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A new type of dyad fuzzy β-covering rough set models base on fuzzy information system and its practical application

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ABSTRACT

In the era of big data, faced with massive and complex information, many mathematical concepts used to make judgments and decisions have emerged. In order to better integrate the multi-level fuzzy information, on the basis of the original covering rough set, we generalize the couple approximate operators defined by L.W. Ma to the information system and propose a new binary model—dyad fuzzy β-covering rough set models. This model can analyze and solve practical problems from multiple angles, so as to make more accurate decisions. In addition, in order to make the model more convenient for large and complex data processing, we use matrix to represent the model and realize it by computer programming. Finally, we illustrate the value of this model by solving a practical problem.

1. Introduction

With the advent of the age of big data, everything tends to be diversified and ambiguous. The judgment of things does not only depend on a single element, but also has different importance among multiple elements. Moreover, the attributes of things are not simply divided into black and white. Therefore, in order to handle with various multivariate fuzzy problems in daily life, scholars have successively proposed some methods, including the fuzzy information system which is obtained by replacing the attribute set on crisp power set in the information system with the attribute set on fuzzy power set [1], in addition to rough sets and their extensions [2–5], Dempster-Shafer Theory [6], fuzzy sets [7,8], granular calculation [9,10], and S-approximation Spaces [11,12].

Rough set theory was initially proposed by Pawlak [2,3], which is a novel mathematical method to handle with ambiguity and uncertainty in information systems. We all know that Pawlak constructs the approximate space over rough set theory according to equivalence relations. Therefore, many scholars generalize Pawlak's rough set to many other rough sets by extending this approximation space to other approximation spaces, such as relation approximation space [13,14], the neighborhood approximation space [15] and the covering approximation space [16].

By extending Pawlak's rough sets into covering approximation spaces, covering-based rough sets [17–19] were proposed to manage the kinds of covering data. In recent years, many scholars have combined covering rough set with other theories for research, including that fuzzy set theory [20–22] has become more and more attractive.

In order to further study the relationship between fuzzy rough set and covering based rough set, Li et al. [23] proposed the notion of fuzzy covering approximate space, which was later developed by Deer et al. [24]. Recently, various fuzzy rough set models have emerged based on this concept [23,25,26]. On the basis of these models, Ma [27] first proposed the fuzzy

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β-covering by introducing the parameter β $(0 < \beta \leq 1)$. Moreover, Ma proposed some neighborhood-related covering rough sets in [28]. In addition, Yang et al. [29] proved some properties of fuzzy β-covering on the basis of Ma's research, and defined a new rough set model by introducing the concept of complementary β-neighborhood.

After the fuzzy β-covering rough set models was defined successfully, many scholars studied it in recent years. By introducing β-neighborhood approximation measures and two kinds of Choquet integral on fuzzy covering rough sets, Zhang et al. [30] proposed a new method to solve attribute reduction and multi-criteria decision making problems. In [31], Yang et al. proposed a fuzzy granular reduction theory and a granular matrix based on the fuzzy β-coverings, and for the first time designed a novel reduction algorithm and a heuristic greedy algorithm. In addition, Zhang et al. [32] defined a reflexive fuzzy α-neighborhood operator and constructed a new fuzzy rough set model based on it, then used this model to define intuitionistic fuzzy-valued information systems and we proposed three different sorting decision-making schemes.

Information system [33,34] is a quad that represents objects with attributes and values of attributes, which refers to a man-machine interaction computer application system with information processing services as the main activity. Information system is one of the indispensable models in artificial intelligence research. In addition, with the advent of information age, the processing, transmission, storage and utilization of information have become research hotspots in the field of information technology. Recently, these problems have attracted the research of many scholars and successfully applied in decision making, knowledge discovery and other fields [35–41].

For an information system $IS = (U, AT, V, f)$, it connects the two sets through the information function f, so that the relationship between the two sets could be studied, and one set could also be used to evaluate the other set. In [42], Yang changes the attribute set AT describing U into a fuzzy set, and introduces the parameter β to limit AT, which becomes a fuzzy β-covering of U, thus forming a fuzzy information system $FIS = (U, \widetilde{AT}, V, f)$. At the same time, in order to study the relationship between information systems, Yang used the concept of fuzzy β-neighborhood to construct a fuzzy β-covering rough set model over fuzzy information system.

In [43], L.W. Ma proposes the notion of couple approximate operators, which is defined in covering rough set theory. In this paper, we generalize the concept of couple approximate operators to fuzzy information systems. For a given critical value β, fuzzy covering $\widetilde{AT}$ can be divided into two parts $\top_x = \left\{\tilde{A}_i \in \widetilde{AT} : \tilde{A}_i(x) \geq \beta\right\}$ and $\bot_x = \left\{\tilde{A}_i \in \widetilde{AT} : \tilde{A}_i(x) < \beta\right\}$. In [42], the model proposed by Yang was intended to study the relationship between the two information systems, and the model he constructed was only dependent on the first part $\top_x$. In this paper, we want to use this model to make information decisions. In real life, some people are more cautious when making decisions, often considering the worst and always hold a negative mind. On the other hand, some people are very optimistic when making decisions and always hold positive psychology. In order to be more in line with the human psychology in the actual situation, considering the differences in decision-making between people of different personalities, the $\bot_x$ part should also be considered. In fact, the two parts $\top_x$ and $\bot_x$ both have very important meanings, because they completely describe things from the positive and negative aspects respectively. So, in this paper, we base on the latter to define another fuzzy β-covering rough set model, and combine it with the model defined in [42] to form a novel kind of dyad fuzzy β-covering rough set models. This model can analyze and solve practical problems from positive and negative aspects, which can better simulate the situation of people in decision-making, so as to make more comprehensive evaluation. Finally, in order to more convenient analyze the big data problems, we use the matrix method to represent the model and realize it through algorithm programming.

The rest of this paper is organized as follows. In Section 2, we introduce some definitions that will be used in this paper. In Section 3, we propose a new rough set model based on fuzzy information system, properties of the new models are investigated. In Section 4, we give the matrix representation of the models defined in Section 3 and implement them with algorithm programming. In Section 5, we use a practical application to verify the value of the model. Finally, we conclude the paper in Section 6.

2. Preliminaries

In this section, we introduce some preliminary definitions of fuzzy set theory and covering rough set theory, which will be used in subsequent research.

Definition 2.1. ([44]) Let U be a universe of discourse and $\boldsymbol{C}$ be a family of subsets of U. If no element in $\boldsymbol{C}$ is empty and $U_{C \in \boldsymbol{C}} C = U$, then $\boldsymbol{C}$ is called a covering of U, and the ordered pair $(U, \boldsymbol{C})$ is called a covering approximation space.

Definition 2.2. ([45]) Let $(U, \boldsymbol{C})$ be a covering approximation space, and if $U - X$ is denoted by X^c in the covering approximation space $(U, \boldsymbol{C})$, then for any $x \in U$, the neighborhood N_x and co-neighborhoods M_x of x are defined as

$$N_x = \cap\{C \in \boldsymbol{C} : x \in C\}, \quad M_x = \cap\{C^c : (C \in \boldsymbol{C}) \wedge (x \notin C)\}$$

respectively, where $M_X = U$ if $x \in C$ for each $C \in \boldsymbol{C}$.

Definition 2.3. ([46]) Let U be a universe of discourse. A fuzzy set $\tilde{A}$, or a fuzzy subset $\tilde{A}$ of U, is defined by a function that assigns each element x of U to a value $\tilde{A}(x) \in [0, 1]$. We use $F(U)$ to denote the family of all fuzzy subsets of U, i.e., the set of all functions from U to $[0, 1]$, which is called the fuzzy power set of U.

For any $\tilde{A}, \tilde{B} \in F(U)$, we say that $\tilde{A}$ is contained in $\tilde{B}$, denoted by $\tilde{A} \subset \tilde{B}$, if $\tilde{A}(x) \leq \tilde{B}(x)$ for all $x \in U$, and we say that $\tilde{A} = \tilde{B}$ if and only if $\tilde{A} \subset \tilde{B}$ and $\tilde{B} \subset \tilde{A}$.

For any family $\alpha_i \in [0,1]$, $i \in I$, $I \subseteq \mathbb{N}^*$ ($\mathbb{N}^*$ is the set of all positive integers), we write $\vee_{i\in I}\alpha_i$ or $\vee\{\alpha_i : i \in I\}$ for the supremum of $\{\alpha_i : i \in I\}$, and $\wedge_{i\in I}\alpha_i$ or $\wedge\{\alpha_i : i \in I\}$ for the infimum of $\{\alpha_i : i \in I\}$. Given $\tilde{A}, \tilde{B} \in F(U)$, the union of $\tilde{A}$ and $\tilde{B}$, represented as $\tilde{A} \cup \tilde{B}$, which is defined as $(\tilde{A} \cup \tilde{B})(x) = \tilde{A}(x) \vee \tilde{B}(x)$ for all $x \in U$; The intersection of $\tilde{A}$ and $\tilde{B}$, denoted as $\tilde{A} \cap \tilde{B}$, which is defined as $(\tilde{A} \cap \tilde{B})(x) = \tilde{A}(x) \wedge \tilde{B}(x)$ for all $x \in U$; The complement of $\tilde{A}$, denoted as $\tilde{A}^c$, is defined by $\tilde{A}^c(x) = 1 - \tilde{A}(x)$ for all $x \in U$.

Definition 2.4. ([27]) Let U be a universe of discourse, and $F(U)$ is the fuzzy power set of U. For each $\beta \in (0,1]$, we call $\tilde{C} = \left\{\tilde{C}_1, \tilde{C}_2, \cdots \tilde{C}_m\right\}$, with $\tilde{C}_i \in F(U)$ $(i = 1, 2, \cdots, m)$ is a fuzzy β-covering of U, if $(\cup_{i=1}^m \tilde{C}_i)(x) \geq \beta$ for each $x \in U$. We also call $(U, \tilde{C})$ a fuzzy β-covering approximation space.

Definition 2.5. ([43]) Let $(U, \tilde{C})$ a fuzzy β-covering approximation space, where $\tilde{C} = \left\{\tilde{C}_1, \tilde{C}_2, \cdots \tilde{C}_m\right\}$ is a fuzzy β-covering of U. For each $x \in U$, we define the fuzzy β-neighborhood $\tilde{N}_x^\beta$ and the fuzzy β-co-neighborhood $\tilde{M}_x^\beta$ of x as:

$$\tilde{N}_x^\beta = \cap\left\{\tilde{C}_i \in \tilde{C} : \tilde{C}_i(x) \geq \beta\right\},$$
$$\tilde{M}_x^\beta = -\cup\left\{\tilde{C}_i : \tilde{C}_i(x) < \beta, \tilde{C}_i \in \tilde{C}\right\} = \cap\left\{\tilde{C}_i^c : \tilde{C}_i(x) < \beta, \tilde{C}_i \in \tilde{C}\right\},$$

where $\tilde{M}_x = \mathbf{1}_U$ (i.e., $\tilde{M}_x^\beta(y) = 1$ for each $y \in U$) if $\tilde{C}_i(x) \geq \beta$ for each $\tilde{C}_i \in \tilde{C}$.

3. Dyad fuzzy β-covering rough set models

In this section, we refer to Yang's idea to define a fuzzy β-covering rough set model over fuzzy information system by using the concept of fuzzy β-co-neighborhood $\tilde{M}_x^\beta$ for the first time. Meanwhile, we also study some properties of this novel type of model. Finally, the concept of couple approximate operators defined in [43] is used to extend to form the dyad fuzzy β-covering rough set models.

In general, information system is a quad $IS = (U, AT, V, f)$ that represents objects with attributes and values of attributes, where U is the set of objects; AT is the set of attributes; $V = \cup_{x\in U} V_x$, V_x is the range of x; $f : U \longrightarrow V$ called information function, where $f(x) = V_x \in V (x \in U)$. Similarly, the definition of fuzzy information system is given as following.

Definition 3.1. ([31]) Let U and V be two finite universes and $f : U \longrightarrow V$ be a mapping from U to V. A fuzzy information system is defined as a quad $FIS = (U, \widetilde{AT}, V, f)$, where $U = \{x_1, x_2, \cdots, x_n\}$ and $V = \{y_1, y_2, \cdots, y_s\}$ are non-empty finite set of objects and $\widetilde{AT} = \left\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_m\right\} \in F(U)$ is a set of attributes describing objects.

For a fuzzy information system $FIS = (U, \widetilde{AT}, V, f)$, if $\wedge_{x\in U}[\vee_{i=1}^m \tilde{A}_i(x)] \neq 0$ for all $x \in U$, then $\widetilde{AT}$ can be seen as a fuzzy β-covering of U, where $\widetilde{AT} = \left\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_m\right\}$ and $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^m \tilde{A}_i(x)]]$. In the following study, we always suppose that a given fuzzy information system satisfies $\wedge_{x\in U}[\vee_{i=1}^m \tilde{A}_i(x)] \neq 0$ for all $x \in U$.

Information system is one of the indispensable models in artificial intelligence research. In order to study various relations between fuzzy information systems, in [42], Yang and Hu used the concept of fuzzy β-neighborhood to define a fuzzy β-covering rough set model.

Definition 3.2. ([42]) Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, where $\widetilde{AT} = \left\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^m \tilde{A}_i(x)]]$. For each $\tilde{X} \in F(U)$, we define the $\mathbf{1^{st}}$ fuzzy covering lower approximation $\underline{APR}(\tilde{X})$ and $\mathbf{1^{st}}$ fuzzy covering upper approximation $\overline{APR}(\tilde{X})$ of $\tilde{X}$ as:

$$\underline{APR}(\tilde{X})(y) = \wedge_{x\in f^{-1}(y)}[(1 - \tilde{N}_x^\beta(x)) \vee \tilde{X}(x)],\ y \in V,$$
$$\overline{APR}(\tilde{X})(y) = \vee_{x\in f^{-1}(y)}[\tilde{N}_x^\beta(x) \wedge \tilde{X}(x)],\ y \in V.$$

If $\underline{APR}(\tilde{X}) \neq \overline{APR}(\tilde{X})$, then $\tilde{X}$ is called $\mathbf{1^{st}}$ fuzzy β-covering rough set model.

Given $\beta \in (0,1)$, for every $x \in U$ in fuzzy information system $FIS = \left(U, \widetilde{AT}, V, f\right)$, we divide $\widetilde{AT}$ into two parts $\top_x = \left\{\tilde{A}_i \in \widetilde{AT} : \tilde{A}_i(x) \geq \beta\right\}$ and $\perp_x = \left\{\tilde{A}_i \in \widetilde{AT} : \tilde{A}_i(x) < \beta\right\}$. The $\mathbf{1^{st}}$ fuzzy β-covering rough set model in Definition 3.2 only depend on the first part $\top_x$. In fact, the two parts $\top_x$ and $\perp_x$ both have very important meanings, because they completely describe x from the positive and negative aspects respectively. Therefore, it is necessary for us to use the latter $\perp_x$ to define another rough set model based on fuzzy covering, and then combine it with the model defined in Definition 3.2 to analyze

Table 1
A fuzzy information system $FIS = (U, \widetilde{AT})$.

	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	$\tilde{A}_5$
x_1	0.6	0.5	0.7	0.3	0.2
x_2	0.3	0.4	0.6	0.5	0.4
x_3	0.5	0.3	0.7	0.4	0.6
x_4	0.7	0.3	0.4	0.4	0.5
x_5	0.4	0.7	0.3	0.6	0.8
x_6	0.5	0.6	0.4	0.6	0.3

Table 2
Fuzzy 0.5-neighborhood $\tilde{N}_{x_i}^{0.5}$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\tilde{N}_{x_1}^{0.5}$	0.5	0.3	0.3	0.3	0.3	0.4
$\tilde{N}_{x_2}^{0.5}$	0.3	0.5	0.4	0.4	0.3	0.4
$\tilde{N}_{x_3}^{0.5}$	0.2	0.3	0.5	0.4	0.3	0.3
$\tilde{N}_{x_4}^{0.5}$	0.2	0.3	0.5	0.5	0.4	0.3
$\tilde{N}_{x_5}^{0.5}$	0.2	0.4	0.3	0.3	0.6	0.3
$\tilde{N}_{x_6}^{0.5}$	0.3	0.3	0.3	0.3	0.4	0.5

and solve practical problems from both positive and negative aspects so as to make better judgments. This is the core idea of this paper.

Based on this idea, we defined a fuzzy β-covering rough set model by using the concept of fuzzy β-co-neighborhood $\tilde{M}_x^\beta$ for the first time.

Definition 3.3. Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, where $\widetilde{AT} = \left\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^m \tilde{A}_i(x)]]$. For each $\tilde{X} \in F(U)$, we define the **2^nd^** fuzzy covering lower approximation $\underline{\underline{APR}}(\tilde{X})$ and **2^nd^** fuzzy covering upper approximation $\overline{\overline{APR}}(\tilde{X})$ of $\tilde{X}$ as:

$$\underline{\underline{APR}}(\tilde{X})(y) = \wedge_{x\in f^{-1}(y)}[(1 - \tilde{M}_x^\beta(x)) \vee \tilde{X}(x)],\ y \in V,$$
$$\overline{\overline{APR}}(\tilde{X})(y) = \vee_{x\in f^{-1}(y)}[\tilde{M}_x^\beta(x) \wedge \tilde{X}(x)],\ y \in V.$$

If $\underline{\underline{APR}}(\tilde{X}) \neq \overline{\overline{APR}}(\tilde{X})$, then $\tilde{X}$ is called **2^nd^** fuzzy β-covering rough set model.

Example 3.1. Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\widetilde{AT} = \left\{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5\right\}$ is listed as follows (Table 1).

Then $\widetilde{AT} = \left\{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5\right\}$ is a fuzzy β-covering of U and $\beta \in (0, 0.6]$. Let $\beta = 0.5$.

Let $V = \{y_1, y_2, y_3, y_4\}$, $f : U \longrightarrow V$ be a mapping from U to V, and

$$f(x) = \begin{cases} y_1 & x \in \{x_2, x_4\} \\ y_2 & x = x_1 \\ y_3 & x \in \{x_3, x_5\} \\ y_4 & x = x_6 \end{cases}$$

For

$$\tilde{X} = \frac{0.3}{x_1} + \frac{0.8}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4} + \frac{0.6}{x_5} + \frac{0.5}{x_6}$$

First, we compute the **1^st^** fuzzy covering lower and upper approximation of $\tilde{X}$. It is easy to find that

$$\tilde{N}_{x_1}^{0.5} = \tilde{A}_1 \cap \tilde{A}_2 \cap \tilde{A}_3, \quad \tilde{N}_{x_2}^{0.5} = \tilde{A}_3 \cap \tilde{A}_4, \quad \tilde{N}_{x_3}^{0.5} = \tilde{A}_1 \cap \tilde{A}_3 \cap \tilde{A}_5$$
$$\tilde{N}_{x_4}^{0.5} = \tilde{A}_1 \cap \tilde{A}_5, \quad \tilde{N}_{x_5}^{0.5} = \tilde{A}_2 \cap \tilde{A}_4 \cap \tilde{A}_5, \quad \tilde{N}_{x_6}^{0.5} = \tilde{A}_1 \cap \tilde{A}_2 \cap \tilde{A}_4$$

So, we can obtain the fuzzy β-neighborhood $\tilde{N}_{x_i}^{0.5}$ of x_i $(i = 1, 2, 3, 4, 5, 6)$ as follows (Table 2).

Then we have

$$\underline{APR}\left(\tilde{X}\right) = \frac{0.7}{y_1} + \frac{0.5}{y_2} + \frac{0.5}{y_3} + \frac{0.5}{y_4},$$
$$\overline{APR}\left(\tilde{X}\right) = \frac{0.5}{y_1} + \frac{0.3}{y_2} + \frac{0.6}{y_3} + \frac{0.5}{y_4}.$$

Table 3
Fuzzy 0.5-co-neighborhood $\tilde{M}^{0.5}_{x_i}$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\tilde{M}^{0.5}_{x_1}$	0.7	0.5	0.4	0.5	0.2	0.4
$\tilde{M}^{0.5}_{x_2}$	0.4	0.6	0.4	0.3	0.2	0.4
$\tilde{M}^{0.5}_{x_3}$	0.5	0.5	0.6	0.6	0.3	0.4
$\tilde{M}^{0.5}_{x_4}$	0.3	0.4	0.3	0.6	0.3	0.4
$\tilde{M}^{0.5}_{x_5}$	0.3	0.4	0.3	0.3	0.6	0.5
$\tilde{M}^{0.5}_{x_6}$	0.3	0.4	0.3	0.5	0.2	0.6

Next, we compute the $\mathbf{2^{nd}}$ fuzzy covering lower and upper approximation of $\tilde{X}$. It is easy to find that

$$\tilde{M}^{0.5}_{x_1} = \tilde{A}_4{}^c \cap \tilde{A}_5{}^c, \quad \tilde{M}^{0.5}_{x_2} = \tilde{A}_1{}^c \cap \tilde{A}_2{}^c \cap \tilde{A}_5{}^c, \quad \tilde{M}^{0.5}_{x_3} = \tilde{A}_2{}^c \cap \tilde{A}_4{}^c$$
$$\tilde{M}^{0.5}_{x_4} = \tilde{A}_2{}^c \cap \tilde{A}_3{}^c \cap \tilde{A}_4{}^c, \quad \tilde{M}^{0.5}_{x_5} = \tilde{A}_1{}^c \cap \tilde{A}_3{}^c, \quad \tilde{M}^{0.5}_{x_6} = \tilde{A}_3{}^c \cap \tilde{A}_5{}^c$$

Thus, we can calculate the fuzzy β-co-neighborhood $\tilde{M}^{0.5}_{x_i}$ of x_i $(i = 1, 2, 3, 4, 5, 6)$ as follows (Table 3).

So, we have

$$\underline{\underline{APR}}\left(\tilde{X}\right) = \frac{0.7}{y_1} + \frac{0.3}{y_2} + \frac{0.4}{y_3} + \frac{0.5}{y_4},$$
$$\overline{\overline{APR}}\left(\tilde{X}\right) = \frac{0.6}{y_1} + \frac{0.3}{y_2} + \frac{0.6}{y_3} + \frac{0.5}{y_4}.$$

Furthermore, the properties of the models in above definitions can be proposed through the following propositions. In [42], Yang and Hu have discussed the properties of the model defined in Definition 3.2 in detail and have given the proofs. Therefore, we omit them in this paper and only study the properties of the new model defined in Definition 3.3 and give some proofs.

Proposition 3.1. *Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, $\widetilde{AT} = \left\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^m \tilde{A}_i(x)]]$. For each $\tilde{X}, \tilde{Y} \in F(U)$, we have the following relationships.*

(1) $\underline{\underline{APR}}(\tilde{X}^c) = \left(\overline{\overline{APR}}(\tilde{X})\right)^c$, $\overline{\overline{APR}}(\tilde{X}^c) = \left(\underline{\underline{APR}}(\tilde{X})\right)^c$.

(2) $\overline{\overline{APR}}(\emptyset) = \emptyset$, $\underline{\underline{APR}}(U) = V$.

(3) $\underline{\underline{APR}}(\tilde{X} \cap \tilde{Y}) = \underline{\underline{APR}}(\tilde{X}) \cap \underline{\underline{APR}}(\tilde{Y})$, $\overline{\overline{APR}}(\tilde{X} \cup \tilde{Y}) = \overline{\overline{APR}}(\tilde{X}) \cup \overline{\overline{APR}}(\tilde{Y})$.

(4) If $\tilde{X} \subseteq \tilde{Y}$, then $\underline{\underline{APR}}(\tilde{X}) \subseteq \underline{\underline{APR}}(\tilde{Y})$ and $\overline{\overline{APR}}(\tilde{X}) \subseteq \overline{\overline{APR}}(\tilde{Y})$.

(5) $\underline{\underline{APR}}(\tilde{X} \cup \tilde{Y}) \supseteq \underline{\underline{APR}}(\tilde{X}) \cup \underline{\underline{APR}}(\tilde{Y})$, $\overline{\overline{APR}}(\tilde{X} \cap \tilde{Y}) \subseteq \overline{\overline{APR}}(\tilde{X}) \cap \overline{\overline{APR}}(\tilde{Y})$.

(6) If $1 - \tilde{M}^{\beta}_x(x) \le \tilde{X}(x) \le \tilde{M}^{\beta}_x(x)$ for any $x \in U$, then $\underline{\underline{APR}}(\tilde{X}) \subseteq \overline{\overline{APR}}(\tilde{X})$.

Proof. (1) For any $y \in V$, we have

$$\begin{aligned}\underline{\underline{APR}}(\tilde{X}^c)(y) &= \wedge_{x\in f^{-1}(y)}[(1 - \tilde{M}^{\beta}_x(x)) \vee \tilde{X}^c(x)] = \wedge_{x\in f^{-1}(y)}[1 - (\tilde{M}^{\beta}_x(x) \wedge \tilde{X}(x))] \\ &= 1 - \vee_{x\in f^{-1}(y)}[\tilde{M}^{\beta}_x(x) \wedge \tilde{X}(x)] = 1 - \overline{\overline{APR}}(\tilde{X})(y) = (\overline{\overline{APR}}(\tilde{X}))^c(y)\end{aligned}$$

and

$$\overline{\overline{APR}}(\tilde{X}^c)(y) = 1 - \underline{\underline{APR}}((\tilde{X}^c)^c)(y) = 1 - \underline{\underline{APR}}(\tilde{X})(y) = (\underline{\underline{APR}}(\tilde{X}))^c(y),$$

then $\underline{\underline{APR}}(\tilde{X}^c) = \left(\overline{\overline{APR}}(\tilde{X})\right)^c$, $\overline{\overline{APR}}(\tilde{X}^c) = \left(\underline{\underline{APR}}(\tilde{X})\right)^c$.

(2) Since $U(x) = 1$ and $\emptyset(x) = 0$ for every $x \in U$, we have

$$\overline{\overline{APR}}(\emptyset)(y) = \vee_{x\in f^{-1}(y)}[\tilde{M}^{\beta}_x(x) \wedge \emptyset(x)] = 0$$

and

$$\underline{\underline{APR}}(U)(y) = \wedge_{x\in f^{-1}(y)}[(1 - \tilde{M}^{\beta}_x(x)) \vee U(x)] = 1,$$

therefore $\overline{\overline{APR}}(\emptyset) = \emptyset$, $\underline{\underline{APR}}(U) = V$.

(3) For any $y \in V$, we have

$$\begin{aligned}\underline{\underline{APR}}(\tilde{X} \cap \tilde{Y})\,(y) &= \wedge_{x \in f^{-1}(y)}[(1 - \tilde{M}_x^{\beta}\,(x)) \vee (\tilde{X} \cap \tilde{Y})(x)] \\ &= \wedge_{x \in f^{-1}(y)}[((1 - \tilde{M}_x^{\beta}\,(x)) \vee \tilde{X}(x)) \wedge ((1 - \tilde{M}_x^{\beta}\,(x)) \vee \tilde{Y}(x))] \\ &= (\wedge_{x \in f^{-1}(y)}[(1 - \tilde{M}_x^{\beta}\,(x)) \vee \tilde{X}(x)]) \wedge (\wedge_{x \in f^{-1}(y)}[(1 - \tilde{M}_x^{\beta}\,(x)) \vee \tilde{Y}(x)]) \\ &= \underline{\underline{APR}}(\tilde{X})(y) \wedge \underline{\underline{APR}}(\tilde{Y})(y) = (\underline{\underline{APR}}(\tilde{X}) \cap \underline{\underline{APR}}(\tilde{Y}))(y)\end{aligned}$$

and

$$\begin{aligned}\overline{\overline{APR}}(\tilde{X} \cup \tilde{Y})(y) &= \vee_{x \in f^{-1}(y)}[\tilde{M}_x^{\beta}\,(x) \wedge (\tilde{X} \cup \tilde{Y})(x)] \\ &= \vee_{x \in f^{-1}(y)}[(\tilde{M}_x^{\beta}\,(x) \wedge \tilde{X}\,(x)) \vee (\tilde{M}_x^{\beta}\,(x) \wedge \tilde{Y}\,(x))] \\ &= (\vee_{x \in f^{-1}(y)}[\tilde{M}_x^{\beta}\,(x) \wedge \tilde{X}(x)]) \vee (\vee_{x \in f^{-1}(y)}[\tilde{M}_x^{\beta}\,(x) \wedge \tilde{Y}(x)]) \\ &= \overline{\overline{APR}}(\tilde{X})(y) \vee \overline{\overline{APR}}(\tilde{Y})(y) = (\overline{\overline{APR}}(\tilde{X}) \cup \overline{\overline{APR}}(\tilde{Y})(y),\end{aligned}$$

then we have $\underline{\underline{APR}}(\tilde{X} \cap \tilde{Y}) = \underline{\underline{APR}}(\tilde{X}) \cap \underline{\underline{APR}}(\tilde{Y})$, $\overline{\overline{APR}}(\tilde{X} \cup \tilde{Y}) = \overline{\overline{APR}}(\tilde{X}) \cup \overline{\overline{APR}}(\tilde{Y})$.

(4) It follows from $\tilde{X} \subseteq \tilde{Y}$ that $\tilde{X}(x) \leq \tilde{Y}(x)$ for any $x \in U$. For each $y \in V$, so we have

$$\underline{\underline{APR}}(\tilde{X})(y) = \wedge_{x \in f^{-1}(y)}[(1 - \tilde{M}_x^{\beta}\,(x)) \vee \tilde{X}(x)] \leq \wedge_{x \in f^{-1}(y)}[(1 - \tilde{M}_x^{\beta}\,(x)) \vee \tilde{Y}(x)] = \underline{\underline{APR}}(\tilde{X})(y)$$

and

$$\overline{\overline{APR}}(\tilde{X})\,(y) = \vee_{x \in f^{-1}(y)}[\tilde{M}_x^{\beta}\,(x) \wedge \tilde{X}(x)] \leq \vee_{x \in f^{-1}(y)}[\tilde{M}_x^{\beta}\,(x) \wedge \tilde{Y}(x)] = \overline{\overline{APR}}(\tilde{Y})\,(y)\,.$$

Hence $\underline{\underline{APR}}(\tilde{X}) \subseteq \underline{\underline{APR}}(\tilde{Y})$ *and* $\overline{\overline{APR}}(\tilde{X}) \subseteq \overline{\overline{APR}}(\tilde{Y})$.

(5) Because of $\underline{\underline{APR}}(\tilde{X}) \subseteq \underline{\underline{APR}}(\tilde{X} \cup \tilde{Y})$, $\underline{\underline{APR}}(\tilde{Y}) \subseteq \underline{\underline{APR}}(\tilde{X} \cup \tilde{Y})$, follows from (4), so we have

$$\underline{\underline{APR}}(\tilde{X}) \cup \underline{\underline{APR}}(\tilde{Y}) \subseteq \underline{\underline{APR}}(\tilde{X} \cup \tilde{Y}).$$

Similarly, since $\overline{\overline{APR}}(\tilde{X} \cap \tilde{Y}) \subseteq \underline{\underline{APR}}(\tilde{X})$ and $\overline{\overline{APR}}(\tilde{X} \cap \tilde{Y}) \subseteq \underline{\underline{APR}}(\tilde{Y})$, we have

$$\overline{\overline{APR}}(\tilde{X} \cap \tilde{Y}) \subseteq \overline{\overline{APR}}(\tilde{X}) \cap \overline{\overline{APR}}(\tilde{Y}).$$

(6) For any $y \in V$, if there is $1 - \tilde{M}_x^{\beta}\,(x) \leq \tilde{X}\,(x) \leq \tilde{M}_x^{\beta}\,(x)$ for any $x \in U$, then

$$\underline{\underline{APR}}(\tilde{X})(y) = \wedge_{x \in f^{-1}(y)}[(1 - \tilde{M}_x^{\beta}\,(x)) \vee \tilde{X}(x)] = \wedge_{x \in f^{-1}(y)}\tilde{X}\,(x) \leq \vee_{x \in f^{-1}(y)}\tilde{X}\,(x)$$

and

$$\overline{\overline{APR}}(\tilde{X})(y) = \vee_{x \in f^{-1}(y)}[\tilde{M}_x^{\beta}\,(x) \wedge \tilde{X}\,(x)] = \vee_{x \in f^{-1}(y)}\tilde{X}\,(x)\,.$$

Thus $\underline{\underline{APR}}(\tilde{X}) \subseteq \overline{\overline{APR}}(\tilde{X})$. □

In order to further study the properties of $\underline{\underline{APR}}(\tilde{X})$ and $\overline{\overline{APR}}(\tilde{X})$, let's see an example as follows.

Example 3.2. For the fuzzy information system $FIS = (U, \widetilde{AT}, V, f)$ in Example 3.1, let

$$\begin{aligned}\tilde{A} &= \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.4}{x_5} + \frac{0.3}{x_6} \\ \tilde{B} &= \frac{0.4}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.4}{x_4} + \frac{0.6}{x_5} + \frac{0.7}{x_6}\end{aligned}$$

thus

$$\begin{aligned}\tilde{A} \cap \tilde{B} &= \frac{0.4}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{0.3}{x_6} \\ \tilde{A} \cup \tilde{B} &= \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.7}{x_6}\end{aligned}$$

Then we have

$$\overline{\overline{APR}}\left(\tilde{A}\right)=\frac{0.6}{y_1}+\frac{0.5}{y_2}+\frac{0.5}{y_3}+\frac{0.3}{y_4},\ \overline{\overline{APR}}\left(\tilde{B}\right)=\frac{0.4}{y_1}+\frac{0.4}{y_2}+\frac{0.6}{y_3}+\frac{0.6}{y_4},\ \overline{\overline{APR}}(\tilde{A}\cap\tilde{B})=\frac{0.4}{y_1}+\frac{0.4}{y_2}+\frac{0.5}{y_3}+\frac{0.3}{y_4}$$
$$\underline{\underline{APR}}\left(\tilde{A}\right)=\frac{0.5}{y_1}+\frac{0.5}{y_2}+\frac{0.4}{y_3}+\frac{0.4}{y_4},\ \underline{\underline{APR}}\left(\tilde{B}\right)=\frac{0.4}{y_1}+\frac{0.4}{y_2}+\frac{0.5}{y_3}+\frac{0.7}{y_4},\ \underline{\underline{APR}}(\tilde{A}\cup\tilde{B})=\frac{0.5}{y_1}+\frac{0.5}{y_2}+\frac{0.5}{y_3}+\frac{0.7}{y_4}$$

We can see that $\overline{\overline{APR}}(\tilde{A}\cap\tilde{B})=\overline{\overline{APR}}\left(\tilde{A}\right)\cap\overline{\overline{APR}}\left(\tilde{B}\right)$ and $\underline{\underline{APR}}(\tilde{A}\cup\tilde{B})=\underline{\underline{APR}}\left(\tilde{A}\right)\cup\underline{\underline{APR}}\left(\tilde{B}\right)$ clearly. That means the equalities of item (5) in Proposition 3.1 can be hold under some conditions.

Now we give a sufficient condition under which the equalities of item (5) hold.
First we introduce the consistency of mapping.

Definition 3.4. ([1]) Let U and V be two finite universes, $f:U\longrightarrow V$ be a mapping from U to V, and $\tilde{A},\tilde{B}\in F(U)$. If $[x]_f=\{y\in U: f(y)=f(x)$, then $\left\{[x]_f: x\in U\right\}$ is a partition of U. For any $x\in U$, if one of the following statements holds:

(1) $\tilde{A}(u)\le\tilde{B}(u)$ for any $u\in[x]_f$,
(2) $\tilde{A}(u)\ge\tilde{B}(u)$ for any $u\in[x]_f$,

then f is called consistent with respect to $\tilde{A}$ and $\tilde{B}$.

Proposition 3.2. *Let $FIS=(U,\widetilde{AT},V,f)$ be a fuzzy information system and $f:U\longrightarrow V$ be a surjective mapping from U to V, $\widetilde{AT}=\left\{\tilde{A}_1,\tilde{A}_2,\cdots,\tilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta\in(0,\wedge_{x\in U}[\vee_{i=1}^m\tilde{A}_i(x)]]$. For $\tilde{A},\tilde{B}\in F(U)$, if f is consistent with respect to $\tilde{A}$ and $\tilde{B}$, then $\overline{\overline{APR}}(\tilde{A}\cap\tilde{B})=\overline{\overline{APR}}\left(\tilde{A}\right)\cap\overline{\overline{APR}}\left(\tilde{B}\right)$ and $\underline{\underline{APR}}(\tilde{A}\cup\tilde{B})=\underline{\underline{APR}}\left(\tilde{A}\right)\cup\underline{\underline{APR}}\left(\tilde{B}\right)$.*

Proof. It follows from Proposition 3.1 that $\underline{\underline{APR}}(\tilde{A}\cup\tilde{B})\supseteq\underline{\underline{APR}}(\tilde{A})\cup\underline{\underline{APR}}(\tilde{B})$ and $\overline{\overline{APR}}(\tilde{A}\cap\tilde{B})\subseteq\overline{\overline{APR}}(\tilde{A})\cap\overline{\overline{APR}}(\tilde{B})$. Then we only need to prove that $\underline{\underline{APR}}(\tilde{A}\cup\tilde{B})\subseteq\underline{\underline{APR}}(\tilde{A})\cup\underline{\underline{APR}}(\tilde{B})$ and $\overline{\overline{APR}}(\tilde{A}\cap\tilde{B})\supseteq\overline{\overline{APR}}(\tilde{A})\cap\overline{\overline{APR}}(\tilde{B})$ hold. For any $y\in V$, if f is consistent with respect to $\tilde{A}$ and $\tilde{B}$, it follows from Definition 3.4 that one of the following conditions holds:

(1) $\tilde{A}(x)\le\tilde{B}(x)$ for any $x\in f^{-1}(y)$,
(2) $\tilde{A}(x)\ge\tilde{B}(x)$ for any $x\in f^{-1}(y)$.

Then we have

$$\underline{\underline{APR}}(\tilde{A}\cup\tilde{B})(y)=\wedge_{x\in f^{-1}(y)}[(1-\tilde{M}_x^\beta(x))\vee(\tilde{A}\cup\tilde{B})(x)]=\wedge_{x\in f^{-1}(y)}[(1-\tilde{M}_x^\beta(x))\vee\tilde{A}(x)]=\underline{\underline{APR}}(\tilde{A})(y)$$

or

$$\underline{\underline{APR}}(\tilde{A}\cup\tilde{B})(y)=\wedge_{x\in f^{-1}(y)}[(1-\tilde{M}_x^\beta(x))\vee(\tilde{A}\cup\tilde{B})(x)]=\wedge_{x\in f^{-1}(y)}[(1-\tilde{M}_x^\beta(x))\vee\tilde{B}(x)]=\underline{\underline{APR}}(\tilde{B})(y).$$

Therefore $\underline{\underline{APR}}(\tilde{A}\cup\tilde{B})\subseteq\underline{\underline{APR}}(\tilde{A})\cup\underline{\underline{APR}}(\tilde{B})$.
Similarly, we have

$$\overline{\overline{APR}}(\tilde{A}\cap\tilde{B})(y)=\vee_{x\in f^{-1}(y)}[\tilde{M}_x^\beta(x)\wedge(\tilde{A}\cap\tilde{B})(x)]=\vee_{x\in f^{-1}(y)}[\tilde{M}_x^\beta(x)\wedge\tilde{A}(x)]=\overline{\overline{APR}}(\tilde{A})(y)$$

or

$$\overline{\overline{APR}}(\tilde{A}\cap\tilde{B})(y)=\vee_{x\in f^{-1}(y)}[\tilde{M}_x^\beta(x)\wedge(\tilde{A}\cap\tilde{B})(x)]=\vee_{x\in f^{-1}(y)}[\tilde{M}_x^\beta(x)\wedge\tilde{B}(x)]=\overline{\overline{APR}}(\tilde{B})(y)$$

Therefore $\overline{\overline{APR}}(\tilde{A}\cap\tilde{B})\supseteq\overline{\overline{APR}}(\tilde{A})\cap\overline{\overline{APR}}(\tilde{B})$.
This concludes the proof of this proposition. □

Through Proposition 3.2, we can get the following corollary.

Corollary 3.1. *Let $FIS=(U,\widetilde{AT},V,f)$ be a fuzzy information system and $f:U\longrightarrow V$ be a surjective mapping from U to V, $\widetilde{AT}=\left\{\tilde{A}_1,\tilde{A}_2,\cdots,\tilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta\in(0,\wedge_{x\in U}[\vee_{i=1}^m\tilde{A}_i(x)]]$. For $X_1,X_2,\cdots,X_n\in F(U)$, if f is consistent with any two of fuzzy sets $X_1,X_2,\cdots,X_n$, then $\underline{\underline{APR}}(\cup_{i=1}^n X_i)=\cup_{i=1}^n\underline{\underline{APR}}(X_i)$ and $\overline{\overline{APR}}(\cap_{i=1}^n X_i)=\cap_{i=1}^n\overline{\overline{APR}}(X_i)$.*

In [43], L.W. Ma proposes the notion of couple approximate operators, which is defined in covering rough set theory. In this paper, we generalize this concept and define a novel model——dyad fuzzy β-covering rough set models, which integrates covering rough set theory, fuzzy rough set theory and fuzzy information system.

Definition 3.5. Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, $\widetilde{AT} = \left\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^{m}\tilde{A}_i(x)]]$. For each $\tilde{X} \in F(U)$, $\underline{APR}(\tilde{X})$ and $\overline{APR}(\tilde{X})$ are **1st** fuzzy covering lower and upper approximation of $\tilde{X}$ respectively, $\underline{\underline{APR}}(\tilde{X})$ and $\overline{\overline{APR}}(\tilde{X})$ are **2nd** fuzzy covering lower and upper approximation of $\tilde{X}$ respectively, we call the two-pair of models $[(\underline{APR}, \overline{APR}), (\underline{\underline{APR}}, \overline{\overline{APR}})]$ a dyad fuzzy β-covering rough set models.

The dyad fuzzy β-covering rough set models is on the basis of fuzzy β-neighborhood and fuzzy β-co-neighborhood at the same time, which are not only closely related but also complementary to each other. Thereby, both positive and negative aspects can be integrated to analyze and handle with practical problems so as to make more accurate decisions.

4. Matrix representation and programming realization

In actual situations, the data is usually very large, and the model proposed in Section 3 is very complicated by manual calculation. In this section, we will use the matrix method to represent the model and realize it through computer programming.

4.1. Matrix representation

First, we define a couple of new operations.

Definition 4.1. ([27]) Let $A = (a_{ik})_{n\times m}$ and $B = \left(b_{kj}\right)_{m\times l}$ be two arbitrary matrices. We define two kinds of operations $C = A \cdot B = \left(c_{ij}\right)_{n\times l}$ and $D = A * B = \left(d_{ij}\right)_{n\times l}$ as follows:

$$c_{ij} = \vee_{k=1}^{m}\left(a_{ik} \wedge b_{kj}\right), i = 1, 2, \cdots n, j = 1, 2, \cdots l;$$
$$d_{ij} = \wedge_{k=1}^{m}\left[(1 - a_{ik}) \vee b_{kj}\right], i = 1, 2, \cdots n, j = 1, 2, \cdots l.$$

Definition 4.2. For an arbitrary matrix $A = (a_{ik})_{n\times m}$. Let A^T represent the transpose matrix of A, $-A$ denote the matrix $(1 - a_{ik})_{m\times n}$, and $\overline{A}$ is defined as follow:

$$\overline{A} = (q_{ij})_{m\times n} = \begin{cases} a_{ik} & i = j \\ 0 & i \neq j. \end{cases}$$

For a certain fuzzy information system $FIS = (U, \widetilde{AT}, V, f)$, we can express the fuzzy β-covering of U, the β-cut covering of U, and the information function between U and V in the form of matrix.

Definition 4.3. Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, where $U = \{x_1, x_2, \cdots, x_n\}$ and $V = \{y_1, y_2, \cdots, y_s\}$ are non-empty finite sets, $\widetilde{AT} = \left\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^{m}\tilde{A}_i(x)]]$.

We define $Q = (\tilde{A}_j(x_i))_{n\times m}$ as the matrix representation of $\widetilde{AT}$, and matrix $Q_\beta = \left(t_{ij}\right)_{n\times m}$ is a β-matrix representation of $\widetilde{AT}$, where

$$t_{ij} = \begin{cases} 1 & \tilde{A}_j(x_i) \geq \beta \\ 0 & \tilde{A}_j(x_i) < \beta. \end{cases}$$

We call $P = \left(p_{ij}\right)_{s\times n}$ a connection matrix of U and V, where

$$p_{ij} = \begin{cases} 1 & if\ f(x_j) = y_i \\ 0 & else. \end{cases}$$

The following property shows how to compute the fuzzy β-neighborhood and the fuzzy β-co-neighborhood using matrices.

Proposition 4.1. *Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system, where $U = \{x_1, x_2, \cdots, x_n\}$, $\widetilde{AT} = \left\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_m\right\}$ is a fuzzy β-covering of U for $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^{m}\tilde{A}_i(x)]]$. Q is the matrix representation of $\widetilde{AT}$, and Q_β is the β-matrix representation of $\widetilde{AT}$, then the fuzzy β-neighborhood $\tilde{N}_x^\beta$ and fuzzy β-co-neighborhood $\tilde{M}_x^\beta$ of x can define as:*

$$\tilde{N}_x^\beta = Q_\beta * Q^T,\ \tilde{M}_x^\beta = (-Q_\beta) * (-Q^T).$$

Proof. First, we prove the $\tilde{N}_x^\beta = Q_\beta * Q^T$.

Suppose that $Q_\beta * Q^T = \left(c_{ij}\right)_{n\times n}$, where $Q_\beta = (t_{ik})_{n\times m}$, $Q = (\tilde{A}_j(x_k))_{n\times m}$, then

$$\begin{aligned}
c_{ij} &= \wedge_{k=1}^{m}\left[(1-t_{ik}) \vee \tilde{A}_k\left(x_j\right)\right] \\
&= \left[\wedge_{t_{ik}=0}\left(1 \vee \tilde{A}_k\left(x_j\right)\right)\right] \wedge \left[\wedge_{t_{ik}=1}\left(0 \vee \tilde{A}_k\left(x_j\right)\right)\right] \\
&= 1 \wedge \left[\wedge_{t_{ik}=1}\tilde{A}_k\left(x_j\right)\right] = \wedge_{t_{ik}=1}\tilde{A}_k\left(x_j\right) \\
&= \wedge_{\tilde{A}_k(x_i)\geq\beta}\tilde{A}_k\left(x_j\right) \\
&= \left(\cap_{\tilde{A}_k(x_i)\geq\beta}\tilde{A}_k\right)\left(x_j\right) = \tilde{N}_{x_i}^\beta\left(x_j\right). \qquad (i, j = 1, 2, \cdots, n)
\end{aligned}$$

thus $Q_\beta * Q^T = \tilde{N}_x^\beta$.

Next, we prove the $\tilde{M}_x^\beta = (-Q_\beta) * (-Q^T)$.

Suppose that $(-Q_\beta) * (-Q^T) = \left(d_{ij}\right)_{n\times n}$, where $Q_\beta = (t_{ik})_{n\times m}$, $Q^T = (\tilde{A}_k\left(x_j\right))_{n\times m}$, then

$$\begin{aligned}
d_{ij} &= \wedge_{k=1}^{m}\left[t_{ik} \vee \left(1 - \tilde{A}_k\left(x_j\right)\right)\right] \\
&= \left[\wedge_{t_{ik}=1}\left(1 \vee \left(1 - \tilde{A}_k\left(x_j\right)\right)\right)\right] \wedge \left[\wedge_{t_{ik}=0}\left(0 \vee \left(1 - \tilde{A}_k\left(x_j\right)\right)\right)\right] \\
&= 1 \wedge \left[\wedge_{t_{ik}=0}\left(1 - \tilde{A}_k\left(x_j\right)\right)\right] = \wedge_{t_{ik}=0}\left(1 - \tilde{A}_k\left(x_j\right)\right) \\
&= \wedge_{\tilde{A}_k(x_i)<\beta}\left(1 - \tilde{A}_k\left(x_j\right)\right) \\
&= \left(\cap_{\tilde{A}_k(x_i)<\beta} - \tilde{A}_k\right)\left(x_j\right) = \tilde{M}_{x_i}^\beta\left(x_j\right). \qquad (i, j = 1, 2, \cdots, n)
\end{aligned}$$

thus $\left(-Q_\beta\right) * \left(-Q^T\right) = \tilde{M}_x^\beta$. □

In the following, we will investigate the matrix representations of **1st** and **2nd** fuzzy covering lower and upper approximation.

Proposition 4.2. *Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, where $U = \{x_1, x_2, \cdots, x_n\}$, $V = \{y_1, y_2, \cdots, y_s\}$, $P = \left(p_{ij}\right)_{s\times n}$ is the connection matrix of U and V, $\tilde{N}_x^\beta$ and $\tilde{M}_x^\beta$ are fuzzy β-neighborhood and fuzzy β-co-neighborhood of x respectively. For each $\tilde{X} \in F(U)$, we have*

(1) $\underline{APR}\left(\tilde{X}\right) = P * \left(\overline{N}_x^\beta * \tilde{X}\right)$, $\overline{APR}\left(\tilde{X}\right) = P \cdot \left(\overline{N}_x^\beta \cdot \tilde{X}\right)$,

(2) $\underline{\underline{APR}}\left(\tilde{X}\right) = P * \left(\overline{M}_x^\beta * \tilde{X}\right)$, $\overline{\overline{APR}}\left(\tilde{X}\right) = P \cdot \left(\overline{M}_x^\beta \cdot \tilde{X}\right)$.

Proof. (1) First, we prove $\underline{APR}\left(\tilde{X}\right) = P * \left(\overline{N}_x^\beta * \tilde{X}\right)$.

Suppose that $\overline{N}_x^\beta * \tilde{X} = (a_k)_{n\times 1}$, where $\overline{N}_x^\beta = \left(b_{kj}\right)_{n\times n}$, $\tilde{X} = \left(x_j\right)_{n\times 1}$, then

$$\begin{aligned}
a_k &= \wedge_{j=1}^{n}\left[\left(1 - b_{kj}\right) \vee x_j\right] \\
&= \left[\wedge_{j=k}\left((1 - b_{kk}) \vee x_k\right)\right] \wedge \left[\wedge_{j\neq k}\left(1 \vee x_j\right)\right] \\
&= \wedge_{j=k}\left((1 - b_{kk}) \vee x_k\right) \\
&= \left(1 - \tilde{N}_{x_k}^\beta(x_k)\right) \vee \tilde{X}(x_k),
\end{aligned}$$

thus, for each $i (1 \leq i \leq s)$, we have

$$\begin{aligned}\left(P*\left(\overline{N}_x^{\beta}*\tilde{X}\right)\right)(y_i)&=\left[\wedge_{f(x_k)=y_i}(0\vee((1-b_{kk})\vee x_k))\right]\wedge\left[\wedge_{f(x_k)\neq y_i}(1\vee((1-b_{kk})\vee x_k))\right]\\&=\left[\wedge_{f(x_k)=y_i}((1-b_{kk})\vee x_k)\right]\wedge 1\\&=\wedge_{f(x_k)=y_i}((1-b_{kk})\vee x_k)\\&=\wedge_{x\in f^{-1}(y_i)}\left[\left(1-\tilde{N}_x^{\beta}(x)\right)\vee\tilde{X}(x)\right].\end{aligned}$$

So, $P*\left(\overline{N}_x^{\beta}*\tilde{X}\right)=\underline{APR}\left(\tilde{X}\right)$.

Next, we prove $\overline{APR}\left(\tilde{X}\right)=P\cdot\left(\overline{N}_x^{\beta}\cdot\tilde{X}\right)$.

Suppose that $\overline{N}_x^{\beta}\cdot\tilde{X}=(c_k)_{n\times 1}$, where $\overline{N}_x^{\beta}=\left(b_{kj}\right)_{n\times n}$, $\tilde{X}=\left(x_j\right)_{n\times 1}$, then

$$\begin{aligned}c_k&=\vee_{j=1}^{n}\left(b_{kj}\wedge x_j\right)\\&=\left[\vee_{j=k}(b_{kk}\wedge x_k)\right]\vee\left[\vee_{j\neq k}\left(0\wedge x_j\right)\right]\\&=\vee_{j=k}(b_{kk}\wedge x_k)\\&=\tilde{N}_{x_k}^{\beta}(x_k)\wedge\tilde{X}(x_k),\end{aligned}$$

thus, for each $i(1\le i\le s)$, we have

$$\begin{aligned}\left(P\cdot\left(\overline{N}_x^{\beta}\cdot\tilde{X}\right)\right)(y_i)&=\left[\vee_{f(x_k)=y_i}(1\wedge(b_{kk}\wedge x_k))\right]\vee\left[\vee_{f(x_k)\neq y_i}(0\wedge(b_{kk}\wedge x_k))\right]\\&=\left[\vee_{f(x_k)=y_i}(b_{kk}\wedge x_k)\right]\vee 0\\&=\vee_{f(x_k)=y_i}(b_{kk}\wedge x_k)\\&=\vee_{x\in f^{-1}(y_i)}\left[\tilde{N}_x^{\beta}(x)\wedge\tilde{X}(x)\right].\end{aligned}$$

Therefore, $P\cdot\left(\overline{N}_x^{\beta}\cdot\tilde{X}\right)=\overline{APR}\left(\tilde{X}\right)$.

(2) The method to prove that $\underline{\underline{APR}}\left(\tilde{X}\right)=P*\left(\overline{M}_x^{\beta}*\tilde{X}\right)$, $\overline{\overline{APR}}\left(\tilde{X}\right)=P\cdot\left(\overline{M}_x^{\beta}\cdot\tilde{X}\right)$ are similar to (1), just replace all $\tilde{N}_x^{\beta}(x)$ with $\tilde{M}_x^{\beta}(x)$, so no more detailed proof here. □

Example 4.1. Let $FIS=(U,\widetilde{AT},V,f)$ be the fuzzy information system in Example 3.1. Then according to Definition 4.3, for $\beta=0.5$,

$$Q=\begin{pmatrix}0.6&0.5&0.7&0.3&0.2\\0.3&0.4&0.6&0.5&0.4\\0.5&0.3&0.7&0.4&0.6\\0.7&0.3&0.4&0.4&0.5\\0.4&0.7&0.3&0.6&0.8\\0.5&0.6&0.4&0.6&0.3\end{pmatrix},\quad Q_{0.5}=\begin{pmatrix}1&1&1&0&0\\0&0&1&1&0\\1&0&1&0&1\\1&0&0&0&1\\0&1&0&1&1\\1&1&0&1&0\end{pmatrix}$$

are the matrix representation and 0.5-matrix representation of $\widetilde{AT}$.

First, we can use Proposition 4.1 to calculate the fuzzy 0.5-neighborhood $\tilde{N}_x^{0.5}$ and the fuzzy 0.5-co-neighborhood $\tilde{M}_x^{0.5}$.

$$\tilde{N}_x^{0.5}=Q_{0.5}*Q^T=\begin{pmatrix}0.5&0.3&0.3&0.3&0.3&0.4\\0.3&0.5&0.4&0.4&0.3&0.4\\0.2&0.3&0.5&0.4&0.3&0.3\\0.2&0.3&0.5&0.5&0.4&0.3\\0.2&0.4&0.3&0.3&0.6&0.3\\0.3&0.3&0.3&0.3&0.4&0.5\end{pmatrix},$$

$$\tilde{M}_x^{0.5}=(-Q_{0.5})*\left(-Q^T\right)=\begin{pmatrix}0.7&0.5&0.4&0.5&0.2&0.4\\0.4&0.6&0.4&0.3&0.2&0.4\\0.5&0.5&0.6&0.6&0.3&0.4\\0.3&0.4&0.3&0.6&0.3&0.4\\0.3&0.4&0.3&0.3&0.6&0.5\\0.3&0.4&0.3&0.5&0.2&0.6\end{pmatrix}.$$

Thus, for $\tilde{X} = \frac{0.3}{x_1} + \frac{0.8}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4} + \frac{0.6}{x_5} + \frac{0.5}{x_6}$, matrix representation is $\tilde{X} = (0.3\ 0.8\ 0.4\ 0.7\ 0.6\ 0.5)^T$, and

$$f(x) = \begin{cases} y_1 & x \in \{x_2, x_4\} \\ y_2 & x = x_1 \\ y_3 & x \in \{x_3, x_5\} \\ y_4 & x = x_6 \end{cases},$$

according to Definition 4.3, we can get the connection matrix of U and V is

$$P = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\underline{APR}\left(\tilde{X}\right)$, $\overline{APR}\left(\tilde{X}\right)$ and $\underline{\underline{APR}}\left(\tilde{X}\right)$, $\overline{\overline{APR}}\left(\tilde{X}\right)$ can be computed according to Proposition 4.2 as follows:

$$\underline{APR}\left(\tilde{X}\right) = P * \left(\overline{N}_x^{\beta} * \tilde{X}\right) = P * \left[\begin{pmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{pmatrix} * \begin{pmatrix} 0.3 \\ 0.8 \\ 0.4 \\ 0.7 \\ 0.6 \\ 0.5 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 0.3 \\ 0.8 \\ 0.4 \\ 0.7 \\ 0.6 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}.$$

$$\overline{APR}\left(\tilde{X}\right) = P \cdot \left(\overline{N}_x^{\beta} \cdot \tilde{X}\right) = P \cdot \left[\begin{pmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{pmatrix} \cdot \begin{pmatrix} 0.3 \\ 0.8 \\ 0.4 \\ 0.7 \\ 0.6 \\ 0.5 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.3 \\ 0.8 \\ 0.4 \\ 0.7 \\ 0.6 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \\ 0.6 \\ 0.5 \end{pmatrix}.$$

$$\underline{\underline{APR}}\left(\tilde{X}\right) = P * \left(\overline{M}_x^{\beta} * \tilde{X}\right) = P * \left[\begin{pmatrix} 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6 \end{pmatrix} * \begin{pmatrix} 0.3 \\ 0.8 \\ 0.4 \\ 0.7 \\ 0.6 \\ 0.5 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 0.3 \\ 0.8 \\ 0.4 \\ 0.7 \\ 0.6 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.3 \\ 0.4 \\ 0.5 \end{pmatrix}.$$

$$\overline{\overline{APR}}\left(\tilde{X}\right) = P \cdot \left(\overline{M}_x^{\beta} \cdot \tilde{X}\right) = P \cdot \left[\begin{pmatrix} 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 0.3 \\ 0.8 \\ 0.4 \\ 0.7 \\ 0.6 \\ 0.5 \end{pmatrix}\right]$$

$$=\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.3 \\ 0.8 \\ 0.4 \\ 0.7 \\ 0.6 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.3 \\ 0.6 \\ 0.5 \end{pmatrix}.$$

Therefore,

$$\underline{APR}\left(\tilde{X}\right) = \frac{0.7}{y_1} + \frac{0.5}{y_2} + \frac{0.5}{y_3} + \frac{0.5}{y_4}, \overline{APR}\left(\tilde{X}\right) = \frac{0.5}{y_1} + \frac{0.3}{y_2} + \frac{0.6}{y_3} + \frac{0.5}{y_4},$$

$$\underline{\underline{APR}}\left(\tilde{X}\right) = \frac{0.7}{y_1} + \frac{0.3}{y_2} + \frac{0.4}{y_3} + \frac{0.5}{y_4}, \overline{\overline{APR}}\left(\tilde{X}\right) = \frac{0.6}{y_1} + \frac{0.3}{y_2} + \frac{0.6}{y_3} + \frac{0.5}{y_4}.$$

By comparing the results of Example 3.1 and Example 4.1, the conclusion of Proposition 4.2 is verified, from which we obtain the matrix representation of the dyad fuzzy β-covering rough set models $[(\underline{APR}, \overline{APR}), (\underline{\underline{APR}}, \overline{\overline{APR}})]$.

4.2. Programming realization

For a fuzzy information system $FIS = (U, \widetilde{AT}, V, f)$, where $U = \{x_1, x_2, \cdots, x_n\}$ and $V = \{y_1, y_2, \cdots, y_s\}$ are non-empty finite sets, $\widetilde{AT} = \left\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_m\right\} \in F(U)$ is a fuzzy β-covering of U and $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^{m} \tilde{A}_i(x)]]$, $f : U \longrightarrow V$ be a surjective mapping from U to V. For each $\tilde{X} \in F(U)$, calculate the dyad fuzzy β-covering rough set models of $\tilde{X}$. We now present an algorithm to solve the above problems.

Algorithm.
Step 1.
Input the fuzzy β-covering of U: Q, the connection matrix of U and V: P, the matrix representation of $\tilde{X}$, and several parameters: the critical value β, the number of objects in set U: n, the number of objects in set V: s, and the number of fuzzy sets in $\widetilde{AT}$: m.

Step 2.
Compute the β-matrix representation of $\widetilde{AT}$: $Q_\beta = \left(t_{ij}\right)_{n\times m}$, where

$$t_{ij} = \begin{cases} 1 & \tilde{A}_j(x_i) \geq \beta \\ 0 & \tilde{A}_j(x_i) < \beta, \end{cases}$$

and $-Q_\beta, -Q^T$.

Step 3.
Define the sub-functions of two operations separately: dot product $C = A \cdot B = \left(c_{ij}\right)_{n\times l}$ and cross product $D = A * B = \left(d_{ij}\right)_{n\times l}$, where

$$c_{ij} = \vee_{k=1}^{m} \left(a_{ik} \wedge b_{kj}\right), i = 1, 2, \cdots n, j = 1, 2, \cdots l;$$

$$d_{ij} = \wedge_{k=1}^{m} \left[(1 - a_{ik}) \vee b_{kj}\right], i = 1, 2, \cdots n, j = 1, 2, \cdots l.$$

Step 4.
Calculate the fuzzy β-neighborhood and the fuzzy β-co-neighborhood:

$$\tilde{N}_x^\beta = Q_\beta * Q^T,\ \tilde{M}_x^\beta = (-Q_\beta) * (-Q^T).$$

Step 5.
Compute the main diagonal matrix of the fuzzy β-neighborhood and the fuzzy β-co-neighborhood: $\overline{N}_x^\beta$, $\overline{M}_x^\beta$.

Step 6.
Calculate the dyad fuzzy β-covering rough set models of $\tilde{X}$: $[(\underline{APR}, \overline{APR}), (\underline{\underline{APR}}, \overline{\overline{APR}})]$.

$$\underline{APR}\left(\tilde{X}\right) = P * \left(\overline{N}_x^\beta * \tilde{X}\right), \overline{APR}\left(\tilde{X}\right) = P \cdot \left(\overline{N}_x^\beta \cdot \tilde{X}\right), \underline{\underline{APR}}\left(\tilde{X}\right) = P * \left(\overline{M}_x^\beta * \tilde{X}\right), \overline{\overline{APR}}\left(\tilde{X}\right) = P \cdot \left(\overline{M}_x^\beta \cdot \tilde{X}\right).$$

Table 4
Part of professional requirements.

Major \ Subject	Chinese	Mathematics	English	Physics	Chemistry	Biology	History	Geography	Politics
Theoretical and Applied Mechanics	0.68	0.82	0.53	0.95	0.34	0.48	0.19	0.14	0.33
Science of Business Administration	0.81	0.82	0.80	0.64	0.32	0.28	0.42	0.15	0.85
Biological Sciences	0.69	0.72	0.65	0.67	0.85	0.82	0.51	0.66	0.13
Applied Linguistics	0.79	0.53	0.90	0.24	0.17	0.14	0.68	0.33	0.74
Environmental Science	0.75	0.70	0.56	0.78	0.67	0.90	0.76	0.92	0.57
Chinese Language and Literature	0.85	0.55	0.80	0.15	0.16	0.23	0.95	0.12	0.66
Educational Studies	0.90	0.56	0.83	0.21	0.24	0.14	0.73	0.22	0.90
Urban Planning	0.80	0.68	0.54	0.45	0.53	0.48	0.74	0.95	0.72
Mathematics and Applied Mathematics	0.62	0.95	0.46	0.81	0.42	0.30	0.32	0.15	0.33
Food Science and Engineering	0.65	0.76	0.71	0.64	0.90	0.69	0.29	0.31	0.23
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 5
Part of the final exam scores.

	Chinese	Mathematics	English	Physics	Chemistry	Biology	History	Geography	Politics
1	122	113	45	38	45	50	79	93	62
2	117	90	52	61	34	61	89	78	62
3	102	114	91	61	41	56	50	69	56
4	109	104	48	36	71	70	73	75	54
5	108	85	48	78	75	31	77	75	61
6	109	113	40	54	47	35	84	95	60
7	108	101	55	33	74	71	75	60	58
8	114	111	38	50	50	45	79	91	57
9	121	85	44	78	68	32	78	87	41
10	124	122	43	20	50	40	88	89	58
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

5. Application example

When setting up and adjusting majors, colleges and universities must not only adapt to the needs of knowledge innovation, scientific and technological progress, and discipline development, but also need to comprehensively consider the matching degree between professional curriculum and students' acceptance, so that students can better master professional knowledge and technology, optimize the curriculum system, accelerate the academic quality, and increase the number of talents absorbed.

In this section, we will use the dyad fuzzy β-covering rough set models constructed above to evaluate and make decisions on the matching degree between the adjusted new major and students.

In the database, the professional requirements of the existing 50 majors (including, Theoretical and Applied Mechanics, Science of Business Administration, Biological Sciences, etc.) for 9 subjects (including, Chinese, Mathematics, English, Physics, Chemistry, Biology, History, Geography and Politics) are shown in Table 4. For the convenience of writing later, the 9 subjects are denoted as $U = \{x_1, x_2, \cdots, x_9\}$, and the 50 majors are represented as $\widetilde{AT} = \left\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_{50}\right\}$, where $Q\,(i, j) = \tilde{A}_j\,(x_i)$ is recorded as the degree of professional $\tilde{A}_j$'s requirements for subject x_i, the value is between $[0, 1]$, the larger the value, the higher the requirement of the major for this subject.

At the same time, we collected the final exam scores of fresh students in a high middle school, and randomly selected 500 samples, which are denoted as $V = \{y_1, y_2, \cdots, y_{500}\}$, part of the data are shown in Table 5. The value of students' test scores/total scores of this subject is a number between 0 and 1, which can be regarded as the degree of students' mastery of this subject. The higher the score is, the higher the value will be. For the seven subjects $x_2, x_4 \cdots x_9$, students with the value greater than or equal to 0.6 are considered to master these subjects, that is $P\,(i, j) = x_j\,(y_i) = 1\ \ (j = 2, 4, \cdots, 9; i \in \{1, 2, \cdots, 500\})$, and those less than 0.6 are recorded as $P\,(i, j) = x_j\,(y_i) = 0\ \ (j = 2, 4, \cdots, 9; i \in \{1, 2, \cdots, 500\})$; For x_1, considering that Chinese is the most basic subject and students generally get high grades, so students with the value greater than or equal to 0.7 are considered to master this subject, that is, $P\,(i, 1) = x_1\,(y_i) = 1\ \ (i \in \{1, 2, \cdots, 500\})$, and $(i, 1) = x_1\,(y_i) = 0\ \ (i \in \{1, 2, \cdots, 500\})$ if less than 0.7; For x_3, due to the high difficulty coefficient of this English subject test, students generally get low grades, therefore students with this value greater than or equal to 0.5 think that they have mastered the subject, that is, $P\,(i, 3) = x_3\,(y_i) = 1\ \ (i \in \{1, 2, \cdots, 500\})$, the value less than 0.5 is recorded as $P\,(i, 3) = x_3\,(y_i) = 1\ \ (i \in \{1, 2, \cdots, 500\})$. Finally, a table of students' mastery of each subject can be obtained, part of which are shown in Table 6.

The enrollment rate of a major in a university is low, and students' professional knowledge and technical mastery are not ideal, so the curriculum arrangement has been adjusted. After the adjustment, the requirement for each subject of the major

Table 6
Part of students' mastery of each subject.

	Chinese	Mathematics	English	Physics	Chemistry	Biology	History	Geography	Politics
1	1	1	0	0	0	0	1	1	1
2	1	1	1	1	0	1	1	1	1
3	0	1	1	1	0	0	0	1	1
4	1	1	0	0	1	1	1	1	1
5	1	0	0	1	1	0	1	1	1
6	1	1	0	0	0	0	1	1	1
7	1	1	1	0	1	1	1	1	1
8	1	1	0	0	0	0	1	1	1
9	1	0	0	1	1	0	1	1	0
10	1	1	0	0	0	0	1	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 7
The dyad fuzzy β-covering rough set models of $\tilde{X}$.

	$\underline{APR}$	$\overline{APR}$	$\underline{\underline{APR}}$	$\overline{\overline{APR}}$		$\underline{APR}$	$\overline{APR}$	$\underline{\underline{APR}}$	$\overline{\overline{APR}}$		$\underline{APR}$	$\overline{APR}$	$\underline{\underline{APR}}$	$\overline{\overline{APR}}$		$\underline{APR}$	$\overline{APR}$	$\underline{\underline{APR}}$	$\overline{\overline{APR}}$
1	0.48	0.54	0.47	0.72	14	0.48	0.51	0.47	0.50	27	0.43	0.54	0.47	0.72	40	0.43	0.54	0.47	0.72
2	0.43	0.61	0.47	0.72	15	0.43	0.54	0.47	0.72	28	0.43	0.54	0.47	0.72	41	0.43	0.53	0.47	0.54
3	0.48	0.61	0.49	0.54	16	0.43	0.61	0.47	0.72	29	0.49	0.51	0.49	0.50	42	0.49	0.54	0.47	0.72
4	0.43	0.54	0.47	0.72	17	0.43	0.61	0.47	0.50	30	0.49	0.54	0.47	0.72	43	0.48	0.61	0.47	0.50
5	0.49	0.61	0.47	0.72	18	0.48	0.54	0.47	0.72	31	0.49	0.61	0.49	0.72	44	0.43	0.54	0.47	0.72
6	0.48	0.54	0.47	0.72	19	0.43	0.54	0.47	0.72	32	0.49	0.54	0.49	0.72	45	0.43	0.54	0.47	0.72
7	0.43	0.54	0.47	0.72	20	0.43	0.61	0.47	0.50	33	0.49	0.61	0.47	0.50	46	0.43	0.54	0.48	0.72
8	0.48	0.54	0.47	0.72	21	0.43	0.61	0.47	0.72	34	0.43	0.61	0.47	0.50	47	0.43	0.53	0.48	0.54
9	0.49	0.61	0.47	0.72	22	0.43	0.61	0.47	0.50	35	0.43	0.49	0.47	0.49	48	0.43	0.54	0.48	0.72
10	0.48	0.54	0.47	0.72	23	0.43	0.51	0.47	0.50	36	0.43	0.49	0.48	0.49	49	0.43	0.54	0.48	0.72
11	0.43	0.54	0.47	0.72	24	0.43	0.53	0.47	0.54	37	0.48	0.54	0.49	0.72	50	0.48	0.40	0.47	0.40
12	0.48	0.54	0.47	0.72	25	0.43	0.61	0.47	0.72	38	0.43	0.54	0.47	0.72	51	0.43	0.49	0.48	0.49
13	0.49	0.54	0.47	0.72	26	0.48	0.54	0.47	0.72	39	0.49	0.54	0.47	0.72	⋮	⋮	⋮	⋮	⋮

is $\tilde{X}$, and the critical value $\beta = 0.51$. Then we can use the model to consider the matching degree between the adjusted major and students.

For

$$\tilde{X} = \frac{0.72}{x_1} + \frac{0.40}{x_2} + \frac{0.88}{x_3} + \frac{0.64}{x_4} + \frac{0.50}{x_5} + \frac{0.30}{x_6} + \frac{0.21}{x_7} + \frac{0.49}{x_8} + \frac{0.76}{x_9}$$

We use Matlab to program the algorithm in Section 4.2, and substitute the data for operation to get the dyad fuzzy β-covering rough set models of $\tilde{X}$, part of which are shown in Table 7.

First, we can make a positive decision just by analyzing $\underline{APR}\left(\tilde{X}\right)$ and $\overline{APR}\left(\tilde{X}\right)$ under the critical value $\beta = 0.51$:

(1). Since $\underline{APR}\left(\tilde{X}\right)(y_i) \geq 0.51$ and $\overline{APR}\left(\tilde{X}\right)(y_i) \geq 0.51$ $(i = 58, 198, 449, 477, 478, 486)$, we conclude that these 6 students have a high degree of matching with this adjusted major.

(2). As $\overline{APR}\left(\tilde{X}\right)\left(y_j\right) \geq 0.51$ and $\underline{APR}\left(\tilde{X}\right)\left(y_j\right) < 0.51$ $(j = 1 - 34, 37 - 49, 52, 53, 55 - 57, etc.)$, we conclude that this adjusted major has a moderate match with the 455 students.

(3). According to $\underline{APR}\left(\tilde{X}\right)(y_k) < 0.51$ and $\overline{APR}\left(\tilde{X}\right)(y_k) < 0.51$ $(k = 35, 36, 50, 51, 54, 59, 100, 111, etc.)$, we conclude that these 39 students have a low degree of matching with this adjusted major.

If we think that the matching degree is moderate or above, then the student is considered to be matched with the adjusted major, then the positive matching degree between this adjusted major and the students is 92.2%.

Then, we can make a negative decision just by analyzing $\underline{\underline{APR}}\left(\tilde{X}\right)$ and $\overline{\overline{APR}}(\tilde{X})$ under the critical value $\beta = 0.51$:

(1). Since $\underline{\underline{APR}}\left(\tilde{X}\right)(y_i) \geq 0.51$ and $\overline{\overline{APR}}\left(\tilde{X}\right)(y_i) \geq 0.51$ $(i = 58, 198, 477, 478, 486)$, we conclude that these 5 students have a high degree of matching with this adjusted major.

(2). As $\overline{\overline{APR}}\left(\tilde{X}\right)\left(y_j\right) \geq 0.51$ and $\underline{\underline{APR}}\left(\tilde{X}\right)\left(y_j\right) < 0.51$ $(j = 1 - 13, 15, 16, 18, 19, 21, 24 - 28, 30 - 32, etc.)$, we conclude that this adjusted major has a moderate match with the 362 students.

(3). According to $\underline{\underline{APR}}\left(\tilde{X}\right)(y_k) < 0.51$ and $\overline{\overline{APR}}\left(\tilde{X}\right)(y_k) < 0.51$ $(k = 14, 17, 20, 22, 23, 29, 33 - 36, etc.)$, we conclude that these 133 students have a low degree of matching with this adjusted major.

Similarly, if the matching degree is moderate or above, it is considered that the student to be matched with the adjusted major, then we can get the negative matching degree between this adjusted major and the students is 73.4%.

We can see that there have some differences between the positive and negative decisions. In order to make a more comprehensive evaluation. Now we use dyad fuzzy β-covering rough set models $[(\underline{APR}, \overline{APR}), (\underline{\underline{APR}}, \overline{\overline{APR}})]$ to analyze this

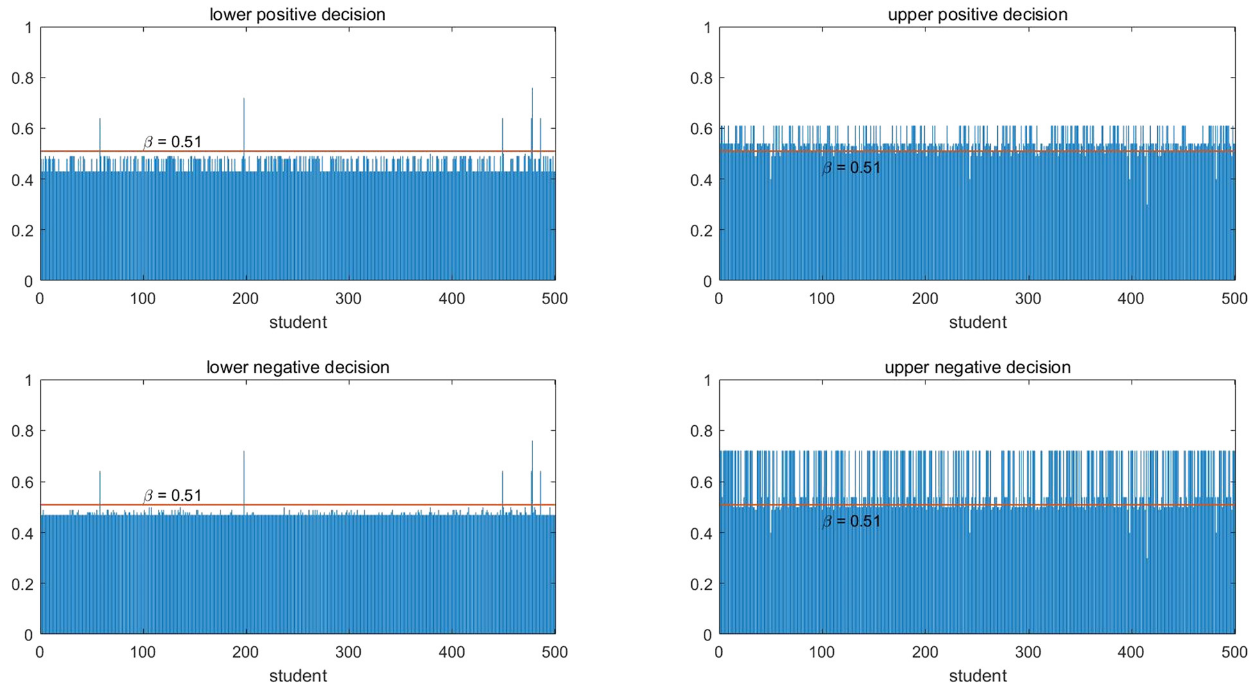

Fig. 1. Four types of decision under the critical value $\beta = 0.51$.

problem. By computing the $\underline{APR}\left(\tilde{X}\right)$ (we say lower positive decision), $\overline{APR}(\tilde{X})$ (we say upper positive decision), $\underline{\underline{APR}}\left(\tilde{X}\right)$ (we say lower negative decision), $\overline{\overline{APR}}(\tilde{X})$ (we say upper negative decision), under the critical value $\beta = 0.51$, relationship among these decisions can be described in Fig. 1.

(1). Since all the decisions of students y_i $(i = 58, 198, 477, 478, 486)$ are not less than the critical value, then this adjusted major is highly compatible with these 5 students.

(2). Because the three decisions of student y_{449} are not less than the critical value, then this adjusted major has a high matching degree with student y_{449}.

(3). According to two decisions of students y_i $(i = 1 - 13, 15, 16, 18, 19, 21, 24 - 28, 30 - 32, etc.)$ are not less than the critical value, then this adjusted major has a moderate match with the 361 students.

(4). Since only one decision of students y_i $(i = 14, 17, 20, 22, 23, 29, 33, 34, 43, 55, 57, 60, etc.)$ are not less than the critical value, then this adjusted major has a low matching degree with these 94 students.

(5). As all the decisions of students y_i $(i = 35, 36, 50, 51, 54, 59, 100, 111, etc.)$ are lower than the critical value, then this adjusted major do not match these 39 students.

If the matching degree is considered moderate or above, the student is considered to be matched with the adjusted major, then the matching degree between the adjusted major and the students is 73.4% under the dyad fuzzy β-covering rough set models, which is consistent with the negative matching degree. We define this value as the lower bound of the matching degree. If only the students who do not match at all are excluded, the matching degree obtained under this model is 92.2%, which is consistent with the positive matching degree. We define this value as the upper bound of the matching degree. That, we can obtain the matching degree under different standards through the dyad fuzzy β-covering rough set models, so as to get a matching degree interval. The matching degree of students to major $\tilde{X}$ under different models is shown in Fig. 2 and Table 8.

This result is more effective for colleges and universities to adjust professional curriculum requirements. Through the dyad fuzzy β-covering rough set models, all students are divided into five categories, so we can improve the matching degree of students more specifically by analyzing the mastery degree of each category's students to the subject.

We continue to adjust the professional requirements of individual subjects on the adjusted major $\tilde{X}$, so as to observe the changes of major matching degree.

Firstly, we reduce the professional requirement of major $\tilde{X}$ for English from 0.88 to 0.48, so as to get $\tilde{X}_1$

$$\tilde{X}_1 = \frac{0.72}{x_1} + \frac{0.40}{x_2} + \frac{0.48}{x_3} + \frac{0.64}{x_4} + \frac{0.50}{x_5} + \frac{0.30}{x_6} + \frac{0.21}{x_7} + \frac{0.49}{x_8} + \frac{0.76}{x_9}$$

Then we can obtain the matching degree of students to major $\tilde{X}_1$ under different models, as shown in Table 9.

By comparing the results in Table 8 and Table 9, it can be found that after reducing the professional requirement for English, the upper bound and the lower bound of the major matching degree both decrease, indicating that the high

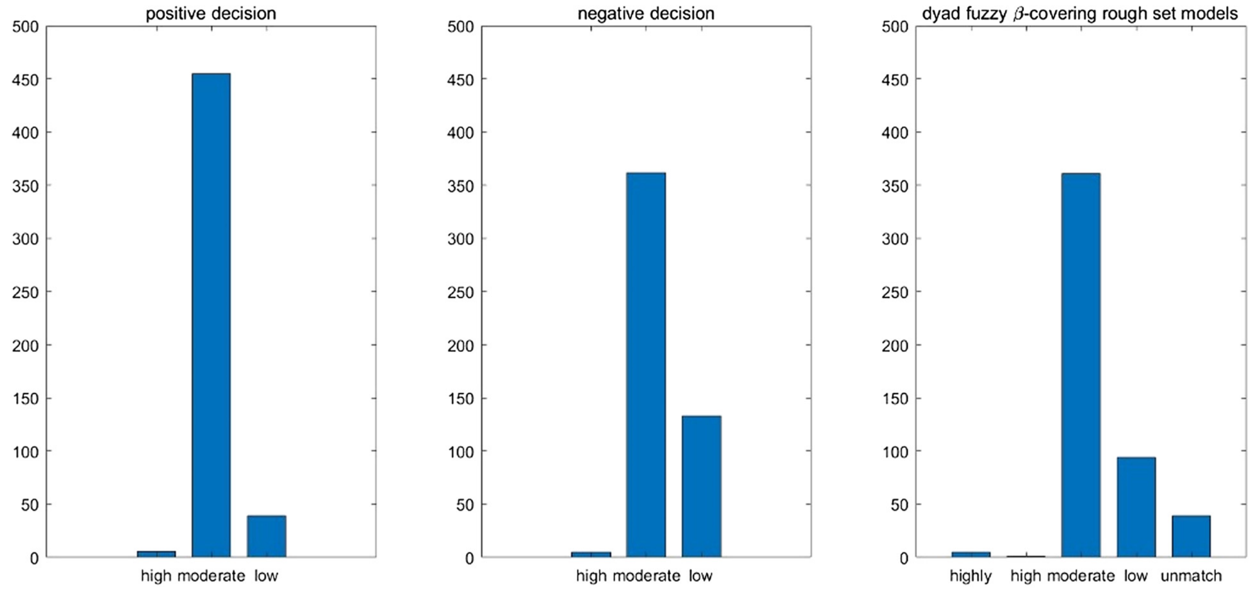

Fig. 2. The division of students' matching degree to major $\tilde{X}$ under different models.

Table 8
The matching degree of students to major $\tilde{X}$ under different models with $\beta = 0.51$.

	Highly	High	Moderate	Low	unmatch	Matching degree
Positive decision	/	6	455	39	/	92.2%
Negative decision	/	5	362	133	/	73.4%
Dyad fuzzy β-covering rough set models	5	1	361	94	39	[73.4%, 92.2%]

Table 9
The matching degree of students to major $\tilde{X}_1$ under different models with $\beta = 0.51$.

	Highly	High	Moderate	Low	unmatch	Matching degree
Positive decision	/	3	429	68	/	86.4%
Negative decision	/	2	263	235	/	53%
Dyad fuzzy β-covering rough set models	2	1	262	167	68	[53%, 86.4%]

Table 10
The matching degree of students to major $\tilde{X}_2$ under different models with $\beta = 0.51$.

	Highly	High	Moderate	Low	unmatch	Matching degree
Positive decision	/	14	471	15	/	97%
Negative decision	/	13	354	133	/	73.4%
Dyad fuzzy β-covering rough set models	13	1	353	118	15	[73.4%, 97%]

requirement for English should be maintained in this major in view of the demand for English from other majors and the mastery of students.

Then, we improve the professional requirement of major $\tilde{X}$ for Mathematics from 0.4 to 0.6, so as to get $\tilde{X}_2$

$$\tilde{X}_2 = \frac{0.72}{x_1} + \frac{0.60}{x_2} + \frac{0.48}{x_3} + \frac{0.64}{x_4} + \frac{0.50}{x_5} + \frac{0.30}{x_6} + \frac{0.21}{x_7} + \frac{0.49}{x_8} + \frac{0.76}{x_9}$$

Therefore, the matching degree of students to major $\tilde{X}_2$ under different models is shown in Table 10.

Comparing the results of Table 8 and Table 10, it can be found that after increasing the professional requirement for Mathematics, the lower bound of major matching has not changed, but the upper bound has increased. After taking into account the requirements of other majors and students' mastery of mathematics, this major should appropriately improve the teaching requirements of mathematics subjects, such as adding courses related to mathematics, improving the assessment standards of related courses in mathematics, etc.

Finally, we improve the critical value from $\beta = 0.51$ to $\beta' = 0.54$, then we can obtain the new matching degree of students to major $\tilde{X}$ under different models, which is shown in Table 11.

By comparing the data in Table 8 and Table 11, it can be seen that after increasing the critical value β, the upper bound of major matching degree has not changed, but the lower bound has decreased significantly. This shows that it is not

Table 11
The matching degree of students to major $\tilde{X}$ under different models with $\beta' = 0.54$.

	Highly	High	Moderate	Low	unmatch	Matching degree
Positive decision	/	6	455	39	/	92.2%
Negative decision	/	3	264	233	/	53.4%
Dyad fuzzy β'-covering rough set models	3	3	261	194	39	[53.4%, 92.2%]

possible to blindly improve the requirements of the major for all subjects, and that the professional arrangements should be adjusted according to the students in a targeted manner.

Through this application example, we can see that the same problem from positive and negative aspects respectively usually leads to different decisions, while considering both aspects at the same time can get a more comprehensive decision. In fact, in order to solve practical problems and make more accurate decisions, if we can analyze the issues using different methods from more aspects, then we can reach better conclusions. This fully reflects the advantages of dyad fuzzy β-covering rough set models to deal with practical problems.

6. Conclusion

In this paper, we extend the idea of couple approximate operators to the fuzzy information system, and defined a new fuzzy β-covering rough set models. Then we combine this model with the model proposed by Yang in [39], and construct the dyad fuzzy β-covering rough set models. This binary model can analyze and solve practical problems from both positive and negative aspects, which can better simulate the situation of people in decision-making, so as to make more comprehensive evaluation. The main points of this article are summarized as follows.

(1) Firstly, **1^st^** fuzzy β-covering rough set model and **2^nd^** fuzzy β-covering rough set model are redefined by replacing the β-neighborhood of Yang [39] model with β-co-neighborhood, properties are investigated, and also constructed the dyad fuzzy β-covering rough set models.

(2) Then, the matrix representation formulas of the model are proved, and verified through an example, then realize the machine operation by using MATLAB algorithm programming.

(3) Finally, an application example is given to illustrate the practical value of the model, and the effectiveness and superiority of the model are verified through data analysis and comparative study.

There are still some problems worthy of further study in fuzzy information system decision-making. For example, whether there is a more scientific method to determine the critical value β instead of just based on experience, and how to use other mathematical concepts to simplify the model proposed in this paper, so that it is faster and more convenient when dealing with practical big data applications. In addition, the application of dyad fuzzy β-covering rough set models under multi-source information systems is also an area worth exploring. To this end, further studies will be carried out on this topic in the future.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.ijar.2021.11.001.

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