepsilon}\times\tilde{\varepsilon}} \end{bmatrix}$. The submatix $C_{\varepsilon\times\varepsilon} = [c_{ij}]_{\varepsilon\times\varepsilon}$ expresses the fuzzy tolerance similarity matrix of $U \times U$ associated with B where $[c_{ij}]_{\varepsilon\times\varepsilon} = [tm_{ij}^{ISM_B^\lambda}]_{\varepsilon\times\varepsilon}$. The submatix $D_{\varepsilon\times\tilde{\varepsilon}} = [d_{ij}]_{\varepsilon\times\tilde{\varepsilon}}$ expresses the fuzzy tolerance similarity matrix of $U \times U_\Delta$ associated with B where $[d_{ij}]_{\varepsilon\times\tilde{\varepsilon}} = ISM_B^\lambda(u_i, u_j)$ if and only if $1 \le i \le \varepsilon$ and $\varepsilon+1 \le j \le \varepsilon+\tilde{\varepsilon}$. The submatix $E_{\tilde{\varepsilon}\times\varepsilon} = [e_{ij}]_{\tilde{\varepsilon}\times\varepsilon}$ expresses the fuzzy tolerance similarity matrix of $U_\Delta \times U$ associated with B where $[e_{ij}]_{\tilde{\varepsilon}\times\varepsilon} = ISM_B^\lambda(u_i, u_j)$ if and only if $\varepsilon+1 \le i \le \varepsilon+\tilde{\varepsilon}$ and $1 \le j \le \varepsilon$. The submatix $G_{\tilde{\varepsilon}\times\tilde{\varepsilon}} = [g_{ij}]_{\tilde{\varepsilon}\times\tilde{\varepsilon}}$ expresses the fuzzy tolerance similarity matrix of $U_\Delta \times U_\Delta$ associated with B where $[g_{ij}]_{\tilde{\varepsilon}\times\tilde{\varepsilon}} = ISM_B^\lambda(u_i, u_j)$ if and only if $\varepsilon+1 \le i \le \varepsilon+\tilde{\varepsilon}$ and $\varepsilon+1 \le j \le \varepsilon+\tilde{\varepsilon}$. Based on the above analysis, we obtain the following forms

$$\tilde{m}_{ij}^{ISM_B^\lambda} = \begin{cases} tm_{ij}^{ISM_B^\lambda}, & 1 \le i, j \le \varepsilon, \\ ISM_B^\lambda(u_i, u_j), & \varepsilon+1 \le i \le \varepsilon+\tilde{\varepsilon} \quad or \quad \varepsilon+1 \le j \le \varepsilon+\tilde{\varepsilon}. \end{cases}$$

Therefore, Eq. (15) has been confirmed.

Example 4 (*Continuation of Example 3*). After adding $U_\Delta = \{u_7, u_8\}$ to Table 1, as shown in the Table 3. The new object set is represented as $\tilde{U} = \{u_1, \ldots, u_8\}$. Based on Eqs. (14) and (15), fuzzy decision value matrix and fuzzy tolerance similarity matrix are updated as

$$FM(\tilde{d}) = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.4 \\ 0.3 \\ 0.2 \end{bmatrix}_{8\times 1}$$

$$M_{\tilde{U}}^{ISM_A^\lambda} = \begin{bmatrix} C_{6\times 6} & D_{6\times 2} \\ E_{2\times 6} & G_{2\times 2} \end{bmatrix}$$

Table 4
The time complexity of AOFTA-IFFD algorithm.

Steps	Time complexity
2 – 4	$\mathcal{O}(\tilde{\varepsilon})$
5 – 15	$\mathcal{O}(\varepsilon \cdot \tilde{\varepsilon} + \tilde{\varepsilon}^2)$
16 – 19	$\mathcal{O}(\varepsilon + \tilde{\varepsilon})$
$Total$	$\mathcal{O}(\varepsilon + 2\tilde{\varepsilon} + \tilde{\varepsilon}^2 + \varepsilon\tilde{\varepsilon})$

$$= \left[\begin{array}{cccccc|cc} 1 & 0.5577 & 0 & 0 & 0 & 0.4469 & 0 & 0 \\ 0.5577 & 1 & 0 & 0 & 0 & 0.5569 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0.4469 & 0.5569 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]_{8\times 8}$$

Based on Eqs. (12) and (13), fuzzy tolerance lower and upper approximation matrix can be computed as $UI = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.4 \\ 0.3 \\ 0.2 \end{bmatrix}_{8\times 1}$, $OI = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.4 \\ 0.3 \\ 0.2 \end{bmatrix}_{8\times 1}$. Based on Definition 5, the fuzzy tolerance lower and upper approximation can be presented as

$$\underline{ISM_B^{\lambda}}d = 0.3/u_1 + 0.4/u_2 + 0.2/u_3 + 0.2/u_4 + 0.4/u_5 + 0.4/u_6 + 0.3/u_7 + 0.2/u_8$$

$$\overline{ISM_B^{\lambda}}d = 0.4/u_1 + 0.4/u_2 + 0.2/u_3 + 0.2/u_4 + 0.4/u_5 + 0.4/u_6 + 0.3/u_7 + 0.2/u_8.$$

On basis of the above analysis, we design a dynamic algorithm (AOFTA-IFFD) for calculating fuzzy tolerance approximations when inserting numerous objects and discuss its time complexity.

- AOFTA-IFFD algorithm (See Algorithm 2): In Algorithm 2, step 1 is to initialize fuzzy decision value matrix, fuzzy tolerance similarity matrix in IFFD. Steps 2–4 are to update the fuzzy decision value matrix after inserting objects under decision attribute in IFFD. In steps 5–15, fuzzy tolerance similarity matrix after inserting objects in IFFD is updated. In steps 16–19, the fuzzy tolerance lower and upper approximation matrix in IFFD are calculated. In step 20, the tolerance-based intuitionistic fuzzy approximation sets in IFFD are returned.
- Time complexity: The time complexity of AOFTA-IFFD algorithm is given in Table 4.
- Comparative analysis of time complexity: The time complexity of FTA-IFFD and AOFTA-IFFD algorithm is compared in Table 5. In the Table 5, we can obtain $|\tilde{U}| = |U| + |U_{\Delta}| = \varepsilon + \tilde{\varepsilon}$ by $\tilde{U} = U \cup U_{\Delta}$. Therefore, it is obviously that the time complexity of the FTA-IFFD algorithm is much larger than that of the AOFTA-IFFD algorithm. The reason is that FTA-IFFD algorithm recalculates new fuzzy tolerance approximations and ignores previously acquired knowledge in IFFD. However, AOFTA-IFFD algorithm utilizes prior knowledge to accelerate the acquisition of tolerance-based intuitionistic fuzzy rough approximations. Therefore, compared with FTA-IFFD algorithm, AOFTA-IFFD algorithm saves time cost.

Algorithm 2 AOFTA-IFFD algorithm

Input: An initial IFFD $\daleth = (U, A \bigcup \{d\}, V, f)$, $|U| = \varepsilon$, parameter λ, initial fuzzy decision value matrix $FM(d)$, initial fuzzy tolerance similarity matrix $M_U^{ISM_A^{\lambda}}$, and the inserting object set $U_{\Delta} = \{u_{\varepsilon+1}, u_{\varepsilon+2}, \cdots, u_{\varepsilon+\tilde{\varepsilon}}\}$.
Output: The new TIFA $\underline{ISM_A^{\lambda}}d(\widetilde{u_i}), \overline{ISM_A^{\lambda}}d(\widetilde{u_i})$ on $U \cup U_{\Delta}$

1: Initialize $\tilde{U} \leftarrow U \cup U_{\Delta}$, $FM(\tilde{d}) \leftarrow FM(\tilde{d})$, $M_{\tilde{U}}^{ISM_A^{\lambda}} \leftarrow M_U^{ISM_A^{\lambda}}$
2: **for** $i = \varepsilon + 1$ to $\varepsilon + \tilde{\varepsilon}$ **do**
3: Calculate $FM(\tilde{d}) = [\tilde{fm}_{i1}]_{(\varepsilon+\tilde{\varepsilon})\times 1}$ by Proposition 1
4: **end for**
5: **for** $i = \varepsilon + 1$ to $\varepsilon + \tilde{\varepsilon}$ **do**
6: **for** $j = 1$ to ε **do**
7: Calculate $M_{\tilde{U}}^{ISM_A^{\lambda}} = [\widetilde{m}_{ij}^{ISM_A^{\lambda}}]_{(\varepsilon+\tilde{\varepsilon})\times(\varepsilon+\tilde{\varepsilon})}$ by Proposition 2
8: $\widetilde{m}_{ji}^{ISM_A^{\lambda}} = \widetilde{m}_{ij}^{ISM_A^{\lambda}}$
9: **end for**
10: **end for**
11: **for** $i = \varepsilon + 1$ to $\varepsilon + \tilde{\varepsilon}$ **do**
12: **for** $j = \varepsilon + 1$ to $\varepsilon + \tilde{\varepsilon}$ **do**
13: Calculate $M_{\tilde{U}}^{ISM_A^{\lambda}} = [\widetilde{m}_{ij}^{ISM_A^{\lambda}}]_{(\varepsilon+\tilde{\varepsilon})\times(\varepsilon+\tilde{\varepsilon})}$ by Proposition 2
14: **end for**
15: **end for**
16: **for** $i = 1$ to $\varepsilon + \tilde{\varepsilon}$ **do**
17: Calculate $UI = [\underline{\underline{ISM_A^{\lambda}}}d_{i1}]_{(\varepsilon+\tilde{\varepsilon})\times 1}$ via using Eq.(12)
18: Calculate $OI = [\overline{ISM_A^{\lambda}}d_{i1}]_{(\varepsilon+\tilde{\varepsilon})\times 1}$ via using Eq.(13)
19: **end for**
20: **return** $\underline{ISM_A^{\lambda}}d(\widetilde{u_i}), \overline{ISM_A^{\lambda}}d(\widetilde{u_i})$

Table 5
Comparison of time complexity of FTA-IFFD and AOFTA-IFFD algorithm.

Algorithms	Time complexity
$FTA-IFFD$	$\mathcal{O}(2\varepsilon + \varepsilon^2)$
$AOFTA-IFFD$	$\mathcal{O}(\varepsilon + 2\tilde{\varepsilon} + \tilde{\varepsilon}^2 + \varepsilon\tilde{\varepsilon})$

4.2. The updating mechanisms of FTRS with the deletion of objects

This subsection researches updating mechanism of FTRS in dynamic IFFD when deleting numerous objects while the attribute set remains constant. Similar to last subsection, the corresponding principle for updating the fuzzy decision value matrix and fuzzy tolerance similarity matrix with the deletion of objects are proposed in IFFD.

Proposition 3. *Supposed that $\daleth = (U, A \bigcup \{d\}, V, f)$ is IFFD, $|U| = \varepsilon$, $U_{\nabla} = \{u_{q_1}, u_{q_2}, \ldots, u_{q_{\tilde{\varepsilon}}}\}$ from $\daleth$ is the deleting object set, and the changed object set is $\tilde{U} = U - U_{\nabla}$ in which $1 \leq q_1 < q_2 < \cdots < q_{\tilde{\varepsilon}}$, $1 \leq r \leq \tilde{\varepsilon}$ and $q_0 = 0$. For any $B \subseteq A$, the previous fuzzy decision value matrix is $FM(d) = [fm_{i1}]_{\varepsilon\times 1}$, which is updated to $FM(\tilde{d}) = [\tilde{fm}_{i1}]_{(\varepsilon-\tilde{\varepsilon})\times 1}$ after deleting objects, where*

$$\tilde{fm}_{i1} = \begin{cases} fm_{(i+r-1,1)}, & q_{r-1} - r + 2 \leq i < q_r - r + 1, \\ fm_{(i+\tilde{\varepsilon},1)}, & q_{\tilde{\varepsilon}} - \tilde{\varepsilon} + 1 \leq i \leq \varepsilon - \tilde{\varepsilon}. \end{cases} \tag{16}$$

Proof. By Definition 6, $FM(d)$ is updated to a new fuzzy decision value matrix $FM(\tilde{d})$ after deleting object set $U_{\nabla} = \{u_{q_1}, u_{q_2}, \ldots, u_{q_{\tilde{\varepsilon}}}\}$ from $\daleth$. For $1 \leq r \leq \tilde{\varepsilon}$, $q_{r-1} + 1 \leq i < q_r$, if u_{q_r} is deleted from $\daleth$, the element fm_{i1} of $FM(d)$ need to be shifted forward by $r - 1$ positions respectively, i.e., $\tilde{fm}_{i1} = fm_{(i+r-1,1)}$. Furthermore, for $q_{\tilde{\varepsilon}} + 1 \leq i \leq \varepsilon$, fm_{i1} of $FM(d)$ also need to be shifted forward by $\tilde{\varepsilon}$ positions, that is, $\tilde{fm}_{i1} = fm_{(i+\tilde{\varepsilon},1)}$.

Proposition 4. *Supposed that $\daleth = (U, A \bigcup \{d\}, V, f)$ is IFFD, $|U| = \varepsilon$, $U_{\nabla} = \{u_{q_1}, u_{q_2}, \ldots, u_{q_{\tilde{\varepsilon}}}\}$ from $\daleth$ is deleting object set, and the changed object set is $\tilde{U} = U - U_{\nabla}$ where $1 \leq q_1 < q_2 < \cdots < q_{\tilde{\varepsilon}}$, $1 \leq r \leq \tilde{\varepsilon}$ and $q_0 = 0$. For any $B \subseteq A$, the previous fuzzy tolerance similarity matrix is $M_U^{ISM_B^{\lambda}} = [tm_{ij}^{ISM_B^{\lambda}}]_{\varepsilon\times\varepsilon}$, which is updated to $M_{\tilde{U}}^{ISM_B^{\lambda}} = [\widetilde{m}_{ij}^{ISM_B^{\lambda}}]_{(\varepsilon-\tilde{\varepsilon})\times(\varepsilon-\tilde{\varepsilon})}$*

Table 6
A new IFFD after deleting objects.

	a_1	a_2	a_3	a_4	a_5	d
u_1	$\langle 0.4, 0.5\rangle$	$\langle 0.3, 0.5\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.7, 0.1\rangle$	0.3
~~u_2~~	~~$\langle 0.3, 0.5\rangle$~~	~~$\langle 0.4, 0.5\rangle$~~	~~$\langle 0.6, 0.1\rangle$~~	~~$\langle 0.4, 0.5\rangle$~~	~~$\langle 0.7, 0.3\rangle$~~	~~0.4~~
~~u_3~~	~~$\langle 0.3, 0.5\rangle$~~	~~$\langle 0.1, 0.8\rangle$~~	~~$\langle 0.8, 0.1\rangle$~~	~~$\langle 0.4, 0.5\rangle$~~	~~$\langle 0.7, 0.3\rangle$~~	~~0.2~~
u_4	$\langle 0.1, 0.8\rangle$	$\langle 0.1, 0.8\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.1, 0.8\rangle$	$\langle 0.8, 0.2\rangle$	0.2
u_5	$\langle 0.7, 0.3\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.8, 0.1\rangle$	0.4
u_6	$\langle 0.3, 0.6\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.5, 0.5\rangle$	$\langle 0.8, 0.2\rangle$	0.4

Table 7
The time complexity of DOFTA-IFFD algorithm.

Steps	Time complexity
$2-9$	$\mathcal{O}(\sum_{r=1}^{\tilde{\varepsilon}}(q_r - q_{r-1} - 2) + \varepsilon - q_{\tilde{\varepsilon}} - 1)$
$10-21$	$\mathcal{O}((\sum_{r=1}^{\tilde{\varepsilon}}(q_r - q_{r-1} - 2)^2) + (\varepsilon - q_{\tilde{\varepsilon}} - 1)^2)$
$22-25$	$\mathcal{O}(\varepsilon - \tilde{\varepsilon})$
Total	$\mathcal{O}(\varepsilon - \tilde{\varepsilon} + \sum_{r=1}^{\tilde{\varepsilon}}((q_r - q_{r-1} - 2) + (q_r - q_{r-1} - 2)^2) + \varepsilon - q_{\tilde{\varepsilon}} - 1 + (\varepsilon - q_{\tilde{\varepsilon}} - 1)^2)$

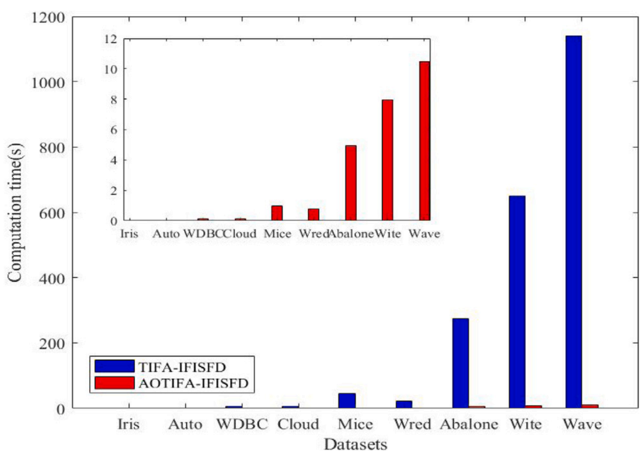


Fig. 1. The computational time of FTA-IFFD and AOFTA-IFFD algorithm under different datasets.

after deleting objects, where

$$\tilde{m}_{ij}^{ISM_B^\lambda} = \begin{cases} tm_{(i+r-1,j+r-1)}^{ISM_B^\lambda}, & q_{r-1} - r + 2 \le i, j < q_r - r + 1, \\ tm_{(i+\tilde{\varepsilon},j+\tilde{\varepsilon})}^{ISM_B^\lambda}, & q_{\tilde{\varepsilon}} - \tilde{\varepsilon} + 1 \le i, j \le \varepsilon - \tilde{\varepsilon}. \end{cases} \tag{17}$$

Proof. By Definition 4, $M_U^{ISM_B^\lambda} = [tm_{ij}^{ISM_B^\lambda}]_{\varepsilon\times\varepsilon}$ is updated to the new fuzzy tolerance similarity matrix $M_{\tilde{U}}^{ISM_B^\lambda} = [\tilde{m}_{ij}^{ISM_B^\lambda}]_{(\varepsilon-\tilde{\varepsilon})\times(\varepsilon-\tilde{\varepsilon})}$ after deleting object set $U_\nabla = \{u_{q_1}, u_{q_2}, \dots, u_{q_{\tilde{\varepsilon}}}\}$ from $\daleth$. For $1 \le r \le \tilde{\varepsilon}$, $q_{r-1} + 1 \le i, j < q_r$, if u_{q_r} is deleted from $\daleth$, the element $tm_{ij}^{ISM_B^\lambda}$ of $M_U^{ISM_B^\lambda}$ need to be shifted forward by $r-1$ positions for row and column respectively, i.e., $\tilde{m}_{ij}^{ISM_B^\lambda} = tm_{(i+r-1,j+r-1)}^{ISM_B^\lambda}$. Meanwhile, for $q_{\tilde{\varepsilon}} + 1 \le i, j \le \varepsilon$, the element $tm_{ij}^{ISM_B^\lambda}$ of $M_U^{ISM_B^\lambda}$ also need to be shifted forward by $\tilde{\varepsilon}$ positions for row and column respectively, that is, $\tilde{m}_{ij}^{ISM_B^\lambda} = tm_{(i+\tilde{\varepsilon},j+\tilde{\varepsilon})}^{ISM_B^\lambda}$. Based on the above analysis, we obtain the following forms

$$\tilde{m}_{ij}^{ISM_B^\lambda} = \begin{cases} tm_{(i+r-1,j+r-1)}^{ISM_B^\lambda}, & q_{r-1} - r + 2 \le i, j < q_r - r + 1, \\ tm_{(i+\tilde{\varepsilon},j+\tilde{\varepsilon})}^{ISM_B^\lambda}, & q_{\tilde{\varepsilon}} - \tilde{\varepsilon} + 1 \le i, j \le \varepsilon - \tilde{\varepsilon}. \end{cases}$$

Therefore, Eq. (17) has been confirmed.

Example 5 (*Continuation of Example 3*)**.** After deleting $U_\nabla = \{u_2, u_3\}$ from Table 1, as shown in the Table 6. The new object set is represented as $\tilde{U} = \{u_1, u_4, u_5, u_6\}$. Based on the Eqs. (16) and (17), fuzzy decision value matrix and fuzzy tolerance similarity matrix are updated as

$$FM(\tilde{d}) = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.4 \\ 0.4 \end{bmatrix}_{4\times1}$$

$$M_{\tilde{U}}^{ISM_A^\lambda} = \begin{bmatrix} 1 & \cancel{0.5577} & \cancel{0} & 0 & 0 & 0.4469 \\ \cancel{0.5577} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0.5569} \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & \cancel{0} & \cancel{0} & 1 & 0 & 0 \\ 0 & \cancel{0} & \cancel{0} & 0 & 1 & 0 \\ 0.4469 & \cancel{0.5569} & \cancel{0} & 0 & 0 & 1 \end{bmatrix}_{6\times6} = \begin{bmatrix} 1 & 0 & 0 & 0.4469 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.4469 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$$

Based on the Eqs. (12) and (13), fuzzy tolerance lower and upper approximation matrix can be computed as $UI = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.4 \\ 0.4 \end{bmatrix}_{4\times1}$ $OI = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \\ 0.4 \end{bmatrix}_{4\times1}$. Based on Definition 5, the fuzzy tolerance lower and upper approximation can be presented as

$$\underline{ISM_B^\lambda} d = 0.3/u_1 + 0.2/u_4 + 0.4/u_5 + 0.4/u_6$$

$$\overline{ISM_B^\lambda} d = 0.4/u_1 + 0.2/u_4 + 0.4/u_5 + 0.4/u_6$$

Now, we design a dynamic algorithm (DOFTA-IFFD) for calculating fuzzy tolerance approximations when deleting numerous objects and analyze its time complexity in IFFD.

- DOFTA-IFFD algorithm (See Algorithm 3): In Algorithm 3, steps 2–9 are to update fuzzy decision value matrix after deleting objects in IFFD. Steps 10–21 are to update fuzzy tolerance similarity matrix after deleting objects in IFFD. Steps 22–25 are to calculate the fuzzy tolerance lower and upper approximation matrix after deleting objects in IFFD. In step 26, the fuzzy tolerance approximation sets in IFFD are returned.
- Time complexity: The time complexity of DOFTA-IFFD algorithm is given in Table 7.
- Comparative analysis of time complexity: The time complexity of FTA-IFFD and DOFTA-IFFD algorithm is compared in Table 8. Obviously, the time complexity of the FTA-IFFD algorithm is much larger than that of the DOFTA-IFFD algorithm. The reason is that DOFTA-IFFD algorithm makes use of prior knowledge to compute fuzzy tolerance approximations and avoids acquired knowledge from scratch in IFFD. Thus, it is very time-consuming for computing new fuzzy tolerance approximations by FTA-IFFD algorithm.

5. Experimental evaluations

This section evaluates the performance of two dynamic algorithms and verifies their effectiveness and efficiency through a series of experiments. Meanwhile, we illustrate the feasibility of the proposed FTRS model in dealing with IFFD. All algorithms in this paper are implemented by MatLabR 2016b. The computer configuration used in this experiment is as follows: CPU is Intel(R) Core(TM) i7-1165G7. Clock Speed is 2.80 GHz. Memory is 16.0 GB. The operating system is 64-bit Windows 10. It is difficulty to obtain intuitionistic fuzzy datasets directly from UCI, which is located at "http://archive.ics.uci.edu/ml/datasets.php". In this paper, nine numerical datasets are downloaded from the UCI and summarized in the Table 9. The intuitionistic fuzzy

Table 8
Comparison of time complexity of FTA-IFFD and DOFTA-IFFD algorithm.

Algorithms	Time complexity
$FTA-IFFD$	$\mathcal{O}(2\varepsilon+\varepsilon^2)$
$DOFTA-IFFD$	$\mathcal{O}(\varepsilon-\tilde{\varepsilon}+\sum_{r=1}^{\tilde{\varepsilon}}((q_r-q_{r-1}-2)+(q_r-q_{r-1}-2)^2)+\varepsilon-q_{\tilde{\varepsilon}}-1+(\varepsilon-q_{\tilde{\varepsilon}}-1)^2)$

Algorithm 3 DOFTA-IFFD algorithm

Input: An initial IFFD $\daleth=(U,A\bigcup\{d\},V,f)$, $|U|=\varepsilon$, parameter λ, initial fuzzy decision value matrix $FM(d)$, initial fuzzy tolerance similarity matrix $M_U^{ISM_A^\lambda}$, and the deleting object set $U_\nabla=\{u_{q_1},u_{q_2},\cdots,u_{q_{\tilde{\varepsilon}}}\}$.

Output: The new TIFA $\underline{ISM_A^\lambda}d(\widetilde{u_i}),\overline{ISM_A^\lambda}d(\widetilde{u_i})$ on $U-U_\nabla$

Let $q_0=0$
for $r=1$ to $\tilde{\varepsilon}$ **do**
  for $i=q_{r-1}-r+2$ to q_r-r **do**
    $\tilde{fm}_{i1}=fm_{(i+r-1,1)}$
  end for
end for
for $i=q_{\tilde{\varepsilon}}-\tilde{\varepsilon}+1$ to $\varepsilon-\tilde{\varepsilon}$ **do**
  $\tilde{fm}_{i1}=fm_{(i+\tilde{\varepsilon},1)}$
end for
for $r=1$ to $\tilde{\varepsilon}$ **do**
  for $i=q_{r-1}-r+2$ to q_r-r **do**
    for $j=q_{r-1}-r+2$ to q_r-r **do**
      $\widetilde{m}_{ij}^{ISM_A^\lambda}=tm_{(i+r-1,j+r-1)}^{ISM_A^\lambda}$
    end for
  end for
end for
for $i=q_{\tilde{\varepsilon}}-\tilde{\varepsilon}+1$ to $\varepsilon-\tilde{\varepsilon}$ **do**
  for $j=q_{\tilde{\varepsilon}}-\tilde{\varepsilon}+1$ to $\varepsilon-\tilde{\varepsilon}$ **do**
    $\widetilde{m}_{ij}^{ISM_A^\lambda}=tm_{(i+\tilde{\varepsilon},j+\tilde{\varepsilon})}^{ISM_A^\lambda}$,
  end for
end for
for $i=1$ to $\varepsilon-\tilde{\varepsilon}$ **do**
  Calculate $UI=[\underline{\underline{ISM_A^\lambda}}d_{i1}]_{(\varepsilon-\tilde{\varepsilon})\times 1}$ via using Eq.(12)
  Calculate $OI=[\overline{\overline{ISM_A^\lambda}}d_{i1}]_{(\varepsilon-\tilde{\varepsilon})\times 1}$ via using Eq.(13)
end for
return $\underline{ISM_A^\lambda}d(\widetilde{u_i}),\overline{ISM_A^\lambda}d(\widetilde{u_i})$

Table 9
The datasets.

No.	Dataset	Abbreviation	Sample	Features
1	Iris	Iris	150	4
2	Auto MPG	Auto	398	7
3	Wisconsin Diagnostic Breast Cancer	WDBC	569	30
4	Cloud	Cloud	1024	10
5	Mice Protein Expression	Mice	1077	68
6	Wine Quality-red	Wred	1599	11
7	Abalone	Abalone	4177	7
8	Wine Quality-white	Wite	4898	11
9	Waveform	Wave	5000	21

information system can be constructed by normalizing and redefining these numerical datasets. In the whole experiment, we set parameter λ as 0.44.

Now, we propose the constructed IFIS method from numerical information systems. Assumed that (U,A,V,f) is a numerical information system and $f(u,a)$ is a real-valued. For any $u\in U$ and $a\in A$, the normalized data can be computed by the formula

$$\grave{f}(u,a)=\frac{f(u,a)-min(V_a)}{max(V_a)-min(V_a)}. \tag{18}$$

Next, we transform the normalized data set into intuitionistic fuzzy number concerning threshold value ω_a as follows:

$$m^a(u)=\grave{f}(u,a), \tag{19}$$

$$n^a(u)=\omega_a*(1-\grave{f}(u,a)). \tag{20}$$

Therefore, the real-valued $f(u,a)$ is converted to intuitionistic fuzzy number $\langle m^a(u),n^a(u)\rangle$ by the above formulas and the corresponding IFIS can be also acquired from (U,A,V,f). Meanwhile, fuzzy decisions of IFIS can be generated by Matlab R2016b, where fuzzy decision is fuzzy set under the universe. In the whole experiment, threshold value ω_a can be ruled as 0.6 for each $a\in A$.

5.1. Performance evaluations of AOFTA-IFFD algorithm with the insertion of objects

The performance of the AOFTA-IFFD algorithm in aspects of effectiveness and efficiency is evaluated by designing two groups of comparative experiments in this subsection.

In terms of effectiveness, the top fifty percent of objects are sampled as the corresponding initial object set U for each dataset in Table 9. The remaining fifty percent of the objects can be treated as the inserting objects U_Δ for each dataset. Next, we calculate the fuzzy tolerance approximation sets by using FTA-IFFD algorithm and AOFTA-IFFD algorithm in IFFD when U_Δ is inserted to U for each dataset. After comparing the computational time of FTA-IFFD algorithm and AOFTA-IFFD algorithm, the effectiveness of the AOFTA-IFFD algorithm is demonstrated and the experimental results are shown in Fig. 1, where horizontal and vertical coordinates stand for the different datasets and the running time of algorithms, respectively. It is worth noting that the computational time of AOFTA-IFFD algorithm is far less than that of FTA-IFFD algorithm. Hence, the effectiveness of AOFTA-IFFD algorithm is verified.

In terms of efficiency, the top fifty percent of the objects are also selected as the initial object set U for each dataset. Different ratios of objects sampled from the remaining objects U_Δ are inserted to U to obtain dynamic datasets, that is to say, 10%, 20%, 30%, 40%, 50%, 60%, 70%, 80%, 90% and 100% of U_Δ are inserted to U. The experimental results are shown in Fig. 2, where horizontal coordinate refers to the ratio of inserting objects and vertical coordinate embodies the computational time to obtain the fuzzy tolerance approximation sets in IFFD. In each subplot of Fig. 2, it can be seen that the FTA-IFFD algorithm takes significantly more time to compute than the AOFTA-IFFD algorithm. The main reason is that AOFTA-IFFD algorithm obtains fuzzy tolerance approximations on basis of the previously fuzzy tolerance similarity matrix and fuzzy decision value matrix in IFFD, and FTA-IFFD algorithm has been repeatedly calculating acquired knowledge. As the number of inserted object set increases, the time consuming trend of FTA-IFFD algorithm increases significantly faster than that of the AOFTA-IFFD algorithm. Therefore, the efficiency of AOFTA-IFFD algorithm is proved.

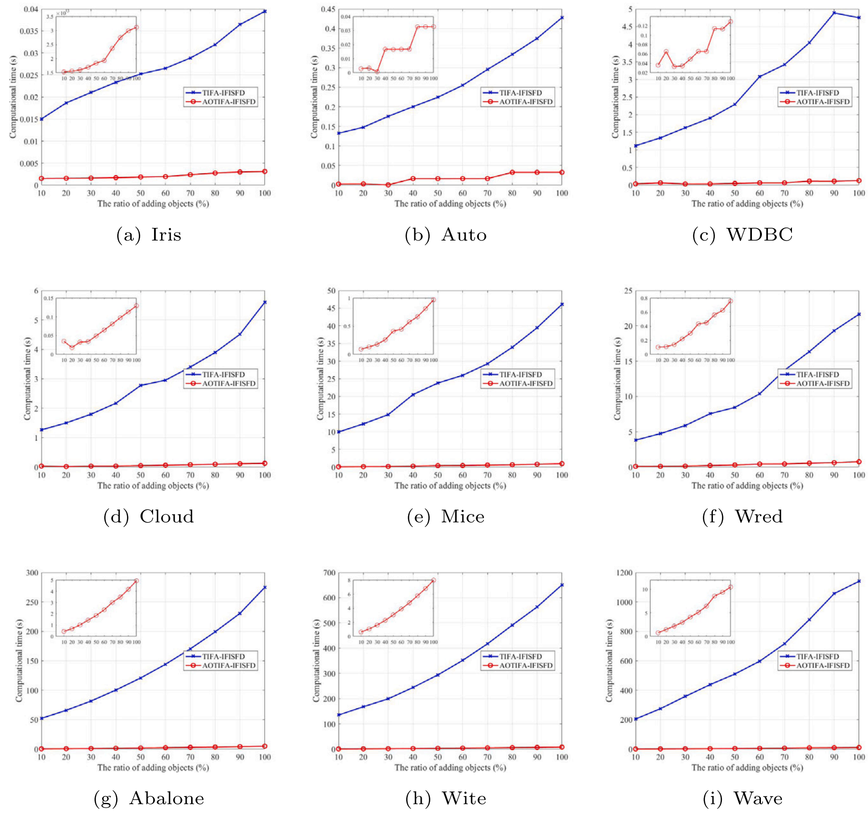


Fig. 2. The computational time of FTA-IFFD and AOFTA-IFFD algorithm under different ratios associated with inserting objects.

5.2. Performance evaluations of DOFTA-IFFD algorithm with the deletion of objects

Similar to the evaluation method in Section 5.1, the effectiveness and efficiency of the DOFTA-IFFD algorithm are evaluated in this subsection.

In terms of effectiveness, we default object set of every detaset in Table 9 to the initial object set U and randomly select fifty percent of the initial objects as the set of deleted objects U_{∇}. DOFTA-IFFD and FTA-IFFD algorithm can be used to calculate new fuzzy tolerance approximation sets in IFFD when objects are deleted. Afterwards, we compare the running time of DOFTA-IFFD and FTA-IFFD algorithm and confirm the effectiveness of DOFTA-IFFD algorithm. The experimental results given in Fig. 3 where horizontal and vertical coordinates stand for the different datasets and the running time of algorithms, respectively. It is obvious that the computational time of FTA-IFFD algorithm is far more than that of DOFTA-IFFD algorithm. As the number of sampled object set increases, the time consuming trend of FTA-IFFD algorithm increases significantly faster than that of the DOFTA-IFFD algorithm. Therefore, the experimental results indicate that DOFTA-IFFD algorithm is effective.

In terms of efficiency, object set of every dataset in Table 9 can be viewed as the initial object set U. The deleting object set U_{∇} can be obtained by randomly drawing different ratios of objects from the initial object set U, i.e. 5%, 10%, 15%, 20%, 25%, 30%, 35%, 40%, 45%, 50% of U are removed to create new tested object set $U - U_{\nabla}$. In Fig. 4, it can be seen that the trend lines of the variation of the FTA-IFFD and DOFTA-IFFD algorithms where horizontal coordinate refers to the ratio of deleting objects and vertical coordinate embodies the computational time to obtain the fuzzy tolerance approximation sets in IFFD. Obviously, the computational time of FTA-IFFD algorithm is far more than DOFTA-IFFD algorithm from every subfigure of Fig. 4. Similarly, as the number of deleting object set increases, the time consuming trend of FTA-IFFD algorithm increases significantly faster than that of the DOFTA-IFFD algorithm. The DOFTA-IFFD algorithm obtains the fuzzy tolerance approximation sets on basis of original fuzzy tolerance similarity matrix and original fuzzy decision value matrix in order to avoid repeated calculations. Therefore, the efficiency of DOFTA-IFFD algorithm is proved.

5.3. Influence of the parameter

The parameter λ plays a crucial role in determining the tolerance radius for similarity measurement in the FTRS model. In this subsection, we will investigate the impact of different parameters on the approximation accuracy of FTRS model and the perturbation of approximation accuracy for different noise levels under various parameters.

Each dataset in Table 9 is considered as the original dataset. The value of parameter λ is ranged from 0.1 to 1 with a step size of 0.1. Table 10 illustrates the variation in approximate accuracy of the FTRS model with respect to the parameter λ across 9 datasets. By executing the FTRS model with different parameter configurations on specific datasets, we obtain different upper and lower approximations, thereby observing varying degrees of approximate accuracy across different parameters. Encouragingly, the FTRS model demonstrates relatively high levels of approximate accuracy on most parameters for the majority of the datasets. Based on the analysis presented above, our proposed FTA-IFFD model exhibits feasibility in addressing IFFD-related issues.

Meanwhile, we investigate the perturbation of approximation accuracy with different parameters and noise levels using four arbitrarily selected datasets from Table 9. For each dataset, we employ Eqs. (18), (19) and (20) to transform the real-valued data into intuitionistic fuzzy numbers. Based on these intuitionistic fuzzy numbers, noisy datasets are generated by introducing random noise at varying proportions. Subsequently, we ascertain the approximation accuracy of FTRS model under different parameter settings by employing datasets with varying

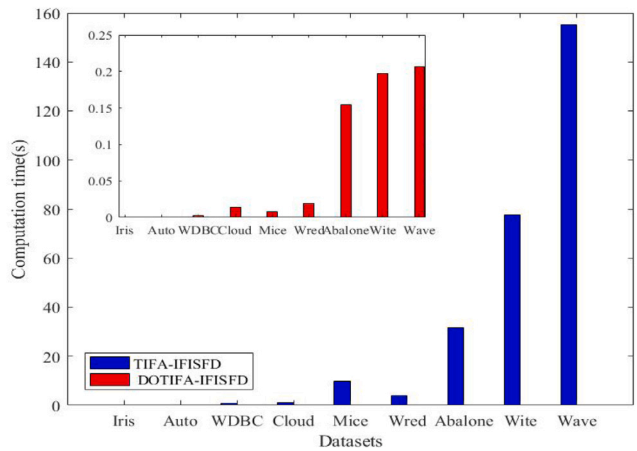


Fig. 3. The computational time of FTA-IFFD and DOFTA-IFFD algorithm under different datasets.

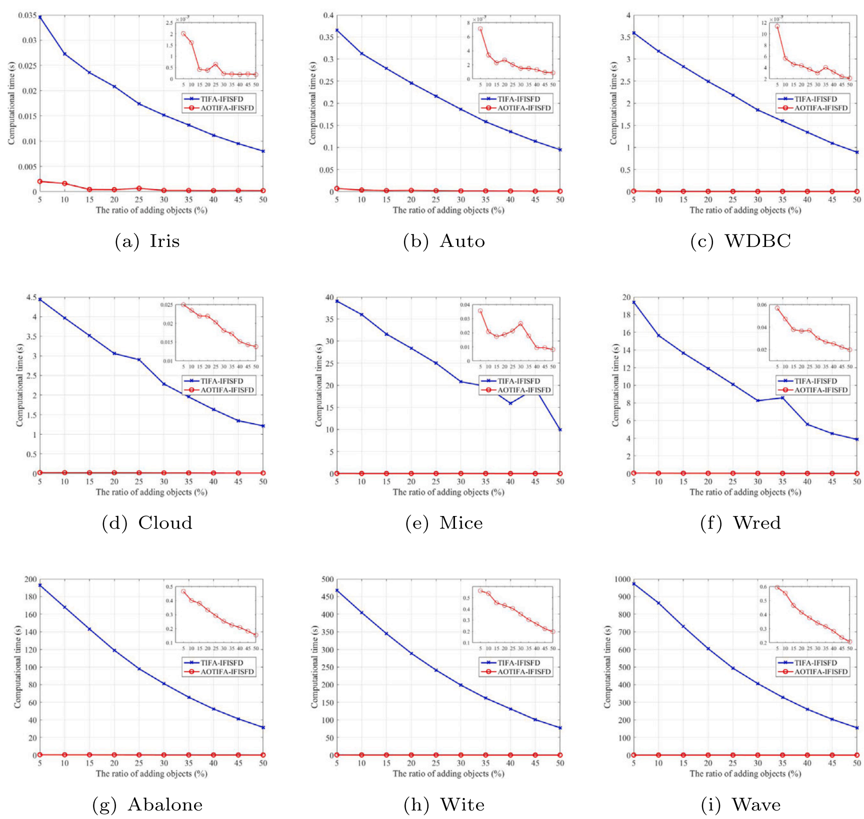
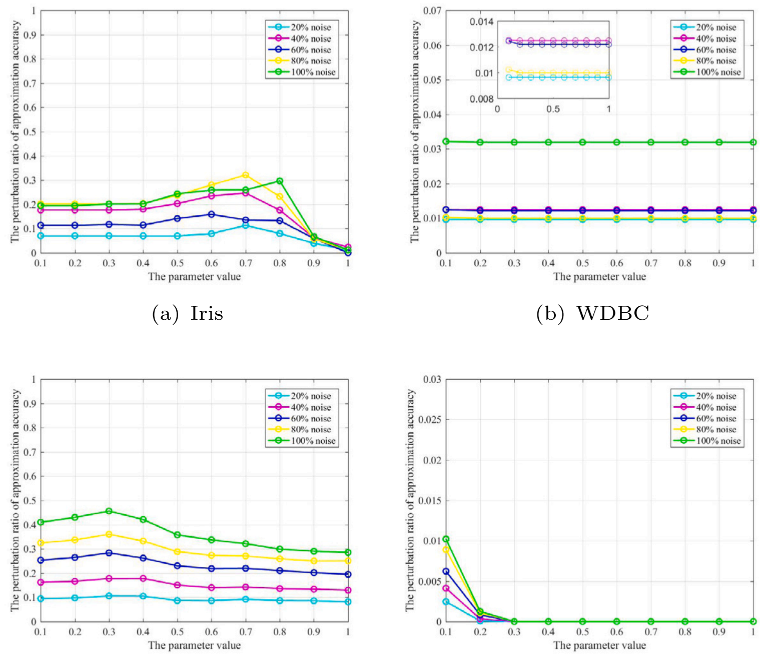


Fig. 4. The computational time of FTA-IFFD and DOFTA-IFFD algorithm under different ratios associated with deleting objects.

Table 10
The approximation accuracy of FTRS model versus different parameter λ.

Dataset	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$	$\lambda = 0.6$	$\lambda = 0.7$	$\lambda = 0.8$	$\lambda = 0.9$	$\lambda = 1$
Iris	0.2800	0.2800	0.2800	0.2825	0.2836	0.3045	0.3542	0.5005	0.8177	0.9223
Auto	0.3974	0.3974	0.3975	0.3990	0.4073	0.4510	0.5941	0.7633	0.8586	0.8606
WDBC	0.8813	0.8815	0.8815	0.8815	0.8815	0.8815	0.8815	0.8815	0.8815	0.8815
Cloud	0.3988	0.3989	0.3992	0.4011	0.4100	0.4700	0.6249	0.8266	0.8752	0.8779
Mice	0.9976	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Wred	0.5425	0.5437	0.5549	0.5868	0.6281	0.6494	0.6618	0.6771	0.6848	0.6898
Abalone	0.1882	0.1882	0.1882	0.1882	0.1884	0.1895	0.1956	0.2514	0.7980	0.9991
Wite	0.5350	0.5356	0.5408	0.5648	0.6129	0.6622	0.7013	0.7171	0.7376	0.7513
Wave	0.9736	0.9985	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000



(a) Iris (b) WDBC

(c) Wred (d) Wave

Fig. 5. The perturbation of approximation accuracy for different noise levels under various parameters.

levels of noise. Then, the perturbation ratios of approximation accuracy (p) under various parameters were determined by utilizing the approximation accuracy of noise data (nda) and original data (oda), i.e. $p = \frac{|nda-oda|}{oda}$. The experimental results, depicted in Fig. 5, showcase the relationship between parameter values and perturbation ratios of approximation accuracy. The $x-$axis represents the parameter values, while the $y-$axis denotes the perturbation ratio of approximation accuracy. Each subplot illustrates five lines representing the perturbation ratio of approximation accuracy of the model under different levels of noise. It is readily apparent that, following the introduction of noise, the perturbation of approximation accuracy in model remain within the proportion of the added noise. This observation underscores the model's ability to adapt to noise perturbations.

5.4. The comparison analysis of approximation accuracy under dynamic environment

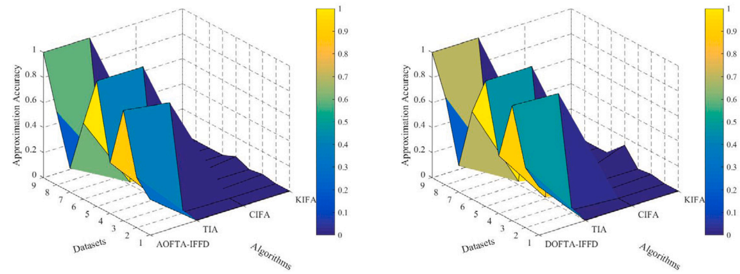
In this subsection, three other relevant algorithms proposed by other researchers have been selected as comparative methods to assess their performance against the algorithm proposed in this study. The following are descriptions of these three relevant algorithms:

- Algorithm TIA. Tiwari introduced the concept of tolerance classes and present tolerance-based intuitionistic fuzzy lower and upper approximations in IFIS [33]. In light of pertinent information, we have devised an algorithm named TIA to obtain corresponding upper and lower approximation sets. The model incorporates the parameters, namely $\alpha = 0.5$, $\beta = 0.4$, $\gamma = 0.1$, $\lambda = 0.44$ and $\kappa = 0.2$.
- Algorithm CIFA. Liu defined the conflict distance by using the idea of measuring intuitionistic fuzzy similarity and gave the intuitionistic fuzzy rough set model in IFIS [51]. We have designed the algorithm CIFA based on the corresponding model, incorporating parameters $\lambda = 0.44$ and $\kappa = 0.2$.
- Algorithm KIFA. Li established the kernel similarity relation under the background of IFIS and introduced a novel rough model [52]. Building upon this concept, we developed the KIFA algorithm by incorporating relevant parameters, specifically setting $\alpha = 0.7$, $\lambda = 0.44$ and $\kappa = 0.2$.

By comparing their approximation accuracy at a fixed increment ratio, we further validate the effectiveness of the AOFTA-IFFD and DOFTA-IFFD algorithms. In the data dynamic addition experiment, we consider the first fifty percent of objects from each dataset in Table 9 as the original object set, while the remaining objects are regarded as additional objects. In the data dynamic deletion experiment, we use the object sets from each dataset in Table 9 as the original object set, and then randomly select fifty percent of objects from each dataset as the deletion objects. Subsequently, we employ different algorithms to compute the upper and lower approximation sets corresponding to various rough set models. Meanwhile, we record the approximation accuracy for each experiment, and the experimental results are presented in Fig. 6. The $x-$axis represents the sequence number of nine different datasets in Table 9, while the $y-$axis denotes four algorithms. The $z-$axis represents the corresponding level of approximate accuracy. Clearly, from Fig. 6, it is evident that our proposed algorithm achieves higher approximation accuracy than the other algorithms for each dataset. This observation highlights the effectiveness of the proposed algorithm in this study.

5.5. Statistical test

In this subsection, the statistical significance of the algorithms used in the statistical experiment was assessed using Friedman test (F-test) and the Nemenyi post-hoc test (N-test). The algorithms in this experiment were divided into two categories: Type 1 included algorithms AOFTA-IFFD, TIA, CIFA, and KIFA, with a data background where fifty percent of objects in each dataset were labeled as original objects while the remaining objects were labeled as added experimental objects. Type 2 included algorithms DOFTA-IFFD, TIA, CIFA, and KIFA, with a data background where all objects in the datasets were considered original objects, and fifty percent of objects were randomly selected as deleted experimental objects. By conducting the F-test, we aimed to determine if these algorithms have similar performance. In the F-test, we set the significance level α to 0.1 and the null hypothesis as "all algorithms have the same performance". According to the F-test results presented in Table 11, all the different types of τ_F values exceeded the critical



(a) The addition of objects

(b) The deletion of objects

Fig. 6. The comparison of approximation accuracy of different algorithms in dynamic environment.

(a) Type 1

(b) Type 2

Fig. 7. N-test results of different types of algorithms.

Table 11
F-test results of different types of algorithms.

Category	τ_F	*Critical value*($\alpha = 0.1$)
Type 1	82	3.619
Type 2	17.8	3.619

value. Thus, we rejected the null hypothesis, indicating that there are significant differences among the algorithms.

Since the F-test can only determine whether there are significant differences among the measurement results of multiple models and cannot assess the differences between any two models, we employed the N-test to examine the significant differences between any two algorithms. The N-test results are depicted in Fig. 7, where the numerical lines indicate the average rankings of each algorithm. If algorithms are connected by a horizontal line, it signifies that there is no significant difference between the respective algorithms. From Fig. 7(a), algorithm AOFTA-IFFD exhibits significant differences compared to algorithms CIFA and KIFA, but shows no significant difference compared to algorithm TIA. On the other hand, Fig. 7(b) demonstrates that our algorithm DOFTA-IFFD exhibits significant differences when compared to algorithms TIA, CIFA, and KIFA. Therefore, this statistical test also corroborates that the proposed algorithms, AOFTA-IFFD and DOFTA-IFFD, demonstrate relatively higher computational efficiency.

6. Conclusion

This study presents a novel fuzzy tolerance rough set model in IFFD, aiming to enhance the efficiency of computing fuzzy tolerance approximations in IFFD when objects are added or deleted. To achieve this objective, we propose two mechanisms to update the fuzzy tolerance rough approximations based on IFFD from a matrix perspective. Additionally, we develop corresponding dynamic algorithms to dynamically update the fuzzy tolerance rough approximations in IFFD. To validate the feasibility, effectiveness, and efficiency of the proposed incremental algorithms, a series of experiments are conducted in this paper using nine UCI datasets as test samples. The experimental results demonstrate the favorable feasibility of our proposed model in handling IFFD. Moreover, compared to three other relevant models, our method exhibits higher approximation accuracy and computational efficiency. Furthermore, the performance of the dynamic algorithm proposed in this paper outperforms that of the static algorithm, further highlighting the effectiveness and efficiency advantages of our proposed update mechanism. In the future, we will consider incremental updating fuzzy tolerance rough set approaches in IFFD with time-evolving attributes and attribute values.

CRediT authorship contribution statement

Lu Wang: Methodology, Writing – original draft, Funding acquisition. **Zheng Pei:** Conceptualization, Methodology. **Keyun Qin:** Conceptualization, Funding acquisition. **Lei Yang:** Data curation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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