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Foundational theories of hesitant fuzzy sets and hesitant fuzzy information systems and their applications for multi-strength intelligent classifiers

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A B S T R A C T

Hesitant fuzzy sets find extensive application in specific scenarios involving uncertainty and hesitation. In the context of set theory, the concept of inclusion relationship holds significant importance as a fundamental definition. Consequently, as a type of sets, hesitant fuzzy sets necessitate a clear and explicit definition of the inclusion relationship. Based on the discrete form of hesitant fuzzy membership degrees, this study proposes multiple types of inclusion relationships for hesitant fuzzy sets. Subsequently, this paper introduces foundational propositions related to hesitant fuzzy sets, as well as propositions concerning families of hesitant fuzzy sets. Furthermore, this research presents foundational propositions regarding parameter reduction of hesitant fuzzy information systems. An example and an algorithm are provided to demonstrate the parameter reduction processes. Lastly, a multi-strength intelligent classifier is proposed for diagnosing the health states of complex systems.

1. Introduction

The foundation of modern mathematics lies in set theory established by Cantor [1] (referred to as classical sets hereafter). Moreover, modern branches of mathematics, including group theory [2,3], topology [4], graph theory [5], among others, form the underpinnings of computer science. Subsequent to the refinement of classical set theory, various types of sets emerged, encompassing fuzzy sets [6,7], interval-valued fuzzy sets [8], intuitionistic fuzzy sets [9], and hesitant fuzzy sets [10], among others. With the exception of hesitant fuzzy sets, several types of sets have established their foundational theoretical frameworks. Notably, the definition of the inclusion relationship between two hesitant fuzzy sets, which is one of the fundamental definitions, lacks adequate clarity. This article develops the foundational theories of hesitant fuzzy sets and hesitant fuzzy information systems, and introduces a multi-strength intelligent classifier for diagnosing the health states of complex systems.

In the case of the classical set [1], the membership degree of an element is described as either 0% or 100%, and the element's relationship with the set is determined by whether it belongs to the set or not. Nevertheless, the real world is fraught with uncertainties. To address such uncertain scenarios, Zadeh introduced the concept of fuzzy sets [6], where the membership degree of an element in

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a set is a value ranging from 0 to 1. The membership degree of a fuzzy set denotes the probability of an element belonging to the set [11], thereby extending beyond the binary values of 0% and 100%. Later, Gorzalczany [8] introduced the notion of interval-valued fuzzy sets, where the membership degree of an element in a set is represented by an interval included in $[0,1]$. Interval-valued fuzzy sets can handle situations where the precise probability of an element belonging to a set cannot be determined, but an interval can be identified within which the probability must lie [12]. Additionally, Atanassov [9] introduced the notion of intuitionistic fuzzy sets, where the membership degree of an element in a set is represented by a bipartite array. Intuitionistic fuzzy sets can capture entities with dual characteristics, with one value in the bipartite array representing the positive aspect and the other value representing the negative aspect [13,14].

Torra [10] introduced the concept of hesitant fuzzy sets, where the membership degree of an element in a set is represented by a multidimensional array. Hesitant fuzzy sets are designed to address situations of uncertainty and hesitation, which are prevalent in real-world scenarios. For example, in scenarios involving data sampling for equipment monitoring and decision scoring by expert teams, the feedback often manifests as arrays with multiple diverse values, leading to indecisiveness. Consequently, hesitant fuzzy sets with a membership degree represented by a discrete array are better suited for capturing the indecisiveness in decision-making compared to classical, fuzzy, interval-valued fuzzy, and intuitionistic fuzzy sets. Hesitant fuzzy sets find extensive applications in various domains, including decision-making [15,16], attribute reduction [17], classification [18], linguistic perceptual [19], and forecasting [20], among others.

Nevertheless, a sufficiently clear definition of the inclusion relationship between two hesitant fuzzy sets is currently lacking. The definition of the inclusion relationship serves as a crucial foundation for sets, as it establishes equivalence between two sets when one is a subset of the other and vice versa. Based on the discrete form of hesitant fuzzy membership degrees, this study introduces multiple types of inclusion relationships for hesitant fuzzy sets and subsequently presents foundational propositions concerning hesitant fuzzy sets and their families. It is important to highlight that some foundational propositions applicable to classical sets do not hold in the case of hesitant fuzzy sets.

In recent years, the study of information systems dealing with uncertain information has gained significant attention due to the increasing prevalence of uncertainty in real-world scenarios. Numerous studies have been conducted in this field, including research on fuzzy information systems [21], fuzzy incomplete information systems [22], interval-valued information systems [23,24], and intuitionistic fuzzy information systems [25]. Building upon the foundational propositions of hesitant fuzzy sets, this study explores various theories concerning parameter reductions in hesitant fuzzy information systems.

The analysis of high-dimensional data with limited sample sizes presents a challenging task, making it difficult to diagnose the health status of complex systems. Drawing upon the foundational propositions of hesitant fuzzy information systems, we propose a multi-strength intelligent classifier in knowledge bases [26] to facilitate health state diagnoses for complex systems. The multi-strength intelligent classifier serves as a promising tool for various tasks, including dimensionality reduction of high-dimensional data, classification, diagnosis, evaluation, decision-making, and more.

The remainder of this paper is organized as follows. Section 2 introduces various types of inclusion relationships among hesitant fuzzy sets, along with their foundational propositions, and shows whether certain rules that apply to classical sets also hold for hesitant fuzzy sets. In Section 3, we put forth propositions concerning families of hesitant fuzzy sets. Section 4 presents propositions related to parameter reductions in hesitant fuzzy information systems and offers an algorithm for performing these reductions. In Section 5, we propose a multi-strength intelligent classifier for diagnosing the health states of complex systems. Section 6 provides a conclusion.

2. Foundations of hesitant fuzzy sets

In this section, we provide a brief overview of fundamental concepts related to hesitant fuzzy sets and introduce various types of inclusion relationships that are applicable to the discrete form of hesitant fuzzy membership degrees. Our focus is on examining whether certain rules that apply to classical sets also hold for hesitant fuzzy sets. In further research on hesitant fuzzy sets, it is crucial for researchers to refrain from relying on the intuitions derived from classical sets and to be mindful of the rules that are valid in classical sets but not in hesitant fuzzy sets.

If two sets A and B are classical sets, $A \sqcap B$ and $A \sqcup B$ represent the intersection and union of A and B, respectively. Furthermore, $A \sqsubset B$ represents that A is a subset of the classical set B. Let U be a universal set and E be a set of parameters.

2.1. Reviews of fuzzy sets and hesitant fuzzy sets

Definition 2.1. [6] A fuzzy set F on U is a mapping $F : U \to [0,1]$.

Definition 2.2. [6] F_1 and F_2 are two fuzzy sets on U, F_1 is a subset of F_2 if $F_1(x) \leqslant F_2(x)$ for all $x \in U$, denoted as $F_1 \subset F_2$.

Definition 2.3. [27] A hesitant fuzzy element is a non-empty, finite subset of $[0,1]$.

Definition 2.4. [10] A hesitant fuzzy set on U is defined as a function that when applied to U returns a subset of $[0,1]$.

In the following, $HF(U)$ denotes the set of all hesitant fuzzy sets defined over U.

Definition 2.5. [10] For each $x \in U$, and a hesitant fuzzy set H, the lower bound and upper bound of $H(x)$ are defined
lower bound $H^-(x) = inf\ H(x)$,
upper bound $H^+(x) = sup\ H(x)$.

Definition 2.6. [10] Given two hesitant fuzzy sets represented by their membership functions H_1 and H_2, their union and intersection are defined
union $(H_1 \cup H_2)(x) = \{h \in H_1(x) \sqcup H_2(x) : h \geqslant sup(H_1^-(x), H_2^-(x))\}$,
intersection $(H_1 \cap H_2)(x) = \{h \in H_1(x) \sqcup H_2(x) : h \leqslant inf(H_1^+(x), H_2^+(x))\}$.

Definition 2.7. [10] For $x \in U$ and $H \in HF(U)$, the complement of H is denoted as H^c, where
$H^c(x) = \sqcup_{\gamma \in H(x)}\{1-\gamma\}$.

Theorem 2.8. *[10,27] The following statements hold for $A, B, C \in HF(U)$,*
(1) $(A^c)^c = A$.
(2) $(A \cap B)^c = A^c \cup B^c$.
(3) $(A \cup B)^c = A^c \cap B^c$.
(4) $A \cap B = B \cap A$, $A \cup B = B \cup A$.
(5) $(A \cap B) \cap C = A \cap (B \cap C)$, $(A \cup B) \cup C = A \cup (B \cup C)$.

Let $U = \{x, y\}$, $A(x) = \{0.6, 0.5, 0.3\}$ and $A(y) = \{0.5, 0.3, 0.2\}$. There are three common expression types for the hesitant fuzzy set A, shown as follows,

$$\begin{cases} (type\ 1) \quad A = \{\frac{\{0.6,0.5,0.3\}}{x}, \frac{\{0.5,0.3,0.2\}}{y}\}, \\ (type\ 2) \quad A = \frac{\{0.6,0.5,0.3\}}{x} + \frac{\{0.5,0.3,0.2\}}{y}, \\ (type\ 3) \quad A = \{\langle x, (0.6, 0.5, 0.3)\rangle, \langle y, (0.5, 0.3, 0.2)\rangle\}. \end{cases}$$

Furthermore, $A = B$ means that $A(x)$ and $B(x)$ are perfectly consistent for each $x \in U$. For example, $B = \frac{\{0.3,0.6,0.5\}}{x} + \frac{\{0.2,0.5,0.3\}}{y}$, then $B = A$; $C = \frac{\{0.3,0.3,0.6,0.5\}}{x} + \frac{\{0.2,0.5,0.3\}}{y}$, then $C \neq A$.

2.2. The proposed inclusion definitions of hesitant fuzzy sets and their relationships

For two hesitant fuzzy sets H_1 and H_2, Babitha et al. [28] defined that H_1 is a hesitant fuzzy subset of H_2 if $H_1(x) \subset H_2(x)$ for all $x \in U$. However, the reference [28] lacks a detailed description of $H_1(x) \subset H_2(x)$.

For two fuzzy sets F_1 and F_2, we have $F_1 \subset F_2 \Leftrightarrow F_1 \cup F_2 = F_2$. Carlos et al. [29] introduced a definition of inclusion relationship for hesitant fuzzy sets, i.e., $H_1 \subset H_2 \Leftrightarrow H_1 \cup H_2 = H_2$. This definition (in [29]) inherits the idea of fuzzy sets, which is convenient for scholars to refer to the existing theory of fuzzy sets to study hesitant fuzzy sets. However, we hold two different views on this definition (in [29]), shown as (i) and (ii),

(i) The inclusion relationship of sets is also the ordering relationship of sets. A foundational and important proposition of inclusion relationship, $A \subset B$ and $B \subset A$ if and only if $A = B$, i.e., the equivalent sets contain each other. We assume that $H_1 = H_2$ means that $H_1(x)$ and $H_2(x)$ are perfectly consistent for each $x \in U$. By the description in [29], $H_1(x) \subset H_2(x) \Leftrightarrow \{h : h \in H_2(x)\} = H_2(x) = (H_1 \cup H_2)(x) = \{h \in H_1(x) \sqcup H_2(x) : h \geqslant sup(H_1^-(x), H_2^-(x))\}$, i.e., $H_1(x) \subset H_2(x)$ implies $h < H_2^-(x)$ for all $h \in H_1(x)$. If $H_1(x) \subset H_2(x)$ and $H_2(x) \subset H_1(x)$, then a contradiction is produced, i.e., $H_1^+(x) < H_2^-(x) \leqslant H_2^+(x) < H_1^-(x) \leqslant H_1^+(x)$.

(ii) The definition of inclusion relationship in [29] is too specific to have a small range of applications. For example, let $H_1(x) = \{0.1, 0.2, 0.5\}$, $H_2(x) = \{0.6, 0.7, 0.9\}$ and $H_3(x) = \{0.1, 0.5, 0.8\}$, then $H_1(x) \subset H_2(x)$ is obvious. However, this definition cannot describe the relationship of $H_2(x)$ and $H_3(x)$ and is not applicable for many cases of hesitant fuzzy sets.

This study introduces various types of inclusion relations for hesitant fuzzy sets determined by the strength of the information contained in hesitant fuzzy membership degrees. Initially, a comprehensive example is provided to clarify Definition 2.10.

Example 2.9. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a set of decision-making schemes and H be an expert team that consists of three experts. $H(U) = \frac{\{0.9,0.2\}}{x_1} + \frac{\{0.6,0.6,0.5\}}{x_2} + \frac{\{0.7,0.5,0.5\}}{x_3} + \frac{\{0.8,0.6,0.5\}}{x_4} + \frac{\{0.9,0.3,0.1\}}{x_5} + \frac{\{0.9,0.8,0.7\}}{x_6}$ are the estimated values of schemes provided by experts, in which the estimated values for x_1 are 0.9 and 0.2 that are obtained through the evaluations made by experts. One of the three experts fails to evaluate the scheme x_1.

(1) Here, x_1 has an estimated value of 0.9, which is greater than or equal to all the estimated values for x_2, it is possible that the scheme x_1 is better than the scheme x_2, denoted as $H(x_2) \subset_p H(x_1)$.

(2) Here, $0.55 = mean[H(x_1)] < mean[H(x_2)] = 0.567$, where $mean[\cdot]$ is the mean value operator. In comparing the mean values of estimated values, we find that the scheme x_2 is better than the scheme x_1, denoted as $H(x_1) \subset_m H(x_2)$.

(3) On the one hand, the best estimated value of x_3 is 0.7, which is greater than or equal to the best estimated value of x_2. On the other hand, the worst estimated value of x_3 is 0.5, which is greater than or equal to the worst estimated value of x_2. To compare the respective best and worst cases of schemes x_2 and x_3, it is acceptable that the scheme x_3 is better than the scheme x_2, denoted as $H(x_2) \subset_a H(x_3)$.

(4) To compare the estimated values for schemes x_3 and x_4 one by one ($0.7 \leqslant 0.8$; $0.5 \leqslant 0.6$; and $0.5 \leqslant 0.5$), it is strongly credible that the scheme x_4 is better than the scheme x_3, denoted as $H(x_3) \subset_s H(x_4)$.

(5) We can obtain $H(x_1) \subset_s H(x_5)$ after truncating the tail estimated value of x_5, i.e., deleting the estimated value 0.1 of x_5. This case is denoted as $H(x_1) \subset_{st} H(x_5)$ and is recorded briefly as $H(x_1) \subset_t H(x_5)$.

(6) The worst estimated value of x_6 is greater than or equal to the best estimated value of x_3. Thus, it is necessary that the scheme x_6 is better than the scheme x_3, which is denoted as $H(x_3) \subset_n H(x_6)$.

Comparing the mean value of estimated values is a common approach for decision-making; however, while doing so, some important information may be lost, such as the best and the worst estimated values.

Definition 2.10. Let H_1 and H_2 be two hesitant fuzzy sets on U. Several kinds of inclusion relationships of two hesitant fuzzy sets are defined as follows,

(1) If $H_1^+(x) \leqslant H_2^+(x)$, then $H_1(x) \subset_p H_2(x)$. If $H_1(x) \subset_p H_2(x)$ for all $x \in U$, then $H_1 \subset_p H_2$. If $H_1 \subset_p H_2$ and $H_2 \subset_p H_1$, then $H_1 =_p H_2$.

(2) If $H_1^+(x) \leqslant H_2^+(x)$ and $H_1^-(x) \leqslant H_2^-(x)$, then $H_1(x) \subset_a H_2(x)$. If $H_1(x) \subset_a H_2(x)$ for all $x \in U$, then $H_1 \subset_a H_2$. If $H_1 \subset_a H_2$ and $H_2 \subset_a H_1$, then $H_1 =_a H_2$.

(3) If $mean[H_1(x)] \leqslant mean[H_2(x)]$, then $H_1(x) \subset_m H_2(x)$. If $H_1(x) \subset_m H_2(x)$ for all $x \in U$, then $H_1 \subset_m H_2$. If $H_1 \subset_m H_2$ and $H_2 \subset_m H_1$, then $H_1 =_m H_2$.

(4) Let $H_1(x) = V = \{v_1, v_2, \cdots, v_k\}$ and $H_2(x) = W = \{w_1, w_2, \cdots, w_l\}$ be two descending sequences. If $k \geqslant l$ and $w_i \geqslant v_i$ for $1 \leqslant i \leqslant l$, then $H_1(x) \subset_s H_2(x)$. If $H_1(x) \subset_s H_2(x)$ for all $x \in U$, then $H_1 \subset_s H_2$. If $H_1 \subset_s H_2$ and $H_2 \subset_s H_1$, then $H_1 =_s H_2$.

(5) Let $H_1(x) = V = \{v_1, v_2, \cdots, v_k\}$ and $H_2(x) = W = \{w_1, w_2, \cdots, w_l\}$ be two descending sequences. If $k < l$ and $w_i \geqslant v_i$ for $1 \leqslant i \leqslant k$, then $H_1(x) \subset_t H_2(x)$. If $H_1(x) \subset_t H_2(x)$ for all $x \in U$, then $H_1 \subset_t H_2$. It is obvious that $H_1 \subset_t H_2$ and $H_2 \subset_t H_1$ cannot hold simultaneously.

(6) If $H_1^+(x) \leqslant H_2^-(x)$, then $H_1(x) \subset_n H_2(x)$. If $H_1(x) \subset_n H_2(x)$ for all $x \in U$, then $H_1 \subset_n H_2$. If $H_1 \subset_n H_2$ and $H_2 \subset_n H_1$, then $H_1 =_n H_2$.

Symbols. (1) Let $W = \{w_1, w_2, \cdots, w_l\}$ and $W' = \{w'_1, w'_2, \cdots, w'_l\}$ be two descending sequences and $|W| = |W'|$. If $w'_1 \geqslant w_1$, $w'_2 \geqslant w_2$, $\cdots$, $w'_l \geqslant w_l$, then the relationship of W and W' is denoted as $W \preccurlyeq W'$.

(2) Let q be a positive integer. To take q larger numbers of W and construct a best q-subsequence of W, denoted as $W(q)$, i.e., $|W(q)| = q$ and $w \leqslant inf(W(q))$ for $w \in W - W(q)$. For example, let $W = \{0.9, 0.8, 0.7, 0.65, 0.6, 0.5\}$, when $q = 2$, $W(q) = \{0.9, 0.8\}$; when $q = 3$, $W(q) = \{0.9, 0.8, 0.7\}$.

Remark 2.11. (1) If W is a subsequence of $H_2(x)$, $H_1(x) \preccurlyeq W$ and $|H_1(x)| < |H_2(x)|$, then $H_1(x) \subset_t H_2(x)$, vice versa.

(2) If W is a subsequence of $H_2(x)$, $H_1(x) \preccurlyeq W$ and $|H_1(x)| = |H_2(x)|$, then $H_1(x) \subset_s H_2(x)$.

(3) Let $q = inf\{|H_1(x)|, |H_2(x)|\}$, $H_1(x)(q)$ and $H_2(x)(q)$ be the best q-subsequences of $H_1(x)$ and $H_2(x)$, respectively. Thereafter, $H_1(x)(q) \preccurlyeq H_2(x)(q)$ if and only if one of $H_1(x) \subset_s H_2(x)$ and $H_1(x) \subset_t H_2(x)$ holds.

(4) Let $K = H_1(x) \sqcap H_2(x)$, if $sup(H_1(x) - K) \leqslant inf(K) \leqslant sup(K) \leqslant inf(H_2(x) - K)$, then one of $H_1(x) \subset_s H_2(x)$ and $H_1(x) \subset_t H_2(x)$ holds.

Proposition 2.12. *The following statements hold for $A, B \in HF(U)$,*

(1) If $A \subset_a B$, then $A \subset_p B$.
(2) If $A \subset_s B$, then $A \subset_p B$.
(3) If $A \subset_s B$, then $A \subset_a B$.
(4) If $A \subset_s B$, then $A \subset_m B$.
(5) If $A \subset_t B$, then $A \subset_p B$.
(6) If $A \subset_n B$, then $A \subset_p B$.
(7) If $A \subset_n B$, then $A \subset_a B$.
(8) If $A \subset_n B$, then $A \subset_m B$.
(9) If $A \subset_n B$, then one of $A(x) \subset_s B(x)$ and $A(x) \subset_t B(x)$ holds for all $x \in U$.

Proof. (1)-(8) are obvious.

(9) If $A \subset_n B$, then $A^+(x) \leqslant B^-(x)$ for all $x \in U$. If $|A(x)| \geqslant |B(x)|$, then $A(x) \subset_s B(x)$. If $|A(x)| < |B(x)|$, then $A(x) \subset_t B(x)$. ■

A brief summary for Proposition 2.12 is shown as follows,

$$\begin{cases} \text{“}\subset_n\text{”} \Rightarrow \text{“}\subset_s\text{”} \Rightarrow \text{“}\subset_a\text{”} \Rightarrow \text{“}\subset_p\text{”}, \\ \text{“}\subset_n\text{”} \Rightarrow \text{“}\subset_s\text{”} \Rightarrow \text{“}\subset_m\text{”}, \\ \text{“}\subset_n\text{”} \Rightarrow \text{“}\subset_t\text{”} \Rightarrow \text{“}\subset_p\text{”}. \end{cases}$$

2.3. Some propositions about the intersection and union of two hesitant fuzzy sets

Let $\bar{A}$ and $\bar{B}$ be two classical sets, we know that $\bar{A}\sqcap\bar{B}\sqsubset\bar{A}$, $\bar{A}\sqcap\bar{B}\sqsubset\bar{B}$, $\bar{A}\sqsubset\bar{A}\sqcup\bar{B}$, $\bar{B}\sqsubset\bar{A}\sqcup\bar{B}$, and $\bar{A}\sqcap\bar{B}\sqsubset\bar{A}\sqcup\bar{B}$. However, these rules are not necessarily true for two hesitant fuzzy sets A and B (shown as Propositions 2.13-2.20).

Proposition 2.13. *The following statements hold for $A_1, A_2 \in HF(U)$,*
(1) $A_1 \cap A_2 \subset_p A_i$ for $i=1,2$.
(2) $A_i \subset_p A_1 \cup A_2$ for $i=1,2$.
(3) $A_1 \cap A_2 \subset_p A_1 \cup A_2$.

Proof. (1) and (2) are obvious.
(3) $inf\{A_1^+(x), A_2^+(x)\} \leqslant sup\{A_1^+(x), A_2^+(x)\}$. ■

Proposition 2.14. *The following statements hold for $A_1, A_2 \in HF(U)$,*
(1) $A_1 \cap A_2 \subset_a A_i$ for $i=1,2$.
(2) $A_i \subset_a A_1 \cup A_2$ for $i=1,2$.
(3) $A_1 \cap A_2 \subset_a A_1 \cup A_2$.

Proof. (1) and (2) are obvious.
(3) $inf\{A_1^+(x), A_2^+(x)\} \leqslant sup\{A_1^+(x), A_2^+(x)\}$ and $inf\{A_1^-(x), A_2^-(x)\} \leqslant sup\{A_1^-(x), A_2^-(x)\}$. ■

Proposition 2.15. *The following statements hold for $x \in U$ and $A, B \in HF(U)$,*
(1) At least one of $(A\cap B)(x) \subset_m A(x)$ and $(A\cap B)(x) \subset_m B(x)$ holds.
(2) At least one of $A(x) \subset_m (A\cup B)(x)$ and $B(x) \subset_m (A\cup B)(x)$ holds.
(3) $A\cap B \subset_m A\cup B$.
(4) If $A \subset_m B$, then $A\cap B \subset_m B$.
(5) If $A \subset_m B$, then $A \subset_m A\cup B$.

Proof. For $x\in U$, denote $k_1=|A(x)|$, $k_2=|B(x)|$, $\bar{a}=mean[A(x)]$, and $\bar{b}=mean[B(x)]$, where $mean[\cdot]$ is the mean operator. Then $\sum_{h\in A(x)} h + \sum_{h\in B(x)} h = k_1\bar{a}+k_2\bar{b}$.

(1) For each $x\in U$, we can divide the relationships of $A^+(x)$ and $B^+(x)$ into two cases (c1) and (c2) as follows,

(c1) $A^+(x)=B^+(x)$.

$\sum_{h\in(A\cap B)(x)} h = \sum_{h\in A(x)} h + \sum_{h\in B(x)} h = k_1\bar{a}+k_2\bar{b} \leqslant sup(\bar{a},\bar{b})(k_1+k_2)$. $mean[(A\cap B)(x)] = \sum_{h\in(A\cap B)(x)} h/(k_1+k_2) \leqslant sup(\bar{a},\bar{b})(k_1+k_2)/(k_1+k_2) = sup(\bar{a},\bar{b})$. Then, one of $(A\cap B)(x)\subset_m A(x)$ and $(A\cap B)(x)\subset_m B(x)$ holds.

(c2) $A^+(x)\neq B^+(x)$.

$A^+(x)\neq B^+(x)$, let $V=\{h: h\in A(x)\ or\ h\in B(x), h\notin (A\cap B)(x)\}$ and $h'=inf(V)$, then $h' > inf(A^+(x),B^+(x)) = sup\{h: h\in (A\cap B)(x)\}$.

Since $h' > sup\{h: h\in(A\cap B)(x)\}$ and $|(A\cap B)(x)|+|V|=k_1+k_2$, then $\sum_{h\in(A\cap B)(x)} h/|(A\cap B)(x)| < (\sum_{h\in(A\cap B)(x)} h + h'|V|)/(k_1+k_2)$. Since $h'=inf(V)$, then $(\sum_{h\in(A\cap B)(x)} h + h'|V|)/(k_1+k_2) \leqslant (k_1\bar{a}+k_2\bar{b})/(k_1+k_2)$.

Then, we can obtain $mean[(A\cap B)(x)] = \sum_{h\in(A\cap B)(x)} h/|(A\cap B)(x)| < (\sum_{h\in(A\cap B)(x)} h + h'|V|)/(k_1+k_2) \leqslant (k_1\bar{a}+k_2\bar{b})/(k_1+k_2) \leqslant sup(\bar{a},\bar{b})(k_1+k_2)/(k_1+k_2) = sup(\bar{a},\bar{b})$. Then, one of $(A\cap B)(x)\subset_m A(x)$ and $(A\cap B)(x)\subset_m B(x)$ holds.

(2) For each $x\in U$, we can divide the relationships of $A^-(x)$ and $B^-(x)$ into two cases (c1) and (c2) as follows,

(c1) $A^-(x)=B^-(x)$.

$\sum_{h\in(A\cup B)(x)} h = \sum_{h\in A(x)} h + \sum_{h\in B(x)} h = k_1\bar{a}+k_2\bar{b} \geqslant inf(\bar{a},\bar{b})(k_1+k_2)$.

$\sum_{h\in(A\cup B)(x)} h/(k_1+k_2) \geqslant inf(\bar{a},\bar{b})(k_1+k_2)/(k_1+k_2) = inf(\bar{a},\bar{b})$. Then, one of $A(x)\subset_m (A\cup B)(x)$ and $B(x)\subset_m (A\cup B)(x)$ holds.

(c2) $A^-(x)\neq B^-(x)$.

$A^-(x)\neq B^-(x)$, let $V=\{h: h\in A(x)\ or\ h\in B(x), h\notin (A\cup B)(x)\}$ and $h'=sup(V)$. $h' < sup(A^-(x),B^-(x)) = inf\{h: h\in (A\cup B)(x)\}$.

Since $h'=sup(V)$ and $|(A\cup B)(x)|+|V|=k_1+k_2$, then $(\sum_{h\in(A\cup B)(x)} h + h'|V|)/(k_1+k_2) \geqslant (k_1\bar{a}+k_2\bar{b})/(k_1+k_2)$. Since $h' < inf\{h: h\in(A\cup B)(x)\}$, then $\sum_{h\in(A\cup B)(x)} h/|(A\cup B)(x)| > (\sum_{h\in(A\cup B)(x)} h + h'|V|)/(k_1+k_2)$.

Based on the results above, then $mean[(A\cup B)(x)] = \sum_{h\in(A\cup B)(x)} h/|(A\cup B)(x)| > (\sum_{h\in(A\cup B)(x)} h + h'|V|)/(k_1+k_2) \geqslant (k_1\bar{a}+k_2\bar{b})/(k_1+k_2) \geqslant inf(\bar{a},\bar{b})(k_1+k_2)/(k_1+k_2) = inf(\bar{a},\bar{b})$. Then, one of $A(x)\subset_m (A\cup B)(x)$ and $B(x)\subset_m (A\cup B)(x)$ holds.

(3) For each $x\in U$, we can divide the relationships of $A^-(x)$, $A^+(x)$, $B^-(x)$ and $B^+(x)$ into four cases (c1), (c2), (c3) and (c4) as follows,

(c1) $A^-(x) = B^-(x)$ and $A^+(x) = B^+(x)$.

In this case, $(A \cap B)(x) = (A \cup B)(x)$, thus $mean[(A \cap B)(x)] = mean[(A \cup B)(x)]$, then $(A \cap B)(x) \subset_m (A \cup B)(x)$ is obvious.

(c2) $A^-(x) = B^-(x)$ and $A^+(x) \neq B^+(x)$.

In this case, $|(A \cap B)(x)| < k_1 + k_2$ and $|(A \cup B)(x)| = k_1 + k_2$. Let $V = \{h : h \in (A \cup B)(x), h \notin (A \cap B)(x)\}$ and $h' = inf(V)$. $h' > inf(A^+(x), B^+(x)) = sup\{h : h \in (A \cap B)(x)\}$.

Since $|(A \cap B)(x)| + |V| = |(A \cup B)(x)| = k_1 + k_2$ and $h' > sup\{h : h \in (A \cap B)(x)\}$, then $\sum_{h\in(A\cap B)(x)} h/|(A \cap B)(x)| < (\sum_{h\in(A\cap B)(x)} h + h'|V|)/(k_1 + k_2)$. Since $h' = inf(V)$, then $(\sum_{h\in(A\cap B)(x)} h + h'|V|)/(k_1 + k_2) \leqslant \sum_{h\in(A\cup B)(x)} h/(k_1 + k_2)$.

Based on the results above, then $mean[(A \cap B)(x)] = \sum_{h\in(A\cap B)(x)} h/|(A \cap B)(x)| < (\sum_{h\in(A\cap B)(x)} h + h'|V|)/(k_1 + k_2) \leqslant \sum_{h\in(A\cup B)(x)} h/(k_1 + k_2) = mean[(A \cup B)(x)]$, i.e., $(A \cap B)(x) \subset_m (A \cup B)(x)$.

(c3) $A^-(x) \neq B^-(x)$ and $A^+(x) = B^+(x)$.

In this case, $|(A \cap B)(x)| = k_1 + k_2$ and $|(A \cup B)(x)| < k_1 + k_2$. Let $V = \{h : h \in (A \cap B)(x), h \notin (A \cup B)(x)\}$ and $h' = sup(V)$. $h' < sup(A^-(x), B^-(x)) = inf\{h : h \in (A \cup B)(x)\}$.

Since $|(A \cup B)(x)| + |V| = |(A \cap B)(x)| = k_1 + k_2$ and $h' < inf\{h : h \in (A \cup B)(x)\}$, then $\sum_{h\in(A\cup B)(x)} h/|(A \cup B)(x)| > (\sum_{h\in(A\cup B)(x)} h + h'|V|)/(k_1 + k_2)$. Since $h' = sup(V)$, then $(\sum_{h\in(A\cup B)(x)} h + h'|V|)/(k_1 + k_2) \geqslant \sum_{h\in(A\cap B)(x)} h/(k_1 + k_2)$.

Based on the results above, then $mean[(A \cup B)(x)] = \sum_{h\in(A\cup B)(x)} h/|(A \cup B)(x)| > (\sum_{h\in(A\cup B)(x)} h + h'|V|)/(k_1 + k_2) \geqslant \sum_{h\in(A\cap B)(x)} h/(k_1 + k_2) = mean[(A \cap B)(x)]$, i.e., $(A \cap B)(x) \subset_m (A \cup B)(x)$.

(c4) $A^-(x) \neq B^-(x)$ and $A^+(x) \neq B^+(x)$.

Based on the condition $A^-(x) \neq B^-(x)$, let $V = \{h : h \in (A \cap B)(x), h \notin (A \cup B)(x)\}$ and $h' = sup(V)$, then $h' < sup(A^-(x), B^-(x)) = inf\{h : h \in (A \cup B)(x)\}$, and $|V| + |(A \cup B)(x)| = k_1 + k_2$.

Based on the results above, then $mean[(A \cup B)(x)] = \sum_{h\in(A\cup B)(x)} h/|(A \cup B)(x)| > (\sum_{h\in(A\cup B)(x)} h + h'|V|)/(k_1 + k_2) \geqslant (k_1\bar{a} + k_2\bar{b})/(k_1 + k_2)$.

Based on the condition $A^+(x) \neq B^+(x)$, let $W = \{h : h \in (A \cup B)(x), h \notin (A \cap B)(x)\}$ and $h'' = inf(W)$, then $h'' > inf(A^+(x), B^+(x)) = sup\{h : h \in (A \cap B)(x)\}$, and $|W| + |(A \cap B)(x)| = k_1 + k_2$.

Based on the results above, then $mean[(A \cap B)(x)] = \sum_{h\in(A\cap B)(x)} h/|(A \cap B)(x)| < (\sum_{h\in(A\cap B)(x)} h + h''|W|)/(k_1 + k_2) \leqslant (k_1\bar{a} + k_2\bar{b})/(k_1 + k_2) < mean[(A \cup B)(x)]$, i.e., $(A \cap B)(x) \subset_m (A \cup B)(x)$.

To sum up, $A \cap B \subset_m A \cup B$.

(4) If $A \subset_m B$, by (1), then $A \cap B \subset_m B$ is obtained.

(5) If $A \subset_m B$, by (2), then $A \subset_m A \cup B$ is obtained. ■

Example 2.16. Let $U = \{x, y, z\}$, $A = \frac{\{0.1,0.8\}}{x} + \frac{\{0.1,0.8\}}{y} + \frac{\{0.7,0.9\}}{z}$, $B = \frac{\{0.7,0.9\}}{x} + \frac{\{0.1,0.9\}}{y} + \frac{\{0.1,0.8\}}{z}$.

$(A \cap B)(x) = \{0.1, 0.7, 0.8\}$, $mean[(A \cap B)(x)] = 0.53$. $mean[A(x)] = 0.45$. $(A \cap B)(x) \not\subset_m A(x)$.

$(A \cup B)(y) = \{0.1, 0.1, 0.8, 0.9\}$, $mean[(A \cup B)(y)] = 0.475$. $mean[B(y)] = 0.5$. $B(y) \not\subset_m (A \cup B)(y)$.

$(A \cap B)(y) \subset_m A(y)$ and $(A \cap B)(y) \subset_m B(y)$.

$A(x) \subset_m (A \cup B)(x)$ and $B(x) \subset_m (A \cup B)(x)$.

For two random hesitant fuzzy sets A and B, Example 2.16 shows that $(A \cap B)(x) \subset_m A(x)$ and $(A \cap B)(x) \subset_m B(x)$ do not hold simultaneously, $A(y) \subset_m (A \cup B)(y)$ and $B(y) \subset_m (A \cup B)(y)$ do not hold simultaneously.

$(A \cap B)(x) \subset_m B(x)$ and $(A \cap B)(x) \not\subset_m A(x)$. Furthermore, $(A \cap B)(z) \subset_m A(z)$ and $(A \cap B)(z) \not\subset_m B(z)$. It means that (1) of Proposition 2.15 cannot be written as that one of $A \cap B \subset_m A$ and $A \cap B \subset_m B$ holds. When we discuss the case "$\subset_m$" we need to note that

$$\begin{cases} A_1 \cap A_2 \subset_m A_i \text{ is uncertain for } i = 1, 2, \\ A_i \subset_m A_1 \cup A_2 \text{ is uncertain for } i = 1, 2. \end{cases}$$

Proposition 2.17. *The following statements hold for $x \in U$ and $A_1, A_2 \in HF(U)$,*

(1) At least one of $A_i(x) \subset_s (A_1 \cup A_2)(x)$ and $A_i(x) \subset_t (A_1 \cup A_2)(x)$ holds for $i = 1, 2$.

(2) At least one of $(A_1 \cap A_2)(x) \subset_s (A_1 \cup A_2)(x)$ and $(A_1 \cap A_2)(x) \subset_t (A_1 \cup A_2)(x)$ holds.

Proof. (1) We prove the case that one of $A_1(x) \subset_s (A_1 \cup A_2)(x)$ and $A_1(x) \subset_t (A_1 \cup A_2)(x)$ holds in follows.

For each $x \in U$, if $A_1^-(x) \geqslant A_2^-(x)$, then $h \in (A_1 \cup A_2)(x)$ for all $h \in A_1(x)$, i.e., $A_1(x)$ is a subsequence of $(A_1 \cup A_2)(x)$. $A_1(x) \preccurlyeq A_1(x)$ and $|A_1(x)| \leqslant |(A_1 \cup A_2)(x)|$, by Remark 2.11 (1) and (2), then one of $A_1(x) \subset_s (A_1 \cup A_2)(x)$ and $A_1(x) \subset_t (A_1 \cup A_2)(x)$ holds.

If $A_1^-(x) < A_2^-(x)$, let $K = A_1(x) \sqcap (A_1 \cup A_2)(x)$, then $sup(A_1(x) - K) < inf(K) \leqslant sup(K) < sup((A_1 \cup A_2)(x) - K)$, by Remark 2.11 (4), then one of $A_1(x) \subset_s (A_1 \cup A_2)(x)$ and $A_1(x) \subset_t (A_1 \cup A_2)(x)$ holds.

With the same manner, we can prove the case that one of $A_2(x) \subset_s (A_1 \cup A_2)(x)$ and $A_2(x) \subset_t (A_1 \cup A_2)(x)$ holds.

(2) Let $K = (A_1 \cap A_2)(x) \sqcap (A_1 \cup A_2)(x)$, then $sup((A_1 \cap A_2)(x) - K) < inf(K) \leqslant sup(K) < inf((A_1 \cup A_2)(x) - K)$, by Remark 2.11 (4), one of $(A_1 \cap A_2)(x) \subset_s (A_1 \cup A_2)(x)$ and $(A_1 \cap A_2)(x) \subset_t (A_1 \cup A_2)(x)$ holds. ■

If two inclusion relationships $H_1(x) \subset_s H_2(x)$ and $H_1(x) \subset_t H_2(x)$ in Definition 2.10 are combined in $H_1(x) \subset_{sot} H_2(x)$ ($\subset_{sot}$ means $\subset_s$ or $\subset_t$), then Proposition 2.17 can be described as follows:

Proposition 2.17 ′ The following statements hold for $A_1, A_2 \in HF(U)$,
(1) $A_i \subset_{sot} A_1 \cup A_2$ for $i = 1, 2$.
(2) $A_1 \cap A_2 \subset_{sot} A_1 \cup A_2$.

Proposition 2.18. *The following statements hold for $x \in U$ and $A, B \in HF(U)$,*
(1) If $A \subset_p B$, then at least one of $A(x) \subset_s (A \cap B)(x)$ and $A(x) \subset_t (A \cap B)(x)$ holds.
(2) If $A \subset_a B$, then at least one of $A(x) \subset_s (A \cap B)(x)$ and $A(x) \subset_t (A \cap B)(x)$ holds.
(3) If $A \subset_s B$, then at least one of $A(x) \subset_s (A \cap B)(x)$ and $A(x) \subset_t (A \cap B)(x)$ holds.
(4) If $A \subset_t B$, then at least one of $A(x) \subset_s (A \cap B)(x)$ and $A(x) \subset_t (A \cap B)(x)$ holds.
(5) If $A \subset_n B$, then at least one of $A(x) \subset_s (A \cap B)(x)$ and $A(x) \subset_t (A \cap B)(x)$ holds.

Proof. (1) $A \subset_p B$ implies that $A^+(x) \leqslant B^+(x)$ for all $x \in U$. So we can deduce that if $h \in A(x)$ then $h \in (A \cap B)(x)$, i.e., $A(x)$ is a subsequence of $(A \cap B)(x)$. $A(x) \preccurlyeq A(x)$ and $|A(x)| \leqslant |(A \cap B)(x)|$, by Remark 2.11 (1) and (2), then one of $A(x) \subset_s (A \cap B)(x)$ and $A(x) \subset_t (A \cap B)(x)$ holds.
(2) $A \subset_a B$ implies $A \subset_p B$, by the result of (1), then (2) holds.
(3) $A \subset_s B$ implies $A \subset_p B$, by the result of (1), then (3) holds.
(4) $A \subset_t B$ implies $A \subset_p B$, by the result of (1), then (4) holds.
(5) $A \subset_n B$ implies $A \subset_p B$, by the result of (1), then (5) holds. ■

Proposition 2.18′ The following statements hold for $A, B, C \in HF(U)$,
(1) If $A \subset_p B$, then $A \subset_{sot} A \cap B$.
(2) If $A \subset_a B$, then $A \subset_{sot} A \cap B$.
(3) If $A \subset_{sot} B$, then $A \subset_{sot} A \cap B$.
(4) If $A \subset_n B$, then $A \subset_{sot} A \cap B$.

Example 2.19. Let $U = \{x, y\}$, $A = \frac{\{0.1,0.2,0.5,0.6,0.9\}}{x} + \frac{\{0.1,0.7\}}{y}$, $B = \frac{\{0.05,0.3,0.4,0.7,0.8\}}{x} + \frac{\{0.8,0.9,0.9\}}{y}$.
$(A \cap B)(x) = \{0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$, then $(A \cap B)(x) \not\subset_s A(x)$, $(A \cap B)(x) \not\subset_s B(x)$, $(A \cap B)(x) \not\subset_t A(x)$, and $(A \cap B)(x) \not\subset_t B(x)$.
$(A \cup B)(y) = \{0.8, 0.9, 0.9\}$, then $A(y) \subset_t (A \cup B)(y)$ and $B(y) \subset_s (A \cup B)(y)$.

Proposition 2.20. *The following statements hold for $A, B \in HF(U)$,*
(1) If $A \subset_n B$, $A \cap B \subset_n B$.
(2) If $A \subset_n B$, $A \subset_n A \cup B$.
(3) If $A \subset_n B$, $A \cap B \subset_n A \cup B$.

Proof. Since $A \subset_n B$, then $A^+(x) \leqslant B^-(x)$ for all $x \in U$, then (1), (2) and (3) are obvious. ■

Example 2.21. Let $U = \{x\}$, $A = \frac{\{0.1,0.3,0.5\}}{x}$, $B = \frac{\{0.2,0.4,0.6\}}{x}$.

$(A \cap B)(x) = \{0.1, 0.2, 0.3, 0.4, 0.5\}$, $(A \cup B)(x) = \{0.2, 0.3, 0.4, 0.5, 0.6\}$. $A \cap B \not\subset_n A$, $A \cap B \not\subset_n B$, $A \not\subset_n A \cup B$, $B \not\subset_n A \cup B$, $A \cap B \not\subset_n A \cup B$.

2.4. Some propositions about a pair of hesitant fuzzy sets (one being a subset of the other) and a random additional hesitant fuzzy set

For three classical sets $\bar{A}$, $\bar{B}$ and $\bar{C}$, if $\bar{A} \sqsubset \bar{B}$, then $\bar{A} \sqsubset \bar{B} \sqcup \bar{C}$, $\bar{A} \sqcap \bar{C} \sqsubset \bar{B} \sqcap \bar{C}$ and $\bar{A} \sqcup \bar{C} \sqsubset \bar{B} \sqcup \bar{C}$. These rules in the cases of three hesitant fuzzy sets A, B and C are investigated, shown as Proposition 2.22 and 2.24.

Proposition 2.22. *The following statements hold for $x \in U$ and $A, B, C \in HF(U)$,*
(1) If $A \subset_p B$, then $A \subset_p B \cup C$.
(2) If $A \subset_a B$, then $A \subset_a B \cup C$.
(3) If $A \subset_s B$, then at least one of $A(x) \subset_s (B \cup C)(x)$ and $A(x) \subset_t (B \cup C)(x)$ holds.
(4) If $A \subset_t B$, then at least one of $A(x) \subset_s (B \cup C)(x)$ and $A(x) \subset_t (B \cup C)(x)$ holds.
(5) If $A \subset_n B$, then at least one of $A(x) \subset_s (B \cup C)(x)$ and $A(x) \subset_t (B \cup C)(x)$ holds.

Proof. (1) and (2) are obvious.
(3) We prove this proposition after dividing the relationships of $h \in C(x)$ and the interval $[B^-(x), B^+(x)]$ into two cases (i) and (ii) for each $x \in U$.

(i) $h \notin [B^-(x), B^+(x)]$ for all $h \in C(x)$.

This case includes two sub-cases, $B^+(x) < C^-(x)$ and $C^+(x) < B^-(x)$.

If $B^+(x) < C^-(x)$, then $A(x) \subset_s B(x) \subset_n C(x) = (B \cup C)(x)$, then $A^+(x) \leqslant B^+(x) < C^-(x)$, i.e., $A(x) \subset_n C(x) = (B \cup C)(x)$. $A(x) \subset_n (B \cup C)(x)$, by Proposition 2.12 (9), then one of $A(x) \subset_s (B \cup C)(x)$ and $A(x) \subset_t (B \cup C)(x)$ holds.

If $C^+(x) < B^-(x)$, then $A(x) \subset_s B(x) = (B \cup C)(x)$.

(ii) $h \in [B^-(x), B^+(x)]$ for some $h \in C(x)$.

This case includes two sub-cases, $C^-(x) \leqslant B^-(x)$ and $B^-(x) < C^-(x)$. Suppose $A(x) = \{o_1, o_2, \cdots, o_a\} = O$, $B(x) = \{v_1, v_2, \cdots, v_b\} = V$ and $C(x) = \{w_1, w_2, \cdots, w_c\} = W$, and O, V and W are descending sequences.

When $C^-(x) \leqslant B^-(x)$.

It is obvious that $h \in (B \cup C)(x)$ for all $h \in B(x)$. We can divide the elements of $A(x)$ into two descending sequences $\{o_1, o_2, \cdots, o_b\}$ and $\{o_{b+1}, o_{b+2}, \cdots, o_a\}$.

Since $A(x) \subset_s B(x)$, for $o_i \in \{o_1, o_2, \cdots, o_b\}$, we have $v_i \geqslant o_i$ and $v_i \in (B \cup C)(x)$, where $i = 1, 2, \cdots, b$.

For $o_j \in \{o_{b+1}, o_{b+2}, \cdots, o_a\} \sqsubset A(x)$ and all $h \in (B \cup C)(x)$, we have $h \geqslant B^-(x) = v_b \geqslant o_b \geqslant o_{b+1} \geqslant o_j$, where $j = b+1, b+2, \cdots, a$.

Based on the illustrations of sub-case $C^-(x) \leqslant B^-(x)$, we have $A(x)(q) \preccurlyeq (B \cup C)(x)(q)$, where $q = inf\{|A(x)|, |(B \cup C)(x)|\}$, by Remark 2.11 (3), one of $A(x) \subset_s (B \cup C)(x)$ and $A(x) \subset_t (B \cup C)(x)$ holds.

When $B^-(x) < C^-(x)$.

Suppose $v_{m-1} \geqslant w_c = C^-(x)$ and $v_m < w_c$. For all $h \in (B \cup C)(x)$, $h \geqslant w_c = C^-(x)$. We can divide the elements of $A(x)$ into two descending sequences $\{o_1, o_2, \cdots, o_{m-1}\}$ and $\{o_m, o_{m+1}, \cdots, o_a\}$.

For $o_i \in \{o_1, o_2, \cdots, o_{m-1}\}$, we have $v_i \geqslant o_i$ and $v_i \in (B \cup C)(x)$, where $i = 1, 2, \cdots, m-1$.

For $o_j \in \{o_m, o_{m+1}, \cdots, o_a\} \sqsubset A(x)$ and all $h \in (B \cup C)(x)$, we have $h \geqslant C^-(x) = w_c > v_m \geqslant o_m \geqslant o_j$, where $j = m, m+1, \cdots, a$.

Based on the illustrations of sub-case $B^-(x) < C^-(x)$, we have $A(x)(q) \preccurlyeq (B \cup C)(x)(q)$, where $q = inf\{|A(x)|, |(B \cup C)(x)|\}$, by Remark 2.11 (3), one of $A(x) \subset_s (B \cup C)(x)$ and $A(x) \subset_t (B \cup C)(x)$ holds.

(4) We prove this proposition after dividing the relationships of $h \in C(x)$ and the interval $[B^-(x), B^+(x)]$ into two cases (i) and (ii) for each $x \in U$.

(i) $h \notin [B^-(x), B^+(x)]$ for all $h \in C(x)$.

This case includes two sub-cases, $B^+(x) < C^-(x)$ and $C^+(x) < B^-(x)$.

When $B^+(x) < C^-(x)$, we have $A(x) \subset_t B(x) \subset_n C(x) = (B \cup C)(x)$, then $A^+(x) \leqslant B^+(x) < C^-(x)$, i.e., $A(x) \subset_n C(x) = (B \cup C)(x)$. $A(x) \subset_n (B \cup C)(x)$, by Proposition 2.12 (9), then one of $A(x) \subset_s (B \cup C)(x)$ and $A(x) \subset_t (B \cup C)(x)$ holds.

When $C^+(x) < B^-(x)$, $A(x) \subset_t B(x) = (B \cup C)(x)$.

(ii) $h \in [B^-(x), B^+(x)]$ for some $h \in C(x)$.

This case includes two sub-cases, $C^-(x) \leqslant B^-(x)$ and $B^-(x) < C^-(x)$.

When $C^-(x) \leqslant B^-(x)$.

We have $h \in (B \cup C)(x)$ for all $h \in B(x)$, i.e., $B(x) \sqsubset (B \cup C)(x)$. Since $A(x) \subset_t B(x)$, by Remark 2.11 (1), there is a subsequence of $B(x)$, denoted as $\overline{B(x)}$, satisfying $A(x) \preccurlyeq \overline{B(x)}$. Since $B(x) \sqsubset (B \cup C)(x)$, then there is a subsequence of $(B \cup C)(x)$, denoted as $\overline{(B \cup C)(x)}$, satisfying $A(x) \preccurlyeq \overline{(B \cup C)(x)}$. $|A(x)| < |B(x)| \leqslant |(B \cup C)(x)|$ is obvious. By Remark 2.11 (1), $A(x) \subset_t (B \cup C)(x)$ holds.

When $B^-(x) < C^-(x)$.

Suppose $A(x) = \{o_1, o_2, \cdots, o_a\} = O$, $B(x) = \{v_1, v_2, \cdots, v_b\} = V$ and $C(x) = \{w_1, w_2, \cdots, w_c\} = W$, and O, V and W are descending sequences.

Suppose $v_{m-1} \geqslant w_c$ and $v_m < w_c = C^-(x)$. We can divide the elements of $A(x)$ into two descending sequences $\{o_1, o_2, \cdots, o_{m-1}\}$ and $\{o_m, o_{m+1}, \cdots, o_a\}$.

Since $A(x) \subset_t B(x)$ and $v_{m-1} \geqslant w_c = C^-(x)$, then $\{o_1, o_2, \cdots, o_{m-1}\} \preccurlyeq \{v_1, v_2, \cdots, v_{m-1}\}$ for $\{o_1, o_2, \cdots, o_{m-1}\} \sqsubset A(x)$ and $\{v_1, v_2, \cdots, v_{m-1}\} \sqsubset (B \cup C)(x)$.

Since $o_m \leqslant v_m < w_c = C^-(x)$, for $o_j \in \{o_m, o_{m+1}, \cdots, o_a\} \sqsubset A(x)$ and all $h \in (B \cup C)(x)$, it implies $h > o_j$, where $j = m, m+1, \cdots, a$.

Based on the illustrations of sub-case $B^-(x) < C^-(x)$, $A(x)(q) \preccurlyeq (B \cup C)(x)(q)$ is obtained, where $q = inf\{|A(x)|, |(B \cup C)(x)|\}$, by Remark 2.11 (3), one of $A(x) \subset_s (B \cup C)(x)$ and $A(x) \subset_t (B \cup C)(x)$ holds.

(5) By Proposition 2.12 (9), $A \subset_n B$ implies $A(x) \subset_s B(x)$ or $A(x) \subset_t B(x)$ for all $x \in U$. By the results of (3) and (4) of Proposition 2.22, (5) holds. ■

(3) and (4) of Proposition 2.22 can be combined as (1) of Proposition 2.22′. (5) of Proposition 2.22 can be written into (2) of Proposition 2.22′.

Proposition 2.22′ The following statements hold for $A, B, C \in HF(U)$,

(1) If $A \subset_{sot} B$, then $A \subset_{sot} B \cup C$.

(2) If $A \subset_n B$, then $A \subset_{sot} B \cup C$.

Proposition 2.23. *The following statements hold for $A, B, C \in HF(U)$,*

(1) If $A \subset_p B$, $A \cap C \subset_p B \cap C$.

(2) If $A \subset_a B$, $A \cap C \subset_a B \cap C$.

(3) If $A \subset_s B$, $A \cap C \subset_a B \cap C$.

(4) If $A \subset_t B$, $A \cap C \subset_p B \cap C$.

(5) If $A \subset_n B$, $A \cap C \subset_a B \cap C$.

Proof. (1)-(5) are obvious.

Proposition 2.24. *The following statements hold for $x \in U$ and $A, B, C \in HF(U)$,*

(1) If $A \subset_p B$, $A \cup C \subset_p B \cup C$.
(2) If $A \subset_a B$, $A \cup C \subset_a B \cup C$.
(3) If $A \subset_s B$, $A \cup C \subset_a B \cup C$.
(4) If $A \subset_t B$, $A \cup C \subset_p B \cup C$.
(5) If $A \subset_n B$, $A \cup C \subset_a B \cup C$.
(6) If $A \subset_s B$, then at least one of $(A \cup C)(x) \subset_s (B \cup C)(x)$ and $(A \cup C)(x) \subset_t (B \cup C)(x)$ holds.
(7) If $A \subset_t B$, then at least one of $(A \cup C)(x) \subset_s (B \cup C)(x)$ and $(A \cup C)(x) \subset_t (B \cup C)(x)$ holds.
(8) If $A \subset_n B$, then at least one of $(A \cup C)(x) \subset_s (B \cup C)(x)$ and $(A \cup C)(x) \subset_t (B \cup C)(x)$ holds.

Proof. (1)-(5) are obvious.

(6) For a random $x \in U$, suppose $A(x) = \{o_1, o_2, \cdots, o_a\} = O$, $B(x) = \{v_1, v_2, \cdots, v_b\} = V$ and $C(x) = \{w_1, w_2, \cdots, w_c\} = W$, and O, V and W are descending sequences. Since $A \subset_s B$, then $b \leqslant a$, and $o_i \leqslant v_i$ for $1 \leqslant i \leqslant b$.

To sort and investigate the two sequences $W^1 = \{o_1, o_2, \cdots, o_b\} \sqcup W$ and $W^2 = \{v_1, v_2, \cdots, v_b\} \sqcup W$, it is obvious that $W^1 \preccurlyeq W^2$. $v_b = B^-(x) \leqslant (B \cup C)^-(x)$, then $(B \cup C)(x)$ is a subsequence of W^2. Let $q = inf\{|(A \cup C)(x)|, |(B \cup C)(x)|\}$, then $q \leqslant |W^1| = |W^2|$. $w_j^1 \leqslant w_j^2$ for $1 \leqslant j \leqslant q = inf\{|(A \cup C)(x)|, |(B \cup C)(x)|\}$, where $w_j^1 \in W^1$ and $w_j^2 \in W^2$.

By Remark 2.11 (3), one of $(A \cup C)(x) \subset_s (B \cup C)(x)$ and $(A \cup C)(x) \subset_t (B \cup C)(x)$ holds.

(7) For a random $x \in U$, suppose $A(x) = \{o_1, o_2, \cdots, o_a\} = O$, $B(x) = \{v_1, v_2, \cdots, v_b\} = V$ and $C(x) = \{w_1, w_2, \cdots, w_c\} = W$, and O, V and W are descending sequences. Since $A \subset_t B$, then $a < b$, and $o_i \leqslant v_i$ for $1 \leqslant i \leqslant a$.

To sort and investigate the two sequences $W^1 = \{o_1, o_2, \cdots, o_a\} \sqcup W$ and $W^2 = \{v_1, v_2, \cdots, v_a\} \sqcup W$, it is obvious that $W^1 \preccurlyeq W^2$. $o_a = A^-(x) \leqslant (A \cup C)^-(x)$, then $(A \cup C)(x)$ is a subsequence of W^1. Let $q = inf\{|(A \cup C)(x)|, |(B \cup C)(x)|\}$, then $q \leqslant |W^1| = |W^2|$. $w_j^1 \leqslant w_j^2$ for $1 \leqslant j \leqslant q = inf\{|(A \cup C)(x)|, |(B \cup C)(x)|\}$, where $w_j^1 \in W^1$ and $w_j^2 \in W^2$.

By Remark 2.11 (3), one of $(A \cup C)(x) \subset_s (B \cup C)(x)$ and $(A \cup C)(x) \subset_t (B \cup C)(x)$ holds.

(8) By Proposition 2.12 (9), $A \subset_n B$ implies $A(x) \subset_s B(x)$ or $A(x) \subset_t B(x)$ for $x \in U$. By the results of (6) and (7) of Proposition 2.24, then (8) holds. ■

(6) and (7) of Proposition 2.24 can be combined as (1) of Proposition 2.24′. (8) of Proposition 2.24 can be written into (2) of Proposition 2.24′.

Proposition 2.24′ The following statements hold for $A, B, C \in HF(U)$,
(1) If $A \subset_{sot} B$, then $A \cup C \subset_{sot} B \cup C$.
(2) If $A \subset_n B$, then $A \cup C \subset_{sot} B \cup C$.

Example 2.25. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, $A = \frac{\{0.2,0.4\}}{x_1} + \frac{\{0.2,0.5\}}{x_2} + \frac{\{0.3,0.5\}}{x_3} + \frac{\{0.5,0.6\}}{x_4} + \frac{\{0.6,0.7\}}{x_5} + \frac{\{0.3,0.4\}}{x_6} + \frac{\{0.3,0.4\}}{x_7} + \frac{\{0.3,0.4\}}{x_8}$, $B = \frac{\{0.1,0.1,0.5\}}{x_1} + \frac{\{0.1,0.8\}}{x_2} + \frac{\{0.4,0.41\}}{x_3} + \frac{\{0.6,0.7\}}{x_4} + \frac{\{0.8,0.9\}}{x_5} + \frac{\{0.1,0.1,0.3,0.5\}}{x_6} + \frac{\{0.1,0.1,0.3,0.5\}}{x_7} + \frac{\{0.5,0.8\}}{x_8}$, $C = \frac{\{0.1,0.2\}}{x_1} + \frac{\{0.5\}}{x_2} + \frac{\{0.45,0.45\}}{x_3} + \frac{\{0.7,0.8\}}{x_4} + \frac{\{0.1,0.7\}}{x_5} + \frac{\{0.3,0.4\}}{x_6} + \frac{\{0.1,0.4\}}{x_7} + \frac{\{0.8,0.9\}}{x_8}$.

(1) $A(x_1) \subset_p B(x_1)$, $(A \cap C)(x_1) = \{0.1, 0.2, 0.2\}$, $(B \cap C)(x_1) = \{0.1, 0.1, 0.1, 0.2\}$. $(A \cap C)(x_1) \not\subset_m (B \cap C)(x_1)$. $(A \cap C)(x_1) \not\subset_t (B \cap C)(x_1)$.

$A(x_2) \subset_p B(x_2)$, $(A \cap C)(x_2) = \{0.2, 0.5, 0.5\}$, $(B \cap C)(x_2) = \{0.1, 0.5\}$, $(A \cap C)(x_2) \not\subset_a (B \cap C)(x_2)$.

$A(x_1) \subset_p B(x_1)$, $(A \cup C)(x_1) = \{0.2, 0.2, 0.4\}$, $(B \cup C)(x_1) = \{0.1, 0.1, 0.1, 0.2, 0.5\}$. $(A \cup C)(x_1) \not\subset_m (B \cup C)(x_1)$. $(A \cup C)(x_1) \not\subset_a (B \cup C)(x_1)$. $(A \cup C)(x_1) \not\subset_t (B \cup C)(x_1)$.

(2) $A(x_5) \subset_a B(x_5)$, $(A \cap C)(x_5) \not\subset_m (B \cap C)(x_5)$. $(A \cap C)(x_5) \not\subset_t (B \cap C)(x_5)$.

$A(x_8) \subset_a B(x_8)$, $(A \cup C)(x_8) \not\subset_m (B \cup C)(x_8)$.

$A(x_5) \subset_a B(x_5)$, $(A \cup C)(x_5) \not\subset_t (B \cup C)(x_5)$.

(3) $A(x_3) \subset_m B(x_3)$, $(A \cap C)(x_3) = \{0.3, 0.45, 0.45\}$, $(B \cap C)(x_3) = \{0.4, 0.41\}$. $(A \cap C)(x_3) \not\subset_p (B \cap C)(x_3)$.

$A(x_2) \subset_m B(x_2)$, $(A \cap C)(x_2) = \{0.2, 0.5, 0.5\}$, $(B \cap C)(x_2) = \{0.1, 0.5\}$. $(A \cap C)(x_2) \not\subset_m (B \cap C)(x_2)$.

$A(x_3) \subset_m B(x_3)$, $(A \cup C)(x_3) = \{0.45, 0.45, 0.5\}$, $(B \cup C)(x_3) = \{0.45, 0.45\}$. $(A \cup C)(x_3) \not\subset_p (B \cup C)(x_3)$.

$A(x_4) \subset_m B(x_4)$, $(A \cup C)(x_4) = \{0.7, 0.8\}$, $(B \cup C)(x_4) = \{0.7, 0.7, 0.8\}$. $(A \cup C)(x_4) \not\subset_m (B \cup C)(x_4)$.

(4) $A(x_5) \subset_s B(x_5)$, $(A \cap C)(x_5) = \{0.1, 0.6, 0.7, 0.7\}$, $(B \cap C)(x_5) = \{0.1, 0.7\}$. $(A \cap C)(x_5) \not\subset_m (B \cap C)(x_5)$. $(A \cap C)(x_5) \not\subset_t (B \cap C)(x_5)$.

$A(x_4) \subset_s B(x_4)$, $(A \cup C)(x_4) \not\subset_m (B \cup C)(x_4)$.

$A(x_5) \subset_s B(x_5)$, $(A \cup C)(x_5) \not\subset_t (B \cup C)(x_5)$.

(5) $A(x_6) \subset_t B(x_6)$, $(A \cap C)(x_6) = \{0.3, 0.3, 0.4, 0.4\}$, $(B \cap C)(x_6) = \{0.1, 0.1, 0.3, 0.3, 0.4\}$. $(A \cap C)(x_6) \not\subset_m (B \cap C)(x_6)$. $(A \cap C)(x_6) \not\subset_a (B \cap C)(x_6)$. $(A \cap C)(x_6) \not\subset_t (B \cap C)(x_6)$.

$A(x_7) \subset_t B(x_7)$, $(A \cup C)(x_7) = \{0.3, 0.4, 0.4\}$, $(B \cup C)(x_7) = \{0.1, 0.1, 0.1, 0.3, 0.4, 0.5\}$. $(A \cup C)(x_7) \not\subset_m (B \cup C)(x_7)$. $(A \cup C)(x_7) \not\subset_a (B \cup C)(x_7)$.

$A(x_6) \subset_t B(x_6)$, $(A \cup C)(x_6) \not\subset_t (B \cup C)(x_6)$.

(6) $A(x_5) \subset_n B(x_5)$, $(A \cap C)(x_5) \not\subset_m (B \cap C)(x_5)$. $(A \cap C)(x_5) \not\subset_t (B \cap C)(x_5)$.

$A(x_8) \subset_n B(x_8)$, $(A \cup C)(x_8) = \{0.8, 0.9\}$, $(B \cup C)(x_8) = \{0.8, 0.8, 0.9\}$. $(A \cup C)(x_8) \not\subset_m (B \cup C)(x_8)$.

$A(x_5) \subset_n B(x_5)$, $(A \cup C)(x_5) \not\subset_t (B \cup C)(x_5)$.

For three classical sets $\bar{A}$, $\bar{B}$ and $\bar{C}$, $\bar{A} \sqsubset \bar{B}$ and $\bar{A} \sqsubset \bar{C}$ if and only if $\bar{A} \sqsubset \bar{B} \sqcap \bar{C}$. This rule in the cases of three hesitant fuzzy sets A, B and C is investigated, shown as Proposition 2.26.

Proposition 2.26. *The following statements hold for $A, B, C \in HF(U)$,*

(1) $A \subset_p B$ and $A \subset_p C$ if and only if $A \subset_p B \cap C$.
(2) $A \subset_a B$ and $A \subset_a C$ if and only if $A \subset_a B \cap C$.
(3) $A \subset_t B$ and $A \subset_t C$, then $A \subset_t B \cap C$.
(4) $A \subset_n B$ and $A \subset_n C$ if and only if $A \subset_n B \cap C$.

Proof. (1) For all $x \in U$, $A^+(x) \leqslant B^+(x)$ and $A^+(x) \leqslant C^+(x)$ if and only if $A^+(x) \leqslant inf\{B^+(x), C^+(x)\}$.

(2) For all $x \in U$, $A^-(x) \leqslant B^-(x)$ and $A^-(x) \leqslant C^-(x)$ if and only if $A^-(x) \leqslant inf\{B^-(x), C^-(x)\}$. To combine the result of (1), then (2) holds.

(3) For a random $x \in U$, the relationships of $h \in C(x)$ and interval $[B^-(x), B^+(x)]$ are divided into two cases (i) and (ii) as follows,

(i) $h \notin [B^-(x), B^+(x)]$ for all $h \in C(x)$.

This case includes two sub-cases, $B^+(x) < C^-(x)$ and $C^+(x) < B^-(x)$. When $B^+(x) < C^-(x)$, $A(x) \subset_t B(x) = (B \cap C)(x)$. When $C^+(x) < B^-(x)$, $A(x) \subset_t C(x) = (B \cap C)(x)$.

(ii) $h \in [B^-(x), B^+(x)]$ for some $h \in C(x)$.

This case includes two sub-cases, $B^+(x) \leqslant C^+(x)$ and $C^+(x) < B^+(x)$.

When $B^+(x) \leqslant C^+(x)$, $h \in (B \cap C)(x)$ for all $h \in B(x)$, i.e., $B(x) \sqsubset (B \cap C)(x)$. Since $A(x) \subset_t B(x)$, by Remark 2.11 (1), there is a subsequence of $B(x)$, denoted as $\overline{B(x)}$, satisfying $A(x) \preccurlyeq \overline{B(x)}$. Since $B(x) \sqsubset (B \cap C)(x)$, thus $\overline{B(x)}$ is a subsequence of $(B \cap C)(x)$, then $A(x) \subset_t (B \cap C)(x)$.

For another sub-case $C^+(x) < B^+(x)$, $h \in (B \cap C)(x)$ for all $h \in C(x)$, i.e., $C(x) \sqsubset (B \cap C)(x)$. Since $A(x) \subset_t C(x)$, by Remark 2.11 (1), there is a subsequence of $C(x)$, denoted as $\overline{C(x)}$, satisfying $A(x) \preccurlyeq \overline{C(x)}$. Since $C(x) \sqsubset (B \cap C)(x)$, thus $\overline{C(x)}$ is a subsequence of $(B \cap C)(x)$, then $A(x) \subset_t (B \cap C)(x)$.

To sum up, $A(x) \subset_t (B \cap C)(x)$ for random $x \in U$, then $A \subset_t B \cap C$.

(4) For all $x \in U$, $A^+(x) \leqslant B^-(x)$ and $A^+(x) \leqslant C^-(x)$ if and only if $A^+(x) \leqslant inf\{B^-(x), C^-(x)\}$. ■

Suppose $A(x) = \{0.3, 0.5, 0.7\}$, $B(x) = \{0.8, 0.9\}$, $C(x) = \{0.6, 0.8, 0.9\}$. $A(x) \subset_t (B \cap C)(x)$, but $A(x) \not\subset_t B(x)$. The converse of Proposition 2.26 (3) does not hold.

2.5. The proposition of the complement of hesitant fuzzy set and the transitivity of the inclusion relationships in hesitant fuzzy sets

Proposition 2.27. *The following statements hold for $A, B \in HF(U)$,*

(1) $A \subset_a B$, then $B^c \subset_a A^c$.
(2) $A \subset_m B$, then $B^c \subset_m A^c$.
(3) $A \subset_s B$, then $B^c \subset_{sot} A^c$.
(4) $A \subset_n B$, then $B^c \subset_n A^c$.

Proof. (1) For all $x \in U$, since $A \subset_a B$, then $A^+(x) \leqslant B^+(x)$ and $A^-(x) \leqslant B^-(x)$. Then $(B^c)^-(x) = 1 - B^+(x) \leqslant 1 - A^+(x) = (A^c)^-(x)$ and $(B^c)^+(x) = 1 - B^-(x) \leqslant 1 - A^-(x) = (A^c)^+(x)$. Then $B^c \subset_a A^c$.

(2) For each $x \in U$, and each $H \in HF(U)$, $mean[H^c(x)] = (|H(x)| \times 1 - |H(x)| \times mean[H(x)])/|H^c(x)|$, since $|H(x)| = |H^c(x)|$, then $mean[H^c(x)] = 1 - mean[H(x)]$. $A \subset_m B$, thus $mean[A(x)] \leqslant mean[B(x)]$, then $1 - mean[B(x)] \leqslant 1 - mean[A(x)]$, it implies $B^c(x) \subset_m A^c(x)$.

(3) For each $x \in U$, we can sort the elements of $A(x)$ and $B(x)$ at first. Let $A(x) = \{h_1, h_2, \cdots, h_l\}$ and $B(x) = \{h'_1, h'_2, \cdots, h'_k\}$ be two descending sequences.

Since $A \subset_s B$, then $h'_1 \geqslant h_1$, $h'_2 \geqslant h_2$, $\cdots$, $h'_k \geqslant h_k$ and $k \leqslant l$. Then, $1 - h'_1 \leqslant 1 - h_1$, $1 - h'_2 \leqslant 1 - h_2$, $\cdots$, $1 - h'_k \leqslant 1 - h_k$. There is a subsequence of $A^c(x)$, denoted as $\overline{A^c(x)}$, satisfying $B^c(x) \preccurlyeq \overline{A^c(x)}$.

Since $|A^c(x)| = |A(x)| \geqslant |B(x)| = |B^c(x)|$, by Remark 2.11 (1) and (2), then $B^c(x) \subset_{sot} A^c(x)$.

(4) For all $x \in U$, since $A \subset_n B$, then $A^+(x) \leqslant B^-(x)$. $(A^c)^-(x) = 1 - A^+(x) \geqslant 1 - B^-(x) = (B^c)^+(x)$, then $B^c(x) \subset_n A^c(x)$. ■

Example 2.28. Let $U = \{x, y\}$, $A = \frac{\{0.4, 0.4\}}{x} + \frac{\{0.2, 0.25\}}{y}$, $B = \frac{\{0.1, 0.1, 0.41\}}{x} + \frac{\{0.1, 0.2, 0.3\}}{y}$.

(1) $A(x) \subset_p B(x)$. $A^c(x) = \{0.6, 0.6\}$, $B^c(x) = \{0.9, 0.9, 0.59\}$. $B^c(x) \not\subset_p A^c(x)$, $B^c(x) \not\subset_m A^c(x)$.

(2) $A(y) \subset_t B(y)$. $A^c(y) = \{0.8, 0.75\}$, $B^c(y) = \{0.9, 0.8, 0.7\}$. $B^c(y) \not\subset_p A^c(y)$, $B^c(y) \not\subset_a A^c(y)$, $B^c(y) \not\subset_m A^c(y)$, $B^c(y) \not\subset_s A^c(y)$, $B^c(y) \not\subset_t A^c(y)$, $B^c(y) \not\subset_n A^c(y)$.

It should be noted that

$$A \subset_p B \nRightarrow B^c \subset_p A^c.$$

Proposition 2.29. *The following statements hold for $A, B, C \in HF(U)$,*

(1) If $A \subset_p B$ and $B \subset_p C$, then $A \subset_p C$.
(2) If $A \subset_a B$ and $B \subset_a C$, then $A \subset_a C$.
(3) If $A \subset_m B$ and $B \subset_m C$, then $A \subset_m C$.
(4) If $A \subset_s B$ and $B \subset_s C$, then $A \subset_s C$.
(5) If $A \subset_t B$ and $B \subset_t C$, then $A \subset_t C$.
(6) If $A \subset_n B$ and $B \subset_n C$, then $A \subset_n C$.

Proof. (1) and (2) are obvious.

(3) $A \subset_m B$ and $B \subset_m C$ imply that $mean[A(x)] \leqslant mean[B(x)] \leqslant mean[C(x)]$ for all $x \in U$, where $mean[\cdot]$ is the mean value operator. Then $A \subset_m C$.

(4) $A \subset_s B$ and $B \subset_s C$ imply that $|A(x)| \geqslant |B(x)| \geqslant |C(x)|$ for all $x \in U$.

Let $C(x) = \{h_1^C, h_2^C, \cdots, h_{|C(x)|}^C\}$, $B(x) = \{h_1^B, h_2^B, \cdots, h_{|B(x)|}^B\}$ and $A(x) = \{h_1^A, h_2^A, \cdots, h_{|A(x)|}^A\}$ be three descending sequences, since $A(x) \subset_s B(x)$ and $B(x) \subset_s C(x)$, then $h_1^C \geqslant h_1^B \geqslant h_1^A$, $h_2^C \geqslant h_2^B \geqslant h_2^A$, $\cdots$, and $h_{|C(x)|}^C \geqslant h_{|C(x)|}^B \geqslant h_{|C(x)|}^A$. Hence, $A \subset_s C$.

(5) $A \subset_t B$ and $B \subset_t C$ imply that $|A(x)| < |B(x)| < |C(x)|$ for all $x \in U$.

Let $C(x) = \{h_1^C, h_2^C, \cdots, h_{|C(x)|}^C\}$, $B(x) = \{h_1^B, h_2^B, \cdots, h_{|B(x)|}^B\}$ and $A(x) = \{h_1^A, h_2^A, \cdots, h_{|A(x)|}^A\}$ be three descending sequences, since $A(x) \subset_t B(x)$ and $B(x) \subset_t C(x)$, then $h_1^C \geqslant h_1^B \geqslant h_1^A$, $h_2^C \geqslant h_2^B \geqslant h_2^A$, $\cdots$, and $h_{|A(x)|}^C \geqslant h_{|A(x)|}^B \geqslant h_{|A(x)|}^A$. Hence, $A \subset_t C$.

(6) $A \subset_n B$ and $B \subset_n C$ imply that $A^+(x) \leqslant B^-(x) \leqslant B^+(x) \leqslant C^-(x)$ for all $x \in U$, then $A \subset_n C$. ■

2.6. The situation of equality in hesitant fuzzy sets

Theorem 2.30. *The following statements hold for $x \in U$ and $A, B \in HF(U)$,*

(1) $A = B$, then $A =_p B$, $A =_a B$ and $A =_m B$.
(2) $A =_s B \Leftrightarrow A = B$.
(3) $A =_s B$, then $A =_p B$, $A =_a B$ and $A =_m B$.
(4) If $A =_n B$, then $h' = h''$ for all $h' \in A(x)$ and all $h'' \in B(x)$.
(5) If $A =_n B$, then $A =_p B$, $A =_a B$ and $A =_m B$.

Proof. (1) Obvious.

(2) It is obvious that $A = B$ implies $A =_s B$.

On the other hand, for each $x \in U$, we can sort $A(x)$ and $B(x)$ to obtain two descending sequences $A(x) = \{h_1, h_2, \cdots, h_l\}$ and $B(x) = \{g_1, g_2, \cdots, g_k\}$. Since $A =_s B$, i.e., $A(x) \subset_s B(x)$ and $B(x) \subset_s A(x)$. Then $|A(x)| = |B(x)|$, $h_i \leqslant g_i$ and $g_i \leqslant h_i$ for $1 \leqslant i \leqslant |A(x)|$. It means that $A = B$.

(3) By (2) and (1), then (3) is obvious.

(4) For each $x \in U$, if $A =_n B$, i.e., $A(x) \subset_n B(x)$ and $B(x) \subset_n A(x)$, then $B^+(x) \geqslant B^-(x) \geqslant A^+(x) \geqslant A^-(x) \geqslant B^+(x)$, then $h' = h''$ for all $h' \in A(x)$ and all $h'' \in B(x)$.

(5) By (4), (5) is obvious. ■

Theorem 2.30 indicates that if one of $A =_p B$, $A =_a B$ and $A =_m B$ does not hold then $A =_n B$ and $A = B$ ($A =_s B$) do not hold.

Theorem 2.31. *The following statements hold for $A, B, C \in HF(U)$,*

(1) $A \cap A =_p A$, $A \cup A =_p A$.
(2) $A \cap A =_a A$, $A \cup A =_a A$.
(3) $A \cap A =_m A$, $A \cup A =_m A$.
(4) $(A \cup B) \cap A =_p A$, $(A \cap B) \cup A =_p A$.
(5) $(A \cup B) \cap A =_a A$, $(A \cap B) \cup A =_a A$.
(6) $(A \cup B) \cap C =_p (C \cap A) \cup (C \cap B)$, $(A \cap B) \cup C =_p (C \cup A) \cap (C \cup B)$.
(7) $(A \cup B) \cap C =_a (C \cap A) \cup (C \cap B)$, $(A \cap B) \cup C =_a (C \cup A) \cap (C \cup B)$.

Proof. (1)-(3) are obvious.

Let A', A'', B', B'', C' and C'' be six fuzzy sets. For all $x \in U$, let $A'(x) = A^+(x)$, $A''(x) = A^-(x)$, $B'(x) = B^+(x)$, $B''(x) = B^-(x)$, $C'(x) = C^+(x)$ and $C''(x) = C^-(x)$.

For the proofs of (4)-(7), we give a simple example as follows. By the results of fuzzy sets [6], $(A'(x) \cup B'(x)) \cap A'(x) = A'(x)$ and $(A''(x) \cup B''(x)) \cap A''(x) = A''(x)$, i.e., $(A^+(x) \cup B^+(x)) \cap A^+(x) = A^+(x)$ and $(A^-(x) \cup B^-(x)) \cap A^-(x) = A^-(x)$, then $(A \cup B) \cap A =_a A$.

(4)-(7) are easy to be proved by the results of fuzzy sets in [6]. ■

Example 2.32. Let $U = \{x\}$, $A = \frac{\{0.1,0.2,0.3\}}{x}$, $B = \frac{\{0.3,0.4,0.5\}}{x}$, $C = \frac{\{0.3,0.45,0.5\}}{x}$.

$(A \cap B)(x) = \{0.1, 0.2, 0.3, 0.3\}$, $((A \cap B) \cup A)(x) = \{0.1, 0, 1, 0.2, 0.2, 0.3, 0.3, 0.3\} \neq_m A(x)$.

$(A \cup B)(x) = \{0.3, 0.3, 0.4, 0.5\}$, $((A \cup B) \cap A)(x) = \{0.1, 0.2, 0.3, 0.3, 0.3\} \neq_m A(x)$.

$((A \cup B) \cap C)(x) = \{0.3, 0.3, 0.3, 0.4, 0.45, 0.5\}$, $((A \cap C) \cup (B \cap C))(x) = \{0.3, 0.3, 0.3, 0.3, 0.4, 0.45, 0.5\}$, $((A \cup B) \cap C)(x) \neq_m ((A \cap C) \cup (B \cap C))(x)$.

$((A \cap B) \cup C)(x) = \{0.3, 0.3, 0.3, 0.45, 0.5\}$, $((A \cup C) \cap (B \cup C))(x) = \{0.3, 0.3, 0.3, 0.3, 0.4, 0.45, 0.45, 0.5, 0.5\}$, $((A \cap B) \cup C)(x) \neq_m ((A \cup C) \cap (B \cup C))(x)$.

In addition, $0.4 \notin ((A \cap B) \cup C)(x)$, however, $0.4 \in ((A \cup C) \cap (B \cup C))(x)$.

Inspired by Theorem 2.30 and Example 2.32, it should be noted that

$$\begin{cases} (A \cap B) \cup A \neq A, \\ (A \cup B) \cap A \neq A, \\ (A \cup B) \cap C \neq (A \cap C) \cup (B \cap C), \\ (A \cap B) \cup C \neq (A \cup C) \cap (B \cup C), \\ (A \cap B) \cup A \neq_n A, \\ (A \cup B) \cap A \neq_n A, \\ (A \cup B) \cap C \neq_n (A \cap C) \cup (B \cap C), \\ (A \cap B) \cup C \neq_n (A \cup C) \cap (B \cup C). \end{cases}$$

When considering the $\subset_p$ and $=_p$ relationships, the primary focus lies on determining the upper bound for an object $x \in U$ and a hesitant fuzzy set defined over U. Consequently, several inclusion and equal situations that are applicable to fuzzy sets can also be extended to hesitant fuzzy sets in the form of $\subset_p$ and $=_p$ relationships. Nevertheless, the absence of the implication $A \subset_p B \not\Rightarrow B^c \subset_p A^c$ indicates that the $\subset_p$ and $=_p$ relationships do not align with the inclusion relationships and equal situations observed in fuzzy sets.

Sections 2.2 and 2.3 reveal that certain commonly observed inclusion and equal situations do not apply to hesitant fuzzy sets but are valid for classical sets.

3. Foundations of families of hesitant fuzzy sets

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two families of hesitant fuzzy sets. If $H_i \in \mathscr{H}_2$ holds for all $H_i \in \mathscr{H}_1$, we say that $\mathscr{H}_1$ is the classical subset of $\mathscr{H}_2$, denoted as $\mathscr{H}_1 \sqsubset \mathscr{H}_2$.

Theorem 3.1. *Let $\mathscr{H} \sqsubset HF(U)$ be a family of hesitant fuzzy sets over U. The following statements hold,*

(1) $\bigcap\{H : H \in \mathscr{H}\} \subset_p \bigcup\{H : H \in \mathscr{H}\}$.

(2) $\bigcap\{H : H \in \mathscr{H}\} \subset_a \bigcup\{H : H \in \mathscr{H}\}$.

(3) $\bigcap\{H : H \in \mathscr{H}\} \subset_m \bigcup\{H : H \in \mathscr{H}\}$.

(4) $\bigcap\{H : H \in \mathscr{H}\} \subset_{sot} \bigcup\{H : H \in \mathscr{H}\}$.

Proof. (1) For each $x \in U$, $inf\{H^+(x) : H \in \mathscr{H}\} \leqslant sup\{H^+(x) : H \in \mathscr{H}\}$.

(2) For each $x \in U$, $inf\{H^-(x) : H \in \mathscr{H}\} \leqslant sup\{H^-(x) : H \in \mathscr{H}\}$. To combine the result of (1), then (2) holds.

(3) For each $x \in U$, let $H^+_{inf}(x) = inf\{H^+(x) : H \in \mathscr{H}\}$, $H^-_{sup}(x) = sup\{H^-(x) : H \in \mathscr{H}\}$.

If $H^+_{inf}(x) \leqslant H^-_{sup}(x)$, $(\bigcap\{H : H \in \mathscr{H}\})(x) \subset_m (\bigcup\{H : H \in \mathscr{H}\})(x)$ is obvious.

If $H^-_{sup}(x) < H^+_{inf}(x)$, we construct a set of numbers $V = \{h : h \in H(x), H \in \mathscr{H}, H^-_{sup}(x) \leqslant h \leqslant H^+_{inf}(x)\}$.

If $(\bigcap\{H : H \in \mathscr{H}\})(x) - V \neq \emptyset$, then $w < inf(V)$ for all $w \in (\bigcap\{H : H \in \mathscr{H}\})(x) - V$. If $(\bigcup\{H : H \in \mathscr{H}\})(x) - V \neq \emptyset$, then $w' > sup(V)$ for all $w' \in (\bigcup\{H : H \in \mathscr{H}\})(x) - V$. Then, $mean[(\bigcap\{H : H \in \mathscr{H}\})(x)] \leqslant mean[V] \leqslant mean[(\bigcup\{H : H \in \mathscr{H}\})(x)]$, i.e., $(\bigcap\{H : H \in \mathscr{H}\})(x) \subset_m (\bigcup\{H : H \in \mathscr{H}\})(x)$.

(4) For each $x \in U$, $H^+_{inf}(x)$ and $H^-_{sup}(x)$ are constructed as (2) above.

If $H^+_{inf}(x) \leqslant H^-_{sup}(x)$, by Proposition 2.12 (9), $(\bigcap\{H : H \in \mathscr{H}\})(x) \subset_{sot} (\bigcup\{H : H \in \mathscr{H}\})(x)$ is obvious.

If $H^-_{sup}(x) < H^+_{inf}(x)$, let $V = (\bigcap\{H : H \in \mathscr{H}\})(x) \sqcap (\bigcup\{H : H \in \mathscr{H}\})(x)$, $inf((\bigcup\{H : H \in \mathscr{H}\})(x) - V) > H^+_{inf}(x) > H^-_{sup}(x) > sup((\bigcap\{H : H \in \mathscr{H}\})(x) - V)$, by Remark 2.11 (4), $(\bigcap\{H : H \in \mathscr{H}\})(x) \subset_{sot} (\bigcup\{H : H \in \mathscr{H}\})(x)$. ■

Theorem 3.2. *Let $A \in HF(U)$ and $\mathscr{H} \sqsubset HF(U)$ be a family of hesitant fuzzy sets over U. The following statements hold,*

(1) $A \subset_p H$ for all $H \in \mathscr{H}$ if and only if $A \subset_p \bigcap\{H : H \in \mathscr{H}\}$.

(2) If $A \subset_p H_\alpha$ for a $H_\alpha \in \mathscr{H}$, then $A \subset_p \bigcup\{H : H \in \mathscr{H}\}$.

(3) $A \subset_a H$ for all $H \in \mathscr{H}$ if and only if $A \subset_a \bigcap\{H : H \in \mathscr{H}\}$.

(4) If $A \subset_a H_\alpha$ for a $H_\alpha \in \mathscr{H}$, then $A \subset_a \bigcup\{H : H \in \mathscr{H}\}$.

(5) If $A \subset_t H$ for all $H \in \mathscr{H}$, then $A \subset_t \bigcap\{H : H \in \mathscr{H}\}$.

(6) If $A \subset_t H_\alpha$ for a $H_\alpha \in \mathscr{H}$, and $|A(x)| < |H(x)|$ for all $H \in \mathscr{H}$ and $x \in U$, then $A \subset_t \bigcup\{H : H \in \mathscr{H}\}$.

(7) $A \subset_n H$ for all $H \in \mathscr{H}$ if and only if $A \subset_n \bigcap\{H : H \in \mathscr{H}\}$.

(8) If $A \subset_n H_\alpha$ for a $H_\alpha \in \mathscr{H}$, then $A \subset_n \bigcup\{H : H \in \mathscr{H}\}$.

Proof. (1) For each $x \in U$, $A^+(x) \leqslant inf\{H^+(x) : H \in \mathscr{H}\}$ if and only if $A^+(x) \leqslant H^+(x)$ for all $H \in \mathscr{H}$.

(2) $A \subset_p H_\alpha \in \mathscr{H}$, $A^+(x) \leqslant H_\alpha^+(x) \leqslant sup\{H^+(x) : H \in \mathscr{H}\}$ for all $x \in U$.

(3) For each $x \in U$, $A^-(x) \leqslant inf\{H^-(x) : H \in \mathscr{H}\}$ if and only if $A^-(x) \leqslant H^-(x)$ for all $H \in \mathscr{H}$. To combine the result of (1), then (3) holds.

(4) $A \subset_a H_\alpha \in \mathscr{H}$, $A^+(x) \leqslant H_\alpha^+(x) \leqslant sup\{H^+(x) : H \in \mathscr{H}\}$ and $A^-(x) \leqslant H_\alpha^-(x) \leqslant sup\{H^-(x) : H \in \mathscr{H}\}$ for all $x \in U$.

(5) For each $x \in U$, $A \subset_t H$ for all $H \in \mathscr{H}$, then $|A(x)| < |H(x)|$ and $|A(x)| < |(\bigcap\{H : H \in \mathscr{H}\})(x)|$ are obvious.

$H_\gamma(x)$ can be sorted into a descending sequence $H_\gamma(x) = \{h_\gamma^1, h_\gamma^2, \cdots\}$ for each $H_\gamma \in \mathscr{H}$ and $\gamma \in \Gamma$, where Γ is an order set and $|\Gamma| = |\mathscr{H}|$.

Let $\overline{H} = \{\overline{h}^1, \overline{h}^2, \cdots, \overline{h}^{|A(x)|}\}$ be a subsequence of $(\bigcap\{H_\gamma : H_\gamma \in \mathscr{H}, \gamma \in \Gamma\})(x)$, where $\overline{h}^j = inf\{h_\gamma^j : h_\gamma^j \in H_\gamma(x), \gamma \in \Gamma\}$ for $1 \leqslant j \leqslant |A(x)|$.

$A(x)$ can be sorted into a descending sequence $A(x) = \{v_1, v_2, \cdots, v_{|A(x)|}\}$. Since $h_\gamma^j \geqslant v_j$ for all $\gamma \in \Gamma$ and $1 \leqslant j \leqslant |A(x)|$, then $\overline{h}^j = inf\{h_\gamma^j : h_\gamma^j \in H_\gamma(x), \gamma \in \Gamma\} \geqslant v_j$ for $1 \leqslant j \leqslant |A(x)|$, i.e., $A(x) \preccurlyeq \overline{H}$.

$|A(x)| < |(\bigcap\{H : H \in \mathscr{H}\})(x)|$, by Remark 2.11 (1), then $A(x) \subset_t (\bigcap\{H : H \in \mathscr{H}\})(x)$.

(6) $|A(x)| < |H(x)|$ for all $H \in \mathscr{H}$ and $x \in U$. Let $H_* = \{h_*^1, h_*^2, \cdots, h_*^{|A(x)|}\}$ be a subsequence of $(\bigcup\{H_\gamma : H_\gamma \in \mathscr{H}, \gamma \in \Gamma\})(x)$, where $h_*^j = sup\{h_\gamma^j : h_\gamma^j \in H_\gamma(x), \gamma \in \Gamma\}$ for $1 \leqslant j \leqslant |A(x)|$.

For two descending sequences $A(x) = \{v_1, v_2, \cdots, v_{|A(x)|}\}$ and $H_\alpha(x) = \{h_\alpha^1, h_\alpha^2, \cdots\}$, since $A \subset_t H_\alpha$, then $v_j \leqslant h_\alpha^j \leqslant h_*^j = sup\{h_\gamma^j : h_\gamma^j \in H_\gamma(x), \gamma \in \Gamma\}$ for $1 \leqslant j \leqslant |A(x)|$, i.e., $A(x) \preccurlyeq H_*$.

$|A(x)| < |(\bigcup\{H : H \in \mathscr{H}\})(x)|$, by Remark 2.11 (1), then $A(x) \subset_t (\bigcup\{H : H \in \mathscr{H}\})(x)$.

(7) For each $x \in U$, $A^+(x) \leqslant inf\{H^-(x) : H \in \mathscr{H}\}$ if and only if $A^+(x) \leqslant H^-(x)$ for all $H \in \mathscr{H}$.

(8) $A \subset_n H_\alpha \in \mathscr{H}$, then $A^+(x) \leqslant H_\alpha^-(x) \leqslant sup\{H^-(x) : H \in \mathscr{H}\}$ for all $x \in U$. ■

Theorem 3.3. *Let $\mathscr{H}_1, \mathscr{H}_2 \sqsubset HF(U)$ be two families of hesitant fuzzy sets over U. The following statements hold,*

(1) If $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $\bigcap\{H : H \in \mathscr{H}_2\} \subset_p \bigcap\{H : H \in \mathscr{H}_1\}$.

(2) If $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $\bigcup\{H : H \in \mathscr{H}_1\} \subset_p \bigcup\{H : H \in \mathscr{H}_2\}$.

(3) If $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $\bigcap\{H : H \in \mathscr{H}_1\} \subset_p \bigcup\{H : H \in \mathscr{H}_2\}$.

(4) If $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $\bigcap\{H : H \in \mathscr{H}_2\} \subset_a \bigcap\{H : H \in \mathscr{H}_1\}$.

(5) If $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $\bigcup\{H : H \in \mathscr{H}_1\} \subset_a \bigcup\{H : H \in \mathscr{H}_2\}$.

(6) If $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $\bigcap\{H : H \in \mathscr{H}_1\} \subset_a \bigcup\{H : H \in \mathscr{H}_2\}$.

(7) If $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $\bigcup\{H : H \in \mathscr{H}_1\} \subset_{sot} \bigcup\{H : H \in \mathscr{H}_2\}$.

(8) If $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $\bigcap\{H : H \in \mathscr{H}_1\} \subset_{sot} \bigcup\{H : H \in \mathscr{H}_2\}$.

Proof. (1) For each $x \in U$, if $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $\{H^+(x) : H \in \mathscr{H}_1\} \sqsubset \{H^+(x) : H \in \mathscr{H}_2\}$. $inf\{H^+(x) : H \in \mathscr{H}_1\} \geqslant inf\{H^+(x) : H \in \mathscr{H}_2\}$, then $(\bigcap\{H : H \in \mathscr{H}_2\})(x) \subset_p (\bigcap\{H : H \in \mathscr{H}_1\})(x)$.

(2) For each $x \in U$, if $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $\{H^+(x) : H \in \mathscr{H}_1\} \sqsubset \{H^+(x) : H \in \mathscr{H}_2\}$. $sup\{H^+(x) : H \in \mathscr{H}_1\} \leqslant sup\{H^+(x) : H \in \mathscr{H}_2\}$, then $(\bigcup\{H : H \in \mathscr{H}_1\})(x) \subset_p (\bigcup\{H : H \in \mathscr{H}_2\})(x)$.

(3) If $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, by Theorem 3.1 (1) and Theorem 3.3 (2), $\bigcap\{H : H \in \mathscr{H}_1\} \subset_p \bigcup\{H : H \in \mathscr{H}_1\} \subset_p \bigcup\{H : H \in \mathscr{H}_2\}$.

(4) For each $x \in U$, if $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $\{H^-(x) : H \in \mathscr{H}_1\} \sqsubset \{H^-(x) : H \in \mathscr{H}_2\}$. $inf\{H^-(x) : H \in \mathscr{H}_1\} \geqslant inf\{H^-(x) : H \in \mathscr{H}_2\}$. To combine the result of (1), then (4) holds.

(5) For each $x \in U$, if $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $\{H^-(x) : H \in \mathscr{H}_1\} \sqsubset \{H^-(x) : H \in \mathscr{H}_2\}$. $sup\{H^-(x) : H \in \mathscr{H}_1\} \leqslant sup\{H^-(x) : H \in \mathscr{H}_2\}$. To combine the result of (2), then (5) holds.

(6) If $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, by Theorem 3.1 (2) and Theorem 3.3 (5), $\bigcap\{H : H \in \mathscr{H}_1\} \subset_a \bigcup\{H : H \in \mathscr{H}_1\} \subset_a \bigcup\{H : H \in \mathscr{H}_2\}$.

(7) For each $x \in U$, let $sup_1^- = sup\{H^-(x) : H \in \mathscr{H}_1\} = (\bigcup\{H : H \in \mathscr{H}_1\})^-(x)$, $sup_2^- = sup\{H^-(x) : H \in \mathscr{H}_2\} = (\bigcup\{H : H \in \mathscr{H}_2\})^-(x)$.

Let $V = (\bigcup\{H : H \in \mathscr{H}_1\})(x) \sqcap (\bigcup\{H : H \in \mathscr{H}_2\})(x)$. Since $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, then $sup_1^- \leqslant sup_2^-$. If $h \in (\bigcup\{H : H \in \mathscr{H}_1\})(x)$ and $sup_2^- \leqslant h$, then $h \in (\bigcup\{H : H \in \mathscr{H}_2\})(x)$. It means that $sup((\bigcup\{H : H \in \mathscr{H}_1\})(x) - V) < sup_2^-$. On the other hand, $sup_2^- < inf((\bigcup\{H : H \in \mathscr{H}_2\})(x) - V)$, by Remark 2.11 (4), then $(\bigcup\{H : H \in \mathscr{H}_1\})(x) \subset_{sot} (\bigcup\{H : H \in \mathscr{H}_2\})(x)$.

(8) For each $x \in U$, let $inf_1^+ = inf\{H^+(x) : H \in \mathscr{H}_1\}$, $sup_2^- = sup\{H^-(x) : H \in \mathscr{H}_2\}$.

If $inf_1^+ \leqslant sup_2^-$, by Proposition 2.12 (9), $(\bigcap\{H : H \in \mathscr{H}_1\})(x) \subset_{sot} (\bigcup\{H : H \in \mathscr{H}_2\})(x)$ is obvious.

If $sup_2^- < inf_1^+$. Let $V = (\bigcap\{H : H \in \mathscr{H}_1\})(x) \sqcap (\bigcup\{H : H \in \mathscr{H}_2\})(x)$.

Base on the condition $\mathscr{H}_1 \sqsubset \mathscr{H}_2$, if $h \in (\bigcap\{H : H \in \mathscr{H}_1\})(x)$ and $sup_2^- \leqslant h$, then $h \in (\bigcup\{H : H \in \mathscr{H}_2\})(x)$. It means that $sup((\bigcap\{H : H \in \mathscr{H}_1\})(x) - V) < sup_2^-$. On the other hand, $sup_2^- < inf((\bigcup\{H : H \in \mathscr{H}_2\})(x) - V)$, by Remark 2.11 (4), then $(\bigcap\{H : H \in \mathscr{H}_1\})(x) \subset_{sot} (\bigcup\{H : H \in \mathscr{H}_2\})(x)$. ■

Theorem 3.4. *Let $\mathscr{H}_1, \mathscr{H}_2 \sqsubset HF(U)$ be two families of hesitant fuzzy sets over U. The following statements hold,*

(1) There is a $H_{\gamma^*} \in \mathscr{H}_2$ satisfied $H_\alpha \subset_p H_{\gamma^*}$ for all $H_\alpha \in \mathscr{H}_1$, then $\bigcup\{H_\alpha : H_\alpha \in \mathscr{H}_1\} \subset_p \bigcup\{H_\gamma : H_\gamma \in \mathscr{H}_2\}$.

(2) There is a $H_{\alpha^*} \in \mathscr{H}_1$ satisfied $H_{\alpha^*} \subset_p H_\gamma$ for all $H_\gamma \in \mathscr{H}_2$, then $\bigcap\{H_\alpha : H_\alpha \in \mathscr{H}_1\} \subset_p \bigcap\{H_\gamma : H_\gamma \in \mathscr{H}_2\}$.

(3) If $H_\alpha \subset_p H_\gamma$ for all $H_\alpha \in \mathscr{H}_1$ and all $H_\gamma \in \mathscr{H}_2$, then $\bigcup\{H_\alpha : H_\alpha \in \mathscr{H}_1\} \subset_p \bigcap\{H_\gamma : H_\gamma \in \mathscr{H}_2\}$.

(4) There is a $H_{\gamma^*} \in \mathscr{H}_2$ satisfied $H_\alpha \subset_a H_{\gamma^*}$ for all $H_\alpha \in \mathscr{H}_1$, then $\bigcup\{H_\alpha : H_\alpha \in \mathscr{H}_1\} \subset_a \bigcup\{H_\gamma : H_\gamma \in \mathscr{H}_2\}$.

(5) There is a $H_{\alpha^*} \in \mathscr{H}_1$ satisfied $H_{\alpha^*} \subset_a H_\gamma$ for all $H_\gamma \in \mathscr{H}_2$, then $\bigcap\{H_\alpha : H_\alpha \in \mathscr{H}_1\} \subset_a \bigcap\{H_\gamma : H_\gamma \in \mathscr{H}_2\}$.

(6) If $H_\alpha \subset_a H_\gamma$ for all $H_\alpha \in \mathscr{H}_1$ and all $H_\gamma \in \mathscr{H}_2$, then $\bigcup\{H_\alpha : H_\alpha \in \mathscr{H}_1\} \subset_a \bigcap\{H_\gamma : H_\gamma \in \mathscr{H}_2\}$.

(7) There is a $H_{\gamma^*} \in \mathscr{H}_2$ satisfied $H_\alpha \subset_n H_{\gamma^*}$ for all $H_\alpha \in \mathscr{H}_1$, then $\bigcup\{H_\alpha : H_\alpha \in \mathscr{H}_1\} \subset_n \bigcup\{H_\gamma : H_\gamma \in \mathscr{H}_2\}$.
(8) There is a $H_{\alpha^*} \in \mathscr{H}_1$ satisfied $H_{\alpha^*} \subset_n H_\gamma$ for all $H_\gamma \in \mathscr{H}_2$, then $\bigcap\{H_\alpha : H_\alpha \in \mathscr{H}_1\} \subset_n \bigcap\{H_\gamma : H_\gamma \in \mathscr{H}_2\}$.
(9) If $H_\alpha \subset_n H_\gamma$ for all $H_\alpha \in \mathscr{H}_1$ and all $H_\gamma \in \mathscr{H}_2$, then $\bigcup\{H_\alpha : H_\alpha \in \mathscr{H}_1\} \subset_n \bigcap\{H_\gamma : H_\gamma \in \mathscr{H}_2\}$.

Proof. (1) For each $x \in U$, there is $H_{\gamma^*} \in \mathscr{H}_2$ such that $H^+_\alpha(x) \leqslant H^+_{\gamma^*}(x)$ for all $H_\alpha \in \mathscr{H}_1$, then $sup\{H^+_\alpha(x) : H_\alpha \in \mathscr{H}_1\} \leqslant H^+_{\gamma^*}(x) \leqslant sup\{H^+_\gamma(x) : H_\gamma \in \mathscr{H}_2\}$. Then $(\bigcup\{H_\alpha : H_\alpha \in \mathscr{H}_1\})(x) \subset_p (\bigcup\{H_\gamma : H_\gamma \in \mathscr{H}_2\})(x)$.

(2) For each $x \in U$, $H^+_{\alpha^*}(x) \leqslant H^+_\gamma(x)$ for all $H_\gamma \in \mathscr{H}_2$, then $inf\{H^+_\alpha(x) : H_\alpha \in \mathscr{H}_1\} \leqslant H^+_{\alpha^*}(x) \leqslant inf\{H^+_\gamma(x) : H_\gamma \in \mathscr{H}_2\}$. Then $(\bigcap\{H_\alpha : H_\alpha \in \mathscr{H}_1\})(x) \subset_p (\bigcap\{H_\gamma : H_\gamma \in \mathscr{H}_2\})(x)$.

(3) For each $x \in U$, $H^+_\alpha(x) \leqslant H^+_\gamma(x)$ for all $H_\alpha \in \mathscr{H}_1$ and all $H_\gamma \in \mathscr{H}_2$, thus $sup\{H^+_\alpha(x) : H_\alpha \in \mathscr{H}_1\} \leqslant H^+_\gamma(x)$ for all $H_\gamma \in \mathscr{H}_2$, then $sup\{H^+_\alpha(x) : H_\alpha \in \mathscr{H}_1\} \leqslant inf\{H^+_\gamma(x) : H_\gamma \in \mathscr{H}_2\}$. Then $(\bigcup\{H_\alpha : H_\alpha \in \mathscr{H}_1\})(x) \subset_p (\bigcap\{H_\gamma : H_\gamma \in \mathscr{H}_2\})(x)$.

(4) For each $x \in U$, there is $H_{\gamma^*} \in \mathscr{H}_2$ such that $H^-_\alpha(x) \leqslant H^-_{\gamma^*}(x)$ for all $H_\alpha \in \mathscr{H}_1$, then $sup\{H^-_\alpha(x) : H_\alpha \in \mathscr{H}_1\} \leqslant H^-_{\gamma^*}(x) \leqslant sup\{H^-_\gamma(x) : H_\gamma \in \mathscr{H}_2\}$. To combine the result of (1), then (4) holds.

(5) For each $x \in U$, $H^-_{\alpha^*}(x) \leqslant H^-_\gamma(x)$ for all $H_\gamma \in \mathscr{H}_2$, then $inf\{H^-_\alpha(x) : H_\alpha \in \mathscr{H}_1\} \leqslant H^-_{\alpha^*}(x) \leqslant inf\{H^-_\gamma(x) : H_\gamma \in \mathscr{H}_2\}$. To combine the result of (2), then (5) holds.

(6) For each $x \in U$, $H^-_\alpha(x) \leqslant H^-_\gamma(x)$ for all $H_\alpha \in \mathscr{H}_1$ and all $H_\gamma \in \mathscr{H}_2$, thus $sup\{H^-_\alpha(x) : H_\alpha \in \mathscr{H}_1\} \leqslant H^-_\gamma(x)$ for all $H_\gamma \in \mathscr{H}_2$, then $sup\{H^-_\alpha(x) : H_\alpha \in \mathscr{H}_1\} \leqslant inf\{H^-_\gamma(x) : H_\gamma \in \mathscr{H}_2\}$. To combine the result of (3), then (6) holds.

(7) For each $x \in U$, there is $H_{\gamma^*} \in \mathscr{H}_2$ such that $H^+_\alpha(x) \leqslant H^-_{\gamma^*}(x)$ for all $H_\alpha \in \mathscr{H}_1$, then $sup\{H^+_\alpha(x) : H_\alpha \in \mathscr{H}_1\} \leqslant H^-_{\gamma^*}(x) \leqslant sup\{H^-_\gamma(x) : H_\gamma \in \mathscr{H}_2\}$. Then $(\bigcup\{H_\alpha : H_\alpha \in \mathscr{H}_1\})(x) \subset_n (\bigcup\{H_\gamma : H_\gamma \in \mathscr{H}_2\})(x)$.

(8) For each $x \in U$, $H^+_{\alpha^*}(x) \leqslant H^-_\gamma(x)$ for all $H_\gamma \in \mathscr{H}_2$, then $inf\{H^+_\alpha(x) : H_\alpha \in \mathscr{H}_1\} \leqslant H^+_{\alpha^*}(x) \leqslant inf\{H^-_\gamma(x) : H_\gamma \in \mathscr{H}_2\}$. Then $(\bigcap\{H_\alpha : H_\alpha \in \mathscr{H}_1\})(x) \subset_n (\bigcap\{H_\gamma : H_\gamma \in \mathscr{H}_2\})(x)$.

(9) For each $x \in U$, $H^+_\alpha(x) \leqslant H^-_\gamma(x)$ for all $H_\alpha \in \mathscr{H}_1$ and all $H_\gamma \in \mathscr{H}_2$, thus $sup\{H^+_\alpha(x) : H_\alpha \in \mathscr{H}_1\} \leqslant H^-_\gamma(x)$ for all $H_\gamma \in \mathscr{H}_2$, then $sup\{H^+_\alpha(x) : H_\alpha \in \mathscr{H}_1\} \leqslant inf\{H^-_\gamma(x) : H_\gamma \in \mathscr{H}_2\}$. Then $(\bigcup\{H_\alpha : H_\alpha \in \mathscr{H}_1\})(x) \subset_n (\bigcap\{H_\gamma : H_\gamma \in \mathscr{H}_2\})(x)$. ■

4. Parameter reductions of hesitant fuzzy information systems

This section focuses on examining various propositions related to parameter reductions in hesitant fuzzy information systems (HFISs), followed by the introduction of an algorithm specifically designed for parameter reductions in such systems.

4.1. *Parameter reductions based on several kinds of β-partition in hesitant fuzzy information systems*

First, we review the definition of the hesitant fuzzy information system as follows.

Definition 4.1. [30] A HFIS is a quadruple (U, F, E, V), where U is a non-empty finite set of objects, E is a non-empty finite set of attributes, $F : U \times E \to V$ is an information function and $F(e)(x) = v \in V$ is a hesitant fuzzy element for $x \in U$ and $e \in E$.

In cases where the information system encounters a missing value, it is designated as null ($v = null$).

In the initial information systems, if an object has the characteristic with respect to attribute e, then $F(e)(x)$ is denoted as 1 (i.e., $F(e)(x) = 1$), otherwise, $F(e)(x)$ is denoted as 0 (i.e., $F(e)(x) = 0$). However, hesitant fuzzy elements are not limited to binary values of 1 or 0. A key challenge lies in determining whether an object $x \in U$ possesses attribute $e \in E$. We address this challenge based on the analysis of Example 2.9 and Definition 2.10.

Considering the scenario where $\beta(e_1) = \{0.5, 0.4, 0.3\}$ represents an evaluation indicator provided by three experts to assess the objects $x_i \in U$ on the attribute e_1. If $\beta(e_1) \subset_p F(e_1)(x_1) = \{0.6, 0.3, 0.2\}$, an adventurous decision-maker would conclude that x_1 possesses the attribute e_1 (or that object x_1 is acceptable to him concerning attribute e_1). A risk-neutral decision-maker might reject the weaker inclusion relationship "$\subset_p$" and instead prefer the stronger inclusion relationship "$\subset_a$" (Proposition 2.12, "$\subset_a$"⇒"$\subset_p$"). If $\beta(e_1) \subset_a F(e_1)(x_2) = \{0.6, 0.3, 0.3\}$, the risk-neutral decision-maker might find object x_2 acceptable and not x_1 when considering the attribute e_1. A conservative decision-maker may accept the inclusion relationship "$\subset_s$" but reject "$\subset_p$" and "$\subset_a$". If $\beta(e_1) \subset_s F(e_1)(x_3) = \{0.6, 0.5, 0.3\}$, the conservative decision-maker might find object x_3 acceptable, while rejecting x_1 and x_2 for the attribute e_1.

Based on the preceding discussion, the object set U can be segmented according to the evaluation indicator β and various inclusion relationships. As illustrated in Example 4.12, $\beta(e_1) \not\subset_p F(e_1)(x)$ for $x \in \{x_1, x_2, x_3\}$, whereas $\beta(e_1) \subset_p F(e_1)(y)$ for $y \in \{x_4, x_5, x_6\}$. Consequently, the object set U is divided into two subsets: $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$. The objects $\{x_4, x_5, x_6\}$ possess the attribute e_1 according to the evaluation indicator β and "$\subset_p$" inclusion relationships, while $\{x_1, x_2, x_3\}$ do not. The partition derived from "$\subset_p$" and β is henceforth denoted as the $p\beta$-partition, with analogous notations for the cases of other inclusion relationships (in Definition 2.10), $a\beta$-partition, $m\beta$-partition, $s\beta$-partition, $t\beta$-partition, and $n\beta$-partition.

By applying the aforementioned operations, it is possible to determine whether an object $x \in U$ possesses the attribute $e \in E$ under the indicator β and various inclusion relationships. The HFIS is thereby converted into a 0-1 binary value information system, facilitating more straightforward information analysis.

Hereafter, $N_{e_1}^{p\beta}$ is referred to as the $p\beta$-partition of U with respect to e_1, defined as $N_{e_1}^{p\beta} = \{\{x : \beta(e_1) \not\subset_p F(e_1)(x), x \in U\}, \{y : \beta(e_1) \subset_p F(e_1)(y), y \in U\}\} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}\}$. Moreover, $N_{e_1}^{p\beta}(x_1)$ represents an element of $N_{e_1}^{p\beta}$ containing x_1, where $N_{e_1}^{p\beta}(x_1) = \{x_1, x_2, x_3\}$. Additionally, $N_{e_1}^{p\beta}(x_2) = N_{e_1}^{p\beta}(x_3) = \{x_1, x_2, x_3\}$, while $N_{e_1}^{p\beta}(x_4) = N_{e_1}^{p\beta}(x_5) = N_{e_1}^{p\beta}(x_6) = \{x_4, x_5, x_6\}$.

Definition 4.2. Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $A \sqsubset E$.

(1) Let $N_e^{p\beta} = \{\{x : \beta(e) \subset_p F(e)(x), x \in U\}, \{y : \beta(e) \not\subset_p F(e)(y), y \in U\}\}$ be a $p\beta$-partition of U with respect to e, $N_e^{p\beta}(x)$ be an element of $N_e^{p\beta}$ which contains x, $N_A^{p\beta}(x) = \sqcap\{N_e^{p\beta}(x) : e \in A\}$, and $Cov^{p\beta}(A) = \{N_A^{p\beta}(x) : x \in U\}$.

(2) Let $N_e^{a\beta} = \{\{x : \beta(e) \subset_a F(e)(x), x \in U\}, \{y : \beta(e) \not\subset_a F(e)(y), y \in U\}\}$ be a $a\beta$-partition of U with respect to e, $N_e^{a\beta}(x)$ be an element of $N_e^{a\beta}$ which contains x, $N_A^{a\beta}(x) = \sqcap\{N_e^{a\beta}(x) : e \in A\}$, and $Cov^{a\beta}(A) = \{N_A^{a\beta}(x) : x \in U\}$.

(3) Let $N_e^{m\beta} = \{\{x : \beta(e) \subset_m F(e)(x), x \in U\}, \{y : \beta(e) \not\subset_m F(e)(y), y \in U\}\}$ be a $m\beta$-partition of U with respect to e, $N_e^{m\beta}(x)$ be an element of $N_e^{m\beta}$ which contains x, $N_A^{m\beta}(x) = \sqcap\{N_e^{m\beta}(x) : e \in A\}$, and $Cov^{m\beta}(A) = \{N_A^{m\beta}(x) : x \in U\}$.

(4) Let $N_e^{s\beta} = \{\{x : \beta(e) \subset_s F(e)(x), x \in U\}, \{y : \beta(e) \not\subset_s F(e)(y), y \in U\}\}$ be a $s\beta$-partition of U with respect to e, $N_e^{s\beta}(x)$ be an element of $N_e^{s\beta}$ which contains x, $N_A^{s\beta}(x) = \sqcap\{N_e^{s\beta}(x) : e \in A\}$, and $Cov^{s\beta}(A) = \{N_A^{s\beta}(x) : x \in U\}$.

(5) Let $N_e^{t\beta} = \{\{x : \beta(e) \subset_t F(e)(x), x \in U\}, \{y : \beta(e) \not\subset_t F(e)(y), y \in U\}\}$ be a $t\beta$-partition of U with respect to e, $N_e^{t\beta}(x)$ be an element of $N_e^{t\beta}$ which contains x, $N_A^{t\beta}(x) = \sqcap\{N_e^{t\beta}(x) : e \in A\}$, and $Cov^{t\beta}(A) = \{N_A^{t\beta}(x) : x \in U\}$.

(6) Let $N_e^{n\beta} = \{\{x : \beta(e) \subset_n F(e)(x), x \in U\}, \{y : \beta(e) \not\subset_n F(e)(y), y \in U\}\}$ be a $n\beta$-partition of U with respect to e, $N_e^{n\beta}(x)$ be an element of $N_e^{n\beta}$ which contains x, $N_A^{n\beta}(x) = \sqcap\{N_e^{n\beta}(x) : e \in A\}$, and $Cov^{n\beta}(A) = \{N_A^{n\beta}(x) : x \in U\}$.

Proposition 4.3. *Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $e \in E$. The following statements hold,*

(1) Suppose $N_e^{p\beta} = \{K_1, K_2\}$, then $K_1 \sqcap K_2 = \emptyset$ and $K_1 \sqcup K_2 = U$.
(2) Suppose $N_e^{a\beta} = \{K_1, K_2\}$, then $K_1 \sqcap K_2 = \emptyset$ and $K_1 \sqcup K_2 = U$.
(3) Suppose $N_e^{m\beta} = \{K_1, K_2\}$, then $K_1 \sqcap K_2 = \emptyset$ and $K_1 \sqcup K_2 = U$.
(4) Suppose $N_e^{s\beta} = \{K_1, K_2\}$, then $K_1 \sqcap K_2 = \emptyset$ and $K_1 \sqcup K_2 = U$.
(5) Suppose $N_e^{t\beta} = \{K_1, K_2\}$, then $K_1 \sqcap K_2 = \emptyset$ and $K_1 \sqcup K_2 = U$.
(6) Suppose $N_e^{n\beta} = \{K_1, K_2\}$, then $K_1 \sqcap K_2 = \emptyset$ and $K_1 \sqcup K_2 = U$.

Proof. (1)-(6) are obvious.

Proposition 4.4. *Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $e \in E$. The following statements hold,*

(1) If $y \in N_e^{p\beta}(x)$, then $N_e^{p\beta}(y) = N_e^{p\beta}(x)$.
(2) If $y \in N_e^{a\beta}(x)$, then $N_e^{a\beta}(y) = N_e^{a\beta}(x)$.
(3) If $y \in N_e^{m\beta}(x)$, then $N_e^{m\beta}(y) = N_e^{m\beta}(x)$.
(4) If $y \in N_e^{s\beta}(x)$, then $N_e^{s\beta}(y) = N_e^{s\beta}(x)$.
(5) If $y \in N_e^{t\beta}(x)$, then $N_e^{t\beta}(y) = N_e^{t\beta}(x)$.
(6) If $y \in N_e^{n\beta}(x)$, then $N_e^{n\beta}(y) = N_e^{n\beta}(x)$.

Proof. (1)-(6) are obvious.

Definition 4.5. Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $a \in A \sqsubset E$.

(1) If $Cov^{p\beta}(A) = Cov^{p\beta}(A - \{a\})$, then a is called a reducible parameter of the HFIS (U, F, A, V) based on $p\beta$-partition, denoted as $a \in R^{p\beta}(A)$; if $Cov^{p\beta}(A) \neq Cov^{p\beta}(A - \{a\})$, then a is called an irreducible parameter of (U, F, A, V) based on $p\beta$-partition, denoted as $a \in Core^{p\beta}(A)$.

(2) If $Cov^{a\beta}(A) = Cov^{a\beta}(A - \{a\})$, then a is called a reducible parameter of the HFIS (U, F, A, V) based on $a\beta$-partition, denoted as $a \in R^{a\beta}(A)$; if $Cov^{a\beta}(A) \neq Cov^{a\beta}(A - \{a\})$, then a is called an irreducible parameter of (U, F, A, V) based on $a\beta$-partition, denoted as $a \in Core^{a\beta}(A)$.

(3) If $Cov^{m\beta}(A) = Cov^{m\beta}(A - \{a\})$, then a is called a reducible parameter of the HFIS (U, F, A, V) based on $m\beta$-partition, denoted as $a \in R^{m\beta}(A)$; if $Cov^{m\beta}(A) \neq Cov^{m\beta}(A - \{a\})$, then a is called an irreducible parameter of (U, F, A, V) based on $m\beta$-partition, denoted as $a \in Core^{m\beta}(A)$.

(4) If $Cov^{s\beta}(A) = Cov^{s\beta}(A - \{a\})$, then a is called a reducible parameter of the HFIS (U, F, A, V) based on $s\beta$-partition, denoted as $a \in R^{s\beta}(A)$; if $Cov^{s\beta}(A) \neq Cov^{s\beta}(A - \{a\})$, then a is called an irreducible parameter of (U, F, A, V) based on $s\beta$-partition, denoted as $a \in Core^{s\beta}(A)$.

(5) If $Cov^{t\beta}(A) = Cov^{t\beta}(A - \{a\})$, then a is called a reducible parameter of the HFIS (U, F, A, V) based on $t\beta$-partition, denoted as $a \in R^{t\beta}(A)$; if $Cov^{t\beta}(A) \neq Cov^{t\beta}(A - \{a\})$, then a is called an irreducible parameter of (U, F, A, V) based on $t\beta$-partition, denoted as $a \in Core^{t\beta}(A)$.

(6) If $Cov^{n\beta}(A) = Cov^{n\beta}(A - \{a\})$, then a is called a reducible parameter of the HFIS (U, F, A, V) based on $n\beta$-partition, denoted as $a \in R^{n\beta}(A)$; if $Cov^{n\beta}(A) \neq Cov^{n\beta}(A - \{a\})$, then a is called an irreducible parameter of (U, F, A, V) based on $n\beta$-partition, denoted as $a \in Core^{n\beta}(A)$.

Theorem 4.6. *Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $A \sqsubset E$. The following statements hold,*

(1) $\sqcup\{C : C \in Cov^{p\beta}(A)\} = U$.
(2) $\sqcup\{C : C \in Cov^{a\beta}(A)\} = U$.
(3) $\sqcup\{C : C \in Cov^{m\beta}(A)\} = U$.
(4) $\sqcup\{C : C \in Cov^{s\beta}(A)\} = U$.
(5) $\sqcup\{C : C \in Cov^{t\beta}(A)\} = U$.
(6) $\sqcup\{C : C \in Cov^{n\beta}(A)\} = U$.

Proof. (1) For all $x \in U$, $x \in N_e^{p\beta}(x)$, then $x \in \sqcap\{N_e^{p\beta}(x) : e \in A\} = N_A^{p\beta}(x) \in Cov^{p\beta}(A)$. Then, $U \sqsubset \sqcup\{C : C \in Cov^{p\beta}(A)\}$.

On the other hand, $C \sqsubset U$ for all $C \in Cov^{p\beta}(A)$, then $\sqcup\{C : C \in Cov^{p\beta}(A)\} \sqsubset U$.

Hence, $\sqcup\{C : C \in Cov^{p\beta}(A)\} = U$.

The proofs of (2)-(6) are similar to the proof of (1). ■

Theorem 4.7. *Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $A \sqsubset E$. The following statements hold,*

(1) If $C_i \neq C_j$, then $C_i \sqcap C_j = \emptyset$ for all $C_i, C_j \in Cov^{p\beta}(A)$.
(2) If $C_i \neq C_j$, then $C_i \sqcap C_j = \emptyset$ for all $C_i, C_j \in Cov^{a\beta}(A)$.
(3) If $C_i \neq C_j$, then $C_i \sqcap C_j = \emptyset$ for all $C_i, C_j \in Cov^{m\beta}(A)$.
(4) If $C_i \neq C_j$, then $C_i \sqcap C_j = \emptyset$ for all $C_i, C_j \in Cov^{s\beta}(A)$.
(5) If $C_i \neq C_j$, then $C_i \sqcap C_j = \emptyset$ for all $C_i, C_j \in Cov^{t\beta}(A)$.
(6) If $C_i \neq C_j$, then $C_i \sqcap C_j = \emptyset$ for all $C_i, C_j \in Cov^{n\beta}(A)$.

Proof. (1) If there are $C_i, C_j \in Cov^{p\beta}(A)$ such that $C_i \sqcap C_j \neq \emptyset$, then there is $x \in C_i \sqcap C_j$.

Since $C_i \neq C_j$, then one of $C_i - C_j \neq \emptyset$ and $C_j - C_i \neq \emptyset$ holds. These two cases can derive the following two cases, case 1, $y \in C_i$ and $y \notin C_j$; and case 2, $y \notin C_i$ and $y \in C_j$. The proof of case 1 is shown as follows,

(i) Since $x, y \in C_i \in Cov^{p\beta}(A)$, then $\beta(e) \subset_p F(e)(x)$ if and only if $\beta(e) \subset_p F(e)(y)$ for $e \in \overline{A} \sqsubset A$; and $\beta(e') \not\subset_p F(e')(x)$ if and only if $\beta(e') \not\subset_p F(e')(y)$ for others $e' \in A - \overline{A}$.

(ii) Since $x \in C_j$, $y \notin C_j$ and $C_j \in Cov^{p\beta}(A)$, then there is at least one $e^* \in A$ such that $\beta(e^*) \subset_p F(e^*)(x)$ and $\beta(e^*) \not\subset_p F(e^*)(y)$, or $\beta(e^*) \not\subset_p F(e^*)(x)$ and $\beta(e^*) \subset_p F(e^*)(y)$.

There is a contradiction in (i) and (ii). Hence, $C_i \sqcap C_j = \emptyset$.

The proof of case 2 ($y \notin C_i$ and $y \in C_j$) is similar to the proof of case 1 ($y \in C_i$ and $y \notin C_j$), and case 2 also can obtain $C_i \sqcap C_j = \emptyset$. Then, $C_i \sqcap C_j = \emptyset$ for all $C_i, C_j \in Cov^{p\beta}(A)$.

The proofs of (2)-(6) are similar to the proof of (1). ■

Theorem 4.8. *Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $A \sqsubset E$, $x, y \in U$. The following statements hold,*

(1) If $y \in N_A^{p\beta}(x)$, then $N_A^{p\beta}(x) = N_A^{p\beta}(y)$.
(2) If $y \in N_A^{a\beta}(x)$, then $N_A^{a\beta}(x) = N_A^{a\beta}(y)$.
(3) If $y \in N_A^{m\beta}(x)$, then $N_A^{m\beta}(x) = N_A^{m\beta}(y)$.
(4) If $y \in N_A^{s\beta}(x)$, then $N_A^{s\beta}(x) = N_A^{s\beta}(y)$.
(5) If $y \in N_A^{t\beta}(x)$, then $N_A^{t\beta}(x) = N_A^{t\beta}(y)$.
(6) If $y \in N_A^{n\beta}(x)$, then $N_A^{n\beta}(x) = N_A^{n\beta}(y)$.

Proof. (1) Suppose $N_A^{p\beta}(x) \neq N_A^{p\beta}(y)$, since $N_A^{p\beta}(x), N_A^{p\beta}(y) \in Cov^{p\beta}(A)$, let $N_A^{p\beta}(x) = C_i$ and $N_A^{p\beta}(y) = C_j$. By Theorem 4.7, $C_i \sqcap C_j = \emptyset$.

On the other hand, $y \in N_A^{p\beta}(x)$ and $y \in N_A^{p\beta}(y)$, then $\emptyset \neq \{y\} \sqsubset C_i \sqcap C_j$. There is a contradiction.

Hence, $N_A^{p\beta}(x) = N_A^{p\beta}(y)$.

The proofs of (2)-(6) are similar to the proof of (1). ■

Theorem 4.9. *Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $a \in A \sqsubset E$. The following statements hold,*

(1) $a \in R^{p\beta}(A)$ if and only if $N_a^{p\beta}(x) = \sqcup\{N_{A-\{a\}}^{p\beta}(y) : y \in N_a^{p\beta}(x)\}$ for all $x \in U$.

Table 1
A quadruple (U, F, E, V).

U	e_1	e_2	e_3	e_4	e_5	e_6	e_7
x_1	{0.5, 0.3, 0.2}	{0.7, 0.6, 0.5}	{0.8, 0.7}	{0.7, 0.2}	{ 0.6, 0.5, 0.3, 0.3}	{0.6, 0.3, 0.3}	{0.8, 0.8, 0.6, 0.5}
x_2	{0.4, 0.4, 0.3}	{0.7, 0.7}	{0.7, 0.6, 0.5}	{0.7, 0.5, 0.5}	{0.7, 0.5, 0.3, 0.3}	{0.6, 0.3, 0.3}	{0.6, 0.4, 0.3}
x_3	{0.3, 0.2, 0.1}	{0.7, 0.6}	{0.6, 0.5}	{0.7, 0.2}	{ 0.6, 0.5, 0.3, 0.1}	{0.6, 0.3}	{0.3, 0.3}
x_4	{0.8, 0.7, 0.7}	{0.4, 0.3}	{0.3, 0.3, 0.2}	{0.9, 0.7}	{0.5, 0.5, 0.2}	{0.9, 0.8, 0.7, 0.6}	{0.9, 0.7}
x_5	{0.8, 0.7, 0.6}	{0.3, 0.3, 0.2}	{ 0.3, 0.3, 0.1}	{ 0.8, 0.8, 0.7}	{0.6, 0.2, 0.2}	{0.7, 0.6, 0.5, 0.3}	{0.5, 0.3}
x_6	{0.7, 0.7, 0.6}	{0.8, 0.6, 0.5}	{0.2, 0.1, 0.1}	{0.8, 0.7, 0.7}	{0.6, 0.2}	{0.6, 0.5, 0.3, 0.1}	{0.5, 0.3, 0.1, 0.1}

(2) $a \in R^{a\beta}(A)$ if and only if $N_a^{a\beta}(x) = \sqcup\{N_{A-\{a\}}^{a\beta}(y) : y \in N_a^{a\beta}(x)\}$ for all $x \in U$.
(3) $a \in R^{m\beta}(A)$ if and only if $N_a^{m\beta}(x) = \sqcup\{N_{A-\{a\}}^{m\beta}(y) : y \in N_a^{m\beta}(x)\}$ for all $x \in U$.
(4) $a \in R^{s\beta}(A)$ if and only if $N_a^{s\beta}(x) = \sqcup\{N_{A-\{a\}}^{s\beta}(y) : y \in N_a^{s\beta}(x)\}$ for all $x \in U$.
(5) $a \in R^{t\beta}(A)$ if and only if $N_a^{t\beta}(x) = \sqcup\{N_{A-\{a\}}^{t\beta}(y) : y \in N_a^{t\beta}(x)\}$ for all $x \in U$.
(6) $a \in R^{n\beta}(A)$ if and only if $N_a^{n\beta}(x) = \sqcup\{N_{A-\{a\}}^{n\beta}(y) : y \in N_a^{n\beta}(x)\}$ for all $x \in U$.

Proof. (1) The HFIS (U, F, A, V) has one more parameter a than $(U, F, A-\{a\}, V)$. Then $N_A^{p\beta}(x) = \sqcap\{N_e^{p\beta}(x) : e \in A\} = N_{A-\{a\}}^{p\beta}(x) \sqcap N_a^{p\beta}(x) = \sqcap\{N_e^{p\beta}(x) : e \in A - \{a\}\} \sqcap N_a^{p\beta}(x)$.

The proof of necessity.

$a \in R^{p\beta}(A)$, then $N_A^{p\beta}(x) = N_{A-\{a\}}^{p\beta}(x)$ for all $x \in U$. $N_A^{p\beta}(x) = N_{A-\{a\}}^{p\beta}(x) \sqcap N_a^{p\beta}(x)$, then $N_{A-\{a\}}^{p\beta}(x) \sqsubset N_a^{p\beta}(x)$ for all $x \in U$.

By Proposition 4.4, if $y \in N_a^{p\beta}(x)$, then $N_a^{p\beta}(x) = N_a^{p\beta}(y)$.

If $y \in N_a^{p\beta}(x)$, then $N_{A-\{a\}}^{p\beta}(y) \sqsubset N_a^{p\beta}(y) = N_a^{p\beta}(x)$, then $\sqcup\{N_{A-\{a\}}^{p\beta}(y) : y \in N_a^{p\beta}(x)\} \sqsubset N_a^{p\beta}(x)$.

Since $y \in N_{A-\{a\}}^{p\beta}(y)$, then $N_a^{p\beta}(x) \sqsubset \sqcup\{N_{A-\{a\}}^{p\beta}(y) : y \in N_a^{p\beta}(x)\}$ holds.

Then $N_a^{p\beta}(x) = \sqcup\{N_{A-\{a\}}^{p\beta}(y) : y \in N_a^{p\beta}(x)\}$.

The proof of sufficiency.

If $N_a^{p\beta}(x) = \sqcup\{N_{A-\{a\}}^{p\beta}(y) : y \in N_a^{p\beta}(x)\}$, since $x \in N_a^{p\beta}(x)$, then $N_{A-\{a\}}^{p\beta}(x) \sqsubset N_a^{p\beta}(x)$. Then $N_A^{p\beta}(x) = N_{A-\{a\}}^{p\beta}(x) \sqcap N_a^{p\beta}(x) = N_{A-\{a\}}^{p\beta}(x)$, i.e., $a \in R^{p\beta}(A)$.

The proofs of (2)-(6) are similar to the proof of (1). ■

Theorem 4.10. *Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $a, b \in A \sqsubset E$. The following statements hold,*
(1) If $N_a^{p\beta} = N_b^{p\beta}$, then $a \in R^{p\beta}(A)$ ($b \in R^{p\beta}(A)$).
(2) If $N_a^{a\beta} = N_b^{a\beta}$, then $a \in R^{a\beta}(A)$ ($b \in R^{a\beta}(A)$).
(3) If $N_a^{m\beta} = N_b^{m\beta}$, then $a \in R^{m\beta}(A)$ ($b \in R^{m\beta}(A)$).
(4) If $N_a^{s\beta} = N_b^{s\beta}$, then $a \in R^{s\beta}(A)$ ($b \in R^{s\beta}(A)$).
(5) If $N_a^{t\beta} = N_b^{t\beta}$, then $a \in R^{t\beta}(A)$ ($b \in R^{t\beta}(A)$).
(6) If $N_a^{n\beta} = N_b^{n\beta}$, then $a \in R^{n\beta}(A)$ ($b \in R^{n\beta}(A)$).

Proof. (1) $N_a^{p\beta} = N_b^{p\beta}$, i.e., $N_a^{p\beta}(x) = N_b^{p\beta}(x)$ for all $x \in U$, then $N_A^{p\beta}(x) = N_{A-\{a,b\}}^{p\beta}(x) \sqcap N_b^{p\beta}(x) \sqcap N_a^{p\beta}(x) = N_{A-\{a,b\}}^{p\beta}(x) \sqcap N_b^{p\beta}(x) \sqcap N_b^{p\beta}(x) = N_{A-\{a,b\}}^{p\beta}(x) \sqcap N_b^{p\beta}(x) = N_{A-\{a\}}^{p\beta}(x)$, i.e., $a \in R^{p\beta}(A)$.

$b \in R^{p\beta}(A)$ can be proved by the same manner of the case $a \in R^{p\beta}(A)$.

The proofs of (2)-(6) are similar to the proof of (1). ■

$a \in R^{p\beta}(A)$ or $b \in R^{p\beta}(A)$ holds, however, $\{a, b\} \sqsubset R^{p\beta}(A)$ is not determinate. The situations of (2)-(6) are the same as the situations of (1).

Theorem 4.11. *Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $a, b \in A \sqsubset E$. The following statements hold,*
(1) If $b \in R^{p\beta}(A - \{a\})$, then $b \in R^{p\beta}(A)$.
(2) If $b \in R^{a\beta}(A - \{a\})$, then $b \in R^{a\beta}(A)$.
(3) If $b \in R^{m\beta}(A - \{a\})$, then $b \in R^{m\beta}(A)$.
(4) If $b \in R^{s\beta}(A - \{a\})$, then $b \in R^{s\beta}(A)$.
(5) If $b \in R^{t\beta}(A - \{a\})$, then $b \in R^{t\beta}(A)$.
(6) If $b \in R^{n\beta}(A - \{a\})$, then $b \in R^{n\beta}(A)$.

Proof. (1) Since $b \in R^{p\beta}(A-\{a\})$, by Theorem 4.9, $N_b^{p\beta}(x) = \sqcup\{N_{A-\{a,b\}}^{p\beta}(y) : y \in N_b^{p\beta}(x)\}$. Since $N_{A-\{b\}}^{p\beta}(y) = N_{A-\{a,b\}}^{p\beta}(y) \sqcap N_a^{p\beta}(y) \sqsubset N_{A-\{a,b\}}^{p\beta}(y)$, then $N_b^{p\beta}(x) = \sqcup\{N_{A-\{a,b\}}^{p\beta}(y) : y \in N_b^{p\beta}(x)\} \sqsupset \sqcup\{N_{A-\{b\}}^{p\beta}(y) : y \in N_b^{p\beta}(x)\}$.

Since $y \in N_{A-\{b\}}^{p\beta}(y)$, then $N_b^{p\beta}(x) \sqsubset \sqcup\{N_{A-\{b\}}^{p\beta}(y) : y \in N_b^{p\beta}(x)\}$ holds.

Based on the results above, then $N_b^{p\beta}(x) = \sqcup\{N_{A-\{b\}}^{p\beta}(y) : y \in N_b^{p\beta}(x)\}$, by Theorem 4.9, $b \in R^{p\beta}(A)$.

The proofs of (2)-(6) are similar to the proof of (1). ■

Example 4.12. $U = \{x_1, x_2, \cdots, x_6\}$, $E = \{e_1, e_2, \cdots, e_7\}$, a quadruple (U, F, E, V) is shown in Table 1. Let $\beta = \frac{\{0.6,0.5,0.3\}}{e_1} + \frac{\{0.5,0.5,0.3\}}{e_2} + \frac{\{0.4,0.3,0.3\}}{e_3} + \frac{\{0.7,0.6,0.5\}}{e_4} + \frac{\{0.6,0.3,0.2\}}{e_5} + \frac{\{0.6,0.3,0.1\}}{e_6} + \frac{\{0.6,0.3,0.2\}}{e_7}$.

(1) For the Theorem 4.11 (1), $a \in R^{p\beta}(A)$ and $b \in R^{p\beta}(A)$, however $b \notin R^{p\beta}(A-\{a\})$.

$N_{e_1}^{p\beta} = N_{e_3}^{p\beta} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}\}$, $N_{e_2}^{p\beta} = \{\{x_1, x_2, x_3, x_6\}, \{x_4, x_5\}\}$.

Let $a = e_1$, $b = e_3$ and $A = \{e_1, e_2, e_3\}$. $Cov^{p\beta}(A) = \{\{x_1, x_2, x_3\}, \{x_4, x_5\}, \{x_6\}\} = Cov^{p\beta}(A-\{a\}) = Cov^{p\beta}(A-\{b\})$, i.e., $a \in R^{p\beta}(A)$ and $b \in R^{p\beta}(A)$.

$Cov^{p\beta}(A-\{a,b\}) = \{\{x_1, x_2, x_3, x_6\}, \{x_4, x_5\}\}$. $Cov^{p\beta}(A-\{a,b\}) \neq Cov^{p\beta}(A-\{a\})$, i.e., $b \notin R^{p\beta}(A-\{a\})$.

(2) The cases of Theorem 4.11 (2), (3), (4) and (6) are similar to the case of Theorem 4.11 (1).

$a \in R^{p\beta}(A)$ and $b \in R^{p\beta}(A)$, $b \notin R^{p\beta}(A-\{a\})$.

With the similar ways above, $Cov^{a\beta}(A) = Cov^{a\beta}(A-\{a\}) = Cov^{a\beta}(A-\{b\})$, $Cov^{a\beta}(A-\{a,b\}) \neq Cov^{a\beta}(A-\{a\})$, i.e., $a \in R^{a\beta}(A)$ and $b \in R^{a\beta}(A)$, $b \notin R^{a\beta}(A-\{a\})$.

Furthermore, $a \in R^{m\beta}(A)$ and $b \in R^{m\beta}(A)$, $b \notin R^{m\beta}(A-\{a\})$; $a \in R^{s\beta}(A)$ and $b \in R^{s\beta}(A)$, $b \notin R^{s\beta}(A-\{a\})$; and $a \in R^{n\beta}(A)$ and $b \in R^{n\beta}(A)$, $b \notin R^{n\beta}(A-\{a\})$.

(3) For the Theorem 4.11 (5), $a \in R^{t\beta}(A)$ and $b \in R^{t\beta}(A)$, however $b \notin R^{t\beta}(A-\{a\})$.

$N_{e_5}^{t\beta} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}\} = N_{e_6}^{t\beta}$, $N_{e_7}^{t\beta} = \{\{x_1\}, \{x_2, x_3, x_4, x_5, x_6\}\}$.

Let $a = e_5$, $b = e_6$, $A = \{e_5, e_6, e_7\}$. $Cov^{t\beta}(A) = Cov^{t\beta}(A-\{a\}) = Cov^{t\beta}(A-\{b\}) = \{\{x_1\}, \{x_2, x_3\}, \{x_4, x_5, x_6\}\}$. $Cov^{t\beta}(A-\{a,b\}) = \{\{x_1\}, \{x_2, x_3, x_4, x_5, x_6\} \neq Cov^{t\beta}(A-\{a\})$.

$a \in R^{t\beta}(A)$ and $b \in R^{t\beta}(A)$, $b \notin R^{t\beta}(A-\{a\})$.

When a $p\beta$-partition is employed to convert a HFIS (U, F, A, V) into a 0-1 binary value information system, the resulting system is denoted as $S_p = (U, F, A, V)_\beta$, with similar notations applied for S_a, S_m, S_s, S_t, and S_n.

Let $\mathscr{B}^p = \{B^p : Cov^{p\beta}(A) = Cov^{p\beta}(A - B^p)\}$ and $\mathcal{I}^p = \{I^p : I^p = A - B^p, B^p \in \mathscr{B}^p\}$ be two families over A in $S_p = (U, F, A, V)_\beta$. The similar ways to define $(\mathscr{B}^a, \mathcal{I}^a)$, $(\mathscr{B}^m, \mathcal{I}^m)$, $(\mathscr{B}^s, \mathcal{I}^s)$, $(\mathscr{B}^t, \mathcal{I}^t)$ and $(\mathscr{B}^n, \mathcal{I}^n)$ in $S_a = (U, F, A, V)_\beta$, $S_m = (U, F, A, V)_\beta$, $S_s = (U, F, A, V)_\beta$, $S_t = (U, F, A, V)_\beta$ and $S_n = (U, F, A, V)_\beta$, respectively.

Theorem 4.13. *Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $A \sqsubset E$. The following statements hold,*

(1) $Core^{p\beta}(A) = \sqcap_{I^p \in \mathcal{I}^p} I^p$.

(2) $Core^{a\beta}(A) = \sqcap_{I^a \in \mathcal{I}^a} I^a$.

(3) $Core^{m\beta}(A) = \sqcap_{I^m \in \mathcal{I}^m} I^m$.

(4) $Core^{s\beta}(A) = \sqcap_{I^s \in \mathcal{I}^s} I^s$.

(5) $Core^{t\beta}(A) = \sqcap_{I^t \in \mathcal{I}^t} I^t$.

(6) $Core^{n\beta}(A) = \sqcap_{I^n \in \mathcal{I}^n} I^n$.

Proof. (1) For all $B^p \in \mathscr{B}^p$, $Core^{p\beta}(A) \sqsubset A - B^p$, then $Core^{p\beta}(A) \sqsubset \sqcap_{B^p \in \mathscr{B}^p}(A - B^p) = \sqcap_{I^p \in \mathcal{I}^p} I^p$.

For all $e \notin Core^{p\beta}(A)$, $\{e\} \in \mathscr{B}^p$, then $(Core^{p\beta}(A))^c \sqsubset \sqcup_{B^p \in \mathscr{B}^p} B^p$. Then $Core^{p\beta}(A) \sqsupset (\sqcup_{B^p \in \mathscr{B}^p} B^p)^c = \sqcap_{B^p \in \mathscr{B}^p}(B^p)^c = \sqcap_{B^p \in \mathscr{B}^p}(A - B^p) = \sqcap_{I^p \in \mathcal{I}^p} I^p$.

Hence, $Core^{p\beta}(A) = \sqcap_{I^p \in \mathcal{I}^p} I^p$.

The proofs of (2)-(6) are similar to the proof of (1). ■

Theorem 4.14. *Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $A \sqsubset E$. The following statements hold,*

(1) $e \in Core^{p\beta}(A)$ if and only if there is at least a pair of $x, y \in U$ whose relations are different in $S_p^1 = (U, F, A, V)_\beta$ and $S_p^2 = (U, F, A - \{e\}, V)_\beta$.

(2) $e \in Core^{a\beta}(A)$ if and only if there is at least a pair of $x, y \in U$ whose relations are different in $S_a^1 = (U, F, A, V)_\beta$ and $S_a^2 = (U, F, A - \{e\}, V)_\beta$.

(3) $e \in Core^{m\beta}(A)$ if and only if there is at least a pair of $x, y \in U$ whose relations are different in $S_m^1 = (U, F, A, V)_\beta$ and $S_m^2 = (U, F, A - \{e\}, V)_\beta$.

(4) $e \in Core^{s\beta}(A)$ if and only if there is at least a pair of $x, y \in U$ whose relations are different in $S_s^1 = (U, F, A, V)_\beta$ and $S_s^2 = (U, F, A - \{e\}, V)_\beta$.

(5) $e \in Core^{t\beta}(A)$ if and only if there is at least a pair of $x, y \in U$ whose relations are different in $S_t^1 = (U, F, A, V)_\beta$ and $S_t^2 = (U, F, A - \{e\}, V)_\beta$.

(6) $e \in Core^{n\beta}(A)$ if and only if there is at least a pair of $x, y \in U$ whose relations are different in $S_n^1 = (U, F, A, V)_\beta$ and $S_n^2 = (U, F, A - \{e\}, V)_\beta$.

Proof. (1) To denote $A' = A - \{e\}$.

The proof of necessity.

$e \in Core^{p\beta}(A)$, then $Cov^{p\beta}(A) \neq Cov^{p\beta}(A - \{e\}) = Cov^{p\beta}(A')$. There is $x_0 \in U$ satisfying $N_A^{p\beta}(x_0) \neq N_{A'}^{p\beta}(x_0)$. Then there is y_0 satisfying $y_0 \in N_A^{p\beta}(x_0)$ and $y_0 \notin N_{A'}^{p\beta}(x_0)$, or another case $y_0 \notin N_A^{p\beta}(x_0)$ and $y_0 \in N_{A'}^{p\beta}(x_0)$. There is a pair of $x_0, y_0 \in U$ that their relations are different in $S_p^1 = (U, F, A, V)_\beta$ and $S_p^2 = (U, F, A - \{e\}, V)_\beta$.

The proof of sufficiency.

For any $x, y \in U$, if the relations of x and y are different in $S_p^1 = (U, F, A, V)_\beta$ and $S_p^2 = (U, F, A - \{e\}, V)_\beta$, i.e., $y \in N_A^{p\beta}(x)$ and $y \notin N_{A'}^{p\beta}(x)$, or another case $y \notin N_A^{p\beta}(x)$ and $y \in N_{A'}^{p\beta}(x)$. In either case, we can deduce $N_A^{p\beta}(x) \neq N_{A'}^{p\beta}(x)$, i.e., $Cov^{p\beta}(A) \neq Cov^{p\beta}(A - \{e\})$, then $e \in Core^{p\beta}(A)$.

The proofs of (2)-(6) are similar to the proof of (1). ■

4.2. Example and algorithm of parameter reduction in hesitant fuzzy information systems

In this subsection, we provide an example and an algorithm of parameter reduction based on the $p\beta$-partition in HFISs. The parameter reductions based on other kinds of β-partitions can be calculated using a method similar to that of $p\beta$-partition.

Example 4.15. A quadruple (U, F, E, V) and β are shown in the Example 4.12. Let $A = E$, we can thus calculate the families $\mathscr{B}^p$ and $\mathcal{I}^p$ of the HFIS (U, F, A, V) based on $p\beta$-partition.

We show the processes of calculating the $\mathscr{B}^p$ and $\mathcal{I}^p$ in the following steps:

Step 1. Calculate $N_{e_i}^{p\beta}$ for every $e_i \in A$.

$N_{e_1}^{p\beta} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}\}$,
$N_{e_2}^{p\beta} = \{\{x_1, x_2, x_3, x_6\}, \{x_4, x_5\}\}$,
$N_{e_3}^{p\beta} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}\}$,
$N_{e_4}^{p\beta} = \{\{x_1, x_2, x_3, x_4, x_5, x_6\}, \emptyset\}$,
$N_{e_5}^{p\beta} = \{\{x_1, x_2, x_3, x_5, x_6\}, \{x_4\}\}$,
$N_{e_6}^{p\beta} = \{\{x_1, x_2, x_3, x_4, x_5, x_6\}, \emptyset\}$,
$N_{e_7}^{p\beta} = \{\{x_1, x_2, x_4\}, \{x_3, x_5, x_6\}\}$.

Step 2. Obtain an initial $\mathscr{B}^p$, if $e_i \in R^{p\beta}(A)$, then $\{e_i\} \in \mathscr{B}^p$. $|B^p| = 1$ for every B^p in the initial $\mathscr{B}^p$.

In this example, the initial $\mathscr{B}^p = \{\{e_1\}, \{e_3\}, \{e_4\}, \{e_5\}, \{e_6\}\}$.

Step 3. Select each $B_i^p \in \mathscr{B}^p$ one by one, if $e_i \in R^{p\beta}(A - B_i^p)$ for an $e_i \in A - B_i^p$, then remove B_i^p from $\mathscr{B}^p$ and add $B_i^p \sqcup \{e_i\}$ to $\mathscr{B}^p$. Here, e_i may not be the only one that satisfies $e \in R^{p\beta}(A - B_i^p)$ in some cases. Thus, if $e_j \in R^{p\beta}(A - B_i^p)$ for $e_j \in A - B_i^p$, then $B_i^p \sqcup \{e_j\}$ is also added to $\mathscr{B}^p$.

For $B_1^p = \{e_1\}$, $e_4 \in R^{p\beta}(A - B_1^p)$, $e_5 \in R^{p\beta}(A - B_1^p)$ and $e_6 \in R^{p\beta}(A - B_1^p)$, then $\{e_1\}$ is removed from $\mathscr{B}^p$, and $B_1^p \sqcup \{e_4\} = \{e_1, e_4\}$, $B_1^p \sqcup \{e_5\} = \{e_1, e_5\}$ and $B_1^p \sqcup \{e_6\} = \{e_1, e_6\}$ are added to $\mathscr{B}^p$. Based on the results above, then $\mathscr{B}^p = \{\{e_1, e_4\}, \{e_1, e_5\}, \{e_1, e_6\}, \{e_3\}, \{e_4\}, \{e_6\}\}$.

To deal with other $B^p \in \mathscr{B}^p$, then we obtain $\mathscr{B}^p = \{\{e_1, e_4\}, \{e_1, e_5\}, \{e_1, e_6\}, \{e_3, e_4\}, \{e_3, e_5\}, \{e_3, e_6\}, \{e_4, e_5\}, \{e_4, e_6\}, \{e_5, e_6\}\}$.

Step 4. Repeat the similar operations of Step 3 until $e \notin R^{p\beta}(A - B^p)$ for every $B^p \in \mathscr{B}^p$ and every $e \in A - B^p$. Then, the final $\mathscr{B}^p$ is obtained as follows:

$\mathscr{B}^p = \{\{e_1, e_4, e_5, e_6\}, \{e_3, e_4, e_5, e_6\}\}$.

Step 5. Calculate $\mathcal{I}^p$, $\mathcal{I}^p = \{A - B^p : B^p \in \mathscr{B}^p\}$.

$\mathcal{I}^p = \{\{e_2, e_3, e_7\}, \{e_1, e_2, e_7\}\}$.

$Cov^{p\beta}(A) = Cov^{p\beta}(I^p)$ for all $I^p \in \mathcal{I}^p$.

A detailed algorithm of parameter reduction of HFISs based on the $p\beta$-partition is shown as follows,

5. Application example of foundational theories of hesitant fuzzy sets and hesitant fuzzy information systems

Complex systems like large wind turbines and shield tunneling machines often have a limited number of failed samples and numerous factors that can potentially impact system failure. The presence of such systems poses challenges in terms of performing system diagnosis and making decisions regarding system replacement. To address these scenarios, we introduce a multi-strength intelligent classifier as an alternative approach.

We assumed that $|F(e)(x_i)| = |F(e)(x_j)|$ for all $e \in E$ and $x_i, x_j \in U$, then the relationship "$\subset_t$" will not be considered in Section 5.

Algorithm 1 Parameter reduction of HFISs based on $p\beta$-partition.

Input: HFIS (U, F, A, V) and β.
Output: $\mathscr{B}^p$ and $\mathcal{I}^p$.
1. Calculate $N_{e_i}^{p\beta}$ for every $e_i \in A$.
2. Obtain an initial $\mathscr{B}^p$, if $e_i \in R^{p\beta}(A)$, then $\{e_i\} \in \mathscr{B}^p$.
3. **While** There is $B^p \in \mathscr{B}^p$ and $e \in A - B^p$ such that $e \in R^{p\beta}(A - B^p)$ **do**
4. **repeat**
5. $\mathscr{B}^p \leftarrow \mathscr{B}^p - \{B^p\}$.
6. **While** There is $e \in A - B^p$ that satisfied $e \in R^{p\beta}(A - B^p)$ **do**
7. **repeat**
8. $\mathscr{B}^p \leftarrow \mathscr{B}^p \sqcup \{B^p \sqcup \{e\}\}$.
9. **until** all the $B^p \sqcup \{e\}$ are added to $\mathscr{B}^p$ for some $e \in A - B^p$ that satisfy $e \in R^{p\beta}(A - B^p)$.
10. **end while**
11. **until** $e \notin R^{p\beta}(A - B^p)$ for every $B^p \in \mathscr{B}^p$ and every $e \in A - B^p$.
12. **end while**
13. $\mathcal{I}^p \leftarrow \{A - B^p : B^p \in \mathscr{B}^p\}$.

5.1. Knowledge bases and similar degree functions

Definition 5.1. [26] A pair $(U, \mathcal{R})$ is called a knowledge base, where $\mathcal{R} \neq \emptyset$ and $\mathcal{R}$ is a family of equivalence relations on U.

As shown in Example 4.12, let $A = \{e_1, e_2, e_3, e_4\}$, $Cov^{p\beta}(A) = \{\{x_1, x_2, x_3\}, \{x_4, x_5\}, \{x_6\}\}$ can be considered as a knowledge, denoted as $(U, R_{(A,p\beta)})$. Furthermore, $R_{(A,p\beta)}(x_1, x_2) = 1$ or $x_2 \in [x_1]_{R_{(A,p\beta)}}$; $R_{(A,p\beta)}(x_1, x_4) = 0$ or $x_4 \notin [x_1]_{R_{(A,p\beta)}}$. $Cov^{a\beta}(A)$, $Cov^{m\beta}(A)$, $Cov^{s\beta}(A)$ and $Cov^{n\beta}(A)$ are knowledge generated by $R_{(A,a\beta)}$, $R_{(A,m\beta)}$, $R_{(A,s\beta)}$ and $R_{(A,n\beta)}$, respectively.

Let $\mathcal{R}^* = \{R_{(A,p\beta)}, R_{(A,a\beta)}, R_{(A,m\beta)}, R_{(A,s\beta)}, R_{(A,n\beta)}\}$, then $(U, \mathcal{R}^*)$ is a knowledge base.

Definition 5.2. Let (U, F, E, V) be a HFIS and $\beta \in HF(E)$ be an evaluation indicator, $F : U \times E \to V$, $x_i, x_j \in U$, $e \in A \sqsubset E$, and $\neg A = \{\neg e_i : e_i \in A\}$,

(1) Attribute recognition functions:
$\xi^{p\beta}(x_i) = \widehat{A}$, where $\widehat{A} \sqsubset A \sqcup \neg A$, and $e \in \widehat{A}$ if $\beta(e) \subset_p F(e)(x_i)$; otherwise, $\neg e \in \widehat{A}$.
$\xi^{a\beta}(x_i) = \widehat{A}$, where $\widehat{A} \sqsubset A \sqcup \neg A$, and $e \in \widehat{A}$ if $\beta(e) \subset_a F(e)(x_i)$; otherwise, $\neg e \in \widehat{A}$.
$\xi^{m\beta}(x_i) = \widehat{A}$, where $\widehat{A} \sqsubset A \sqcup \neg A$, and $e \in \widehat{A}$ if $\beta(e) \subset_m F(e)(x_i)$; otherwise, $\neg e \in \widehat{A}$.
$\xi^{s\beta}(x_i) = \widehat{A}$, where $\widehat{A} \sqsubset A \sqcup \neg A$, and $e \in \widehat{A}$ if $\beta(e) \subset_s F(e)(x_i)$; otherwise, $\neg e \in \widehat{A}$.
$\xi^{n\beta}(x_i) = \widehat{A}$, where $\widehat{A} \sqsubset A \sqcup \neg A$, and $e \in \widehat{A}$ if $\beta(e) \subset_n F(e)(x_i)$; otherwise, $\neg e \in \widehat{A}$.

(2) Similar degree functions:
$Sim^{p\beta}(x_i, x_j) = \frac{|\xi^{p\beta}(x_i) \sqcap \xi^{p\beta}(x_j)|}{|A|}$.
$Sim^{a\beta}(x_i, x_j) = \frac{|\xi^{a\beta}(x_i) \sqcap \xi^{a\beta}(x_j)|}{|A|}$.
$Sim^{m\beta}(x_i, x_j) = \frac{|\xi^{m\beta}(x_i) \sqcap \xi^{m\beta}(x_j)|}{|A|}$.
$Sim^{s\beta}(x_i, x_j) = \frac{|\xi^{s\beta}(x_i) \sqcap \xi^{s\beta}(x_j)|}{|A|}$.
$Sim^{n\beta}(x_i, x_j) = \frac{|\xi^{n\beta}(x_i) \sqcap \xi^{n\beta}(x_j)|}{|A|}$.

As shown in Example 4.12, $\beta(e_1) \not\subset_p F(e_1)(x_1)$; and $\beta(e_i) \subset_p F(e_i)(x_1)$ for $e_i \in E - \{e_1\}$. It means that the object x_1 has no characteristic concerning e_1 based on $p\beta$-partition. Then, $\xi^{p\beta}(x_1) = \{\neg e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. In the same manner, $\xi^{p\beta}(x_5) = \{e_1, \neg e_2, \neg e_3, e_4, e_5, e_6, \neg e_7\}$. Furthermore, $Sim^{p\beta}(x_1, x_5) = \frac{|\xi^{p\beta}(x_1) \sqcap \xi^{p\beta}(x_5)|}{|E|} = \frac{|\{e_4, e_5, e_6\}|}{|E|} = \frac{3}{7}$.

Theorem 5.3. *For $X \sqsubset U$, the following statements hold in the knowledge base $(U, \mathcal{R}^*)$.*
(1) $Sim^{p\beta}(x, y) = 1$ for $x, y \in X \in (U, R_{(A,p\beta)})$.
(2) $Sim^{a\beta}(x, y) = 1$ for $x, y \in X \in (U, R_{(A,a\beta)})$.
(3) $Sim^{m\beta}(x, y) = 1$ for $x, y \in X \in (U, R_{(A,m\beta)})$.
(4) $Sim^{s\beta}(x, y) = 1$ for $x, y \in X \in (U, R_{(A,s\beta)})$.
(5) $Sim^{n\beta}(x, y) = 1$ for $x, y \in X \in (U, R_{(A,n\beta)})$.

Proof. (1) For $X \in (U, R_{(A,p\beta)}) = Cov^{p\beta}(A)$ and $x \in X$, since $\{x, y\} \sqsubset X = \sqcap\{N_e^{p\beta}(x) : e \in A\}$, then $y \in N_e^{p\beta}(x)$ for all $e \in A$. Therefore, we can obtain that $\xi^{p\beta}(x) = \xi^{p\beta}(y)$, then $Sim^{p\beta}(x, y) = 1$.

The proofs of (2)-(5) are similar to the proof of (1). ■

Table 2
Credible degrees that the equipment is healthy

Kinds of partition	Credible degrees of systems' health states	Health grade	Sample set
$n\beta$-partition	necessary credible	G1	U_n
$s\beta$-partition	strong credible	G2	U_s
$m\beta$-partition	middle credible	G3	U_m
$a\beta$-partition	middle credible	G3	U_a
$p\beta$-partition	weak credible	G4	U_p
other cases	not credible	G5	U_f

5.2. Multi-strength intelligent classifier for health status diagnosis of complex systems

In this section, we introduce a multi-strength intelligent classifier designed to diagnose the health status of complex systems. We provide a description of the health state partitions for the complex system, discuss the conditions for terminating the training of the intelligent classifier, and describe the classification approach employed by the intelligent classifier.

5.2.1. Partitions of health states

In order to use the results of the previous section to diagnose the health status of the system, we divide the health level of the system into five levels according to Definition 2.10, shown in Table 2. For example, $x \in U_n$ means that it is necessary credible that x is a healthy equipment; $y \in U_f$ means that it is not credible that y is a healthy equipment, i.e., y is a failed equipment.

5.2.2. Conditions under which the multi-strength intelligent classifier terminates training

If $Sim^{n\beta}(x_i,x_j) \geqslant PT$ ($Sim^{n\beta}(x_i,x_j) < NT$), then x_i and x_j are (not) considered as two samples in the same category based on $n\beta$-partition, where PT is a positive threshold and NT is a negative threshold.

$$\begin{cases} \frac{1}{C^2_{|U_n^*|}} \sum_{x_i,x_j \in U_n^*} Sim^{n\beta}(x_i,x_j) \geqslant PT, \\ \frac{1}{C^2_{|U_s^*|}} \sum_{x_i,x_j \in U_s^*} Sim^{s\beta}(x_i,x_j) \geqslant PT, \\ \frac{1}{|U_s^*| \times |U_n^*|} \sum_{x_i \in U_s^*, x_j \in U_n^*} Sim^{n\beta}(x_i,x_j) < NT, \\ \frac{1}{C^2_{|U_m^*|}} \sum_{x_i,x_j \in U_m^*} Sim^{m\beta}(x_i,x_j) \geqslant PT, \\ \frac{1}{|U_m^*| \times |U_s^*|} \sum_{x_i \in U_m^*, x_j \in U_s^*} Sim^{s\beta}(x_i,x_j) < NT, \\ \frac{1}{C^2_{|U_a^*|}} \sum_{x_i,x_j \in U_a^*} Sim^{a\beta}(x_i,x_j) \geqslant PT, \\ \frac{1}{|U_a^*| \times |U_s^*|} \sum_{x_i \in U_a^*, x_j \in U_s^*} Sim^{s\beta}(x_i,x_j) < NT, \\ \frac{1}{C^2_{|U_p^*|}} \sum_{x_i,x_j \in U_p^*} Sim^{p\beta}(x_i,x_j) \geqslant PT, \\ \frac{1}{|U_1| \times |U_f^*|} \sum_{x_i \in U_1, x_j \in U_f^*} Sim^{p\beta}(x_i,x_j) < NT, \\ \frac{1}{|U_2| \times |U_f^*|} \sum_{x_i \in U_2, x_j \in U_f^*} Sim^{m\beta}(x_i,x_j) < NT. \end{cases} \tag{1}$$

Based on the Proposition 2.12, $\subset_n \Rightarrow \subset_s \Rightarrow \subset_a \Rightarrow \subset_p$ and $\subset_n \Rightarrow \subset_s \Rightarrow \subset_m$, then we define $U_1 = U_n^* \sqcup U_s^* \sqcup U_a^* \sqcup U_p^*$ and $U_2 = U_n^* \sqcup U_s^* \sqcup U_m^*$ to distinguish U_f^*, where $U_n^* = U_n \sqcap U_{tain}$, $U_s^* = U_s \sqcap U_{tain}$, $U_m^* = U_m \sqcap U_{tain}$, $U_a^* = U_a \sqcap U_{tain}$, $U_p^* = U_p \sqcap U_{tain}$ and $U_f^* = U_f \sqcap U_{tain}$. The conditions for the intelligent classifier to terminate the training are shown in the inequality array (1), where $C_N^2 = N \times (N-1)/2$.

The final two inequalities in the inequality array (1) serve to differentiate failure samples from healthy samples, where the samples in U_1 and U_f^* exhibit fewer shared attributes according to the $p\beta$-partition, while the samples in U_2 and U_f^* demonstrate fewer shared attributes based on the $m\beta$-partition. Consequently, these last two inequalities in inequality array (1) impose restrictions on samples to avoid having numerous features from both healthy and failed samples, thereby preventing the classifier from classifying the same sample as both healthy and failed.

The inequality $\frac{1}{|U_s^*|\times|U_n^*|}\sum_{x_i\in U_s^*,x_j\in U_n^*} Sim^{n\beta}(x_i,x_j) < NT$ restricts samples $x_i \in U_s$ and $x_j \in U_n$ from having many features of both G1 and G2 health grades and prevents a sample $x_i \in U_s$ from being classified as $x_i \in U_n$, i.e., it prevents the classifier from classifying samples of low-level health states into a set of high-level health states. The other inequalities with a condition of "$< NT$" in array (1) serve a similar purpose.

5.2.3. The classification way of the multi-strength intelligent classifier

When the training of a multi-strength intelligent classifier is finished, it is used to classify other samples. Let $Cl(x) = \{Cl_n(x), Cl_s(x), Cl_m(x), Cl_a(x), Cl_p(x), Cl_f(x)\}$, where the calculations of $Cl(x)$ are shown in equation array (2).

$$\begin{cases} Cl_n(x) = \frac{1}{|U_n^*|}\sum_{x_i\in U_n^*} Sim^{n\beta}(x,x_i), \\ Cl_s(x) = \frac{1}{|U_s^*|}\sum_{x_i\in U_s^*} Sim^{s\beta}(x,x_i), \\ Cl_m(x) = \frac{1}{|U_m^*|}\sum_{x_i\in U_m^*} Sim^{m\beta}(x,x_i), \\ Cl_a(x) = \frac{1}{|U_a^*|}\sum_{x_i\in U_a^*} Sim^{a\beta}(x,x_i), \\ Cl_p(x) = \frac{1}{|U_p^*|}\sum_{x_i\in U_p^*} Sim^{p\beta}(x,x_i), \\ Cl_f(x) = \frac{1}{|U_f^*|}\sum_{x_i\in U_f^*} Sim^{p\beta}(x,x_i). \end{cases} \tag{2}$$

The values in $Cl(x)$ have three cases (i), (ii) and (iii) as follows.

(i) One of the values in $Cl(x)$ is greater than or equal to PT, then x is assigned to the class corresponding to the maximum value in $Cl(x)$.

(ii) More than one value in $Cl(x)$ is greater than or equal to PT, then x is assigned to more advanced health states. For example, $Cl_n(x) \geqslant PT$ and $Cl_s(x) \geqslant PT$, then x is assigned to U_n. By Proposition 2.12, $\subset_n \Rightarrow \subset_s$, then case (ii) exists. On the other hand, the inequality $\frac{1}{|U_s^*|\times|U_n^*|}\sum_{x_i\in U_s^*,x_j\in U_n^*} Sim^{n\beta}(x_i,x_j) < NT$ prevents the sample $x \in U_s$ from being classified as $x \in U_n$. Hence, $x \in U_s$ cannot obtain $Cl_n(x) \geqslant PT$, but $x \in U_n$ can obtain $Cl_s(x) \geqslant PT$. If $Cl_n(x) \geqslant PT$ and $Cl_s(x) \geqslant PT$, then $x \in U_n$.

(iii) No value in $Cl(x)$ is greater than or equal to PT, the classifier is invalidated for x. For small sample size and high dimension data, it is difficult for the classifier to learn all the features of the data, case (iii) may be present in a new monitoring sample.

5.2.4. The algorithm of the multi-strength intelligent classifier

The algorithm for training the multi-strength intelligent classifier is shown as Algorithm 2 where AcT is the threshold of the correct classification rate of the testing samples. And, referring back the assumption of Section 5, $|F(e)(x)| = |F(e)(y)|$ for all $x, y \in U$.

When U_{tain}^*, A^* and β^* are obtained, a multi-strength intelligent classifier $(U_{tain}^*, \mathcal{R}^*, \{Sim\})$ is obtained, where $\mathcal{R}^* = \{R_{(A,p\beta)}, R_{(A,a\beta)}, R_{(A,m\beta)}, R_{(A,s\beta)}, R_{(A,n\beta)}\}$ is illustrated below Definition 5.1, $\{Sim\}$ is a family of similar degree functions and $\{Sim\} = \{Sim^{p\beta}(\cdot), Sim^{a\beta}(\cdot), Sim^{m\beta}(\cdot), Sim^{s\beta}(\cdot), Sim^{n\beta}(\cdot)\}$ that is illustrated in the Definition 5.2.

5.2.5. An example of the multi-strength intelligent classifier application

Monitoring data of large wind turbines of the same model with different health states are used to do case analysis as follows. For the raw data of devices, let $U = \{x_1, x_2, \cdots, x_{|U|}\}$ be a sample set and $data(e)(x_i)$ be the raw data of $x_i \in U$ with respect to the attribute e. Let $data(e) = \sqcup_{x_i\in U} data(e)(x_i)$ and $h = \frac{d - inf(data(e))}{sup(data(e)) - inf(data(e))}$, where $d \in data(e)(x_i)$ and $h \in F(e)(x_i)$. Then, we can convert the raw data into the data conformed to the value range of the hesitant fuzzy membership degree.

Example 5.4. $U = \{x_1, x_2, \cdots, x_{17}\}$ is a sample set. $E = \{e_1, e_2, \cdots, e_{26}\}$ is an attribute set. PT, NT and AcT are set to 85%, 50% and 95%, respectively.

The samples are classified based on different health states $U_n = \{x_1, x_6, x_{13}\}$, $U_s = \{x_2, x_{12}, x_{14}\}$, $U_m = U_a = \{x_3, x_5, x_{15}\}$ (the samples x_3, x_5 and x_{15} are in a same health grade G3), $U_p = \{x_7, x_8, x_{11}, x_{16}\}$ and $U_f = \{x_4, x_9, x_{10}, x_{17}\}$. About two-thirds of the samples from various health states are taken as training samples, and the remaining samples are used as testing samples.

In the last iteration, $U_{tain}^* = \{x_2, x_3, x_4, x_6, x_7, x_9, x_{11}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}\}$, $U_{test} = \{x_1, x_5, x_8, x_{10}, x_{12}\}$ and $A^* = \{e_1, e_2, e_6, e_7, e_8, e_{11}, e_{14}, e_{17}, e_{18}, e_{19}, e_{21}, e_{26}\}$. In the following, we demonstrate some numerical calculations of Algorithm 2 with the results of the last iteration.

Algorithm 2 Multi-strength intelligent classifier.

Input: PT, NT, AcT, sample sets U_n, U_s, U_a, U_m, U_p and U_f, and the HFIS (U, F, E, V), where $U = U_n \sqcup U_s \sqcup U_a \sqcup U_m \sqcup U_p \sqcup U_f$.
Output: U^*_{tain}, A^* and β^*.
1. $\beta(e) \leftarrow \{0, 0, \cdots, 0\}$, where $|\beta(e)| = |F(e)(x)|$ for each $e \in E$.
2. $AC \leftarrow 0$.
3. **While** $AC < AcT$ **do**
4. **repeat**
5. Two-thirds of the samples in U are randomly selected as training samples U_{tain} and the other third as testing samples U_{test}.
6. $Sim^{n\beta}(x_i, x_j) \leftarrow 0$ for all $x_i, x_j \in U_n$.
7. **While** One of the inequalities in the inequality array (1) does not hold **do**
8. **repeat**
9. The genetic algorithm is used to iterate an indicator β.
10. Apply Algorithm 1 to calculate $\mathcal{I}^p$, $\mathcal{I}^a$, $\mathcal{I}^m$, $\mathcal{I}^s$ and $\mathcal{I}^n$.
11. $A \leftarrow I_p \sqcup I_a \sqcup I_m \sqcup I_s \sqcup I_n$, where I_p, I_a, I_m, I_s and I_n are the random elements in $\mathcal{I}^p$, $\mathcal{I}^a$, $\mathcal{I}^m$, $\mathcal{I}^s$ and $\mathcal{I}^n$, respectively.
12. Calculate the factors in the array (1) ($\frac{1}{C^2_{|U^*_n|}} \sum_{x_i, x_j \in U^*_n} Sim^{n\beta}(x_i, x_j)$, etc), and judge whether all the inequalities in the inequality array (1) hold.
13. **until** All the inequalities in the inequality array (1) hold.
14. **end while**
15. Calculate $Cl(x)$ for $x \in U_{test}$ and classify x.
16. Calculate AC, where $AC = \frac{N_0}{|U_{tset}|}$ and N_0 is the number of testing samples that were correctly classified.
17. **until** $AC \geqslant AcT$.
18. **end while**
19. $U^*_{tain} \leftarrow U_{tain}$, $A^* \leftarrow A$ and $\beta^* \leftarrow \beta$.

In $(U^*_{tain}, \mathcal{R}^*, \{Sim\})$, $\xi^{n\beta}(x_6) = (\neg e_1, e_2, e_6, e_7, e_8, \neg e_{11}, \neg e_{14}, \neg e_{17}, e_{18}, e_{19}, \neg e_{21}, \neg e_{26})$ and $\xi^{n\beta}(x_{13}) = (\neg e_1, e_2, e_6, e_7, e_8, \neg e_{11}, \neg e_{14}, e_{17}, e_{18}, e_{19}, \neg e_{21}, \neg e_{26})$. Then, $Sim^{n\beta}(x_6, x_{13}) = \frac{|\xi^{n\beta}(x_6) \sqcap \xi^{n\beta}(x_{13})|}{|A^*|} = \frac{11}{12}$. Then, $\frac{1}{C^2_{|U^*_n|}} \sum_{x_i, x_j \in U^*_n} Sim^{n\beta}(x_i, x_j) = \frac{11}{12} > PN = 85\%$.

In the same manner, to calculate the other factors in the inequality array (1), we can verify that U^*_{tain}, A^* and β^* satisfy the other conditions of the inequality array (1). Then, we classify the samples in U_{test} to calculate the classification accuracy.

For $x_1 \in U_{test}$, $Cl(x_1) = \{Cl_n(x_1), Cl_s(x_1), Cl_m(x_1), Cl_a(x_1), Cl_p(x_1), Cl_f(x_1)\} = \{0.9583, 0.9583, 0.7083, 0.7083, 0.4722, 0.0556\}$, x_1 is assigned to U_n (health grade is G1) and classified correctly. In the same manner, we can obtain $x_5 \in U_m = U_a$ (G3), $x_8 \in U_p$ (G4), $x_{10} \in U_f$ (G5), and $x_{12} \in U_s$ (G2). All five testing samples are correctly classified to their classes, then $AC = \frac{5}{|U_{test}|} = 100\% > 95\%$, the intelligent classifier passes the test. And the intelligent classifier $(U^*_{tain}, \mathcal{R}^*, \{Sim\})$ is stored in the knowledge base $(U^*_{tain}, \mathcal{R}^*)$.

When we obtain a new sample x_{new}, we can calculate $Cl(x_{new})$ and classify it by using the multi-strength intelligent classifier $(U^*_{tain}, \mathcal{R}^*, \{Sim\})$ like the way above.

In relation to attribute e, there exists a positive correlation between the system's health status and the value of hesitant fuzzy membership related to e, indicating that higher values related to attribute e correspond to higher levels of system health. Conversely, attribute e exhibits a negative correlation with the system's health status, as lower values of hesitant fuzzy membership related to e correspond to higher levels of system health. The proposed multi-strength intelligent classifier offers the advantage of not necessitating an analysis of the positive or negative correlation between attributes and the system's health state before training the classifier.

Attributes that do not exhibit a positive or negative correlation with the system's health state will influence the similarity functions of the classifier. The classifier can eliminate these attributes during the training process, ensuring compliance with the constraints outlined in the inequality array (1). Consequently, the classifier can effectively reduce the dimensionality of high-dimensional data by eliminating attributes that do not significantly contribute to the outcome.

When considering objects in U as investment candidates, the multi-strength intelligent classifier can evaluate investment projects and identify the most suitable investment opportunities, thereby assisting in making informed investment decisions.

To summarize, the multi-strength intelligent classifier serves as a promising tool for dimensionality reduction in high-dimensional data and can be applied in various domains such as classification, diagnosis, evaluation, decision-making, and more.

6. Conclusion

The definition of the inclusion relationship is one of the most foundational concepts of sets. Based on the discrete form of hesitant fuzzy membership degrees, we propose several kinds of inclusion relationships for hesitant fuzzy sets. Furthermore, we present some propositions of hesitant fuzzy sets and some propositions of the families of hesitant fuzzy sets based on the proposed inclusion relationships. Some rules that hold in classical sets do not apply to hesitant fuzzy sets, as shown in Section 2. In subsequent studies of hesitant fuzzy sets, researchers must avoid the intuitive understanding of classical sets and cannot use these rules.

Furthermore, we investigate some foundational theorems of HFISs with respect to parameter reductions and present an example and an algorithm in HFISs to illustrate the processes of parameter reductions. Finally, we propose a multi-strength intelligent classifier to perform health state diagnoses for complex systems.

In our future work, we will investigate more applications of multi-strength intelligent classifiers, saturation reductions in HFISs, and foundational theories of knowledge bases with varying strengths of knowledge, among other topics.

CRediT authorship contribution statement

Shizhan Lu: Writing – review & editing, Writing – original draft, Software, Methodology, Formal analysis, Data curation. **Zeshui Xu:** Writing – review & editing, Methodology, Formal analysis. **Zhu Fu:** Funding acquisition, Formal analysis. **Longsheng Cheng:** Resources, Formal analysis. **Tongbin Yang:** Formal analysis.

Declaration of competing interest

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Data availability

Data will be made available on request.

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