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An overlap function-based three-way intelligent decision model under interval-valued fuzzy information systems

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ABSTRACT

One of the expression forms of uncertain information are interval-valued fuzzy information systems (IFISs). Whereas, psychological experiments show that psychological behaviors of decision maker (DM) affect the decision results under uncertain information. Consequently, the main research objectives of this paper are two-fold: 1. To study how to portray the influence of psychological behaviors on decision-making under IFISs, so that the decision-making process is more realistic. 2. To investigate how to efficiently solve decision-making problems in the context of IFISs. Three-way decision (3WD) theory and fuzzy rough sets (FRSs) are effective tools for solving uncertain information. For that reason, this paper develops a 3WD model based on prospect theory (PT) under IFISs, which can provide the best decision action for objects from a viewpoint of optimization. First, an attribute-oriented interval fuzzy set is introduced, which establish a bridge between the state sets in 3WD and the reference points in PT. Furthermore, conditional probability of an object is reasonably estimated by interval-valued fuzzy rough sets (IVFRSs). Meanwhile, an attribute-oriented relative value model is proposed, which can get the prospect values of an object under each action. Second, an intelligent 3WD model is developed from a viewpoint of information granularity. Finally, through the search of the existing literature, relevant research data is obtained, and then case research method, qualitative and quantitative combination method, control variable method are adopted in turn to design the case study, comparative and experimental analyses. In short, the developed model not only expands the development of related theories, 3WD, PT, FRSs, but also offers new methods for solving the decision-making problems under IFISs, which has important practical guidance significance.

1. Introduction

As the technology advances rapidly, uncertainty information is increasing. How to use existing uncertain information to make scientific decision analysis and provide effective intelligent decision-making results for practical problems have become important research directions. For instances, existing multi-attribute border approximation area comparison (MABAC) method (Pamučar & Ćirović, 2015; Wang, Wei, Wei, & Wei, 2020), the multi-attribute COmplex PRoportional ASsessment of alternatives (COPRAS) method (Kaklauskas et al., 2006; Zhang, Dai, & Wan, 2023), MultiAttributive Ideal-Real Comparative Analysis (MAIRCA) method (Haq, Saeed, Mateen, Siddiqui, & Ahmed, 2023; Pamučar, Vasin, & Lukovac, 2014), VIKOR method (Sayadi, Heydari, & Shahanaghi, 2009) can provide reasonable ranking results for objects. In addition, 3WD theory (Yao, 2010) incorporates Bayesian decision process that can provide effective decision-making action for objects. This provides theoretical support for solving uncertainty decision problems. A 3WD model developed Zhan, Jiang, and Yao (2021) provides a solution to the enterprise project investment decision-making problem. These methods address the decision-making problem from different perspectives, which are the basis of the work in this paper. The benefits and weaknesses of some existing methods are shown in Table 1.

According to Table 1 it can be seen:

(1) MABAC is a ranking method based on the gap between object and border approximation area. However, there is a weakness in the determination of border approximation area under this methodology. That is, if the evaluation values of most objects under a certain attribute are all large, and only one object has a value of 0, then the value of border approximation area under this attribute is 0 by the action of continued multiplication. Obviously, in the actual decision-making process, this is too absolute.

(2) COPRAS is a ranking method by the relative weight of each object. However, the relative weight value of objects under this method may exceed the range that set by normalization. For example, in order to eliminate dimensions under different attributes, the data is normalized to the interval $[0, 1]$, but according to the calculation method

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Table 1
Characteristics of different methods.

Method	Ranking	Classification	Risk	Ranking/Classification principle
MABAC	Yes	No	No	Distance from border approximation area
COPRAS	Yes	No	No	Relative weight value
MAIRCA	Yes	No	No	Distance between theoretical ponder and actual ponder
VIKOR	Yes	No	Yes	Compromise solution
Zhan et al.'s method	Yes	Yes	Yes	Risk minimization

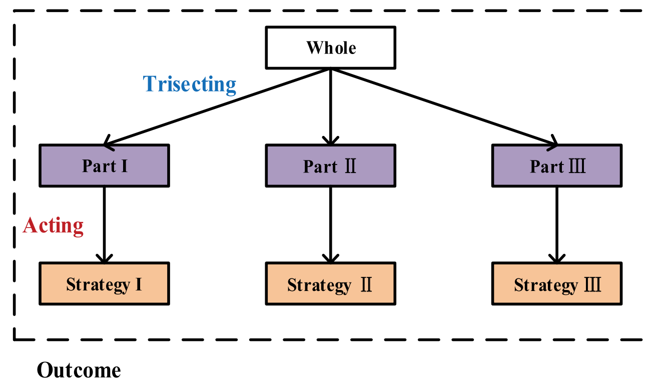

Fig. 1. Trisecting-acting-outcome model.

(addition operation), the relative weight value will exceed 1, which will cause this method to invalid and the ranking result is unreasonable.

(3) MAIRCA is a ranking method according to the gap between theoretical ponder matrix and actual ponder matrix. Nevertheless, the mechanism has same theoretical ponder values for different objects, that is, the determination of the theoretical ponder matrix is only related to preference probability of object and weight value of attribute, and it is independent of the characteristics (evaluation value) of object under each attribute, which makes the theoretical ponder matrix can be designed arbitrarily.

(4) VIKOR is a ranking method that maximizes group utility and minimizes individual regret through a compromise, and is an optimization decision model. It adds a decision-making mechanism coefficient that can be well integrated into the subjective preferences of DM. In addition, this method considers compromise solution with prioritization, which makes for more than one optimal solution or complete ranking result.

(5) A decision-making method based on 3WD with ranking and classification function proposed by Zhan et al. (2021). Its well avoided the weaknesses of MABAC, COPRAS, MAIRCA, VIKOR methods that cannot provide decision-making action for each object. However, this method follows the principle of risk minimization, but Tversky and Kahneman's psychological experiments (Kahneman, 1979; Tversky & Kahneman, 1992) have shown that DMs do not always avoid risk in actual decision-making. That is, the method is inconsistent with some actual situations.

In short, some of the existing decision-making methods can only provide ranking result and cannot obtain classification decision-making result. As for the decision-making problems in the context of big data, the objects themselves are more concerned about their own decision-making results. For example, in a national level exam, candidates will be more concerned about whether they pass this exam or not rather than ranking result. Thus, given the shortcomings of existing research methods, the main objectives of this paper are mainly: 1. How to design a rational decision-making model that can scientifically consider the effects of different psychological behaviors on decision-making. 2. How to build a decision model that can simultaneously achieve classification decision-making and ranking, and thus can provide a new paradigm for solving real-world decision-making problems.

Moreover, uncertain information is also one of the crucial research subjects in RS theory, which has become a valid mathematical tool to solve uncertainty information. In traditional RS models, uncertainty information is represented in the shape of an information system, such as triangular fuzzy information systems (Li, Wang, Liang, & Yi, 2021), IFISs (Liu, Dai, Chen, & Zhang, 2021), and hesitant fuzzy information systems (Zhan, Wang, Ding, & Yao, 2023). In this paper, we prefer to focus on modeling issues under IFISs. For decision-making problems under IFISs, many relevant research achievements (Dymova, Sevastjanov, & Tikhonenko, 2013; Hafezalkotob, Hafezalkotob, & Sayadi, 2016; Jiang & Hu, 2022; Ren & Toniolo, 2018; Sayadi et al., 2009) have been proposed. However, the existing research results still have deficiencies. For example, some works assumes that the decision-making process is perfectly rational (Dymova et al., 2013; Sayadi et al., 2009), and some works ignores irrational psychological behaviors (Jiang & Hu, 2022). Therefore, in order to scientifically and effectively solve the decision-making problem under IFISs, a novel decision model on the basis of PT and 3WD theory will be introduced in this paper. In the paragraphs below, we mainly retrospect the research progress of the relevant theories.

3WD theory was introduced by Yao (2010). It gives a rational semantic explanation for positive region ($Pos(X)$), boundary region ($Bnd(X)$), and negative region ($Neg(X)$) in probabilistic RSs. 3WD theory provides a theoretical support for the application of RS theory in actual decision problems. Later, given the decision advantage that it fits human cognition, 3WD has been widely developed. It can be broadly divided into narrow sense and wide sense. There are two critical components in the narrow sense, namely conditional probability and loss functions. In response to overly strict conditions for conditional probability based on equivalence classes in classical 3WD model (Yao, 2010), Zhan et al. (2021) introduced conditional probability based on outranked class. Conditional probability based on partial-overall dominance class was proposed by Yang, Deng, and Fujita (2020). There are also many studies on loss functions. For instance, the relative loss function was defined by Jia and Liu (2019) from a viewpoint of evaluation value of an object. This not only greatly reduces the subjectivity of 3WD model (Yao, 2010), but also unifies the evaluation information of an object and loss functions. Later, Liu, Wang, Jia, and Fujita (2020) extended Jia and Liu's model (Jia & Liu, 2019) to intuitionistic fuzzy environment. Furthermore, Li, Wang, Sun, et al. (2021) designed a threshold determination approach based on weight entropy, and established a 3WD theory on general information tables. However, the above models (Jia & Liu, 2019; Li, Wang, Sun, et al., 2021; Liu et al., 2020; Yang et al., 2020; Yao, 2010; Zhan et al., 2021) all follow the decision rule of risk minimization. That is, decision makers (DMs) always select an action with the least loss/risk, which is obviously inconsistent with reality. To address the above deficiencies, a 3WD model based on PT was developed Wang, Li, Zhou, Huang, and Zhu (2020), which adequately considered impact of psychological behavior on decision-making outcomes. Nevertheless, prospect matrix in Wang et al.'s model (Wang, Li, et al., 2020) is highly subjective randomness. Afterwards, Deng, Zhan, Ding, Liu, and Pedrycz (2022) improved the shortcomings, and extended it to multi-scale information systems. Additionally, the wide sense 3WD model (Yao, 2021) is a ternary process, as shown in Fig. 1. It mainly solves classification and decision-making problems, for example three-way c−means (Zhang, 2019). In summary, the existing researches on decision making problems under IFISs are not sufficient, so building a three-way intelligent decision model is the focus of this paper.

Table 2
Relationship between various RSs and FSs.

Type	Form of the concept	Relationship
RS	Classical sets	Equivalence relation
Fuzzy sets	Membership function	✗
Rough fuzzy sets	Fuzzy sets	Equivalence relation
FRSs	Fuzzy sets	Fuzzy equivalence relation
Generalized Rough fuzzy sets	Fuzzy sets	General relation
Generalized FRSs	Fuzzy sets	Fuzzy relation
IVRFSs	Interval-valued Fuzzy sets	Equivalence relation
IVFRSs	Interval-valued Fuzzy sets	Interval-valued fuzzy equivalence relation
Generalized IVRFSs	Interval-valued Fuzzy sets	General relation
Generalized IVFRSs	Interval-valued Fuzzy sets	Interval-valued fuzzy relation

RS theory (Pawlak, 1982) serves as a valid tool to address various kinds of incomplete information such as imprecision, inconsistency, and incompleteness. The core of this theory is that the equivalence relation can be used to compute the lower and upper approximations of a concept. Afterwards, due to the mature mathematical foundation and its practicality, RS theory has been extensively developed. For instance, Han (2020) explored the theoretical properties of MW-rough topological approximations, and then developed a MW-topological RS structure. Ye, Sun, Zhan, and Chu (2022) introduced four variable precision multi-decision multi-granulation RT models, and addressed decision problem of medical diagnosis. Although the relevant theory and applications of RSs have been widely researched, the related models can only solve problem under discrete data. After that, given that fuzzy sets (FSs) and RSs have strong complementarity, some researchers integrate these two theories to deal with uncertainty, and its can show strong functionality. For example, (Dubois & Prade, 1990) proposed fuzzy rough sets (FRSs) by combining fuzzy set theory with RS theory, which greatly expanded the application of RSs. (Zhang & Dai, 2022) improved TOPSIS method with the help of decision-theoretic rough fuzzy sets (RFSs). Yao, Zhang, and Zhou (2023) established FRSs based on real-valued hemimetric and applied it to digital surface contour extraction. From a viewpoint of granular, Palangetić, Cornelis, Greco, and Słowiński (2022) considered OWA-based FRSs. Qiao (2021) investigated the (I_O, O)-FRSs via overlap functions. Moreover, Sun, Gong, and Chen (2008) discussed interval-valued fuzzy rough sets (IVFRSs), and investigated the related properties. Since then, many IVFRSs generalized models based on IFISs have been presented. Under IFISs with fuzzy decision, a fuzzy neighborhood RS model and its matrix representation were proposed by Yang, Qin, Sang, and Xu (2021). The relationship between various types of RSs and FSs is shown in Table 2. In conclusion, considering that the main purpose of this paper is to study uncertain decision-making problems under IFISs, this paper plans to choose IVFRSs environment for further discussion.

PT is a research achievement of psychology and behavioral science, developed by Kahneman (1979), Tversky and Kahneman (1992). PT not only scientifically shows the influence of psychological behavior on decision-making outcomes, but also rejects an assumption that the decision-making process is perfectly rational. Thus, PT provides a scientific support for studying uncertainty decision-making. Xu, Zhou, and Xu (2011) put forward a cumulative PT-based decision rule, which resolves travelers' route choice problem. Leoneti and Gomes (2021) derived reasonable parameter values for the exponential function in PT by surveying 100 students from different majors at São Paulo university. In addition, with the help of PT, a variant of TODIM, i.e. ExpTODIM, was proposed, which not only better fitted the actual decision situation but also provided more accurate predictions for individual decisions. Nevertheless, these methods only rank and cannot provide the corresponding decision for objects. This in not perfect. After that, Wang, Li, et al. (2020) integrated 3WD theory and PT and established a PT-based 3WD model, which solved the problem that some methods could not provide decisions. Later, Deng et al. (2022) and Wang, Ma, Xu, Pedrycz, and Zhan (2022) developed PT-based 3WD model under hesitant fuzzy information systems and multi-scale information systems, respectively. This expands the application of PT in the term of decision-making. However, the decision-making problem under IFISs, as one of the uncertain decision-making problems. Thus, it is also essential to further explore the influence of psychological behavior on the decision-making. In brief, considering PT under IFISs is also a research direction of this study.

In light of the above statements, the following are the research motivations of this study.

(1) The decision-making methods (Dymova et al., 2013; Hafezalkotob et al., 2016; Haq et al., 2023; Ren & Toniolo, 2018; Sayadi et al., 2009; Wang, Wei, et al., 2020; Zhang et al., 2023) already proposed under IFISs only obtain ranking results and cannot offer decision-making action for each object. Thus, for decision-making problem in the era of big data, a decision-making model with both ranking and classification functions will be built.

(2) Although the existing 3WD models (Jiang & Hu, 2022; Liang & Liu, 2014; Yang et al., 2020; Zhang & Yang, 2019) under IFISs can provide decision actions for objects, these models always blindly pursue risk minimization. Psychological experiments (Kahneman, 1979; Tversky & Kahneman, 1992) show that in the face of losses DMs will avoid risk; and risk-seeking in the face of gains. Hence, this study intends to build a 3WD model based on PT under IFISs, which can make the decision model more reasonable.

(3) In the framework of 3WD based on PT, an object eventually chooses an action with maximal prospect value. There are many existing definitions (Cornelis, Deschrijver, & Kerre, 2004; Fishburn, 1973; Liu, Wang, & Zhang, 2001; Sengupta & Pal, 2000; Xu & Da, 2002; Yang et al., 2020; Zhang & Yang, 2019; Zhao, Bao, & Guan, 2013) for the order relation between interval values, but all of them are subjective. For example, in θ ranking method (Liu et al., 2001), the determination of the θ is obtained by DM based on social experience. Thus, we aim to establish an intelligent decision model from a viewpoint of information granularity, so as to provide the best decision action for an object.

(4) FRSs are a valid mathematical tool for solving with uncertainty problems. In generalized IVFRS theory, the upper and lower approximations of an object with respect to a concept (i.e. interval-value fuzzy set) can be computed, which indicates the possible and absolute membership degrees of an object belonging to the interval-value fuzzy set, respectively. Therefore, in this study, we attempt to compute conditional probability of an object by using generalized IVFRSs as a tool.

Given the previous motivation, the innovations and contributions of this paper are summarized as follows:

(1) In this paper, we propose an attribute-oriented interval fuzzy set which portrays the minimum requirement of DM on attribute. The introduction of attribute-oriented interval fuzzy set establishes a bridge between the state sets in 3WD framework and the reference points in PT, which has important implications for strengthening the link between 3WD and PT theory.

(2) Inspired by generalized IVFRSs, the interval-value fuzzy lower approximation and interval-value fuzzy upper approximation can describe the degree to which an object is definitely and may belongs to an attribute-oriented interval fuzzy set, respectively. In view of this,

Table 3
Fuzzy order relationships between interval values.

Method	Meaning	Ranking rule
Certain ranking	θ ranking method (Liu et al., 2001)	$m_\theta = (1-\theta)u + \theta v$
	Geometric average ranking method (Zhao et al., 2013)	$G_\eta([\mu_1, \nu_1]) = \mu_1^{(1-\eta)} \nu_1^{\eta}$
	$\preceq_{LR}$ ranking method (Cornelis et al., 2004)	$\mu_1 \leq \mu_2, \nu_1 \leq \nu_2$, then $[\mu_1, \nu_1] \preceq_{LR} [\mu_2, \nu_2]$
	$\preceq_F$ ranking method (Fishburn, 1973)	$\nu_1 \leq \mu_2$, then $[\mu_1, \nu_1] \preceq_F [\mu_2, \nu_2]$
	$\preceq_N$ ranking method (Sengupta & Pal, 2000)	$\mu_2 \leq \mu_1, \nu_1 \leq \nu_2$, then $[\mu_1, \nu_1] \preceq_N [\mu_2, \nu_2]$
Possibility ranking	Inclusion degree ranking (Zhang & Yang, 2019)	$Inc_\vee([\mu_1, \nu_1], [\mu_2, \nu_2]) = 1 - \frac{1}{2}(\mu_1 - \min(\mu_1, \mu_2) + \nu_1 - \min(\nu_1, \nu_2))$ $Inc_\wedge([\mu_1, \nu_1], [\mu_2, \nu_2]) = \begin{cases} 0, [\mu_1, \nu_1] \npreceq_\wedge [\mu_2, \nu_2] \\ 1, [\mu_1, \nu_1] \npreceq_\wedge [\mu_2, \nu_2] \\ \frac{\max(0, \min(\nu_1, \nu_2)) - \max(\mu_1, \mu_2)}{\nu_1 - \mu_1}, \text{otherwise.} \end{cases}$
	Preference degree (Xu & Da, 2002)	$p([\mu_1, \nu_1] \succeq [\mu_2, \nu_2]) = \max\{1 - \max(\frac{\nu_2 - \mu_1}{(\mu_1 - \nu_1) + (\mu_2 - \nu_2)}, 0), 0\}$
	Dominance measure method (Yang et al., 2020)	$IM_{\leq_D}([\mu_1, \nu_1], [\mu_2, \nu_2]) = \begin{cases} \frac{1}{1 + \exp(K([\mu_1, \nu_1] - [\mu_2, \nu_2]))}, \text{ if } Luv = 0, \\ 1 - \frac{1}{2\exp(K(\frac{\nu_2 - \mu_1}{Luv} - 0.5))}, \text{ if } \nu_2 - \mu_1 \geq 0.5Luv, \\ \frac{1}{2\exp(K(0.5 - \frac{\nu_2 - \mu_1}{Luv}))}, \text{otherwise.} \end{cases}$

this paper designs a model for estimating the conditional probability by using the interval-value fuzzy lower approximation and interval-value fuzzy upper approximation, which provides a new estimation method for exploring conditional probability model in 3WD, and furthermore extends the application of generalized IVFRSs.

(3) Based on the conclusions of psychological experiments, this paper improves the relative loss matrix in 3WD to present an attribute-oriented relative value matrix, which not only can calculate prospect value of object, but also is more suitable for different realistic decision-making scenarios.

(4) In this paper, an intelligent 3WD model based on PT is developed, which provides ranking and optimal classification decision-making results for objects. This enriches the theoretical system of PT, and also provides new ideas for the solution decision-making problems under IFISs.

Therefore, the general structures of this paper are as follows: Section 2 reviews some of the basics. An intelligent PT-based 3WD model is developed in Section 3. Section 4 presents a case study of the developed model. In Section 5, we construct comparative analysis for classification and ranking functions of developed model, respectively. The parameters of developed model are performed experiments in Section 6. Finally, the contributions of this paper are concluded and several directions for future investigation are listed in Section 7.

2. Preliminaries

To better understand this study, some relevant concepts are reviewed.

2.1. Interval value operations

In actuality, a certain range is often used to express uncertain fuzzy information. This is the interval value, which is defined as follows.

Definition 2.1 (*Moore & Lodwick, 2003*). Consider a bounded closed interval $A = [\mu, \nu]$, which denotes the set of all real numbers with $\{a \mid \mu \leq a \leq \nu, a \in \mathbf{R}\}$. If $\mu, \nu \in \mathbf{R}$, then A is referred to an interval number; if $\mu, \nu \in [0, 1]$, then A is said to be an interval value.

It is worth noting that when $u = v$, A degenerates to a real number. In particular, note that the set of all interval numbers on $[0, 1]$ is $I_{[0,1]}$. In addition, the operation rules between two interval values are specified as follows.

Suppose that $[\mu_1, \nu_1]$ and $[\mu_2, \nu_2]$ are two interval values, $k \in [0, 1]$ is a real number, then

Table 4
The relative loss functions.

$A \setminus S$	X	$\neg X$
a_P	$\bar{\mathcal{L}}_{PP}$	$\bar{\mathcal{L}}_{PN}$
a_B	$\bar{\mathcal{L}}_{BP}$	$\bar{\mathcal{L}}_{BN}$
a_N	$\bar{\mathcal{L}}_{NP}$	$\bar{\mathcal{L}}_{NN}$

- $[\mu_1, \nu_1] + [\mu_2, \upsilon_2] = [\mu_1 + \mu_2, \nu_1 + \nu_2]$,
- $[\mu_1, \nu_1] - [\mu_2, \nu_2] = [\mu_1 - \nu_2, \nu_1 - \mu_2]$,
- $k[\mu_1, \nu_1] = [k\mu_1, k\nu_1]$,
- $[\mu_1, \nu_1]^k = [\mu_1^k, \nu_1^k]$.

In particular, the complement of $[\mu_1, \nu_1]$ is:

$$[\mu_1, \nu_1]^c = [1 - \nu_1, 1 - \mu_1].$$

In addition, the operations for infimun, supremum between interval values are as follows.

$$\text{Infimun: } \inf([\mu_1, \nu_1], [\mu_2, \nu_2]) = [\min(\mu_1, \mu_2), \min(\nu_1, \nu_2)],$$
$$\text{Supremum: } \sup([\mu_1, \nu_1], [\mu_2, \nu_2]) = [\max(\mu_1, \mu_2), \max(\nu_1, \nu_2)].$$

Consequently, the order relationships between interval values have been studied, as shown in Table 3.

2.2. Three-way decision theory

3WD theory was presented by Yao (2010), which gives a justifiable semantic explanation for the three regions in decision-theory RS model. The main core concepts are loss functions and conditional probability.

3WD theory contains state set $S = \{X, \neg X\}$ and action set $A = \{a_P, a_B, a_N\}$. X and $\neg X$ represents an object in X and in $\neg X$, respectively. a_P, a_B and a_N indicating three actions of an object in classifying, Later, the relative loss function was defined by Jia and Liu (2019). This is expressed as follows.

Let e is the evaluation value of a_i, the loss values under different actions are presented in Table 4.

Among that $\bar{\mathcal{L}}_{\star P}(\star = P, B, N)$ means the loss value of doing $a_\star$ under state X. $\bar{\mathcal{L}}_{\star N}(\star = P, B, N)$ means the loss value of doing $a_\star$ under state $\neg X$. $\bar{\mathcal{L}}_{PP} = 0$, $\bar{\mathcal{L}}_{NP} = e$, $\bar{\mathcal{L}}_{BP} = \zeta e$, $\bar{\mathcal{L}}_{NN} = 0$, $\bar{\mathcal{L}}_{PN} = 1 - e$, $\bar{\mathcal{L}}_{BN} = \zeta(1 - e)$. Based on $\mathcal{P}r(a_i)$ and $\bar{\mathcal{L}}_{\star P}(\star = P, B, N)$, the expected loss $EP(a_\star|[a_i])$ under each action can be calculated as follows.

$$EP(a_\star|[a_i]) = \mathcal{P}r(a_i) \times \bar{\mathcal{L}}_{\star P} + (1 - \mathcal{P}r(a_i)) \times \bar{\mathcal{L}}_{\star N}, (\star = P, B, N).$$

Then, on the basis of the Bayesian minimum risk procedure, the following decision rule is determined:

If $\min\{EP(a_P|[a_i]), EP(a_B|[a_i]), EP(a_N|[a_i])\}$
$= EP(a_P|[a_i])$, then $a_i \in Pos(X)$, i.e. $a_i \Rightarrow a_P$.
If $\min\{EP(a_P|[a_i]), EP(a_B|[a_i]), EP(a_N|[a_i])\}$
$= EP(a_B|[a_i])$, then $a_i \in Bnd(X)$, i.e. $a_i \Rightarrow a_B$.
If $\min\{EP(a_P|[a_i]), EP(a_B|[a_i]), EP(a_N|[a_i])\}$
$= EP(a_N|[a_i])$, then $a_i \in Neg(X)$, i.e. $a_i \Rightarrow a_N$.

2.3. Prospect theory

In response to the long-standing incorrect assumption of "perfect rationality", PT (Kahneman, 1979; Tversky & Kahneman, 1992) reveals the irrational psychology in the actual decision-making from empirical studies. In PT, there are two classic psychological experiments that can truly reflect the non-rational psychology as follows.

The first experiment is:

- $\mathbf{E}_1^1$: The probability of getting wealth a is 100%.
- $\mathbf{E}_2^1$: There is a 50% probability of getting wealth $2a$; there is a 50% chance of getting nothing.

The corresponding experiment is:

- $\mathbf{E}_1^2$: The probability of loosing wealth b is 100%.
- $\mathbf{E}_2^2$: There is a 50% probability of loosing wealth $2b$; there is a 50% chance of loosing nothing.

In the empirical study, for the first experiment, most people chose $\mathbf{E}_1^1$. In the second experiment, most people chose $\mathbf{E}_2^2$. The above experiments accurately demonstrates the influence of psychological behavior on the decision result. It reveals that when faced with gains, DMs prefer less risky options; when faced with losses, DMs are more willing to take risks.

In PT, a value function is used to scientifically measure the subjective emotions of DMs under different actions, the specific formula is as follows:

$$V_i = \begin{cases} (\Delta O_i)^{\alpha}, & \text{if } \Delta O_i \geq 0, \\ -\theta(-\Delta O_i)^{\beta}, & \text{if } \Delta O_i \leq 0, \end{cases}$$

where $\Delta O_i = O_i - O_r$, O_i represents the outcome value of a_i. O_r indicates the reference outcome value. Furthermore, the sensitivity of value function is measured by α and β that satisfies $\alpha, \beta \in (0,1)$. θ denotes the loss aversion parameter, which denotes the loss aversion of DM. Xu et al. (2011) proposed parameters $\alpha = 0.37$, $\beta = 0.59$, $\theta = 1.51$.

It is not difficult to know that when $\Delta O_i \geq 0$, the object a_i is the gain state; when $\Delta O_i \leq 0$, the object a_i is the loss state. Fig. 2 shows the specific graphical representation of the value function.

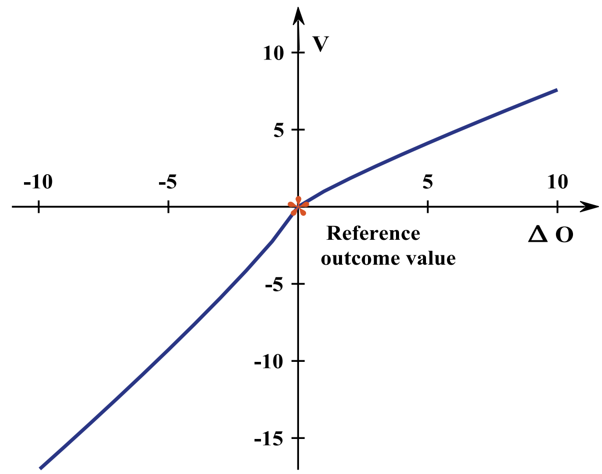

Fig. 2. The value function.

2.4. Interval overlap functions

Overlap functions and grouping functions were presented by Bustince, Fernandez, Mesiar, Montero, and Orduna (2010), Bustince, Pagola, Mesiar, Hullermeier, and Herrera (2012). Later, Qiao and Hu (2017) extended both to interval-valued fuzzy environment. The specific definitions are as follows.

Definition 2.2 (Qiao & Hu, 2017). An interval function $IO : [0,1]^2 \to [0,1]$ satisfies the following conditions:

- IO is said to be commutative. $IO([\mu_1, \nu_1], [\mu_2, \nu_2]) = IO([\mu_2, \nu_2], [\mu_1, \nu_1])$.
- IO satisfies the boundary conditions, i.e $IO([\mu_1, \nu_1], [\mu_2, \nu_2]) = [0,0]$ iff $[\mu_1, \nu_1] = [0,0]$ or $[\mu_2, \nu_2] = [0,0]$.
- IO satisfies the boundary conditions, i.e $IO([\mu_1, \nu_1], [\mu_2, \nu_2]) = [1,1]$ iff $[\mu_1, \nu_1] = [\mu_2, \nu_2] = [1,1]$.
- IO is said to be nondecreasing. $IO([\mu_1, \nu_1], [\mu_2, \nu_2]) \leq IO([\mu_1, \nu_1], [u_3, v_3])$ when $[\mu_2, \nu_2] \leq [u_3, v_3]$.
- IO is said to be Moore-continuous.

then IO is an interval overlap function.

Furthermore, the definition of interval grouping functions was given by Qiao and Hu (2017).

Definition 2.3 (Qiao & Hu, 2017). An interval function $IG : [0,1]^2 \to [0,1]$ satisfies the following conditions:

- IG is said to be commutative. $IG([\mu_1, \nu_1], [\mu_2, \nu_2]) = IG([\mu_2, \nu_2] [\mu_1, \nu_1])$.
- IG satisfies the boundary conditions, i.e $IG([\mu_1, \nu_1], [\mu_2, \nu_2]) = [0,0]$ iff $[\mu_1, \nu_1] = [\mu_2, \nu_2] = [0,0]$.
- IG satisfies the boundary conditions, i.e $IG([\mu_1, \nu_1], [\mu_2, \nu_2]) = [1,1]$ iff $[\mu_1, \nu_1] = [1,1]$ or $[\mu_2, \nu_2] = [1,1]$.
- IG is said to be nondecreasing. $IG([\mu_1, \nu_1], [\mu_2, \nu_2]) \leq IG([\mu_1, \nu_1], [u_3, v_3])$ when $[\mu_2, \nu_2] \leq [u_3, v_3]$.
- IG is said to be Moore-continuous.

then IG is an interval grouping function.

Theoretically, the interval overlap functions and interval grouping functions are a generalization of the interval t-norm[1] and interval t-conorm.[2] Therefore, Definition 2.2 combined with Definition 2.3 and obtained the following relationships.

Proposition 2.1 (Qiao & Hu, 2017). *Supposed that $IO : [0,1]^2 \to [0,1]$ is an interval overlap functions, $[\mu_1, \nu_1]$, $[\mu_2, \nu_2]$ are any two interval values,*

$$IG([\mu_1, \nu_1], [\mu_2, \nu_2]) = [1,1] - IO([1,1] - [\mu_1, \nu_1], [1,1] - [\mu_2, \nu_2]).$$

then IG is an interval grouping functions.

Proposition 2.2 (Qiao & Hu, 2017). *Supposed that $IG : [0,1]^2 \to [0,1]$ is an interval grouping functions, $[\mu_1, \nu_1]$, $[\mu_2, \nu_2]$ are any two interval values,*

$$IO([\mu_1, \nu_1], [\mu_2, \nu_2]) = [1,1] - IG([1,1] - [\mu_1, \nu_1], [1,1] - [\mu_2, \nu_2]).$$

then IO is an interval overlap functions.

[1] An interval function $IF : [0,1]^2 \to [0,1]$ is said to be commutative, associative, increasing and $IF([u,v],[1,1]) = [u,v]$, then IF is an interval t-norm (Bedregal & Takahashi, 2005).

[2] An interval function $IS : [0,1]^2 \to [0,1]$ is said to be commutative, associative, increasing and $IF([u,v],[0,0]) = [u,v]$, then IS is an interval t-conorm (Bedregal & Takahashi, 2006).

Table 5
Interval-valued fuzzy information system.

	C_1	C_2	$\cdots$	C_n
a_1	e_{11}	e_{12}	$\cdots$	e_{1n}
a_2	e_{21}	e_{22}	$\cdots$	e_{2n}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
a_m	e_{m1}	e_{m2}	$\cdots$	e_{mn}

3. An intelligent three-way decision-making model under interval-value fuzzy information systems

In this section, a generalized IVFRSs is first introduced, which can calculate the association degree of an object concerning any interval fuzzy events. Meanwhile, we introduce a new interval-value fuzzy model. Furthermore, the best classification decision-making result for objects are obtained by an intelligent optimization algorithm. Finally, the specific ranking strategy for alternatives will be given.

3.1. Conditional probability based on generalized interval fuzzy rough sets

In decision-theoretic RSs, conditional probability is an important knowledge. It represents to what extent an object belonging to a fuzzy set (or fuzzy event) X. In summary, the determination of a fuzzy set are necessary to calculate conditional probability value of an object. Therefore, this subsection contains the study of both fuzzy sets and conditional probability.

The classical X is a fuzzy set concerning object universe. Differently, a criterion-oriented fuzzy concept from a viewpoint of criterion universe by proposed Zhang, Dai, and Xu (2022). The criterion-oriented fuzzy concept describes the minimum requirement for each attribute. Given the realistic semantic of criterion-oriented fuzzy concept, this section defines attribute-oriented interval fuzzy sets under IFISs.

This is preceded by an introduction for IFIS. For many concrete realistic interval-valued fuzzy decision problems can be expressed by an tuple form $E = \{A, C\}$, where $A = \{a_1, a_2, \ldots, a_m\}$ is a set concerning all objects, called universe of object. $C = \{C_1, C_2, \ldots, C_n\}$ is a set concerning all attributes, called universe of attribute. The evaluation of object a_i $(i = 1, 2, \ldots, m)$ under C_j $(j = 1, 2, \ldots, n)$ is interval-valued fuzzy value $e_{ij} = [u_{ij}, v_{ij}]$. Table 5 denotes a specific IFIS.

Below, we define an interval fuzzy event on attribute universe accordingly.

Definition 3.1. Suppose $E = \{A, C\}$ is an IFIS. A, C are non-empty finite universes. Attribute-oriented interval fuzzy sets X portrays the minimum requirement on attribute C_j by DM. That is, X is identified as an interval-value fuzzy subset of universe of attribute C, as follows:

$$X = \frac{[p_1, q_1]}{C_1} + \frac{[p_2, q_2]}{C_2} + \cdots + \frac{[p_n, q_n]}{C_n}.$$

In general, $[p_j, q_j]$ denotes the minimum requirement concerning C_j $(j = 1, 2, \ldots, n)$ by DM. It is also equivalent to setting the standard value $[p_j, q_j]$ concerning C_j. For example, in house shopping, the public can consider factors such as price C_1, distance from school c_2, and area C_3. And DM thinks that the price of a satisfactory house should be within $[0.5, 0.7]$, the distance from the school should be within $[0.7, 0.8]$, and the area should be within $[0.65, 0.75]$, then the corresponding interval-value fuzzy set is $X = \frac{[0.5,0.7]}{C_1} + \frac{[0.7,0.8]}{C_2} + \frac{[0.65,0.75]}{C_3}$. Definition 3.1 lays the foundation for determining conditional probability value of an object. In the below, conditional probability formula will be discussed on attribute-oriented interval fuzzy sets X.

Under IFISs, different methods for determining conditional probabilities have been studied. For instance, Liang and Liu (2014), Liang, Wang, Xu, and Chen (2021) and Zhang and Yang (2019) determined conditional probability value through subjective experience. Jiang and Hu (2022) calculated conditional probability value using the ideology of interval-valued TOPSIS approach. Moreover, Yang et al. (2020) obtained conditional probability value from a viewpoint of the partial-overall dominance relationship between objects. Different from existing methods, this paper calculates conditional probability of an object with the help of generalized IVFRSs and overlap functions. This paper construct a generalized IVFRSs model based on overlap functions as follows:

Definition 3.2. Suppose $E = \{A, C\}$ is an IFIS. A, C are two non-empty finite sets. R is a binary interval-value fuzzy relation. Then, for any interval-value fuzzy sets X, the interval-value fuzzy lower approximation $R \downarrow_{PL}$ and interval-value fuzzy upper approximation $R \uparrow_{G_0}$ of object $a_i \in A$ are expressed as:

$$R \downarrow_{PL} (X)(a_i) = \sup_{j \in N} \; PL(e_{ij}, X(C_j));$$

$$R \uparrow_{G_0} (X)(a_i) = \inf_{j \in N} \; G_0(e_{ij}, X(C_j)).$$

Where $X(C_j)$ is the minimum requirement interval-value under attribute C_j according to Definition 3.1. e_{ij} denotes the evaluation value. PL and G_0 represent the interval overlap functions and the interval grouping functions, respectively.

Remark 3.1. The reasons for choosing generalized IVFRSs are as follows:

First, for IFISs, the concept X cannot be precisely depicted, so X is often in the form of an interval-valued fuzzy set. Second, in the classical IVFRSs (Sun et al., 2008), the upper and lower approximations are derived using equivalence relations. However, Cock, Cornelis, and Kerre (2007) showed that the equivalence relations may be disappearing in practical situations. Jiang and Hu (2023) also proved this conclusion. So the interval-valued fuzzy relation is chosen. In addition, overlap functions and grouping functions are two classical logical operators with clustering function. They can portray the degree of an object belongs to different fuzzy similar classes. In summary, the generalized IVFRSs is chosen to estimate the conditional probability of an object. It is meaningful to build a generalized IVFRSs model by overlap functions and grouping functions.

Proposition 3.1. *For any a_i, there exists $R \downarrow_{PL} (X)(a_i) \preceq R \uparrow_{G_0} (X)(a_i)$.*

Proof. According ro Definition 3.2, the above proof is converted into a proof that $\sup_{j \in N} \; PL(e_{ij}, X(C_j)) \preceq \inf_{j \in N} \; G_0(e_{ij}, X(C_j))$ holds.

From Definition 2.2, we know that PL is an interval overlap function. According to PL is increasing and Moore-continuous, we get

$$\sup_{j \in N} \; PL(e_{ij}, X(C_j)) \preceq PL(e_{ij}, [1, 1]) = e_{ij}.$$

Similarly, Definition 2.3 shows that G_0 is an interval grouping function. Since G_0 is increasing and Moore-continuous, we have

$$\inf_{j \in N} \; G_0(e_{ij}, X(C_j)) \succeq G_0(e_{ij}, [0, 0]) = e_{ij}.$$

For interval overlap functions and interval group function, Qiao and Hu (2017) have proposed and proved the relevant theory. The calculations in this paper are based on the following equations.

The interval overlap functions (Qiao & Hu, 2017):

$$PL([\mu_1, \nu_1], [\mu_2, \nu_2]) = \begin{cases} [\mu_1 \mu_2 (\mu_1 + \mu_2 - 1), \nu_1 \nu_2], & \text{if } \mu_1 + \mu_2 > 1, \\ [0, \nu_1 \nu_2], & \text{otherwise.} \end{cases}$$

The interval grouping functions (Qiao & Hu, 2017):

$$G_0([\mu_1, \nu_1], [\mu_2, \nu_2]) = [1, 1] - PL(1 - [\mu_1, \nu_1], 1 - [\mu_2, \nu_2]).$$

In the theoretical framework of 3WD, conditional probability of an object describes the degree of correlation with interval fuzzy event X. Furthermore, according to Definition 3.2, $R \downarrow_{PL}$ and $R \uparrow_{G_0}$ of object a_i with respect to interval fuzzy sets X can be obtained. The semantic

Table 6
An interval-valued fuzzy information system E.

$A\backslash C$	C_1	C_2	C_3
a_1	[0.82, 0.93]	[0.41, 0.43]	[0.45, 0.63]
a_2	[0.32, 0.50]	[0.68, 0.84]	[0.29, 0.46]
a_3	[0.55, 0.66]	[0.19, 0.38]	[0.26, 0.36]

interpretation of $R\downarrow_{PL}(a_i)$ represents the degree to which object a_i definitely belongs to X. $R\uparrow_{G_0}(a_i)$ denotes the degree to which object a_i may belong to X. Thus, conditional probability value of an object can be calculated based on $R\downarrow_{PL}(a_i)$ and $R\uparrow_{G_0}(a_i)$ in this paper.

Definition 3.3. Let $E = \{A, C\}$ be an IFIS. A, C are non-empty finite universes. An attribute-oriented interval fuzzy set X portrays the minimum requirement on C_j $(j = 1, 2, \ldots, n)$ by DM. Then, conditional probability value of an object is as follows:

$$\mathcal{P}r(a_i) = \lambda R\downarrow_{PL}(X)(a_i) + (1-\lambda)R\uparrow_{G_0}(X)(a_i),$$

where $\lambda \in [0,1]$ is the risk-appetite coefficient. If DM is optimistic, the value of λ converges to 0. Conversely, if DM is pessimistic, the value of λ converges to 1.

Example 3.1. Consider an IFIS $E = \{A, C\}$ as shown in Table 6. e_{ij} is evaluation value of object $a_i \in A$ $(i = 1, 2, 3)$ under attribute $C_j \in C$ $(j = 1, 2, 3)$. Let interval fuzzy sets $X = \frac{[0.3565,\ 0.4408]}{C_1} + \frac{[0.2671,\ 0.3443]}{C_2} + \frac{[0.2471,\ 0.3582]}{C_3}$, and $\lambda_i = 0.5$. Then, according to Definition 3.1 and Definition 3.2, the conditional probability $\mathcal{P}r(a_i)$ can be calculated. The specific calculation procedure is illustrated by $\mathcal{P}r(a_1)$.

The specific calculation procedures for $R\downarrow_{PL}(X)(a_1)$ and $R\uparrow_{G_0}(X)(a_1)$ are as follows, respectively:

$$\begin{aligned} R\downarrow_{PL}(X)(a_1) &= \sup\ (PL(e_{11}, X(C_1)), PL(e_{12}, X(C_2)), PL(e_{13}, X(C_3))),\\ &= \sup\ ([0.0516, 0.41], [0, 0.1481], [0, 0.2257]),\\ &= [\max\{0.0516, 0, 0\}, \max\{0.41, 0.1481, 0.2257\}],\\ &= [0.0516, 0.41].\end{aligned}$$

$$\begin{aligned} R\uparrow_{G_0}(X)(a_1) &= \inf\ (G_0(e_{11}, X(C_1)), G_0(e_{12}, X(C_2)), G_0(e_{13}, X(C_3))),\\ &= \inf\ ([0.8842, 1], [0.5676, 0.9156], [0.5859, 0.9972]),\\ &= [\min\{0.8842, 0.5676, 0.5859\}, \min\{1, 0.9156, 0.9972\}],\\ &= [0.5676, 0.9156].\end{aligned}$$

Similarly, it can be obtained that:

$$R\downarrow_{PL}(X)(a_2) = [0, 0.2892],\quad R\uparrow_{G_0}(X)(a_2) = [0.4654, 0.9375],$$
$$R\downarrow_{PL}(X)(a_3) = [0, 0.2910],\quad R\uparrow_{G_0}(X)(a_3) = [0.4064, 0.8843].$$

Then, the conditional probability value $\mathcal{P}r(a_1)$ is:

$$\begin{aligned}\mathcal{P}r(a_1) &= \lambda_1 R\downarrow_{G_0}(X)(a_1) + (1-\lambda_1)R\uparrow_{PL}(X)(a_1),\\ &= 0.5 * [0.0516, 0.41] + 0.5 * [0.5676, 0.9156],\\ &= [0.3096, 0.6628].\end{aligned}$$

In addition, the conditional probabilities of the other objects can be obtained as follows:

$$\mathcal{P}r(a_2) = [0.2327, 0.6131], \mathcal{P}r(a_3) = [0.2032, 0.5876].$$

In conclusion, this section presents a new method for computing conditional probabilities from a viewpoint of the generalized IVFRSs, which can effectively reflect the actual uncertainty.

3.2. Attribute-oriented relative value model

In the previous section, conditional probability of an object has been discussed. Later, given the development of behavioral theory, Deng et al. (2022) and Wang, Li, et al. (2020) reveal that DM choose the action with larger prospect value rather than one with the least loss in practical decision-making. Hence, this paper will study the prospect value. That is, in this section, we will introduce attribute-oriented relative value model.

According to Section 2.3, different reference points can affect the value function. The performance of an object is evaluated from multi-attribute. Consequently, an object has corresponding value of the value function under each attribute. In view of the realistic semantics of attribute-oriented interval fuzzy sets X, the value of the value function under each attribute can be determined by X.

Table 7
Attribute-oriented relative outcome matrix of e_{ij}.

ΔO_{ij}	X	$\neg X$
a_P	ΔO_{ij}^{PP}	ΔO_{ij}^{PN}
a_B	ΔO_{ij}^{BP}	ΔO_{ij}^{BN}
a_N	ΔO_{ij}^{NP}	ΔO_{ij}^{NN}

Definition 3.4. Assume $E = \{A, C\}$ is an IFIS. The evaluation of object a_i $(i = 1, 2, \ldots, m)$ under attribute C_j $(j = 1, 2, \ldots, n)$ is $e_{ij} = [\mu_{ij}, \nu_{ij}]$. The evaluation value e_{ij} represents the wealth status of a_i under C_j. Based on attribute-oriented interval fuzzy sets $X = \frac{[p_1, q_1]}{C_1} + \frac{[p_2, q_2]}{C_2} + \cdots + \frac{[p_n, q_n]}{C_n}$, attribute-oriented relative outcome matrix of e_{ij} under each action (a_P, a_B, a_N) can be constructed as presented in Table 7.

$$\Delta O_{ij} = \begin{pmatrix} [u_{ij}, v_{ij}] - X(C_j) & [0,0] \\ \xi([u_{ij}, v_{ij}] - X(C_j)) & \xi(X(C_j) - [u_{ij}, v_{ij}]) \\ [0,0] & X(C_j) - [u_{ij}, v_{ij}] \end{pmatrix}$$

where ξ denotes the profit pursuit coefficient (Wang et al., 2022).

Remark 3.2. The realistic interpretation of attribute-oriented relative outcome is the relative outcome value of an object taking a specific action in a specific state with respect to a reference point. Definition 3.4 sets reference points under each attribute with the help of attribute-oriented interval fuzzy sets X, which relate the fuzzy states in 3WD to the reference points in PT. In the following, we need to determine the outcome value when an object under different fuzzy states.

- In interval-fuzzy state X, taking action a_N does not bring value, thus $\Delta O_{NP} = [0,0]$. Furthermore, for taking a_P action, since the object itself has the wealth of e_{ij}, so the $[u_{ij}, v_{ij}] - X(C_j)$ outcome value can be obtained with respect to reference point $X(C_j)$. Finally, the outcome value under a_B is between a_P and a_N. That is, the outcome value under action a_B is less than the outcome value under action a_P, but more than $[0,0]$. Thus, a profit pursuit coefficient ξ is introduced, and the outcome value under a_B is expressed as $\xi([u_{ij}, v_{ij}] - X(C_j))$.
- Similarly, in interval-fuzzy state $\neg X$, taking action a_P does not bring value, thus $\Delta O_{PN} = [0,0]$. Furthermore, for taking a_N action, since the object itself has the wealth of $[1,1] - e_{ij}$ and the reference point is $[1,1] - X(C_j)$, so the $[1,1] - [u_{ij}, v_{ij}] - ([1,1] - X(C_j)) = X(C_j) - [u_{ij}, v_{ij}]$ outcome value can be obtained with respect to reference point $X(C_j)$. Finally, the outcome value under a_B is between a_P and a_N. That is, the outcome value under action a_B is less than the outcome value under action a_N, but more than $[0,0]$. Thus, a profit pursuit coefficient ξ is introduced, and the outcome value under a_B is expressed as $\xi(X(C_j) - [u_{ij}, v_{ij}])$.
- ξ depends on the performance of the object. If the DM thinks that the object has greater potential, then ξ will be assigned a larger value, and vice versa. Wang et al. (2022) have shown that when $\xi \in [0, 0.5]$ the proposed model is two-way, while when $\xi \in [0.5, 1]$ the proposed model is three-way. Because 3WD is decision model of consistent with human cognitive process, the case of $\xi \in [0.5, 1]$ is considered in this paper.

Table 8
Interval-valued fuzzy value matrix of a_i.

V_i	X	$\neg X$
a_P	V_i^{PP}	V_i^{PN}
a_B	V_i^{BP}	V_i^{BN}
a_N	V_i^{NP}	V_i^{NN}

The above only considers attribute-oriented relative outcome matrix of an object under a single attribute, while the evaluation of an object is considered from multiple attributes. Hence, in the following, the overall outcome matrix of an object under all attributes will be obtained by weighted aggregation approach as follows.

Definition 3.5. Assume $E = \{A, C\}$ is an IFIS. $e_{ij} = [\mu_{ij}, \nu_{ij}]$ is the evaluation value of a_i $(i = 1, 2, \ldots, m)$ under C_j $(j = 1, 2, \ldots, n)$. $W = \{W_1, W_2, \ldots, W_n\}$ means the weight vector of attribute and satisfies $\sum_{j=1}^{n} W_j = 1$. Attribute-oriented interval fuzzy sets $X = \frac{[p_1,q_1]}{C_1} + \frac{[p_2,q_2]}{C_2} + \cdots + \frac{[p_n,q_n]}{C_n}$. The attribute-oriented relative outcome matrix of e_{ij} denotes ΔO_{ij}. The overall outcome matrix O_i of object a_i can be obtained about X under actions a_P, a_B, a_N, as follows.

$$O_i = \begin{bmatrix} & X & \neg X \\ a_P & O_i^{PP} & O_i^{PN} \\ a_B & O_i^{BP} & O_i^{BN} \\ a_N & O_i^{NP} & O_i^{NN} \end{bmatrix} = \begin{bmatrix} & X & \neg X \\ a_P & \sum_{j=1}^{n} W_j * \Delta O_{ij}^{PP} & [0,0] \\ a_B & \sum_{j=1}^{n} W_j * \Delta O_{ij}^{BP} & \sum_{j=1}^{n} W_j * \Delta O_{ij}^{BN} \\ a_N & [0,0] & \sum_{j=1}^{n} W_j * \Delta O_{ij}^{NN} \end{bmatrix}.$$

From Section 2.3, based on attribute-oriented relative outcome matrix of e_{ij}, the corresponding value function matrix can be obtained, as presented in Table 8.

Among them

$$V_i^{*\star} = \begin{cases} (O_i^{*\star})^{\alpha}, & \text{if } O_i^{*\star} \geq 0, \\ -\theta(-O_i^{*\star})^{\beta}, & \text{if } O_i^{*\star} \leq 0, \end{cases}$$

where $*, \star = P, B, N$, $\alpha = 0.37$, $\beta = 0.59$, $\theta = 1.51$.

Next, a trivial instance is given to show the calculation process of the overall value matrix.

Example 3.2 (*Continuation Example* 3.1). Suppose that the weight vector is $W = (0.3672, 0.3740, 0.2588)$, and the profit pursuit coefficient $\xi = (0.5, 0.5, 0.5)$. Then, the above calculation process is explained using a_1 as an example. According to Definition 3.4, we can calculate ΔO_{11} as follows.

$$\Delta O_{11} = \begin{pmatrix} [u_{11}, v_{11}] - X(C_1) & [0,0] \\ \xi([u_{11}, v_{11}] - X(C_1)) & \xi(X(C_1) - [u_{11}, v_{11}]) \\ [0,0] & X(C_1) - [u_{11}, v_{11}] \end{pmatrix} = \begin{pmatrix} [0.3792, 0.5735] & [0,0] \\ [0.1896, 0.2868] & [-0.2868, -0.1896] \\ [0,0] & [-0.5735, -0.3792] \end{pmatrix}.$$

Similarly, ΔO_{12}, ΔO_{13} can be calculated. Then, we can obtain the overall matrix O_1 of object a_1, as follows.

$$O_1 = \begin{bmatrix} & X & \neg X \\ a_P & [0.1875, 0.3706] & [0,0] \\ a_B & [0.0938, 0.1853] & [-0.1853, -0.0938] \\ a_N & [0,0] & [-0.3706, -0.1875] \end{bmatrix}.$$

Continuing from Table 8, V_1 of object a_1 can be calculated.

$$V_1 = \begin{bmatrix} & X & \neg X \\ a_P & [0.5383, 0.6926] & [0,0] \\ a_B & [0.4166, 0.5360] & [-0.5585, -0.3737] \\ a_N & [0,0] & [-0.8407, -0.5625] \end{bmatrix}.$$

In summary, both the interval-valued fuzzy value matrix V_i and conditional probability $\mathcal{P}r(a_i)$ in 3WD theory framework have been established. Therefore, the next section will investigate how to classify and rank for objects.

3.3. Three-way decision model from an intelligent optimization view

The research on 3WD under IFISs has achieved many achievements. For example, some studies (Yang et al., 2020) are performed under the condition that the loss functions is in real number form. Although some work (Jiang & Hu, 2022; Liang et al., 2021) is based on loss functions in interval-valued form, the final decision-making is made by converting the interval-valued to real number. Nevertheless, the evaluation information of object is uncertain information such as interval-valued, so it is difficult to determine the loss functions with exact real number according to the social experience in practice. In addition, there are many methods to convert interval-valued into real number, such as θ ranking method, geometric average ranking method listed in Table 3. How to determine the parameter values θ, η in these methods, it lacks a scientific explanation. Thus, on the one hand, considering that the actual decision-making process does not always seek to minimize risk, this paper investigates value functions with interval-valued form rather than loss functions. On the other hand, we plan to develop an interval-value 3WD model with information granularity and regard them as important and useful assets for extracting optimal decision rules. That is, the interval-values are considered as specific information granularity. For actual decision-making problem under IFISs, this section designs a 3WD model from an intelligent optimization view. It conforms to the following principles:

(1) The interval-value is considered as information granularity in decision-making process.

(2) The action of an object with the greatest prospect value is selected.

(3) A 3WD model is established from a viewpoint of intelligent optimization.

In Sections 3.1 and 3.2, conditional probability and the value matrix with interval-value fuzzy information are introduced. Based on conditional probability and value matrix of object a_i, the prospect value $\mathbb{P}(a_*|a_i)$ can be calculated (Wang, Li, et al., 2020), as follows.

$$\mathbb{P}(a_*|a_i) = \mathcal{P}r(a_i) * V_i^{*P} + (1 - \mathcal{P}r(a_i)) * V_i^{*N}, (*= \{P, B, N\}).$$

The interval-value represents information granularity. For instance, Pedrycz and Song (2014) proposed the granularity assignment method with the help of information granularity. The parameter values that make the objective function optimal are selected as decision-making criteria. In this sense, 3WD under IFISs contains many threshold pairs, and it is necessary to construct an objective function to select the appropriate threshold. In view of this, Liang and Liu (2014) took the minimization of overall uncertainty as the optimization principle and used information granularity to determine the loss function value and extract the optimal decision rule. At this point, both $\mathcal{P}r(a_i)$ and V_i^{*P} are interval values, many prospect values can be calculated. As stated in the second principle, an object will ultimately select the action with greatest prospect value. Hence, we regard both $\mathcal{P}r(a_i)$ and V_i^{*P} as a kind of information granularity with interval-valued information, and extract the final decision rule with the objective function of maximizing the prospect value.

The existing successful optimization algorithms are: particle swarm optimization (PSO), genetic algorithm (GA), etc. In this article, PSO optimization algorithm is chosen to achieve maximizing the prospect value for each object.

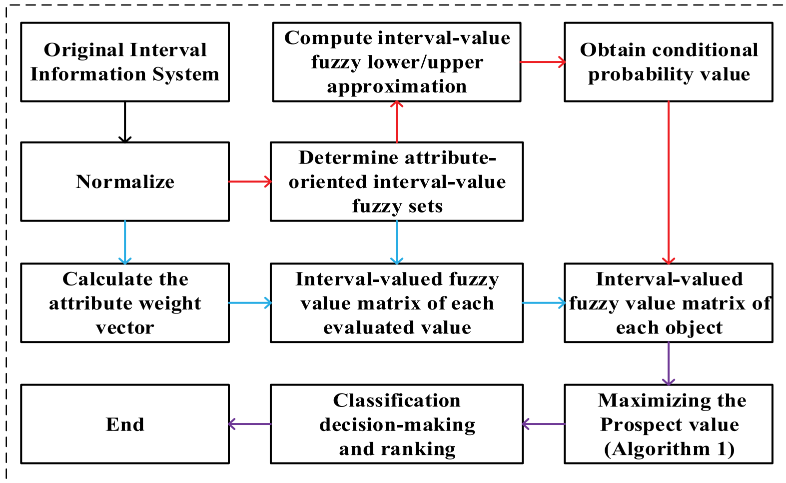

Fig. 3. Flowchart of the developed model.

PSO algorithm has a simple principle and small parameters. In this paper, $\mathbb{P}(a_*|a_i)$ as the objective function and $\mathcal{P}r(a_i)$, V_i^{*P} as the target search space, the specific flow is shown in Algorithm 1.

Algorithm 1: PSO algorithm form of maximizing the prospect value

Input: The number of particles N, the dimension of space De, the number of iterations T, the individual learning factor $c1$, the social learning factor $c2$, the inertia weights W_{max}, W_{min}.
Output: The maximum prospect values of each object, and the classification result.

begin
  for $k = \{1, 2, \dots, N\}$ **do**
    determine the location $\mathbb{L}$ of the initial population.
  end
  for $k = \{1, 2, \dots, N\}$ **do**
    calculate: the individual fitness $\mathbb{P}$ of each particle.
  end
  for $t = \{1, 2, \dots, T\}$ **do**
    Update: the velocity Ve of the particle, and the location of the particle.
    compute: the individual fitness at new location.
    if: the fitness is better at the new location.
    then: update individual optimal value $\mathbb{I}$ and population optimal value $\mathbb{G}$.
  end
  return: the optimal value and classification result of each object.
end

The PSO-based 3WD method not only achieves the best decision-making for objects, but also provides a new paradigm for intelligent decision-making. Below, the PSO-based 3WD method is shown in detail by an example.

Example 3.3 (*Continuation Example* 3.2). Based on Examples 3.1 and 3.2, $\mathcal{P}r(a_i)$ and V_i of object a_i $(i = 1, 2, 3)$ are obtained. Next, the decision result of a_i can be calculated through the PSO-based 3WD method.

According to Algorithm 1, the parameters in the example are specified as follows: $N = 500$, $De = 7$, $T = 500$, $c1 = 1.5$, $c2 = 0.3$, $W_{max} = 0.9$, $W_{min} = 0.4$. The particle velocity update equation is: $Ve = Ve * W_t + c1 * rand * (\mathbb{I} - \mathbb{L}) + c2 * rand * (\mathbb{G} - \mathbb{L})$, where W_t denotes the inertia weight. To avoid missing the optimal value for larger velocity at the end of the iteration, $W_t = W_{max} - \frac{(W_{max} - W_{min}) * t}{T}$ is set. As the inertia velocity becomes smaller with constant iterations, the particle gradually approaches the optimal position. The update of the position is computed in the form $\mathbb{L} = \mathbb{L} + Ve$. For a_1, a_2, a_3, the optimal results are:

$\mathbb{P}(a_P|a_1) = 0.6626 * 0.6924 + (1 - 0.6626) * 0 = 0.4588.$

$\mathbb{P}(a_P|a_2) = 0.6129 * 0.6563 + (1 - 0.6129) * 0 = 0.4022.$

$\mathbb{P}(a_B|a_3) = 0.5870 * 0.4112 + (1 - 0.5870) * 0.2413 = 0.3410.$

Hence, it can be obtained that $a_1 \in Pos(X)$, $a_2 \in Pos(X)$ and $a_3 \in Bnd(X)$. The best decision action for a_1, a_2 is a_P, and the best decision action for a_3 is a_B.

In summary, from intelligent optimization view, the proposed PSO-based 3WD method provides the best decision action for each object by fully considering information granularities $\mathcal{P}r$ and V_i, further achieving the classification goal for objects. In addition, ranking for objects is another goal in actual decision-making, and this problem will be addressed in the next section.

3.4. Ranking strategy

Does the ranking result between objects matter in a decision problem? Obviously this answer is yes. For instance, urgent patients should be prioritized under limited medical resources. Therefore, how to make a reasonable ranking strategy is also a concern issue for DMs.

With the proposed 3WD model, we can divide all objects into $Pos(X)$, $Bnd(X)$, $Neg(X)$. Besides, we use the global prospect function to implement ranking for all objects.

1. $Pos(X) \succ Bnd(X) \succ Neg(X)$. Objects in $Pos(X)$ take precedence over those in $Bnd(X)$, and objects in $Bnd(X)$ take precedence over those in $Neg(X)$. This is consistent with actual decision perception (Deng et al., 2022; Zhang et al., 2022).
2. To rank objects in the same region, we compare them based on the global prospect value GP.

$$GP(a_i) = \begin{cases} \mathbb{P}(a_P|a_i), & \text{if} \quad a_i \in Pos(X), \\ \mathbb{P}(a_B|a_i), & \text{if} \quad a_i \in Bnd(X), \\ \mathbb{P}(a_N|a_i), & \text{if} \quad a_i \in Neg(X). \end{cases}$$

For objects in the same region, we are arranged them in descending order of GP. Namely, if a_1, a_2 belongs to $Pos(X)$ and $GP(a_1) < GP(a_2)$, then we get $a_1 \prec a_2$.

Example 3.4 (*Continuation Example* 3.3). From Example 3.3, prospect value $\mathbb{P}$ and classification results of objects are known. Below we rank for the objects.Based on the first principle of ranking, a_1, a_2 is better than a_3. Then, based on $GP(a_1) = 0.4588 > GP(a_2) = 0.4022$, so $a_1 \succ a_2$. Therefore, the ranking result is $a_1 \succ a_2 \succ a_3$.

3.5. The detailed modeling process

Based on the above discussion, the developed 3WD model based on overlap function has classification and ranking performance. For more clarity, the overall decision-making process is summarized below.

Step 1: A specific IFIS $E = \{A, C\}$. $e_{ij} = [u_{ij}, v_{ij}]$ means the evaluation value of object a_i ($i = 1, 2, \ldots, m$) under attribute C_j ($j = 1, 2, \ldots, n$).

Step 2: Normalized interval-valued fuzzy information matrix E.

$$[\overline{u}_{ij}, \overline{v}_{ij}] = [\frac{u_{ij} - \min_i\{u_{ij}\}}{\max_i\{v_{ij}\} - \min_i\{u_{ij}\}}, \frac{v_{ij} - \min_i\{u_{ij}\}}{\max_i\{v_{ij}\} - \min_i\{u_{ij}\}}],$$
$$\text{for } i \in \{1, 2, \ldots, m\}, j \in C_B.$$
$$[\overline{u}_{ij}, \overline{v}_{ij}] = [\frac{\max_i\{v_{ij}\} - v_{ij}}{\max_i\{v_{ij}\} - \min_i\{u_{ij}\}}, \frac{\max_i\{v_{ij}\} - u_{ij}}{\max_i\{v_{ij}\} - \min_i\{u_{ij}\}}],$$
$$\text{for } i \in \{1, 2, \ldots, m\}, j \in C_C.$$

Where C_B denotes the benefit attribute set; C_C denotes the cost attribute set. Note that $\overline{e}_{ij} = [\overline{u}_{ij}, \overline{v}_{ij}]$.

Step 3: Calculate the attribute weight vector W using maximum deviation method. The steps are as follows:

For $\forall j \in \{1, 2, \ldots, n\}$,

$$D_j(a_i, a_k) = \sqrt{(\overline{u}_{ij} - \overline{u}_{kj})^2 + (\overline{v}_{ij} - \overline{v}_{kj})^2}, i, k \in \{1, 2, \ldots, m\}.$$

Then, $\omega_j = \Sigma_{i=1}^{m} \Sigma_{j=1}^{n} D_j(a_i, a_k)$. Finally, $W_j = \frac{\omega_j}{\Sigma_{j=1}^{n} \omega_j}$.

Step 4: Determine attribute-oriented interval-value fuzzy sets X.

$$X = \frac{[p_1, q_1]}{C_1} + \frac{[p_2, q_2]}{C_2} + \cdots + \frac{[p_n, q_n]}{C_n}.$$

Step 5: According to Definition 3.2, $R \downarrow_{PL} (X)(a_i)$ and $R \uparrow_{G_0} (X)(a_i)$ of objects are calculated.

Step 6: Calculate the conditional probability $\mathcal{P}r(a_i)$ by Definition 3.3.

Step 7: Determine attribute-oriented relative outcome matrix according to Definitions 3.4 and 3.5.

Step 8: The value function matrix can be obtained by Table 8.

Step 9: Based on the PSO algorithm, the maximum prospect value of each object in the 7-dimensional space ($\mathcal{P}r(a_i)$ and the value matrix V_i) can be calculated, and thus classified.

Step 10: From ranking strategy in Section 3.4, the ranking result of objects is obtained.

The detailed flowchart and algorithm of the developed model are presented in Fig. 3 and Algorithm 2, respectively.

Remark 3.3. According to Algorithm 2, the algorithmic complexity of the developed model for each step can be calculated as: **Step 1** is the data collection process, which does not involve complexity. **Step 2** is data preprocessing, whose algorithm complexity is $\mathcal{O}(2mn)$. **Step 3** is the calculation of the weight vector of attributes, whose algorithm complexity is $\mathcal{O}(m^2 n)$. **Step 4** is the determination of attribute-oriented interval-value fuzzy sets X, whose algorithm complexity is $\mathcal{O}(n)$. **Step 5** is the calculation of $R \downarrow_{PL} (X)(a_i)$ and $R \uparrow_{G_0} (X)(a_i)$, whose algorithm complexity is $\mathcal{O}(2mn)$. **Step 6** is the computing of conditional probability, whose algorithm complexity is $\mathcal{O}(m)$. **Step 7** is the determination of attribute-oriented relative outcome matrix, whose algorithm complexity is $\mathcal{O}(2mn)$. **Step 8** is the calculation of the value function matrix, whose algorithm complexity is $\mathcal{O}(6\ m)$. **Step 9** is the computing of the optimal value and classification result, whose algorithm complexity is $\mathcal{O}(m(2N + 7TN))$. **Step 10** is to obtain ranking result for objects, whose algorithm complexity is $\mathcal{O}(m)$. In summary, the complexity of Algorithm 2 is $\mathcal{O}(m^2 n + m(2N + 7TN))$.

Algorithm 2: The developed 3WD model based on overlap function

Input: An IFIS $E = \{A, C\}$.
Output: The classification and ranking results for objects.

begin
  Normalize the decision matrix E to eliminate the dimension.
  for $i = \{1, 2, \ldots, m\}$ **do**
    for $j = \{1, 2, \ldots, n\}$ **do**
      if $j \in C_B$ **then**
        $[\overline{u}_{ij}, \overline{v}_{ij}] = [\frac{u_{ij} - \min_i\{u_{ij}\}}{\max_i\{v_{ij}\} - \min_i\{u_{ij}\}}, \frac{v_{ij} - \min_i\{u_{ij}\}}{\max_i\{v_{ij}\} - \min_i\{u_{ij}\}}]$.
      else
        $[\overline{u}_{ij}, \overline{v}_{ij}] = [\frac{\max_i\{v_{ij}\} - v_{ij}}{\max_i\{v_{ij}\} - \min_i\{u_{ij}\}}, \frac{\max_i\{v_{ij}\} - u_{ij}}{\max_i\{v_{ij}\} - \min_i\{u_{ij}\}}]$.
      end
    end
  end
  for $j = \{1, 2, \ldots, n\}$ **do**
    compute: the weight vector W of attributes.
  end
  for $j = \{1, 2, \ldots, n\}$ **do**
    determine: attribute-oriented interval-value fuzzy sets X.
  end
  for $i = \{1, 2, \ldots, m\}$ **do**
    compute: $R \downarrow_{PL} (X)(a_i) = \sup_{j \in \mathrm{N}} PL(e_{ij}, X(C_j))$
    and $R \uparrow_{G_0} (X)(a_i) = \inf_{j \in \mathrm{N}} G_0(e_{ij}, X(C_j))$.
  end
  for $i = \{1, 2, \ldots, m\}$ **do**
    calculate: $\mathcal{P}r(a_i) = \lambda R \downarrow_{PL} (X)(a_i) + (1 - \lambda) R \uparrow_{G_0} (X)(a_i)$.
  end
  for $i = \{1, 2, \ldots, m\}$ **do**
    compute: attribute-oriented relative outcome matrix by Definition 3.5.
    compute: the value function matrix V_i by Table 8.
  end
  for $i = \{1, 2, \ldots, m\}$ **do**
    calculate: the prospect value $\mathbb{P}$ and the classification result of objects in the light of Algorithm 1.
  end
  for $a_i, a_k \in A$ **do**
    if a_i, a_k belongs to the same region **then**
      $GP(a_i) \le GP(a_k)$, then $a_k \succ a_i$.
    else
      $Pos(X) \succ Bnd(X) \succ Neg(X)$.
    end
  end
  return: the optimal value and classification result.
end

4. Case study

A new decision model is proposed for solving practical problems. Therefore, this section continues to present the practicality of the developed model with cases. By searching the existing literature, we obtain relevant research data form the literature of Yang et al. (2020). The IFIS $E = \{A, C\}$ has 10 objects and 5 attributes, and the specific evaluation values are presented in Table 3 in literature (Yang et al., 2020). The specific IFIS after normalization is shown in Table 9.

Let the risk-appetite coefficient $\lambda = 0.1$, profit pursuit coefficients $\xi = (0.9, 0.9, 0.9, 0.9, 0.9)$. Note that we are not DMs themselves so we cannot determine the coefficients directly. This section only displays the results of the model developed with this coefficient setting. For the effect of different coefficient values on the model results, see Sections 6.1 and 6.2. In addition, for the parameters in PSO algorithm are set as: $N = 500$, $De = 7$, $T = 500$, $c1 = 1.5$, $c2 = 0.3$, $W_{max} = 0.9$, $W_{min} = 0.4$. Fig. 4 presents the classification and ranking results of the developed model.

Table 9
A specific IFIS after normalization.

$A\backslash C$	C_1	C_2	C_3	C_4	C_5
a_1	[0.2690, 0.4276]	[0.3367, 0.4638]	[0.3149, 0.5352]	[0.2939, 0.4351]	[0.3445, 0.4833]
a_2	[0.5448, 0.8621]	[0.5810, 0.9352]	[0.5363, 0.9089]	[0.4447, 0.7824]	[0.5120, 0.8612]
a_3	[0.1908, 0.3908]	[0.1696, 0.4688]	[0.5352, 0.8858]	[0.2462, 0.4580]	[0.2297, 0.4043]
a_4	[0.0805, 0.2575]	[0.0798, 0.2469]	[0, 0.1546]	[0.0382, 0.1813]	[0.1364, 0.2919]
a_5	[0.5655, 1]	[0.5661, 1]	[0.4360, 0.8870]	[0.3989, 0.7691]	[0.5526, 1]
a_6	[0.2966, 0.5586]	[0.3741, 0.5935]	[0.4752, 0.7174]	[0.3111, 0.4885]	[0.4569, 0.6555]
a_7	[0.2805, 0.4759]	[0.3317, 0.5536]	[0.2053, 0.5144]	[0.1889, 0.3836]	[0.3086, 0.5024]
a_8	[0.3471, 0.6989]	[0.3541, 0.7531]	[0.5225, 1]	[0.5286, 1]	[0.4689, 0.8493]
a_9	[0.0552, 0.2299]	[0.0623, 0.2020]	[0.1430, 0.3137]	[0.0878, 0.2557]	[0.1483, 0.2967]
a_{10}	[0, 0.1655]	[0, 0.1796]	[0.1142, 0.3529]	[0, 0.1641]	[0, 0.1770]

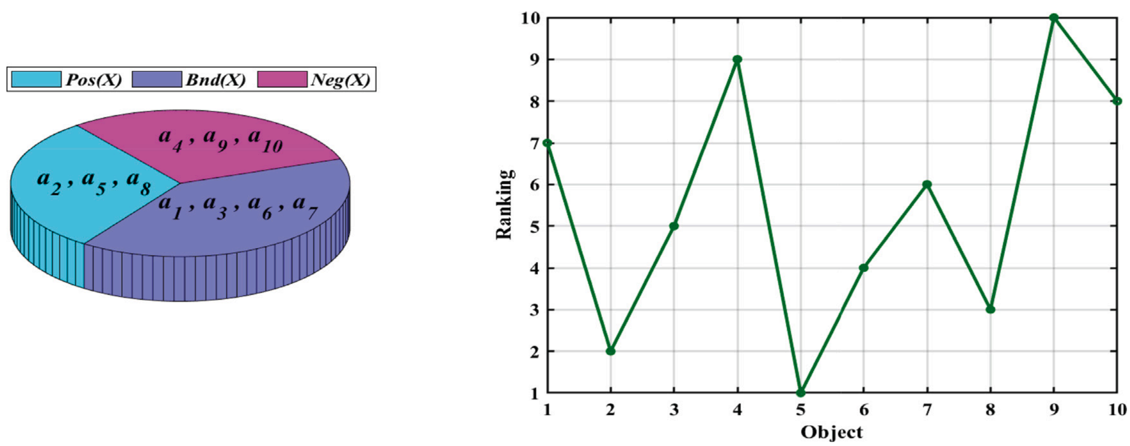

Fig. 4. The classification result and ranking result of the developed model.

Remark 4.1. Based on the results in Fig. 4, the following conclusions can be made: (1) For each object, the optimal results are: $\mathbb{P}(a_B|a_1) = 0.4484$, $\mathbb{P}(a_P|a_2) = 0.7101$, $\mathbb{P}(a_B|a_3) = 0.5727$, $\mathbb{P}(a_N|a_4) = 0.5639$, $\mathbb{P}(a_P|a_5) = 0.7462$, $\mathbb{P}(a_B|a_6) = 0.5911$, $\mathbb{P}(a_B|a_7) = 0.5335$, $\mathbb{P}(a_P|a_8) = 0.7094$, $\mathbb{P}(a_N|a_9) = 0.5363$, $\mathbb{P}(a_N|a_{10}) = 0.5941$. Therefore, the classification results of the developed model for 10 objects are: $Pos(X) = \{a_2, a_5, a_8\}$, $Bnd(X) = \{a_1, a_3, a_6, a_7\}$, $Neg(X) = \{a_4, a_9, a_{10}\}$.

(2) According to the classification results, the best decision action for each object can be obtained as follows: a_2, a_5, a_8 should take the accepting action, i.e. a_P. a_1, a_3, a_6, a_7 should take the pending consideration action, i.e. a_B. a_4, a_9, a_{10} should take the rejecting action, i.e. a_N.

(3) Furthermore, on the basis of the two ranking strategy principles in Section 3.4, the ranking result of the developed model is: $a_5 \succ a_2 \succ a_8 \succ a_6 \succ a_3 \succ a_7 \succ a_1 \succ a_{10} \succ a_4 \succ a_9$.

The above classification and ranking results show that the developed model can address the practical decision-making problem under IFISs. However, the reasonableness of classification and ranking performance of the developed model deserves further discussion.

5. Comparative analysis based on actual cases

As described above, this section constructs a comparative analysis with existing methods to justify the reasonableness of classification and ranking performances of the developed model, respectively.

5.1. Comparative analysis for classification performance

In the interval-valued fuzzy context, the existing classification decision methods are: 3WD model based on the partial-overall dominance relation was established by Yang et al. (2020). An evaluation-based 3WD model from an optimization viewpoint was proposed by Jiang and Hu (2022). An optimization 3WD method from the perspective of information granularity by constructed Liang and Liu (2014). Liang et al. (2021) introduced a risk 3WD model based on regret theory interval information. Zhang and Yang (2019) investigated the 3WD model based in inclusion measures. These methods (Jiang & Hu, 2022; Liang & Liu, 2014; Liang et al., 2021; Yang et al., 2020; Zhang & Yang, 2019) with classification function are studied on various levels. Therefore, this section compares with these five classification decision methods (Jiang & Hu, 2022; Liang & Liu, 2014; Liang et al., 2021; Yang et al., 2020; Zhang & Yang, 2019) to demonstrate the classification performance of the developed model.

In the following, the developed model and the five existing classification methods (Jiang & Hu, 2022; Liang & Liu, 2014; Liang et al., 2021; Yang et al., 2020; Zhang & Yang, 2019) are simultaneously applied to the cases in Section 4. The parameters of different methods are specifically set as follows: the parameters of the developed model are consistent with Section 4. In Yang et al.'s method (Yang et al., 2020), $\alpha = 0.8$, $\beta = 0.4$, $L_{PP} = 0$, $L_{BP} = 2$, $L_{NP} = 8$, $L_{NN} = 1$, $L_{BN} = 5$, $L_{PN} = 13$. The risk aversion coefficients of attributes $\sigma = (0.1, 0.4, 0.1, 0.5, 0.01)$ and $\theta = 0.5$ in Jiang and Hu's method (Jiang & Hu, 2022). In Liang and Liu's method (Liang & Liu, 2014), conditional probability and loss functions of objects are determined subjectively without an objective calculation method. To avoid subjective influence, the results of conditional probability and loss functions of objects in Jiang and Hu's method (Jiang & Hu, 2022) are used in Liang and Liu's method (Liang & Liu, 2014), specifically: $L_{PP} = [0, 0]$, $L_{BP} = [0.0686, 0.1007]$, $L_{NP} = [0.3121, 0.4695]$, $L_{NN} = [0, 0]$, $L_{BN} = [0.1208, 0.1529]$, $L_{PN} = [0.5305, 0.6879]$, $\mathcal{P}r(a_1) = 0.4449$, $\mathcal{P}r(a_2) = 0.8824$, $\mathcal{P}r(a_3) = 0.4561$, $\mathcal{P}r(a_4) = 0.0905$, $\mathcal{P}r(a_5) = 0.9194$, $\mathcal{P}r(a_6) = 0.5942$, $\mathcal{P}r(a_7) = 0.4221$, $\mathcal{P}r(a_8) = 0.8238$, $\mathcal{P}r(a_9) = 0.1385$, $\mathcal{P}r(a_{10}) = 0.0464$. Similarly, conditional probability and loss function values in Liang et al.' method (Liang et al., 2021) the same as in Jiang and Hu's method, and $\theta = 0.02$, $\delta = 0.02$. The conditional probabilities and loss functions of objects in Zhang and Yang's method (Zhang & Yang, 2019) are consistent with Liang and Liu's method (Liang & Liu, 2014). Table 10 presents the classification results of different methods.

Table 10
The classification results of different methods.

Method	$Pos(X)$	$Bnd(X)$	$Neg(X)$
The developed model	a_2, a_5, a_8	a_1, a_3, a_6, a_7	a_4, a_9, a_{10}
Yang et al.'s method (Yang et al., 2020)	a_2, a_5, a_8	a_1, a_3, a_6, a_7	a_4, a_9, a_{10}
Jiang and Hu's method (Jiang & Hu, 2022)	a_2, a_5, a_8	a_1, a_3, a_6, a_7	a_4, a_9, a_{10}
Liang and Liu's method (Liang & Liu, 2014)	a_2, a_5, a_8	a_1, a_3, a_6, a_7	a_4, a_9, a_{10}
Liang et al.'s method (Liang et al., 2021)	a_2, a_5, a_8	a_1, a_3, a_6, a_7	a_4, a_9, a_{10}
Zhang and Yang's method (Zhang & Yang, 2019)	a_2, a_5	a_1, a_3, a_6, a_7, a_8	a_4, a_9, a_{10}

Remark 5.1. As can be seen from the classification results in Table 10:

- The classification results of the developed model are the same as those of Yang et al.'s method (Yang et al., 2020), Jiang and Hu's method (Jiang & Hu, 2022), Liang and Liu's method (Liang & Liu, 2014), Liang et al.'s method (Liang et al., 2021). In Zhang and Yang's method (Zhang & Yang, 2019), only the classification result of a_8 is different, which is considered to belong to $Bnd(X)$.
- The classification result of the developed model is highly concordant with these methods (Jiang & Hu, 2022; Liang & Liu, 2014; Liang et al., 2021; Yang et al., 2020; Zhang & Yang, 2019). This also shows the reasonableness of the classification performance of the developed model.

5.2. Comparative analysis for ranking performance

By comparing with the methods (Jiang & Hu, 2022; Liang & Liu, 2014; Liang et al., 2021; Yang et al., 2020; Zhang & Yang, 2019) with classification function, it demonstrates that the classification functions of the developed model is reasonable. In what follows, this section continues discussion the ranking performance of the developed model.

In the interval-valued fuzzy context, methods with ranking functions include: Dymova et al.'s method (Dymova et al., 2013) is established on the positive and negative ideal solutions, which is an extended TOPSIS method. By improving VIKOR method, the compromise solution can be obtained via Sayadi et al.'s method (Sayadi et al., 2009), which satisfies maximizing "group utility" and minimizing "individual regret". MULTIMOORA method (Hafezalkotob et al., 2016) is developed a ranking method on the basis of fuzzy logic concept and a preference technique. DEMATTEL method (Ren & Toniolo, 2018) is an interval evaluation method on the basis of distance-averaged solution. 3WD model based on the partial-overall dominance relation was established by Yang et al. (2020). An evaluation-based 3WD model from an optimization viewpoint was proposed by Jiang and Hu (2022). MABAC method (Wang, Wei, et al., 2020) accomplishes the ranking based on the distance of object from border approximation area. COPRAS method (Zhang et al., 2023) implements ranking for objects based on the relative weight value. The ranking principle of MAIRCA method (Haq et al., 2023) is that the smaller the gap between theoretical ponder and actual ponder, the better the object.

These methods (Dymova et al., 2013; Hafezalkotob et al., 2016; Haq et al., 2023; Jiang & Hu, 2022; Ren & Toniolo, 2018; Sayadi et al., 2009; Wang, Wei, et al., 2020; Yang et al., 2020; Zhang et al., 2023) can provide corresponding ranking results for decision objects from different principles. Next, these methods are employed in the example in Section 4. The ranking results are presented in Table 11. Likewise, the weight vector of attributes is involved in these methods (Dymova et al., 2013; Hafezalkotob et al., 2016; Haq et al., 2023; Ren & Toniolo, 2018; Sayadi et al., 2009; Wang, Wei, et al., 2020; Zhang et al., 2023). To maintain uniformity, it is set to $W = (0.1975, 0.2048, 0.2041, 0.1951, 0.1985)$.

Remark 5.2. The following conclusions can be obtained from Table 11. Firstly, it is not difficult to find that Sayadi et al.'s method (Sayadi et al., 2009) dose no have ranking result. The ranking result of Sayadi et al.'s method (Sayadi et al., 2009) are presented in Fig. 5. According to the pointing diagram for partial objects in Fig. 5, it is obtained that $a_5 \succ a_8$, $a_8 \succ a_2$ and $a_2 \succ a_8$, which cannot get the ranking result for a_2, a_5, a_8. Moreover, according to the pointing diagram for all objects in Fig. 5, it is known that: Sayadi et al.'s method (Sayadi et al., 2009) is failing under this case.

Secondly, the optimal object of the COPRAS method is a_{10}, and the developed model and other methods (Dymova et al., 2013; Hafezalkotob et al., 2016; Haq et al., 2023; Jiang & Hu, 2022; Ren & Toniolo, 2018; Wang, Wei, et al., 2020; Yang et al., 2020; Zhang et al., 2023) are all a_5. The reason for this result is that the relative weight value of an object is equal to the sum of standard value and a ratio value in COPRAS method, which cannot guarantee that the relative weight value is within the normalized range. That is, in order to eliminate dimensions under different attributes, the data is normalized to the interval $[0,1]$. For instance, $[0,1]+[0,1]=[0,2]$, and $[0,2]$ is not in the range $[0,1]$. According to the calculation method (additive operation), the relative weight value of a_{10} is $[0.1825, 9.8265]$, which is outside the range $[0,1]$. Consequently, the ranking result of COPRAS method is inverted and is invalid. In addition, this also indicates that the ranking performance of the developed model is not inverted. Thus, the ranking performance of the developed model is valid.

Finally, Spearman can measure the correlation for two statistical variables. The correlation coefficients of the developed model with Dymova et al.'s method (Dymova et al., 2013), MULTIMOORA method (Hafezalkotob et al., 2016), DEMATTEL method (Ren & Toniolo, 2018), Yang et al.'s method (Yang et al., 2020), Jiang and Hu's method (Jiang & Hu, 2022), MABAC (Wang, Wei, et al., 2020), MAIRCA (Haq et al., 2023) are 0.9394, 0.9394, 0.9394, 0.9152, 0.9394, 0.8667, 0.9152 respectively. From Spearman threshold table (Zhang & Dai, 2022), it can see that when the total number of objects is 10 and the confidence level is 0.05, if the correlation coefficient of the two methods is greater than 0.648, it indicates that the two variables have high correlation. Therefore, the above Spearman results indicate that the developed model in this paper has high correlation reliability with the existing method in terms of ranking performance, and its reliability is 95%.

In summary, the reasonableness of the developed model in terms of ranking performance is demonstrated.

5.3. Discussion

In Section 5.2, the reasonable and validity of the developed model were shown through comparative analysis. Below, this section analyzes the differences of diverse approaches theoretically and the advantages of the developed model over the existing approaches. The differences of different methods under five aspects are shown in Table 12.

By combining the differences of these methods in Table 12, the advantages of the developed model can be seen as follows:

- In terms of classification and ranking functions, the developed model can provide both classification decision-making and ranking for objects. Except for the Yang et al.'s method (Yang et al., 2020) and Jiang and Hu's method (Jiang & Hu, 2022), the other methods do not have both functions. In practical decision-making, the importance of the classification function is that when the decision problem contains more objects, each object is more concerned with its own classification result. For example, in

Table 11
The ranking results of different methods.

Method	Ranking	Optimal
The developed model	$a_5 \succ a_2 \succ a_8 \succ a_6 \succ a_3 \succ a_7 \succ a_1 \succ a_{10} \succ a_4 \succ a_9$	a_5
Dymova et al.'s method (Dymova et al., 2013)	$a_5 \succ a_2 \succ a_8 \succ a_6 \succ a_3 \succ a_1 \succ a_7 \succ a_9 \succ a_4 \succ a_{10}$	a_5
Sayadi et al.'s method (Sayadi et al., 2009)	None	None
MULTIMOORA method (Hafezalkotob et al., 2016)	$a_5 \succ a_2 \succ a_8 \succ a_6 \succ a_3 \succ a_1 \succ a_7 \succ a_9 \succ a_4 \succ a_{10}$	a_5
DEMATTEL method (Ren & Toniolo, 2018)	$a_5 \succ a_2 \succ a_8 \succ a_6 \succ a_3 \succ a_1 \succ a_7 \succ a_9 \succ a_4 \succ a_{10}$	a_5
Yang et al.'s method (Yang et al., 2020)	$a_5 \succ a_2 \succ a_8 \succ a_6 \succ a_1 \succ a_3 \succ a_7 \succ a_9 \succ a_4 \succ a_{10}$	a_5
Jiang and Hu's method (Jiang & Hu, 2022)	$a_5 \succ a_2 \succ a_8 \succ a_6 \succ a_3 \succ a_1 \succ a_7 \succ a_9 \succ a_4 \succ a_{10}$	a_5
MABAC (Wang, Wei, et al., 2020)	$a_5 \succ a_2 \succ a_8 \succ a_6 \succ a_3 \succ a_1 \succ a_4 \succ a_9 \succ a_7 \succ a_{10}$	a_5
COPRAS (Zhang et al., 2023)	$a_{10} \succ a_4 \succ a_5 \succ a_2 \succ a_8 \succ a_9 \succ a_6 \succ a_1 \succ a_3 \succ a_7$	a_{10}
MAIRCA (Haq et al., 2023)	$a_5 \succ a_2 \succ a_8 \succ a_6 \succ a_1 \succ a_3 \succ a_7 \succ a_9 \succ a_4 \succ a_{10}$	a_5

Table 12
The differences among different methods.

Method	Classification	Ranking	Psychological	Risk	Parameters
The developed model	✓	✓	✓	✓	2
Yang et al.'s method (Yang et al., 2020)	✓	✓	✗	✓	8
Jiang and Hu's method (Jiang & Hu, 2022)	✓	✓	✗	✓	2
Liang and Liu's method (Liang & Liu, 2014)	✓	✗	✗	✓	$6+m$
Liang et al.'s method (Liang et al., 2021)	✓	✗	✓	✓	$7m$
Zhang and Yang's method (Zhang & Yang, 2019)	✓	✗	✗	✓	$7m$
Dymova et al.'s method (Dymova et al., 2013)	✗	✓	✗	✗	n
Sayadi et al.'s method (Sayadi et al., 2009)	✗	✓	✗	✗	n
MULTIMOORA method (Hafezalkotob et al., 2016)	✗	✓	✗	✗	n
DEMATTEL method (Ren & Toniolo, 2018)	✗	✓	✗	✗	n
MABAC (Wang, Wei, et al., 2020)	✗	✓	✗	✗	1
COPRAS (Zhang et al., 2023)	✗	✓	✗	✗	1
MAIRCA (Haq et al., 2023)	✗	✓	✗	✗	$m+1$

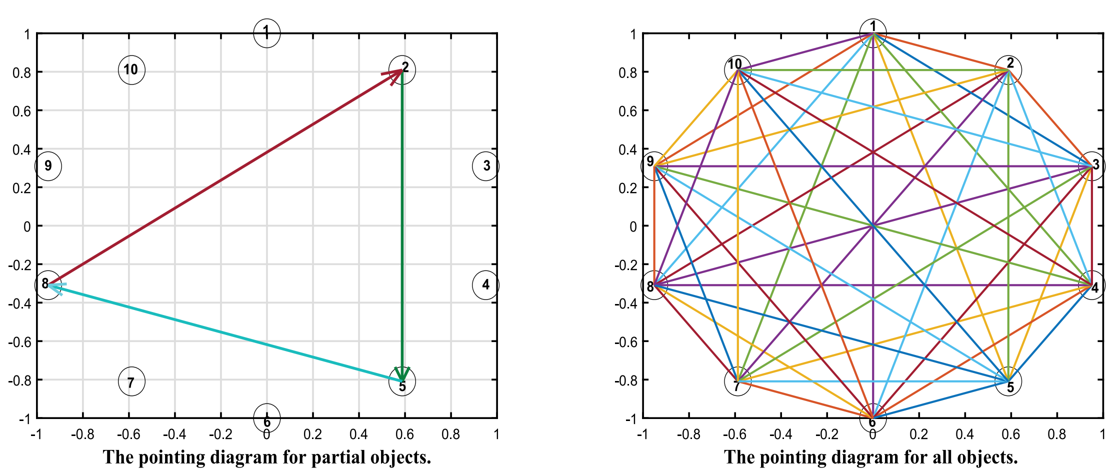

Fig. 5. The pointing diagram for objects in Sayadi et al.'s method (Sayadi et al., 2009).

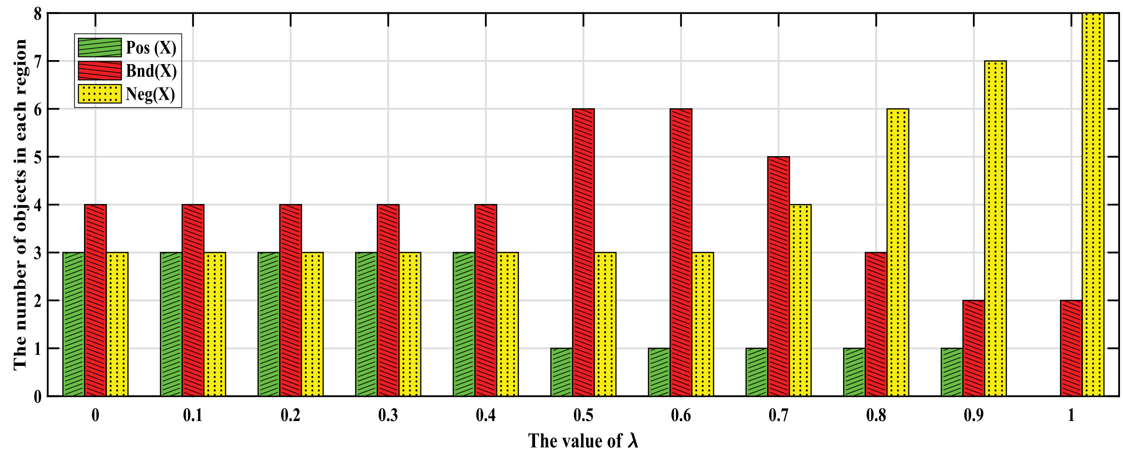

Fig. 6. The classification results of the developed model under different λ values.

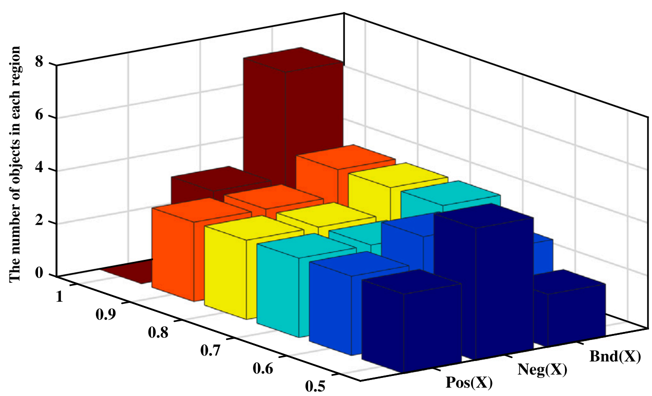

Fig. 7. The classification results of the developed model under different ξ.

an important level exam, candidates are more concerned with whether they pass than with their ranking result. Moreover, for the selection test, the ranking result of objects is important to DM. In summary, the developed model with both classification and ranking functions can satisfy the needs of practical decision making.

- Dymova et al.'s, Sayadi et al.'s, MULTIMOORA, DEMATTEL, MABAC, COPRAS, MAIRCA methods does not consider risk appetite and psychological behaviors. These methods (Dymova et al., 2013; Hafezalkotob et al., 2016; Haq et al., 2023; Ren & Toniolo, 2018; Sayadi et al., 2009; Wang, Wei, et al., 2020; Zhang et al., 2023) assume that DM is fully rational in decision-making process. However, DMs have their own subjective judgments, i.e., irrationality. The developed model introduces the risk-appetite and profit pursuit coefficients to portray the irrational behavior of DMs, which can fit well with the real decision-making situation.
- For the methods of Yang et al.'s (2020), Jiang and Hu's (2022), Liang and Liu's (2014), Zhang and Yang's (2019), although these all fully consider the risk loss in decision-making process, these methods (Jiang & Hu, 2022; Liang & Liu, 2014; Yang et al., 2020; Zhang & Yang, 2019) follow the decision principle of risk minimization, namely, the object chooses the decision action with the least risk. However, the results of psychological experiments (Kahneman, 1979; Tversky & Kahneman, 1992) show that DMs do not pursue risk minimization in actual decision-making process. The developed model in this paper is based on PT to reflect the effect of psychological behavior on decision-making result, which effectively avoids shortcomings of blindly pursuing risk minimization.
- In terms of the number of parameters, the developed model has two parameters, λ and ξ. The parameters values in methods are determined by DM based on social experience. From Table 12, it can be seen that the number of parameters of the developed model is not more than other methods (Dymova et al., 2013; Hafezalkotob et al., 2016; Jiang & Hu, 2022; Liang & Liu, 2014; Liang et al., 2021; Ren & Toniolo, 2018; Sayadi et al., 2009; Yang et al., 2020; Zhang & Yang, 2019). This also indicates that the decision-making result of developed model is less subjective randomness than other methods (Dymova et al., 2013; Hafezalkotob et al., 2016; Jiang & Hu, 2022; Liang & Liu, 2014; Liang et al., 2021; Ren & Toniolo, 2018; Sayadi et al., 2009; Yang et al., 2020; Zhang & Yang, 2019).
- The parameter in MABAC (Wang, Wei, et al., 2020) and COPRAS (Zhang et al., 2023) is set when ranking for interval numbers. In terms of method mechanism, the determination of border approximation area in MABAC method through continued multiplication is too absolute. Furthermore, under COPRAS method, the relative weight value is not ensured to be within the set range, which can make this method invalid. Nevertheless, the developed model fully considers the effects of objects under gain and loss states, and provides optimal classification decision results from an optimization perspective, and ranks them according to the descending order of prospect values, which makes the model closer to reality.

In brief, the qualitative analysis shows that the developed model avoids the shortcomings of the existing methods, and can better solve the decision-making problem under the actual IFISs.

6. Experimental analysis and evaluation

As can be seen from Algorithm 2, the developed model in this paper contains two parameters, λ and ξ. However, the effects of different parameter values on the decision outcomes are not discussed. Consequently, in Section 6.1, we construct an experimental analysis on parameter λ. Furthermore, in Section 6.2, an experimental analysis for parameter ξ are established.

6.1. Experimental analysis on parameter λ

In **Step 6**, when calculating the conditional probability of an object, a parameter, risk-appetite coefficient $\lambda \in [0, 1]$, is involved. Next, we need to understand how different λ values affect the final classification result? In view of this, this section provides an experimental analysis of λ to further verify the feasibility of the classification performance.

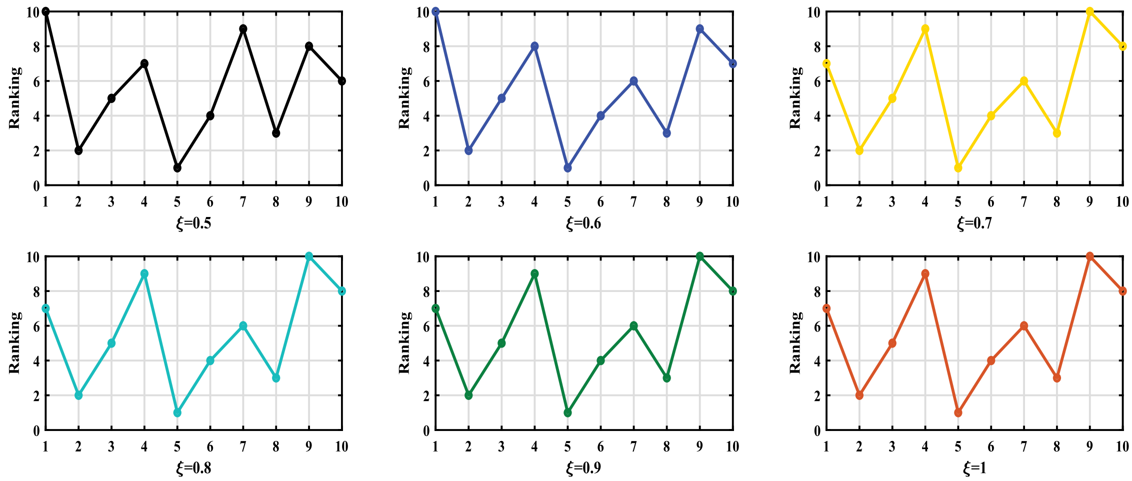

Fig. 8. The ranking results of the developed model under different ξ.

According to Definition 3.3, the larger value of λ represents that DM is more conservative, so the probability is considered to belong to interval fuzzy sets X is close to $R\downarrow_{PL}(X)$, namely, absolute conditional probability. On the contrary, if the value of λ is smaller, it means that DM is very optimistic. Hence, DM believes the probability that an object belongs to X to be close to $R\uparrow_{G_0}(X)$, namely, relative conditional probability. Furthermore, different λ value directly affect conditional probability of an object, which in turn influences the classification result of each object. Thus, an experiment is set up in this section to investigate the effect of different λ values on classification result under the case in Section 4.

For the example in Section 4, the profit pursuit coefficients $\xi = (0.9, 0.9, 0.9, 0.9, 0.9)$ remains fixed and varies the risk-appetite coefficient λ from 0 to 1 in steps of 0.1. Fig. 6 presents the classification results with different λ values.

Remark 6.1. According to the results in Fig. 6, it is obtained that:

(1) With larger value of λ increases, the number of objects in $Pos(X)$ does not increase. Conversely, as the value of λ becomes larger, the number of objects in $Neg(X)$ does not decrease.

(2) When $\lambda = 0$, conditional probability value is the largest, then an object belongs to X with a larger probability. According to the developed model, when an object belongs to X, the corresponding best classification result is $Pos(X)$. Moreover, as λ keeps increasing, conditional probability value becomes smaller. Then objects in $Pos(X)$ does not increase and objects in $Neg(X)$ does not decrease. Furthermore, when $\lambda = 1$, an object has the smallest conditional probability value, then an object belongs to $\neg X$ with a larger probability. According to the developed model, when an object belongs to $\neg X$, the corresponding best classification result is $Neg(X)$.

(3) In summary, the parameter λ variations of the developed model are consistent with the reality. Therefore, the developed model in this paper is feasible.

6.2. Experimental analysis on parameter ξ

The developed model has two parameters. The previous section shows the effect of different λ values on developed model. Thus, this section continues to investigate the effect of profit pursuit coefficient $\xi \in [0.5, 1]$ on decision-making result. Based on Definition 3.4, different ξ values affect the outcome values of an object under a_B action. Different outcome values can change the classification result of an object, which in turn affects the ranking result. Thus, this section build an experiment to investigate the effect of different ξ values on classification and ranking results under the case in Section 4.

For the example in Section 4, the risk-appetite coefficient $\lambda = 0.1$ remains fixed and varies the profit pursuit coefficients ξ from 0.5 to 1 in steps of 0.1. Fig. 7 displays the classification result for different ξ values. In addition, the ranking results of the developed model under different ξ values is displayed in Fig. 8.

Remark 6.2. On the basis of the above results, the following results were obtained.

(1) From the classification result in Fig. 7, it can be obtained that: as the value of ξ increases, the number of objects in $Pos(X)$, $Neg(X)$ does not increase, and the number of objects in $Bnd(X)$ does not decrease. The reason for the law is that the outcome values of an object under a_B action becomes greater when ξ becomes larger. Hence, the prospect value under a_B becomes larger. According to the decision principle of maximizing prospect value, objects in $Bnd(X)$ will not decrease. Since $Pos(X)$, $Bnd(X)$, $Neg(X)$ is a division of all objects, accordingly objects in $Pos(X)$, $Neg(X)$ will not increase when ξ becomes larger. In general, the classification results obtained under the experiments are consistent with the theoretical laws, so the classification performance of the developed model is feasible.

(2) From Algorithm 2, it is known that different classification results can change the final ranking result. Thus, according to the ranking results in Fig. 8, the optimal object is the same, which is a_5. This also indicates that the developed model is robust with respect to coefficients ξ.

(3) Moreover, Spearman correlation coefficients between the ranking results of the developed model under different ξ values is shown in Fig. 9. Spearman correlation coefficients are all greater than 0.648 (Zhang & Dai, 2022), which once again verifies that the developed model is stabile.

7. Conclusion

In conclusion, starting from IFISs, this paper has explored the impact of DM's psychological behavior (gains and losses) on decision-making and established a PT-based 3WD model that can scientifically solve the decision-making problems in the context of actual IFISs. This model takes PT as the theoretical guide and 3WD as the theoretical tool, which can effectively make up for the shortcomings of the existing methods, such as the inability to classification decision-making and the complete rationality of decision-making process. The major **contributions** of this paper are summarized. Firstly, with the help of generalized IVFRSs, we have estimated conditional probability of an object. This not only enhances the connection between IVFRSs and 3WD theory, but also offers a new semantic interpretation of conditional probability in 3WD. Secondly, we have introduced attribute-oriented interval fuzzy sets,

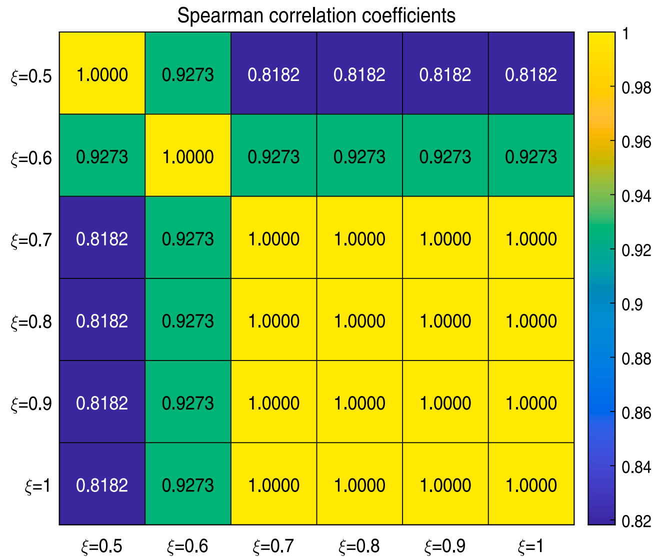

Fig. 9. Spearman correlation coefficients among different ξ values.

which build a bridge between state sets in 3WD and reference points in PT. Finally, combined with PT, this paper has proposed an attribute-oriented relative value model, which fully takes into account the impact of psychological behavior on the decision result and also avoids the shortcomings of the classic 3WD model which blindly pursues risk minimization. Actually, from a viewpoint of information granularity, this paper has established a 3WD intelligent decision model under IFISs, which not only has the functions of classification and ranking, but also can provide the optimal decision results. Hence, the developed model not only expands the development of PT, 3WD theories theoretically, but also provides new ideas for the actual decision-making problems under IFISs, which has certain practical value.

Furthermore, there are some **limitations** in this paper according to the algorithmic process: First, the developed model can only address decision-making problem under a single DM. That is, it is not considered the decision-making problems under group IFISs. Second, the developed model in this paper addresses the decision-making problems under IFISs, and does not address other environments. Finally, the developed model in this paper only takes into account psychological behavior of DMs in the context of gains and losses. However, the effects of regret and rejoicing on the decision-making result are not considered.

Finally, given the above limitations, **future investigation directions** will focus on the following aspects. First, exploring the effects of different psychological behaviors of DMs within a decision group on decision-making and modeling for group decision-making problems under IFISs is a direction of future research (Zhang & Yang, 2019). Second, the 3WD intelligent decision model will be further investigated in other environments, such as triangular fuzzy information systems (Li, Wang, Liang, & Yi, 2021), hesitant fuzzy information systems (Zhan et al., 2023). Finally, it is also a meaningful research topic to consider the effects of regret and rejoicing psychological behaviors on decision results in IFISs (Mondal, Roy, & Pamucar, 2023).

CRediT authorship contribution statement

Jiajia Wang: Conceptualization, Methodology, Writing – original draft. **Xiaonan Li:** Investigation, Writing – review & editing.

Declaration of competing interest

All authors declare that there is no conflict of interest regarding the publication of this manuscript.

Data availability

Data will be made available on request.

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