Information Sciences 698 (2025) 121754

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

A three-way decision model in incomplete ordered information systems with fuzzy pre-decision

Zhao Li [a], Ju Sheng Mi [a,b,*], Lei Jun Li [a,b]

[a] School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, Hebei 050024, China
[b] Hebei Key Laboratory of Computational Mathematics and Applications, Shijiazhuang, Hebei 050024, China

ARTICLE INFO

Keywords:
Three-way decision
Incomplete ordered information systems
TOPSIS
Regret theory
Dominance relation

ABSTRACT

The three-way decision model has been extended by many researchers in recent years to allow its application to a wide range of datasets. The three-way decision model in incomplete ordered information systems is the focus of this paper. First, we estimate the missing values in an incomplete ordered information system to obtain a complete ordered information system with confidence. Second, according to the ideas of TOPSIS method and regret theory, the dominance relations based on PIS and NIS respectively are defined. Then, according to the idea of TOPSIS method, the fuzzy pre-decision and relative loss function matrix are defined. The fuzzy pre-decision and dominance relations are utilized to calculate the degree of objects belonging to PIS class and NIS class. Finally, the expected loss of executing different decisions is calculated and three-way decision rules are induced by Bayesian minimum risk decision theory. The associated properties of the rules are discussed and the rationality of the model is explained. In the experimental section of the paper, we design two sets of experiments aimed at categorizing and ordering the objects according to the proposed algorithm, thus providing a deeper insight into the feasibility and superiority of the three-way decision model.

1. Introduction

In the context of the exponential growth of information in modern society, the nature of data is becoming increasingly diverse. In daily life, incomplete information is generated due to inadequate storage or transmission, resulting in the loss of some values in information systems. Consequently, in recent years, numerous scholars have conducted extensive research on the uncertainty measures of incomplete information and decision-making based on incomplete information.

Liang et al. [12] introduced the concepts of information entropy, rough entropy, knowledge granulation and granularity measure into incomplete information systems, discussed their important properties, and established the relationship between these concepts. Dai et al. [5] constructed the uncertainty measures on the incomplete information systems through pure rough set, and studied the three definitions of upper and lower approximations and their corresponding uncertainty measurement concepts. Sun et al. [23] introduced the concepts of information entropy and uncertainty measures based on information granulation in incomplete information systems, and proposed some variants of information entropy and information granulation based on the maximal consistent block technique. Dai et al. [4] defined the α-weak relation in the incomplete interval-valued information systems, and studied the uncertainty measurement in the incomplete interval-valued information systems according to this binary relation. Kryszkiewicz [11] pioneered

* Corresponding author at: School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, Hebei 050024, China.
E-mail addresses: lizhao_9803@163.com (Z. Li), mijsh@263.net (J.S. Mi), lileijun1985@163.com (L.J. Li).

https://doi.org/10.1016/j.ins.2024.121754
Received 7 October 2024; Received in revised form 5 December 2024; Accepted 13 December 2024

Available online 17 December 2024

a rough set inference method, and suggested that incomplete data can be replaced by evaluated values of arbitrary attributes. Du et al. [7] introduced a new type of dominance relation in incomplete ordered information systems, which they called characteristic-based dominance relation. They also investigated a method of calculating all (relative) reducts based on the discernibility matrix and discernibility function. Tan et al. [24] utilized the belief function and plausibility function in evidence theory to describe the multi-granulation rough set approximations and attribute reduct in incomplete information systems. Lin et al. [13] constructed two distinct types of rough set models, namely the optimistic multi-granulation rough set model and the pessimistic multi-granulation rough set model, within the context of incomplete interval-valued decision information systems. Additionally, they devised algorithms for calculating roughness and degree of dependence.

The three-way decision with decision-theoretic rough sets [36] is a classification model proposed by Yao. This model posits that the universe of discourse can be divided into three parts: the decision positive domain, the decision boundary domain, and the decision negative domain. These three regions correspond to three kinds of semantics, namely, acceptance, delay and rejection. In recent years, a significant body of scholarship has emerged examining the application of the three-way decision model in incomplete information systems ([16], [33], [15], [25], [32], [34], [37], [14], [31]) and other related domains ([9], [18], [19], [21], [35], [40]). Peng et al. [16] defined a probabilistic similarity measure in an incomplete hybrid information system and constructed a novel three-way decision model. Subsequently, this model was combined with two customized ranking rules to address the multi-attribute decision-making problem in incomplete hybrid information systems. Yang et al. [33] introduced intuitionistic fuzzy sets into fuzzy incomplete information systems, and constructed a three-way decision model on fuzzy incomplete information systems. Tu et al. [25] completed the incomplete information systems based on similarity and combined the idea of TOPSIS to obtain the calculation method of conditional probability. Then according to the risk aversion coefficient under different attributes, the interval relative loss function values were calculated, and finally the corresponding three-way decision model was constructed. Zhan et al. [37] introduced a method to determine the value of the relative utility function and constructed a model of three-way decision within an incomplete fuzzy decision system, which was applied to the modeling of incomplete multi-attribute decision-making problems. Liu et al. [14] defined a novel relation to quantify the similarity of incomplete information, derived the loss function using interval numbers, and constructed a three-way decision model specifically tailored for incomplete information systems. In order to improve the ability to deal with complex incomplete information systems, Xin et al. [31] extended the traditional three-way decision model to an intuitionistic fuzzy three-way decision model based on an intuitionistic fuzzy incomplete information system.

Regret theory (RT) [1] was proposed by Bell in 1982, which can be used to explain situations where the actual behavior is contrary to the expected utility. RT reflects the individual's assessment of the expected response to future events or situations, which can be described as emotions generated by comparing the results or states of a given event. In recent years, there have been more and more studies on RT. Wang et al. [26] employed regret theory to formulate the optimal decision rules maximizing utility, and utilized the ideal solution method to compute conditional probability. Subsequently, they constructed a three-way decision model grounded in the concept of regret within an interval-type-2 fuzzy framework. Wang et al. [29] defined a prior probability tolerance dominance relation in fuzzy incomplete information systems, devised a method for calculating objective weights, and integrated these with regret theory to determine comprehensive utility perception. Zhang et al. [39] defined the calculation formulas for regret function and perceived utility and used the idea of LINMAP (Linear Programming Technique for Multidimensional Analysis of Preference) to establish a mathematical programming model that determined the optimal attribute weight and a defuzzified fuzzy ideal solution. Finally, the optimal comprehensive perceived utility value of the decision-maker was calculated, and the ranking of the alternatives was obtained. Deng et al. [6] employed linguistic term sets to transform multi-scale evaluation information into interval fuzzy numbers and subsequently proposed an approximate estimation method for target dependence on incomplete interval fuzzy subsystems. Additionally, a three-point sorting method was established according to the regret-rejoice preference between objects.

The TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) method utilizes a dual-reference framework comprising a positive ideal solution (PIS) and a negative ideal solution (NIS) as benchmarks for evaluating and ranking a collection of decision alternatives. Conceptualizing these alternatives as n-dimensional points within a decision space, the rationale behind TOPSIS's choice mechanism lies in its pursuit of the alternative that simultaneously minimizes its distance from the PIS (the optimal or most desirable state) and maximizes its distance from the NIS (the least favorable or undesirable state). TOPSIS tries to find the alternative with the smallest distance from PIS and the largest distance from NIS. In recent years, many researchers ([38], [2], [17], [30], [3], [10]) have promoted and improved the TOPSIS method. Zhan et al. [38] extended the classical TOPSIS model according to the mean and standard deviation, and proposed a TOPSIS method based on three-way decision. Qian et al. [17] extended and improved the application scope and decision-making ability of TOPSIS method by introducing sequential three-way decision. By obtaining the relative loss function derived from the evaluation value, the interval-valued intuitionistic fuzzy three-way decision-making (IVIF3WD) model was proposed in reference [10], and the ideal solution was introduced to construct the multi-attribute interval-valued intuitionistic fuzzy three-way decision model.

The main contributions of this paper are listed as follows:

(1) In the context of incomplete information systems, this paper addresses the issue of missing values by considering the correlation between attributes and the similarity between objects. Subsequently, the novel dominance relations are proposed, based on the calculation of the regret value generated when an object replaces PIS or is replaced by NIS. This binary relation not only reflects the comparative strengths and weaknesses among objects, but also allows for the quantification of the degree of such advantages and disadvantages.

(2) In [37], the author introduced the concept of pre-decision but did not provide the corresponding calculation method. Therefore, this paper proposes a calculation method for fuzzy pre-decision based on the idea of TOPSIS in incomplete ordered information systems, and defines a relative loss function matrix by using the fuzzy pre-decision.

(3) In accordance with the fuzzy pre-decision and dominance relation, this paper puts forth a methodology for quantifying the degree of object belonging to the PIS class and the NIS class, which is consistent with the semantics of D^+(PIS) and D^-(NIS) states in the relative loss function matrix.

The remainder of this paper is structured as follows: Section 2 introduces the linear regression model, the classical three-way decision model, regret theory, and the definition of the incomplete ordered information systems. Section 3 constructs a new dominance binary relation based on the ideal of TOPSIS method and regret theory, and discusses its related properties. Then, we define the fuzzy pre-decision and relative loss function matrix, and propose a calculation method for the degree of the object belonging to the two states in the relative loss function matrix. Finally, the corresponding three-way decision rules are obtained by the minimum risk decision theory. In Section 4, we compare the algorithm proposed in this paper with the existing algorithms and verify the feasibility and effectiveness of the algorithm in classification and sorting. Finally, Section 5 summarizes the core content of the proposed model and looks forward to future work.

2. Preliminaries

In this section, we briefly review the linear regression model, the three-way decision model, regret theory, and some definitions related to incomplete ordered information systems.

2.1. Linear regression model

Definition 1. [22] Suppose X and Y are two random variables, σ_X and σ_Y denote the standard deviations of X and Y, respectively. σ_{XY} denotes the covariance of X and Y, the Pearson correlation coefficient is defined as:

$$r(X,Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}. \tag{1}$$

If the sample data for X and Y are $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$, then the Pearson correlation coefficient is:

$$r(X,Y) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}, \tag{2}$$

where $\bar{x}$ and $\bar{y}$ represent the mean values of X and Y, respectively.

Definition 2. [22] Suppose the value of the dependent variable Y is different. For a specific observed value, the magnitude of the variation can be represented by $(Y - \bar{Y})$ the difference between the actual observed value and its mean. The total variation of n observations can be expressed by the sum of squares of these deviations, which is called the sum of squares of total variation, that is, $SST = \Sigma(Y - \bar{Y})^2$.

Definition 3. [22] The estimated unary regression equation is $\hat{Y} = b_0 + b_1 X$, and $\Sigma(\hat{Y} - \bar{Y})^2$ is the sum of squares of the deviation between the regression value $\hat{Y}$ and the mean value $\bar{Y}$, which can be regarded as the part of the change caused by the linear relationship between X and Y in the total variation of Y, which can be explained by the regression line, so it is called the sum of squares of regression, which is recorded as SSR.

Definition 4. [22] The ratio of SSR to SST is used to reflect the degree of fitting of the straight line to the observed value. This ratio is called the coefficient of determination and is recorded as R^2, that is,

$$R^2 = \frac{SSR}{SST} = \frac{\Sigma(\hat{Y} - \bar{Y})^2}{\Sigma(Y - \bar{Y})^2} = 1 - \frac{\Sigma(Y - \hat{Y})^2}{\Sigma(Y - \bar{Y})^2}. \tag{3}$$

The value range of the coefficient of determination R^2 is $[0, 1]$. When $R^2 = 1$, the fitting is complete, that is, all the observations are on the straight line. If X has nothing to do with Y, X cannot explain the variation of Y, In this case $\hat{Y} = \bar{Y}$, then $R^2 = 0$. Therefore, the closer R^2 is to 1, the greater the proportion of the sum of squares of regression to the sum of squares of total variation, the closer the regression line is to each observation point, and the better the fitting degree of the regression line is. On the contrary, the closer R^2 is to 0, the worse the fitting degree of the regression line is.

For n samples, the data for dependent variables and p independent variables are collected, respectively. The dependent variables are represented by Y, and the independent variables are represented by X_1, X_2, $\cdots$, X_p. The arrangement of sample data is shown in Table 1.

In Table 1, x_{ij} is the value of the S_i sample under the variable X_j. When $p = 1$, the regression is called unary regression, when $p > 1$, the regression is called multiple regression.

In multiple regression analysis, the most commonly used general form of the multiple linear regression model is:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p + \varepsilon, \tag{4}$$

Table 1
Data structure of multivariate variables.

Sample	Variables and data					
	Y	X_1	X_2	X_3	$\cdots$	X_p
S_1	y_1	x_{11}	x_{12}	x_{13}	$\cdots$	x_{1p}
S_2	y_2	x_{21}	x_{22}	x_{23}	$\cdots$	x_{2p}
S_3	y_3	x_{31}	x_{32}	x_{33}	$\cdots$	x_{3p}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_n	y_n	x_{n1}	x_{n2}	x_{n3}	$\cdots$	x_{np}

where β_0, β_1, β_2, $\cdots$, β_p are $p+1$ unknown parameters, β_0 is a constant term of the regression equation, β_1, β_2, $\cdots$, β_p are called partial regression coefficients, and ε is a random error term [22]. For the random error term, we assume that:

$$\begin{cases} E(\varepsilon)=0, \\ Var(\varepsilon)=\sigma^2. \end{cases} \tag{5}$$

Therefore,

$$Y=\beta_0+\beta_1X_1+\beta_2X_2+\cdots++\beta_pX_p. \tag{6}$$

This formula is called the theoretical regression equation. By using the least squares method, a set of estimated parameters $\hat{\beta}=(\hat{\beta}_0,\hat{\beta}_1,\hat{\beta}_2,\cdots,\hat{\beta}_p)$ can be found, so the estimated multiple regression model can be obtained [22].

$$\hat{Y}=\hat{\beta}_0+\hat{\beta}_1X_1+\hat{\beta}_2X_2+\cdots+\hat{\beta}_pX_p. \tag{7}$$

2.2. Three-way decision with decision-theoretic rough sets

Three-way decision with decision-theoretic rough sets [36] was proposed by Yao. Let $(U,\mathcal{R})$ be an approximation space, where $U=\{x_1,x_2,\cdots,x_n\}$ is a non-empty set and $\mathcal{R}$ is an equivalence relation on U. $\mathbb{S}=\{\mathbb{D},\neg\mathbb{D}\}$ is a set of two states. Let $\mathbb{A}=\{a_{\mathbb{P}},a_{\mathbb{B}},a_{\mathbb{N}}\}$ be a set of three behaviors, where $a_{\mathbb{P}}$, $a_{\mathbb{B}}$, $a_{\mathbb{N}}$ express acceptance, delay and rejection, respectively. $\lambda_{\mathbb{P}P}$, $\lambda_{\mathbb{B}P}$, $\lambda_{\mathbb{N}P}$ are loss function values corresponding to behaviors $a_{\mathbb{P}}$, $a_{\mathbb{B}}$, $a_{\mathbb{N}}$ in the state $\mathbb{D}$, $\lambda_{\mathbb{P}N}$, $\lambda_{\mathbb{B}N}$, $\lambda_{\mathbb{N}N}$ are loss function values corresponding to actions $a_{\mathbb{P}}$, $a_{\mathbb{B}}$, $a_{\mathbb{N}}$ in the state $\neg\mathbb{D}$. $P(\mathbb{D}|[x]_{\mathcal{R}})=\frac{|\mathbb{D}\cap[x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}$, $P(\mathbb{D}|[x]_{\mathcal{R}})+P(\neg\mathbb{D}|[x]_{\mathcal{R}})=1$.

For the object x described as $[x]_{\mathcal{R}}$, the expected losses from different behaviors are as follows:
$EL(a_{\mathbb{P}}|[x]_{\mathcal{R}})=\lambda_{\mathbb{P}P}P(\mathbb{D}|[x]_{\mathcal{R}})+\lambda_{\mathbb{P}N}P(\neg\mathbb{D}|[x]_{\mathcal{R}})$,
$EL(a_{\mathbb{B}}|[x]_{\mathcal{R}})=\lambda_{\mathbb{B}P}P(\mathbb{D}|[x]_{\mathcal{R}})+\lambda_{\mathbb{B}N}P(\neg\mathbb{D}|[x]_{\mathcal{R}})$,
$EL(a_{\mathbb{N}}|[x]_{\mathcal{R}})=\lambda_{\mathbb{N}P}P(\mathbb{D}|[x]_{\mathcal{R}})+\lambda_{\mathbb{N}N}P(\neg\mathbb{D}|[x]_{\mathcal{R}})$.
The minimum risk decision rules for expected losses are as follows:
(**P**) If $EL(a_{\mathbb{P}}|[x]_{\mathcal{R}})\le EL(a_{\mathbb{B}}|[x]_{\mathcal{R}})$ and $EL(a_{\mathbb{P}}|[x]_{\mathcal{R}})\le EL(a_{\mathbb{N}}|[x]_{\mathcal{R}})$, $x\in POS(\mathbb{D})$,
(**B**) If $EL(a_{\mathbb{B}}|[x]_{\mathcal{R}})\le EL(a_{\mathbb{P}}|[x]_{\mathcal{R}})$ and $EL(a_{\mathbb{B}}|[x]_{\mathcal{R}})\le EL(a_{\mathbb{N}}|[x]_{\mathcal{R}})$, $x\in BND(\mathbb{D})$,
(**N**) If $EL(a_{\mathbb{N}}|[x]_{\mathcal{R}})\le EL(a_{\mathbb{P}}|[x]_{\mathcal{R}})$ and $EL(a_{\mathbb{N}}|[x]_{\mathcal{R}})\le EL(a_{\mathbb{B}}|[x]_{\mathcal{R}})$, $x\in NEG(\mathbb{D})$.
In general, loss function values satisfy the following conditions:

$$\lambda_{\mathbb{P}P}\le\lambda_{\mathbb{B}P}<\lambda_{\mathbb{N}P},\lambda_{\mathbb{N}N}\le\lambda_{\mathbb{B}N}<\lambda_{\mathbb{P}N}.$$

The decision rules **P** − **N** can be simplified as follows:
(**P**$_1$) If $P(\mathbb{D}|[x]_{\mathcal{R}})\ge\alpha$ and $P(\mathbb{D}|[x]_{\mathcal{R}})\ge\gamma$, $x\in POS(\mathbb{D})$,
(**B**$_1$) If $P(\mathbb{D}|[x]_{\mathcal{R}})\le\alpha$ and $P(\mathbb{D}|[x]_{\mathcal{R}})\ge\beta$, $x\in BND(\mathbb{D})$,
(**N**$_1$) If $P(\mathbb{D}|[x]_{\mathcal{R}})\le\beta$ and $P(\mathbb{D}|[x]_{\mathcal{R}})\le\gamma$, $x\in NEG(\mathbb{D})$,
where $\alpha=\frac{\lambda_{\mathbb{P}N}-\lambda_{\mathbb{B}N}}{(\lambda_{\mathbb{P}N}-\lambda_{\mathbb{B}N})+(\lambda_{\mathbb{B}P}-\lambda_{\mathbb{P}P})}$,
$\beta=\frac{\lambda_{\mathbb{B}N}-\lambda_{\mathbb{N}N}}{(\lambda_{\mathbb{B}N}-\lambda_{\mathbb{N}N})+(\lambda_{\mathbb{N}P}-\lambda_{\mathbb{B}P})}$,
$\gamma=\frac{\lambda_{\mathbb{P}N}-\lambda_{\mathbb{N}N}}{(\lambda_{\mathbb{P}N}-\lambda_{\mathbb{N}N})+(\lambda_{\mathbb{N}P}-\lambda_{\mathbb{P}P})}$.
When the loss function values also satisfy $(\lambda_{\mathbb{B}P}-\lambda_{\mathbb{P}P})(\lambda_{\mathbb{B}N}-\lambda_{\mathbb{N}N})\le(\lambda_{\mathbb{P}N}-\lambda_{\mathbb{B}N})(\lambda_{\mathbb{N}P}-\lambda_{\mathbb{B}P})$, the decision rules **P**$_1$ − **N**$_1$ can be simplified as follows:
(**P**$_2$) If $P(\mathbb{D}|[x]_{\mathcal{R}})\ge\alpha$, $x\in POS(\mathbb{D})$,
(**B**$_2$) If $\beta\le P(\mathbb{D}|[x]_{\mathcal{R}})\le\alpha$, $x\in BND(\mathbb{D})$,
(**N**$_2$) If $P(\mathbb{D}|[x]_{\mathcal{R}})\le\beta$, $x\in NEG(\mathbb{D})$.

2.3. Regret theory

The regret theory, proposed by Bell [1], posits that during the decision-making process, the decision-maker will compare the outcomes of their chosen option with those of other potential alternatives. If decision-makers find that choosing another option can

Table 2
An incomplete ordered information system.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
x_1	0.5650	0.4400	0.1550	0.9395	0.4275	0.2140	$*$
x_2	0.5900	0.4450	0.1400	0.9310	0.3560	0.2340	0.2800
x_3	0.5650	0.4400	0.1600	0.9150	0.3540	0.1935	0.3200
x_4	0.6200	0.5100	0.1750	1.6150	0.5105	0.1920	0.6750
x_5	0.5700	0.4650	0.1800	1.2950	0.3390	0.2225	0.4400
x_6	0.5750	0.4500	0.1600	0.9775	0.3135	0.2310	0.3300
x_7	0.5550	0.4400	0.1500	0.7550	0.3070	0.1525	0.2600
x_8	0.6050	0.4700	0.1600	1.1735	0.4975	0.2405	0.3450
x_9	0.5200	$*$	0.1550	0.7270	0.2910	0.1835	0.2350

get better results, they will regret it. On the contrary, they will rejoice. Therefore, decision-makers should predict the regret or joy of options in the decision-making process and try to avoid choices that will make them feel regret. Let z_1 and z_2 be the outcomes obtained after selecting choices x_1 and x_2, respectively. The decision-maker's perceived utility for the choice x_1 is

$$v_1 = \mu(x_1) + RE(\Delta\mu), \tag{8}$$

where $\mu(x_i)$ denotes the utility function when taking the option x_i, $RE(\Delta\mu)$ represents the regret value generated by replacing the alternative x_2 with the alternative x_1, $\Delta\mu$ is the utility difference between x_1 and x_2.

$$\mu(x_i) = \frac{1-\exp(-\theta z_i)}{\theta}, \tag{9}$$

$$RE(\Delta\mu) = 1 - \exp(-\sigma \bullet \Delta\mu), \tag{10}$$

where $0 < \theta < 1$ denotes utility preference, $\sigma \in (0, +\infty)$ is the risk preference coefficient.

2.4. Incomplete ordered information systems

Definition 5. [11] An information system can be denoted by a 4-tuple $IS = (U, A, f, V)$, where $U = \{x_1, x_2, \cdots, x_n\}$ is a set containing n objects, and $A = \{a_1, a_2, \cdots, a_m\}$ is a set containing m attributes. $V = \cup_{a_j \in A} V_{a_j}$, where V_{a_j} is the value domain of attribute a_j. $f : U \times A \to V$ is a mapping, that is $f(x_i, a_j) = u_{ij} \in V, \forall x_i \in U, a_j \in A$. If there are some evaluation values in the information system that are vacant, these unknown values are represented as $*$. In other words, $\exists x \in U,\ a \in A$, such that $f(x, a) = *$. An information system with these missing values is called an incomplete information system, which is usually expressed as $IIS = (U, A, f, V \cup \{*\})$.

Definition 6. [7] Let $IS = (U, A, f, V)$ be an information system. The universe of discourse $U = \{x_1, x_2, \cdots, x_n\}$ contains n objects. $A = \{A^+ \cup A^-\} = \{a_1, a_2, \cdots, a_m\}(A^+ \cap A^- = \emptyset)$ is a set of evaluation attributes of objects in U, where A^+ and A^- are benefit attribute set and cost attribute set, respectively. Assuming that there are k benefit attributes and $m - k$ cost attributes, then $A^+ = \{a_1, a_2, \cdots, a_k\}$ is the benefit attribute set, and $A^- = \{a_{k+1}, a_{k+2}, \cdots, a_m\}$ is the cost attribute set. The benefit attribute has a monotonically increasing utility, that is, the larger the attribute value, the greater the utility of the attribute; the cost attribute has a monotonically decreasing utility, that is, the larger the attribute value, the smaller the utility of the attribute. $\forall x_i \in U,\ a_j \in A$, then the value of object x_i under attribute a_j is expressed as u_{ij} or $f(x_i, a_j)$, and all values constitute an evaluation matrix $\mathcal{X}_{n\times m}$. An information system with the above characteristics is called a ordered information system. When a ordered information system contains missing values, the information system is called an incomplete ordered information system $IOIS = (U, A, f, V \cup \{*\})$.

Example 1. Given an incomplete ordered information system $IOIS = (U, A, f, V \cup \{*\})$, $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$, $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$, $f(x_1, a_7) = *$, $f(x_9, a_2) = *$, $S^+ = \{0.62, 0.51, 0.18, 1.615, 0.5105, 0.2405, 0.675\}$, $S^- = \{0.52, 0.44, 0.135, 0.727, 0.291, 0.1495, 0.235\}$. As shown in Table 2.

Definition 7. [20] Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system, and $B \subseteq A$, then the dominance relation with respect to the attribute subset B is defined as follows:

$$DR_B = \{(x, y) \in U \times U \mid f(x, a) \geq f(y, a) \vee f(x, a) = * \vee f(y, a) = *, \forall a \in B\}. \tag{11}$$

Based on the above definition, the dominance relation between x and y is established when the value of x under each attribute is equal to or greater than the value of y under the corresponding attribute, thus satisfying the dominance requirement. When the attribute set contains more attributes, the dominance relation is too strict and loses more information. Therefore, some scholars have proposed the probabilistic dominance relation.

Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system, $\forall a_j \in A$, $V_{a_j} = \{v_1, v_2, \cdots, v_{k_j}\}$ is the value domain of attribute a_j and the elements in V_{a_j} are arranged in order of size, $v_1 < v_2 < \cdots < v_{k_j}$. $P(v_{k_j}) = P_{k_j}$ denotes the frequency of attribute value v_{k_j} in value domain V_{a_j}.

Definition 8. [27] Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system, $\forall x_i, x_l \in U$, $a_j \in A$, $V_{a_j} = \{v_1, v_2, \cdots, v_{k_j}\}$, $v_t \in V_{a_j}$, then the probability that object x_i is superior to object x_l under attribute a_j is defined as follows:

$$UR_{a_j}(x_i, x_l) = \begin{cases} 1, & u_{ij} \geq u_{lj} \wedge u_{ij} \neq * \wedge u_{lj} \neq *, \\ 0, & u_{ij} < u_{lj} \wedge u_{ij} \neq * \wedge u_{lj} \neq *, \\ \sum_{h=1}^{k_j} P_h^2, & u_{ij} = * \wedge u_{lj} = *, \\ \sum_{1 \leq h \leq t} P_h, & u_{ij} = v_t \wedge u_{lj} = *, \\ \sum_{t \leq h \leq k_j} P_h, & u_{ij} = * \wedge u_{lj} = v_t. \end{cases} \tag{12}$$

Definition 9. [27] Given an incomplete ordered information system $IOIS = (U, A, f, V \cup \{*\})$, for all $B \subseteq A$, $\forall x, y \in U$, the weighted probability dominance degree with respect to B is provided below:

$$UR_B(x, y) = \sum_{a_i \in B} w_i UR_{a_i}(x, y), \tag{13}$$

where w_i is the weight of attribute a_i.

Definition 10. [27] Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system. The probabilistic dominance relation with respect to the attribute subset $B \subseteq A$ is as follows:

$$UR_B^{\geq \theta} = \{(x, y) \in U \times U | UR_B(x, y) \geq \theta\}, \tag{14}$$

where θ is called a threshold of the probability dominance relation and meets $0 \leq \theta \leq 1$.

Definition 11. [29] Given an incomplete ordered information system $IOIS = (U, A, f, V \cup \{*\})$. For any $x_i, x_l \in U$, $a_j \in A$ and $v_t \in V_{a_j} (t \in \{1, 2, \cdots, k_j\})$, the probability that the object x_i is worse than the object x_l is:

$$LR_{a_j}(x_i, x_l) = \begin{cases} \frac{u_{lj} - u_{ij}}{u_{lj}}, & u_{ij} < u_{lj} \wedge u_{ij} \neq * \wedge u_{lj} \neq *, \\ 0, & u_{ij} \geq u_{lj} \wedge u_{ij} \neq * \wedge u_{lj} \neq *, \\ \sum_{h=1}^{k_j} P_h^2, & u_{ij} = * \wedge u_{lj} = *, \\ \sum_{1 \leq h \leq t} P_h, & u_{ij} = * \wedge u_{lj} = v_t, \\ \sum_{t \leq h \leq k_j} P_h, & u_{ij} = v_t \wedge u_{lj} = * . \end{cases} \tag{15}$$

Definition 12. [29] Given an incomplete ordered information system $IOIS = (U, A, f, V \cup \{*\})$, for all $B \subseteq A$, $\forall x, y \in U$, the priori probability tolerance dominance degree with respect to the attribute set B is represented below:

$$LR_B(x, y) = \sum_{a_j \in B} LR_{a_j}(x, y) / |B|, \tag{16}$$

where $|\cdot|$ denotes the number of attribute sets.

Afterwards, the priori probability tolerance dominance relation of $IOIS$ under the attribute set B is represented as:

$$LR_B^{\beta} = \{(x, y) \in U \times U | LR_B(x, y) \leq \beta\}, \tag{17}$$

where β is called a tolerance rate for the attribute set B, which satisfies $0 \leq \beta \leq 1$.

3. Three-way decision model based on a novel dominance relation

This section proposes a method for filling in missing values in incomplete information systems. A new type of dominance relation is constructed based on filling values with confidence. The relative loss function and the degree of membership of an object to different states are defined in accordance with the tenets of TOPSIS and regret theory. Ultimately, a three-way decision model is proposed, situated within the framework of minimum risk decision theory.

3.1. A filling method for missing values

Definition 13. Let $IIS = (U, A, f, V \cup \{*\})$ be an incomplete information system. $\forall x, y \in U$, the degree of similarity $S(x, y)$ between x and y is defined as follows:

$$S(x,y) = 1 - \frac{D(x,y)}{M_D}, \tag{18}$$

where $M_D = \max\{D(x,u) | u \in U\}$,

$$D(x,u) = \sqrt{\sum_{a \in B_{(x,u)}} \left(\frac{f(x,a) - f(u,a)}{Z(a)}\right)^2}, \tag{19}$$

where $B_{(x,u)} = \{a \in A | f(x,a) \neq * \wedge f(u,a) \neq *\}$, $Z(a) = Z_a^+ - Z_a^-$, $Z_a^+ = \max\{f(x,a) | x \in U\}$, $Z_a^- = \min\{f(x,a) | x \in U\}$.

Definition 14. Let $IIS = (U, A, f, V \cup \{*\})$ be an incomplete information system. Suppose $f(x_i, a_j) = *$, $x_i \in U$, $a_j \in A$. The filling value of $f(x_i, a_j)$ is as follows:

$$f_S(x_i, a_j) = \left\{(\hat{Y}_*, T_{R^2}), (\bar{V}_{ij}, T_{\bar{S}_{ij}})\right\}, \tag{20}$$

$$T_{R^2} = \frac{\alpha R^2}{\alpha R^2 + \beta \bar{S}_{ij}}, \tag{21}$$

$$T_{\bar{S}_{ij}} = \frac{\beta \bar{S}_{ij}}{\alpha R^2 + \beta \bar{S}_{ij}}, \tag{22}$$

where $\hat{Y}_*$ and $\bar{V}_{ij}$ are the filling values based on the linear correlation between the attribute columns and the similarity between the object rows, respectively. $T_{R^2} \in [0,1]$, $T_{\bar{S}_{ij}} \in [0,1]$ are called the confidence of the corresponding filling values. α is the column preference coefficient and β is the row preference coefficient, $\alpha, \beta \in [0,1]$, $\alpha + \beta = 1$.

$$\hat{Y}_* = \hat{\beta}_0 + \hat{\beta}_1 f(x_i, a_{j_1}) + \cdots + \hat{\beta}_l f(x_i, a_{j_l}), \tag{23}$$

where $a_{j_l} \in A_{\lambda_1} = \{a \in A | r(a_j, a) \geq \lambda_1 \wedge f(x_i, a) \neq *\}$, A_{λ_1} is a set of attributes whose Pearson correlation coefficient with the attribute column a_j is greater than or equal to the threshold λ_1 and $f(x_i, a) \neq *$, R^2 is the coefficient of determination.

$$\bar{V}_{ij} = w_1 f(x_{i_1}, a_j) + w_2 f(x_{i_2}, a_j) + \cdots + w_t f(x_{i_t}, a_j), \tag{24}$$

$$\bar{S}_{ij} = \sum_{x_{i_t} \in X_{\lambda_2}} w_t S(x_i, x_{i_t}), \tag{25}$$

where $w_t = S(x_i, x_{i_t}) / \sum_{x \in X_{\lambda_2}} S(x_i, x)$, $x_{i_t} \in X_{\lambda_2} = \{x \in U | S(x_i, x) \geq \lambda_2 \wedge f(x, a_j) \neq *\}$.

The filling value $f_S(x_i, a_j) = \left\{(\hat{Y}_*, T_{R^2}), (\bar{V}_{ij}, T_{\bar{S}_{ij}})\right\}$ will not necessarily exist in both $\hat{Y}_*$ and $\bar{V}_{ij}$ because of the size of the thresholds λ_1 and λ_2. There will be $f_S(x_i, a_j) = \{(\hat{Y}_*, R^2)\}$ and $f_S(x_i, a_j) = \{(\bar{V}_{ij}, \bar{S}_{ij})\}$. When both $\hat{Y}_*$ and $\bar{V}_{ij}$ cannot be obtained, the thresholds λ_1 and λ_2 are reduced.

In Definition 14, $f_S(x_i, a_j) = \left\{(\hat{Y}_*, T_{R^2}), (\bar{V}_{ij}, T_{\bar{S}_{ij}})\right\}$, where $\hat{Y}_*$ and $\bar{V}_{ij}$ are filling values based on the linear correlation between attributes and the similarity between objects, respectively. Below we explain the rationality of the method.

The Pearson correlation coefficient measures the linear correlation between two sets of random variables. Therefore, we select the attribute columns ($A_{\lambda_1} = \{a \in A | r(a_j, a) \geq \lambda_1 \wedge f(x_i, a) \neq *\}$) that have a high Pearson correlation coefficient with the attribute column containing missing values. Then, we fit a linear regression equation (Formula (23)) using the known data. Finally, this paper predicts the missing values based on the linear regression equation and obtains the coefficient of determination for the regression equation.

In Definition 13, this paper defines the similarity between objects based on Euclidean distance. When the differences between the values of the two objects on each attribute are smaller, the similarity between the two objects is greater. Therefore, in Definition 14, this paper selects complete objects ($X_{\lambda_2} = \{x \in U | S(x_i, x) \geq \lambda_2 \wedge f(x, a_j) \neq *\}$) that are more similar to the object containing missing values, estimates the missing values by taking the weighted average of their attribute values (Formula (24)), and calculates the confidence of the filling value based on their corresponding similarity.

Let $IIS = (U, A, V \cup \{*\}, f)$ be an incomplete information system. Suppose $f(x_i, a_j) = *$, $x_i \in U$, $a_j \in A$. The steps of filling missing values in incomplete information system of continuous data are as follows:

Step1: Calculate the Pearson correlation coefficient between the attribute a_j column and other attribute columns, and get the set $A_{\lambda_1} = \{a \in A | r(a_j, a) \geq \lambda_1 \wedge f(x_i, a) \neq *\}$.

Step2: The filling value $\hat{Y}_*$ and the coefficient of determination R^2 are obtained from the fitted multiple linear regression model.

Step3: The similarity $S(x_i, y)(y \in U - \{x_i\})$ between x_i and other objects is calculated according to Formula (18).

Step4: The filling values $\bar{V}_{ij}$ and $\bar{S}_{ij}$ are calculated from Formulas (24) and (25).

Step5: Combine with the results of steps 2 and 4, the filling value $f_S(x_i, a_j) = \{(\hat{Y}_*, T_{R^2}), (\bar{V}_{ij}, T_{\bar{S}_{ij}})\}$ is obtained.

Step6: If the missing value is not supplied, the thresholds λ_1 and λ_2 are reduced, and steps 1 and 4 are repeated.

Algorithm 1 The filling algorithm of missing values in incomplete information systems of continuous data.

```
input An incomplete information system IIS = (U, A, V ∪ {*}, f), where the data are continuous. f(x_i, a_j) = *, x_i ∈ U, a_j ∈ A.
output The filling value f_S(x_i, a_j) of the missing value f(x_i, a_j).
A_{λ1} = ∅
for every a in A − {a} do
    Calculate r(a_j, a) = σ_{a_j a} / (σ_{a_j} σ_a)
    if r(a_j, a) ≥ λ1 and f(x_i, a) ≠ * then
        A_{λ1} = A_{λ1} ∪ {a}
    end
end
Divide independent variables A_{λ1} and dependent variable {a_j}.
Calculate (β̂_0, β̂_1, ⋯, β̂_l) and 1 − Σ(Y−Ŷ)² / Σ(Y−Ȳ)²  // The parameters in the multivariate linear regression model and the coefficient of determination R².
Ŷ_* = β̂_0
for a_j in A_{λ1} do
    Ŷ_* = Ŷ_* + β̂_j f(x_i, a_j)
end
X_{λ2} = ∅
for every x in U − {x_i} do
    Calculate S(x, x_i)
    if S(x_i, x) ≥ λ2 and f(x, a_j) ≠ * then
        X_{λ2} = X_{λ2} ∪ {x}
    end
end
V̄_ij = 0, S̄_ij = 0
for every x_t in X_{λ2} do
    Calculate: V̄_ij = V̄_ij + w_t f(x_t, a_j) and S̄_ij = S̄_ij + w_t S(x_i, x_t)
end
Calculate T_{R²} = αR² / (αR² + βS̄_ij) and T_{S̄_ij} = βS̄_ij / (αR² + βS̄_ij)
return f_S(x_i, a_j) = {(Ŷ_*, R²), (V̄_ij, S̄_ij)}
```

3.2. A new dominance relation in incomplete ordered information systems

Definitions 8 and 11 only give the probability that object x is superior or inferior to object y under attribute a_j, and does not point out the degree that object x is superior or inferior to object y under attribute a_j. In this paper, a new dominance relation between objects in incomplete ordered information system is defined by principles of TOPSIS and regret theory. This relation offers a sophisticated analysis that not only identifies the advantages and disadvantages between two distinct objects, but also meticulously quantifies the degree of these disparities.

Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system. Assuming that the attribute set $A = \{a_1, a_2, \cdots, a_m\}$ contains only the benefit attribute and the cost attribute, we can find a positive ideal solution $S^+ = (u_1^+, u_2^+, \cdots, u_m^+)$ and a negative ideal solution $S^- = (u_1^-, u_2^-, \cdots, u_m^-)$ under this incomplete ordered information system. According to the filling method of incomplete information systems proposed in Definition 14, we can get a complete ordered information system with confidence $COIS = (\tilde{U}, A, f, \tilde{V}, T)$.

Example 2. According to the method in Definition 14 and the data in Example 1, we can obtain a complete ordered information system with confidence $COIS = (\tilde{U}, A, f, \tilde{V}, T)$ as shown in Table 3. $f(x_1, a_7) = *$, let $\lambda_1 = 0.7$, we can get $A_{\lambda_1} = \{a_2, a_3, a_4\}$, then construct the multivariate linear regression model $\hat{Y}_* = 6 \times f(x_1, a_2) + 1 \times f(x_1, a_3) + 0 \times f(x_1, a_4) - 2.53 = 0.265$, the coefficient of determination $R^2 = 0.8163$. Let $\lambda_2 = 0.6$, we can obtain $S(x_1, x_2) = 0.604$, $S(x_1, x_3) = 0.72$, $S(x_1, x_6) = 0.61$, $A_{\lambda_2} = \{x_2, x_3, x_6\}$. $\bar{V}_{ij} = 0.3123 \times f(x_2, a_7) + 0.3723 \times f(x_3, a_7) + 0.3154 \times f(x_6, a_7) = 0.3107$, $\bar{S}_{ij} = 0.3123 \times 0.604 + 0.3723 \times 0.72 + 0.3154 \times 0.61 = 0.649$. Let $\alpha = 0.485$, $\beta = 0.515$, $f_S(x_1, a_7) = \{(0.265, 0.5422), (0.3107, 0.4578)\}$. Similarly, we can calculate $f_S(x_9, a_2) = \{(0.4392, 0.3925), (0.44, 0.6075)\}$.

Definition 15. Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system, positive ideal solution $S^+ = (u_1^+, u_2^+, \cdots, u_m^+)$, $COIS = (\tilde{U}, A, f, \tilde{V}, T)$ is a complete ordered information system with confidence corresponding to $IOIS$. $\forall x_i \in U$, the distance between x_i and positive ideal solution S^+ is calculated as follows:

Table 3
A complete ordered information system with confidence.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	T
x_1^1	0.5650	0.4400	0.1550	0.9395	0.4275	0.2140	0.2650	0.5422
x_1^2	0.5650	0.4400	0.1550	0.9395	0.4275	0.2140	0.3107	0.4578
x_2	0.5900	0.4450	0.1400	0.9310	0.3560	0.2340	0.2800	1
x_3	0.5650	0.4400	0.1600	0.9150	0.3540	0.1935	0.3200	1
x_4	0.6200	0.5100	0.1750	1.6150	0.5105	0.1920	0.6750	1
x_5	0.5700	0.4650	0.1800	1.2950	0.3390	0.2225	0.4400	1
x_6	0.5750	0.4500	0.1600	0.9775	0.3135	0.2310	0.3300	1
x_7	0.5550	0.4400	0.1500	0.7550	0.3070	0.1525	0.2600	1
x_8	0.6050	0.4700	0.1600	1.1735	0.4975	0.2405	0.3450	1
x_9^1	0.5200	0.4392	0.1550	0.7270	0.2910	0.1835	0.2350	0.3925
x_9^2	0.5200	0.4400	0.1550	0.7270	0.2910	0.1835	0.2350	0.6075

Table 4
Dominance relation r^+.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_1	1	(+)0.0537	(+)0.0329	(−)0.6161	(−)0.2132	(−)0.0170	(+)0.1672	(−)0.2741	(+)0.1739
x_2	(−)0.0537	1	(−)0.0214	(−)0.6367	(−)0.2554	(−)0.0698	(+)0.1200	(−)0.3130	(+)0.1270
x_3	(−)0.0329	(+)0.0214	1	(−)0.6287	(−)0.2391	(−)0.0494	(+)0.1388	(−)0.2980	(+)0.1457
x_4	(+)0.6161	(+)0.6367	(+)0.6287	1	(+)0.5121	(+)0.6094	(+)0.6803	(+)0.4712	(+)0.6828
x_5	(+)0.2132	(+)0.2554	(+)0.2391	(−)0.5121	1	(+)0.1996	(+)0.3447	(−)0.0773	(+)0.3500
x_6	(+)0.0170	(+)0.0698	(+)0.0494	(−)0.6094	(−)0.1996	1	(+)0.1813	(−)0.2615	(+)0.1879
x_7	(−)0.1672	(−)0.1200	(−)0.1388	(−)0.6803	(−)0.3447	(−)0.1813	1	(−)0.3954	(+)0.0080
x_8	(+)0.2741	(+)0.3130	(+)0.2980	(−)0.4712	(+)0.0773	(+)0.2615	(+)0.3954	1	(+)0.4003
x_9	(−)0.1739	(−)0.1270	(−)0.1457	(−)0.6828	(−)0.3500	(−)0.1879	(−)0.0080	(−)0.4003	1

$$d_i^+ = \begin{cases} (\sum_{j=1}^{m}((u_{ij} - u_j^+)/z_j)^2)^{1/2}, \forall a \in A, f(x_i, a) \neq *, \\ \sum_{l_i=1}^{n_i}(\sum_{j=1}^{m}((u_{ij}^{l_i} - u_j^+)/z_j)^2)^{1/2} T(x_i^{l_i}), \exists a \in A, f(x_i, a) = *, \end{cases} \tag{26}$$

where $z_j = max(\tilde{V}_{a_j}) - min(\tilde{V}_{a_j})$, $x_i^{l_i} \in \{x_i^1, x_i^2, \cdots, x_i^{n_i}\} \subseteq \tilde{U}$, $n_i = \prod_{a \in A_{x_i}^*} |f_S(x_i, a)|$, $A_{x_i}^* = \{a \in A | f(x_i, a) = *\}$. When $f_S(x_i, a_j) = \{(\hat{Y}_*, R^2)\}$ or $f_S(x_i, a_j) = \{(\bar{V}_{ij}, \bar{S}_{ij})\}$, in order to reduce the error of d_i^+, we set $R^2 = 1$, $\bar{S}_{ij} = 1$.

Definition 16. Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system, and $COIS = (\tilde{U}, A, f, \tilde{V}, T)$ be a complete ordered information system with confidence corresponding to $IOIS$. $\forall x_i, x_j \in U$ the dominance relation between x_i and x_j relative to PIS is defined as follows:

$$r^+(x_i, x_j) = \begin{cases} (+)\overline{D_{ij}}, RE(d_i^+) < RE(d_j^+), \\ 1, RE(d_i^+) = RE(d_j^+), \\ (-)\underline{D_{ij}}, RE(d_i^+) > RE(d_j^+). \end{cases} \tag{27}$$

$$\overline{D_{ij}} = \frac{|RE(d_i^+) - RE(d_j^+)|}{RE(d_j^+)}, \tag{28}$$

$$\underline{D_{ij}} = \frac{|RE(d_i^+) - RE(d_j^+)|}{RE(d_i^+)}, \tag{29}$$

where (+) denotes that object x_i is better than object x_j under attribute set A, (−) denotes that object x_i is worse than object x_j under attribute set A, and 1 denotes that object x_i is equivalent to object x_j under attribute set A. $\overline{D_{ij}}$ denotes the degree of x_i better than x_j, $\underline{D_{ij}}$ denotes the degree of x_i worse than x_j. $r^+(x_i, x_j) = 1$ can also be expressed by $r^+(x_i, x_j) = (+)0$ or $r^+(x_i, x_j) = (-)0$, that is, the degree of x_i is better or worse than x_j is 0. $RE(d_i^+)$ is the regret value generated by replacing PIS with x_i.

Example 3. According to the data in Example 2 and Definition 16, we can get the dominance relation r^+ on U, as shown in Table 4.

Property 1. Given an incomplete ordered information system $IOIS = (U, A, f, V \cup \{*\})$, then the dominance relation r^+ in $IOIS$ has following properties:

(1) Reflexivity, $\forall x_i \in U$, $r^+(x_i,x_i)=1$,

(2) Symmetry, $\forall x_i,x_j \in U$, $\overline{D_{ij}}=\underline{D_{ji}}$ or $\underline{D_{ij}}=\overline{D_{ji}}$,

(3) Transitivity, $\forall x_i,x_j,x_k \in U$, if x_i is better than x_j and x_j is better than x_k, then $\overline{D_{ik}} \geq \overline{D_{ij}} \wedge \overline{D_{jk}}$. If x_i is worse than x_j and x_j is worse than x_k, then $\underline{D_{ik}} \geq \underline{D_{ij}} \wedge \underline{D_{jk}}$.

Proof. (1) When $RE(d_i^+)=RE(d_j^+)$, we have $r^+(x_i,x_j)=1$, so $r^+(x_i,x_i)=1$.
(2) When $RE(d_i^+)<RE(d_j^+)$, we have $\overline{D_{ij}}=\underline{D_{ji}}=\frac{|RE(d_i^+)-RE(d_j^+)|}{RE(d_j^+)}$. When $RE(d_i^+)=RE(d_j^+)$, we have $\overline{D_{ij}}=\underline{D_{ji}}=\underline{D_{ij}}=\overline{D_{ji}}=0$. When $RE(d_i^+)>RE(d_j^+)$, we have $\underline{D_{ij}}=\overline{D_{ji}}=\frac{|RE(d_i^+)-RE(d_j^+)|}{RE(d_i^+)}$. In summary, $\forall x_i,x_j \in U$, $\overline{D_{ij}}=\underline{D_{ji}}$ or $\underline{D_{ij}}=\overline{D_{ji}}$.
(3) $\forall x_i,x_j,x_k \in U$, when $r^+(x_i,x_j)=(+)\overline{D_{ij}}$, $r^+(x_j,x_k)=(+)\overline{D_{jk}}$, we have $\overline{D_{ij}}=\frac{|RE(d_i^+)-RE(d_j^+)|}{RE(d_j^+)}$, $\overline{D_{jk}}=\frac{|RE(d_j^+)-RE(d_k^+)|}{RE(d_k^+)}$, and $RE(d_i^+)<RE(d_j^+)<RE(d_k^+)$. Hence, $\frac{|RE(d_i^+)-RE(d_k^+)|}{RE(d_k^+)}>\frac{|RE(d_j^+)-RE(d_k^+)|}{RE(d_k^+)}$, $\overline{D_{ik}}>\overline{D_{ij}} \wedge \overline{D_{jk}}$. When $r^+(x_i,x_j)=(+)\overline{D_{ij}}$, $r^+(x_j,x_k)=1$, we have $\overline{D_{ij}}=\frac{|RE(d_i^+)-RE(d_j^+)|}{RE(d_j^+)}$, $\overline{D_{jk}}=0$, and $RE(d_i^+)<RE(d_j^+)=RE(d_k^+)$. Hence, $\overline{D_{ik}}>\overline{D_{ij}} \wedge \overline{D_{jk}}$. When $r^+(x_i,x_j)=1$, $r^+(x_j,x_k)=(+)\overline{D_{jk}}$, we have $\overline{D_{ij}}=0$, $\overline{D_{jk}}=\frac{|RE(d_j^+)-RE(d_k^+)|}{RE(d_k^+)}$, and $RE(d_i^+)=RE(d_j^+)<RE(d_k^+)$. Hence, $\overline{D_{ik}}>\overline{D_{ij}} \wedge \overline{D_{jk}}$.

$\forall x_i,x_j,x_k \in U$, when $r^+(x_i,x_j)=(-)\underline{D_{ij}}$, $r^+(x_j,x_k)=(-)\underline{D_{jk}}$, we have $(-)\underline{D_{ij}}=\frac{|RE(d_i^+)-RE(d_j^+)|}{RE(d_i^+)}$, $(-)\underline{D_{jk}}=\frac{|RE(d_j^+)-RE(d_k^+)|}{RE(d_j^+)}$, and $RE(d_i^+)>RE(d_j^+)>RE(d_k^+)$. Hence, $\frac{|RE(d_i^+)-RE(d_k^+)|}{RE(d_i^+)}>\frac{|RE(d_i^+)-RE(d_j^+)|}{RE(d_i^+)}$, $\underline{D_{ik}}>\underline{D_{ij}} \wedge \underline{D_{jk}}$. when $r^+(x_i,x_j)=(-)\underline{D_{ij}}$, $r^+(x_j,x_k)=1$, we have $(-)\underline{D_{ij}}=\frac{|RE(d_i^+)-RE(d_j^+)|}{RE(d_i^+)}$, $\underline{D_{jk}}=0$, and $RE(d_i^+)>RE(d_j^+)=RE(d_k^+)$. Hence, $\underline{D_{ik}}>\underline{D_{ij}} \wedge \underline{D_{jk}}$. when $r^+(x_i,x_j)=1$, $r^+(x_j,x_k)=(-)\underline{D_{jk}}$, we have $\underline{D_{ij}}=0$, $(-)\underline{D_{jk}}=\frac{|RE(d_j^+)-RE(d_k^+)|}{RE(d_j^+)}$, and $RE(d_i^+)=RE(d_j^+)>RE(d_k^+)$. Hence, $\underline{D_{ik}}>\underline{D_{ij}} \wedge \underline{D_{jk}}$.

$\forall x_i,x_j,x_k \in U$, when $r^+(x_i,x_j)=1$, $r^+(x_j,x_k)=1$, we have $RE(d_i^+)=RE(d_j^+)=RE(d_k^+)$. Hence, $\overline{D_{ik}}=\overline{D_{ij}} \wedge \overline{D_{jk}}$, and $\underline{D_{ik}}=\underline{D_{ij}} \wedge \underline{D_{jk}}$.

In summary, $\forall x_i,x_j,x_k \in U$, $\overline{D_{ik}} \geq \overline{D_{ij}} \wedge \overline{D_{jk}}$, $\underline{D_{ik}} \geq \underline{D_{ij}} \wedge \underline{D_{jk}}$. □

Theorem 1. *Let $IOIS=(U,A,f,V\cup\{*\})$ be an incomplete ordered information system, $\forall x_i,x_j \in U$. $S^+=(u_1^+,u_2^+,\cdots,u_m^+)$ and $S^-=(u_1^-,u_2^-,\cdots,u_m^-)$ are positive ideal solution and negative ideal solution, respectively. When $d_i^+ \in (0,d_j^+]$, $\overline{D_{ij}}$ decreases with the increase of d_i^+.*

Proof. According to Definition 16, we have $\overline{D_{ij}}=\frac{RE(d_j^+)-RE(d_i^+)}{RE(d_j^+)}$. Let $d_i^+=x$, $\overline{D_{ij}}=F(x)$, then $\frac{dF}{dx}=\frac{\exp(-\sigma x)(-\sigma)}{RE(d_j^+)}$. When $x\in(0,d_j^+]$, $\frac{dF}{dx}<0$, so $\overline{D_{ij}}$ decreases with the increase of d_i^+. □

Theorem 2. *Let $IOIS=(U,A,f,V\cup\{*\})$ be an incomplete ordered information system, $\forall x_i,x_j \in U$. $S^+=(u_1^+,u_2^+,\cdots,u_m^+)$ and $S^-=(u_1^-,u_2^-,\cdots,u_m^-)$ are positive ideal solution and negative ideal solution, respectively. When $d_i^+ \in [d_j^+,+\infty)$, $\underline{D_{ij}}$ increase with the increase of d_i^+.*

Proof. According to Definition 16, we have $\underline{D_{ij}}=\frac{RE(d_i^+)-RE(d_j^+)}{RE(d_i^+)}$. Let $d_i^+=x$, $\underline{D_{ij}}=F(x)$, then $\frac{dF}{dx}=RE(d_j^+)(1-\exp(-\sigma x))^{-2}\sigma\times\exp(-\sigma x)$. When $x\in[d_j^+,+\infty)$, $\frac{dF}{dx}>0$, so $\underline{D_{ij}}$ increase with the increase of d_i^+. □

Theorem 1 shows that when $x\in(0,d_j^+]$, the degree of x_i better than x_j decreases with the increase of d_i^+. Based on Theorem 2, the degree to which x_i is inferior to x_j increases as d_i^+ increases with $d_i^+\in[d_j^+,+\infty)$. These two theorems conform to people's cognition in real life.

Definition 17. Let $IOIS=(U,A,f,V\cup\{*\})$ be an incomplete ordered information system, negative ideal solution $S^-=(u_1^-,u_2^-,\cdots,u_m^-)$, $COIS=(\tilde{U},A,f,\tilde{V},T)$ is a complete ordered information system with confidence corresponding to $IOIS$. $\forall x_i \in U$, the distance between x_i and negative ideal solution S^- is calculated as follows:

Table 5
Dominance relation r^-.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_1	1	(−)0.0593	(+)0.1654	(−)0.4260	(−)0.2465	(−)0.0596	(+)0.5627	(−)0.3047	(+)0.4964
x_2	(+)0.0593	1	(+)0.2149	(−)0.3899	(−)0.1990	(−)0.0004	(+)0.5886	(−)0.2609	(+)0.5263
x_3	(−)0.1654	(−)0.2149	1	(−)0.5210	(−)0.3711	(−)0.2152	(+)0.4760	(−)0.4198	(+)0.3966
x_4	(+)0.4260	(+)0.3899	(+)0.5210	1	(+)0.2383	(+)0.3896	(+)0.7490	(+)0.1745	(+)0.7110
x_5	(+)0.2465	(+)0.1990	(+)0.3711	(−)0.2383	1	(+)0.1987	(+)0.6705	(−)0.0773	(+)0.6205
x_6	(+)0.0596	(+)0.0004	(+)0.2152	(−)0.3896	(−)0.1987	1	(+)0.5888	(−)0.2606	(+)0.5265
x_7	(−)0.5627	(−)0.5886	(−)0.4760	(−)0.7490	(−)0.6705	(−)0.5888	1	(−)0.6960	(−)0.1316
x_8	(+)0.3047	(+)0.2609	(+)0.4198	(−)0.1745	(+)0.0773	(+)0.2606	(+)0.6960	1	(+)0.6499
x_9	(−)0.4964	(−)0.5263	(−)0.3966	(−)0.7110	(−)0.6205	(−)0.5265	(+)0.1316	(−)0.6499	1

$$d_i^- = \begin{cases} (\sum_{j=1}^{m}((u_{ij}-u_j^-)/z_j)^2)^{1/2}, \forall a\in A, f(x_i,a)\neq *, \\ \sum_{l_i=1}^{n_i}(\sum_{j=1}^{m}((u_{ij}^{l_i}-u_j^-)/z_j)^2)^{1/2}T(x_i^{l_i}), \exists a\in A, f(x_i,a)=*, \end{cases} \tag{30}$$

where $z_j = max(\tilde{V}_{a_j}) - min(\tilde{V}_{a_j})$, $x_i^{l_i} \in \{x_i^1, x_i^2, \cdots, x_i^{n_i}\} \subseteq \tilde{U}$, $n_i = \prod_{a\in A^*_{x_i}} |f_S(x_i,a)|$, $A^*_{x_i} = \{a\in A | f(x_i,a) = *\}$. When $f_S(x_i,a_j) = \{(\hat{Y}_*, R^2)\}$ or $f_S(x_i,a_j) = \{(\bar{V}_{ij}, \bar{S}_{ij})\}$, in order to reduce the error of d_i^-, we set $R^2 = 1$, $\bar{S}_{ij} = 1$.

Definition 18. Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system, and $COIS = (\tilde{U}, A, f, \tilde{V}, T)$ be a complete ordered information system with confidence corresponding to $IOIS$. $\forall x_i, x_j \in U$ the dominance relation between x_i and x_j relative to NIS is defined as follows:

$$r^-(x_i,x_j) = \begin{cases} (+)\overline{D_{ij}}, RE(d_i^-) > RE(d_j^-), \\ 1, RE(d_i^-) = RE(d_j^-), \\ (-)\underline{D_{ij}}, RE(d_i^-) < RE(d_j^-). \end{cases} \tag{31}$$

$$\overline{D_{ij}} = \frac{|RE(d_i^-) - RE(d_j^-)|}{RE(d_i^-)}, \tag{32}$$

$$\underline{D_{ij}} = \frac{|RE(d_i^-) - RE(d_j^-)|}{RE(d_j^-)}, \tag{33}$$

where (+) denotes that object x_i is better than object x_j under attribute set A, (−) denotes that object x_i is worse than object x_j under attribute set A, and 1 denotes that object x_i is equivalent to object x_j under attribute set A. $\overline{D_{ij}}$ denotes the degree of x_i better than x_j, $\underline{D_{ij}}$ denotes the degree of x_i worse than x_j. $r^-(x_i,x_j) = 1$ can also be expressed by $r^-(x_i,x_j) = (+)0$ or $r^-(x_i,x_j) = (-)0$, that is, the degree of x_i is better or worse than x_j is 0. $RE(d_i^-)$ is the regret value generated by replacing x_i with NIS.

Example 4. According to the data in Example 2 and Definition 18, we can get the dominance relation r^- on U, as shown in Table 5.

Property 2. Given an incomplete ordered information system $IOIS = (U, A, f, V \cup \{*\})$, then the dominance relation r^- in $IOIS$ has following properties:

(1) Reflexivity, $\forall x_i \in U$, $r^-(x_i,x_i) = 1$,

(2) Symmetry, $\forall x_i, x_j \in U$, $\overline{D_{ij}} = \underline{D_{ji}}$, $\underline{D_{ij}} = \overline{D_{ji}}$,

(3) Transitivity, $\forall x_i, x_j, x_k \in U$, if x_i is better than x_j and x_j is better than x_k, then $\overline{D_{ik}} \geq \overline{D_{ij}} \wedge \overline{D_{jk}}$. If x_i is worse than x_j and x_j is worse than x_k, then $\underline{D_{ik}} \geq \underline{D_{ij}} \wedge \underline{D_{jk}}$.

Proof. Similar to the proof of Property 1. □

Theorem 3. *Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system, $\forall x_i, x_j \in U$. $S^+ = (u_1^+, u_2^+, \cdots, u_m^+)$ and $S^- = (u_1^-, u_2^-, \cdots, u_m^-)$ are positive ideal solution and negative ideal solution, respectively. When $d_i^- \in (0, d_j^-]$, $\underline{D_{ij}}$ decreases with the increase of d_i^-.*

Proof. Similar to the proof of Theorem 1. □

Table 6
Fuzzy pre-decision $(d_{PIS}(x), d_{NIS}(x))$.

	x_1	x_2	x_3	x_4	x_5
$(d_{PIS}(x), d_{NIS}(x))$	(0.3852, 0.6148)	(0.3845, 0.6155)	(0.3244, 0.6756)	(0.8121, 0.1879)	(0.5546, 0.4454)
	x_6	x_7	x_8	x_9	
$(d_{PIS}(x), d_{NIS}(x))$	(0.4093, 0.5907)	(0.1563, 0.8437)	(0.6066, 0.3934)	(0.1758, 0.8242)	

Theorem 4. *Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system, $\forall x_i, x_j \in U$. $S^+ = (u_1^+, u_2^+, \cdots, u_m^+)$ and $S^- = (u_1^-, u_2^-, \cdots, u_m^-)$ are positive ideal solution and negative ideal solution, respectively. When $d_i^- \in [d_j^-, +\infty)$, $\overline{D_{ij}}$ increase with the increase of d_i^-.*

Proof. Similar to the proof of Theorem 2. □

Consistent with rational cognitive thinking, Theorems 3 and 4 illustrate that as the distance between x_i and NIS increases, the lesser the degree of x_i's inferiority to x_j becomes, or conversely, the greater the degree of x_i's superiority to x_j.

3.3. The calculation of expected losses

Similar to the classical three-way decision model, in order to calculate the expected loss of different decisions, we need to calculate the degree of objects belonging to the PIS class and the NIS class, i.e. the conditional probability of objects belonging to different states in the classical three-way decision model, and the relative loss function matrix. In order to calculate the degree of objects belonging to different state classes and the relative loss function matrix in incomplete ordered information systems, we introduce the fuzzy pre-decision.

Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system. $S^+ = (u_1^+, u_2^+, \cdots, u_m^+)$ and $S^- = (u_1^-, u_2^-, \cdots, u_m^-)$ are positive ideal solution and negative ideal solution, respectively. $\forall x_i \in U$, d_i^+ and d_i^- are the distance values from object x_i to PIS and NIS. According to Definitions 15 and 17, we can get the fuzzy pre-decision value $d(x_i) = (\frac{d_i^-}{d_i^+ + d_i^-}, \frac{d_i^+}{d_i^+ + d_i^-}) = (d_{PIS}(x_i), d_{NIS}(x_i))$ of the object x_i.

Example 5. According to Table 3 and Definitions 15 and 17, we can get the fuzzy pre-decision $(d_{PIS}(x), d_{NIS}(x))$ as shown in Table 6.

Theorem 5. *Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system. $S^+ = (u_1^+, u_2^+, \cdots, u_m^+)$ and $S^- = (u_1^-, u_2^-, \cdots, u_m^-)$ are positive ideal solution and negative ideal solution, respectively. $\forall x_i \in U$, d_i^+ and d_i^- are the distances from object x_i to PIS and NIS.*
(1) When $d_i^+, d_i^- \in (0, +\infty)$, $d_{PIS}(x_i)$ decreases with the increase of d_i^+ and the decrease of d_i^-.
(2) When $d_i^+, d_i^- \in (0, +\infty)$, $d_{NIS}(x_i)$ increases with the increase of d_i^+ and the decrease of d_i^-.

Proof. (1) $\frac{1}{d_{PIS}(x_i)} = \frac{d_i^+ + d_i^-}{d_i^-} = 1 + \frac{d_i^+}{d_i^-}$, when d_i^+ increase and d_i^- decrease, $\frac{1}{d_{PIS}(x_i)}$ increase, so $d_{PIS}(x_i)$ decrease.
(2) Similar to the proof of (1). □

Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system, $\forall x_i \in U$. $S^+ = (u_1^+, u_2^+, \cdots, u_m^+)$ and $S^- = (u_1^-, u_2^-, \cdots, u_m^-)$ are positive ideal solution and negative ideal solution, respectively. Using fuzzy pre-decision $(d_{PIS}(x_i), d_{NIS}(x_i))$ and dominance relations r^+ and r^-, the degree to which the object x_i belongs to the positive ideal solution class $BD_{PIS}(x_i)$, and the degree to which the object x_i belongs to the negative ideal solution class $BD_{NIS}(x_i)$ are calculated as follows:

$$BD_{PIS}(x_i) = \frac{Q(x_i)}{Q(x_i) + Z(x_i)}, \tag{34}$$

$$BD_{NIS}(x_i) = \frac{Z(x_i)}{Q(x_i) + Z(x_i)}, \tag{35}$$

$$Q(x_i) = \sum_{x_j \in U} q(x_i, x_j), \tag{36}$$

$$Z(x_i) = \sum_{x_j \in U} z(x_i, x_j), \tag{37}$$

Table 7
$BD_{PIS}(x)$ and $BD_{NIS}(x)$.

	x_1	x_2	x_3	x_4	x_5
$(BD_{PIS}(x), BD_{NIS}(x))$	(0.4070,0.5930)	(0.4072,0.5928)	(0.3696,0.6304)	(0.6517,0.3483)	(0.5114,0.4886)
	x_6	x_7	x_8	x_9	
$(BD_{PIS}(x), BD_{NIS}(x))$	(0.4226,0.5774)	(0.2716,0.7284)	(0.5425,0.4575)	(0.2814,0.7186)	

Table 8
Relative loss function matrix for x_i.

	D^+	D^-
$a_{\mathbb{P}}$	0	$\frac{d_i^+}{d_i^+ + d_i^-}$
$a_{\mathbb{B}}$	$\sigma\frac{d_i^-}{d_i^+ + d_i^-}$	$\sigma\frac{d_i^+}{d_i^+ + d_i^-}$
$a_{\mathbb{N}}$	$\frac{d_i^-}{d_i^+ + d_i^-}$	0

$$q(x_i,x_j)=\begin{cases}(1+\overline{D_{ij}})\times d_{PIS}(x_j), r^+(x_i,x_j)=(+)\overline{D_{ij}},\\ d_{PIS}(x_j)/(1+\underline{D_{ij}}),\quad r^+(x_i,x_j)=(-)\underline{D_{ij}},\\ d_{PIS}(x_j),\qquad\qquad\qquad r^+(x_i,x_j)=1.\end{cases} \tag{38}$$

$$z(x_i,x_j)=\begin{cases}d_{NIS}(x_j)/(1+\overline{D_{ij}}),\quad r^-(x_i,x_j)=(+)\overline{D_{ij}},\\ (1+\underline{D_{ij}})\times d_{NIS}(x_j), r^-(x_i,x_j)=(-)\underline{D_{ij}},\\ d_{NIS}(x_j),\qquad\qquad\qquad r^-(x_i,x_j)=1.\end{cases} \tag{39}$$

Example 6. For any object in the universe of discourse, we can calculate $(BD_{PIS}(x), BD_{NIS}(x))$ by Tables 4, 5, and 6. $(BD_{PIS}(x), BD_{NIS}(x))$ as shown in Table 7.

Theorem 6. *Let $IOIS=(U,A,f,V\cup\{*\})$ be an incomplete ordered information system, $\forall x_i\in U$. $S^+=(u_1^+,u_2^+,\cdots,u_m^+)$ is positive ideal solution, $S^-=(u_1^-,u_2^-,\cdots,u_m^-)$ is negative ideal solution. $BD_{PIS}(x_i)$ and $BD_{NIS}(x_i)$ are the degrees of belonging to the positive and negative ideal solution classes, respectively.*
(1) When $d_i^+, d_i^-\in(0,+\infty)$, $BD_{PIS}(x_i)$ decreases with the increase of d_i^+ and the decrease of d_i^-.
(2) When $d_i^+, d_i^-\in(0,+\infty)$, $BD_{NIS}(x_i)$ increases with the increase of d_i^+ and the decrease of d_i^-.

Proof. (1) $\forall x_j\in U$, when $d_i^+\in(0,d_j^+]$, $q(x_i,x_j)=(1+\overline{D_{ij}})\times d_{PIS}(x_j)$. According to Theorem 1, we can know $\overline{D_{ij}}$ decreases with the increase of d_i^+. Hence, $q(x_i,x_j)$ decreases with the increase of d_i^+. When $d_i^+\in[d_j^+,+\infty)$, $q(x_i,x_j)=d_{PIS}(x_j)/(1+\underline{D_{ij}})$. According to Theorem 2, we can know $\underline{D_{ij}}$ increase with the increase of d_i^+. Hence, $q(x_i,x_j)$ decreases with the increase of d_i^+. $Q(x_i)=\sum_{x_j\in U}q(x_i,x_j)$, so when $d_i^+\in(0,+\infty)$, $Q(x_i)$ decreases with the increase of d_i^+.

$\forall x_j\in U$, when $d_i^-\in(0,d_j^-]$, $z(x_i,x_j)=(1+\underline{D_{ij}})\times d_{NIS}(x_j)$. According to Theorem 3, we can know $\underline{D_{ij}}$ decreases with the increase of d_i^-. Hence, $z(x_i,x_j)$ decreases with the increase of d_i^-. When $d_i^-\in[d_j^+,+\infty)$, $q(x_i,x_j)=d_{NIS}(x_j)/(1+\overline{D_{ij}})$. According to Theorem 4, we can know $\overline{D_{ij}}$ increase with the increase of d_i^-. Hence, $z(x_i,x_j)$ decreases with the increase of d_i^-. $Z(x_i)=\sum_{x_j\in U}z(x_i,x_j)$, so when $d_i^+\in(0,+\infty)$, $Z(x_i)$ decreases with the increase of d_i^-.

$BD_{PIS}(x_i)=\frac{Q(x_i)}{Q(x_i)+Z(x_i)}$, then $\frac{1}{BD_{PIS}(x_i)}=\frac{Q(x_i)+Z(x_i)}{Q(x_i)}=1+\frac{Z(x_i)}{Q(x_i)}$. $\frac{1}{BD_{PIS}(x_i)}$ increases with the increase of d_i^+ and the decrease of d_i^-, then $BD_{PIS}(x_i)$ decreases with the increase of d_i^+ and the decrease of d_i^-.
(2) Similar to the proof of (1). □

Let $IOIS=(U,A,f,V\cup\{*\})$ be an incomplete ordered information system. $S^+=(u_1^+,u_2^+,\cdots,u_m^+)$ and $S^-=(u_1^-,u_2^-,\cdots,u_m^-)$ are positive ideal solution and negative ideal solution, respectively. Given an set of two states $\Omega=\{D^+,D^-\}$, $\forall x_i\in U$ then the relative loss function matrix of the $a_{\mathbb{P}}$, $a_{\mathbb{B}}$, and $a_{\mathbb{N}}$ decision-making under the D^+ and D^- states is shown in Table 8, where $\sigma\in(0,1)$. From Table 8 we can get $\lambda^i_{\mathbb{P}P}<\lambda^i_{\mathbb{B}P}<\lambda^i_{\mathbb{N}P}$, $\lambda^i_{\mathbb{N}N}<\lambda^i_{\mathbb{B}N}<\lambda^i_{\mathbb{P}N}$.

Property 3. Given an incomplete ordered information system $IOIS=(U,A,f,V\cup\{*\})$, $S^+=(u_1^+,u_2^+,\cdots,u_m^+)$ and $S^-=(u_1^-,u_2^-,\cdots,u_m^-)$ are positive ideal solution and negative ideal solution, respectively. $\forall x_i\in U$, loss function values have following properties:

Table 9
Expected losses.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$EL_{\mathbb{P}}(x)$	0.3646	0.3649	0.4259	0.0654	0.2176	0.3411	0.6146	0.1800	0.5923
$EL_{\mathbb{B}}(x)$	0.1804	0.1804	0.1889	0.2058	0.1734	0.1779	0.2273	0.1761	0.2221
$EL_{\mathbb{N}}(x)$	0.1568	0.1566	0.1199	0.5292	0.2836	0.1730	0.0425	0.3291	0.0495

(1) When $0 < \sigma < 0.5$, $(\lambda^i_{\mathbb{B}P} - \lambda^i_{\mathbb{P}P})(\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}) < (\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N})(\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{B}P})$.
(2) When $\sigma = 0.5$, $(\lambda^i_{\mathbb{B}P} - \lambda^i_{\mathbb{P}P})(\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}) = (\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N})(\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{B}P})$.
(3) When $0.5 < \sigma < 1$, $(\lambda^i_{\mathbb{B}P} - \lambda^i_{\mathbb{P}P})(\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}) > (\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N})(\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{B}P})$.

Proof. (1) $(\lambda^i_{\mathbb{B}P} - \lambda^i_{\mathbb{P}P})(\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}) = \sigma\frac{d_i^-}{d_i^+ + d_i^-}\sigma\frac{d_i^+}{d_i^+ + d_i^-} = \sigma^2\frac{d_i^- d_i^+}{(d_i^+ + d_i^-)^2}$, $(\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N})(\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{B}P}) = (1-\sigma)\frac{d_i^+}{d_i^+ + d_i^-}(1-\sigma)\frac{d_i^-}{d_i^+ + d_i^-} = (1-\sigma)^2\frac{d_i^- d_i^+}{(d_i^+ + d_i^-)^2}$. $(\lambda^i_{\mathbb{B}P} - \lambda^i_{\mathbb{P}P})(\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}) < (\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N})(\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{B}P}) \Leftrightarrow \sigma^2 < (1-\sigma)^2 \Leftrightarrow \sigma < 0.5$. Hence, when $0 < \sigma < 0.5$, $(\lambda^i_{\mathbb{B}P} - \lambda^i_{\mathbb{P}P})(\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}) < (\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N})(\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{B}P})$.
(2) Similar to the proof of (1).
(3) Similar to the proof of (1). □

Given an incomplete ordered information system $IOIS = (U, A, f, V \cup \{*\})$, $S^+ = (u_1^+, u_2^+, \cdots, u_m^+)$ and $S^- = (u_1^-, u_2^-, \cdots, u_m^-)$ are positive ideal solution and negative ideal solution, respectively. $\forall x_i \in U$, according to $BD_{PIS}(x_i)$, $BD_{NIS}(x_i)$, and the relative loss function matrix in Table 8, we can calculate the expected loss when x_i performs different decisions.
$EL_{\mathbb{P}}(x_i) = \lambda^i_{\mathbb{P}P} \times BD_{PIS}(x_i) + \lambda^i_{\mathbb{P}N} \times BD_{NIS}(x_i)$,
$EL_{\mathbb{B}}(x_i) = \lambda^i_{\mathbb{B}P} \times BD_{PIS}(x_i) + \lambda^i_{\mathbb{B}N} \times BD_{NIS}(x_i)$,
$EL_{\mathbb{N}}(x_i) = \lambda^i_{\mathbb{N}P} \times BD_{PIS}(x_i) + \lambda^i_{\mathbb{N}N} \times BD_{NIS}(x_i)$.

Example 7. According to the data in Tables 7 and 8, $\forall x \in U$, we can calculate the $EL_{\mathbb{P}}(x)$, $EL_{\mathbb{B}}(x)$, and $EL_{\mathbb{N}}(x)$ shown in Table 9.

Theorem 7. *Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system, $\forall x_i \in U$. $S^+ = (u_1^+, u_2^+, \cdots, u_m^+)$ and $S^- = (u_1^-, u_2^-, \cdots, u_m^-)$ are positive ideal solution and negative ideal solution, respectively. When $d_i^+, d_i^- \in (0, +\infty)$, $EL_{\mathbb{P}}(x_i)$ increases with the increase of d_i^+ and the decrease of d_i^-.*

Proof. $EL_{\mathbb{P}}(x_i) = \lambda_{\mathbb{P}P} \times BD_{PIS}(x_i) + \lambda_{\mathbb{P}N} \times BD_{NIS}(x_i)$, we define $\lambda_{\mathbb{P}P} = 0$ then $EL_{\mathbb{P}}(x_i) = \lambda_{\mathbb{P}N} \times BD_{NIS}(x_i)$. According to Theorems 5 and 6, we can get $EL_{\mathbb{P}}(x_i)$ increases with the increase of d_i^+ and the decrease of d_i^-. □

Theorem 8. *Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system, $\forall x_i \in U$. $S^+ = (u_1^+, u_2^+, \cdots, u_m^+)$ and $S^- = (u_1^-, u_2^-, \cdots, u_m^-)$ are positive ideal solution and negative ideal solution, respectively. When $d_i^+, d_i^- \in (0, +\infty)$, $EL_{\mathbb{N}}(x_i)$ decreases with the increase of d_i^+ and the decrease of d_i^-.*

Proof. Similar to the proof of Theorem 7. □

Theorems 7 and 8 show that when an object in the universe of discourse is farther from PIS and closer to NIS, the expected loss of being classified into the positive domain is greater, and the expected loss of being classified into the negative domain is smaller. These two theorems are in line with actual and rational thinking.

3.4. Three-way decision rules based on minimum expected loss

Additionally, the following are the classification rules based on the Bayesian minimum risk decision theory:
(P_1) If $EL_{\mathbb{P}}(x_i) < EL_{\mathbb{B}}(x_i)$ and $EL_{\mathbb{P}}(x_i) < EL_{\mathbb{N}}(x_i)$, decide $x_i \in POS(U)$,
(B_1) If $EL_{\mathbb{B}}(x_i) \le EL_{\mathbb{P}}(x_i)$ and $EL_{\mathbb{B}}(x_i) \le EL_{\mathbb{N}}(x_i)$, decide $x_i \in BND(U)$,
(N_1) If $EL_{\mathbb{N}}(x_i) < EL_{\mathbb{P}}(x_i)$ and $EL_{\mathbb{N}}(x_i) < EL_{\mathbb{B}}(x_i)$, decide $x_i \in NEG(U)$.
Because the loss function values satisfy $\lambda^i_{\mathbb{P}P} < \lambda^i_{\mathbb{B}P} < \lambda^i_{\mathbb{N}P}$, $\lambda^i_{\mathbb{N}N} < \lambda^i_{\mathbb{B}N} < \lambda^i_{\mathbb{P}N}$, and $BD_{PIS}(x_i) + BD_{NIS}(x_i) = 1$, the decision rules (P_1), (B_1) and (N_1) are simplified as follows:
(P_2) If $BD_{PIS}(x_i) > \alpha_i$, $BD_{PIS}(x_i) > \gamma_i$, decide $x_i \in POS(U)$,
(B_2) If $BD_{PIS}(x_i) \le \alpha_i$, $BD_{PIS}(x_i) \ge \beta_i$, decide $x_i \in BND(U)$,
(N_2) If $BD_{PIS}(x_i) < \beta_i$, $BD_{PIS}(x_i) < \gamma_i$, decide $x_i \in NEG(U)$,
where $\alpha_i = \frac{\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N}}{(\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N}) + (\lambda^i_{\mathbb{B}P} - \lambda^i_{\mathbb{P}P})}$, $\beta_i = \frac{\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}}{(\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}) + (\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{B}P})}$, $\gamma_i = \frac{\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{N}N}}{(\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{N}N}) + (\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{P}P})}$.
Similar to the classical decision-theoretic rough set, when $(\lambda^i_{\mathbb{B}P} - \lambda^i_{\mathbb{P}P})(\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}) \le (\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N})(\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{B}P})$, the decision rules (P_2), (B_2) and (N_2) are simplified as follows:

(P_3) If $BD_{PIS}(x_i) > \alpha_i$, decide $x_i \in POS(U)$,
(B_3) If $\beta_i \le BD_{PIS}(x_i) \le \alpha_i$, decide $x_i \in BND(U)$,
(N_3) If $BD_{PIS}(x_i) < \beta_i$, decide $x_i \in NEG(U)$.

Remark 1. (1) When $\alpha \ge \beta$, the model is three-way decision:
If $BD_{PIS}(x_i) > \alpha_i$, decide $x_i \in POS(U)$,
If $\beta_i \le BD_{PIS}(x_i) \le \alpha_i$, decide $x_i \in BND(U)$,
If $BD_{PIS}(x_i) < \beta_i$, decide $x_i \in NEG(U)$.
(2) When $\alpha < \beta$, the mode is two-way decision:
If $BD_{PIS}(x_i) \ge \gamma_i$, decide $x_i \in POS(U)$,
If $BD_{PIS}(x_i) < \gamma_i$, decide $x_i \in NEG(U)$.

Theorem 9. *For relative loss function matrix based on fuzzy pre-decision, the model is a three-way decision if* $0 < \sigma \le 0.5$*, and it is a two-way decision model if* $0.5 < \sigma < 1$.

Proof. Three-way decision models request $\alpha \ge \beta$, so $\forall x_i \in U$, $\alpha_i - \beta_i \ge 0$, then we have $(1-2\sigma)d_i^+ d_i^- \ge 0$. As $d_i^+ > 0$ and $d_i^- > 0$, then $1 - 2\sigma \ge 0$. So we can obtain $0 < \sigma \le 0.5$.

Similarly, it is a two-way decision model if $0.5 < \sigma < 1$. □

Theorem 10. *Given an incomplete ordered information system* $IOIS = (U, A, f, V \cup \{*\})$*,* $S^+ = (u_1^+, u_2^+, \cdots, u_m^+)$ *and* $S^- = (u_1^-, u_2^-, \cdots, u_m^-)$ *are positive ideal solution and negative ideal solution, respectively.* $\forall x_i \in U$*,* $\sigma \in (0, 1)$*, then*

(1) $\alpha_i = \frac{\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N}}{(\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N}) + (\lambda^i_{\mathbb{B}P} - \lambda^i_{\mathbb{P}P})}$ *decreases monotonically with respect to* σ.

(2) $\beta_i = \frac{\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}}{(\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}) + (\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{B}P})}$ *increases monotonically with respect to* σ.

(3) $\gamma_i = \frac{\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{N}N}}{(\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{N}N}) + (\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{P}P})}$ *has nothing to do with* σ.

Proof. (1) According to Table 8, we can get $\alpha_i = \frac{\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N}}{(\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N}) + (\lambda^i_{\mathbb{B}P} - \lambda^i_{\mathbb{P}P})} = \left(1 + \frac{\sigma d_i^-}{(1-\sigma)d_i^+}\right)^{-1}$. Let $F(\sigma) = \left(1 + \frac{\sigma d_i^-}{(1-\sigma)d_i^+}\right)^{-1}$, then $\frac{dF}{d\sigma} = -\left(1 + \frac{\sigma d_i^-}{(1-\sigma)d_i^+}\right)^{-2} \times \frac{d_i^- d_i^+}{((1-\sigma)d_i^+)^2}$. By $\sigma \in (0,1)$, we can obtain $\frac{dF}{d\sigma} < 0$. Thus, we have $\alpha_i = \frac{\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N}}{(\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{B}N}) + (\lambda^i_{\mathbb{B}P} - \lambda^i_{\mathbb{P}P})}$ decreases monotonically with respect to σ.

(2) According to Table 8, we can get $\beta_i = \frac{\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}}{(\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}) + (\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{B}P})} = \left(1 + \frac{(1-\sigma)d_i^-}{\sigma d_i^+}\right)^{-1}$. Let $F(\sigma) = \left(1 + \frac{(1-\sigma)d_i^-}{\sigma d_i^+}\right)^{-1}$, then $\frac{dF}{d\sigma} = \left(1 + \frac{(1-\sigma)d_i^-}{\sigma d_i^+}\right)^{-2} \times \frac{d_i^- d_i^+}{(\sigma d_i^+)^2}$. By $\sigma \in (0,1)$, we can obtain $\frac{dF}{d\sigma} > 0$. Thus, we have $\beta_i = \frac{\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}}{(\lambda^i_{\mathbb{B}N} - \lambda^i_{\mathbb{N}N}) + (\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{B}P})}$ increases monotonically with respect to σ.

(3) According to Table 8, we can get $\gamma_i = \frac{\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{N}N}}{(\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{N}N}) + (\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{P}P})} = \frac{d_i^+}{d_i^+ + d_i^-}$. Thus, $\gamma_i = \frac{\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{N}N}}{(\lambda^i_{\mathbb{P}N} - \lambda^i_{\mathbb{N}N}) + (\lambda^i_{\mathbb{N}P} - \lambda^i_{\mathbb{P}P})}$ has nothing to do with σ. □

Theorem 11. *Given an incomplete ordered information system* $IOIS = (U, A, f, V \cup \{*\})$*,* $S^+ = (u_1^+, u_2^+, \cdots, u_m^+)$ *and* $S^- = (u_1^-, u_2^-, \cdots, u_m^-)$ *are positive ideal solution and negative ideal solution, respectively.* $\forall x_i \in U$*,* $\sigma \in (0, 0.5]$*, we then yield:*

(1) *The range of* $POS(U)$ *increases with the increase of* σ.
(2) *The range of* $BND(U)$ *decreases with the increase of* σ.
(3) *The range of* $NEG(U)$ *increases with the increase of* σ.

Proof. The theorem can be proof by Theorem 10 and rules (P_3), (B_3), (N_3). □

$\forall x_i \in U$, then the related loss of object x_i classified according to decision rules P_3, B_3, and N_3 is calculated as follows:

$$cost(x_i) = \begin{cases} EL_{\mathbb{P}}(x_i), x_i \in POS(U), \\ EL_{\mathbb{B}}(x_i), x_i \in BND(U), \\ EL_{\mathbb{N}}(x_i), x_i \in NEG(U). \end{cases} \tag{40}$$

Based on the semantic interpretation of decision rules P_3, B_3, and N_3, we consider an order relation $POS(U) \succ BND(U) \succ NEG(U)$. The order relation shows that the object in the positive domain is superior to the object in the boundary domain, and the object in the positive domain and the boundary domain are superior to the object in the negative domain. $\forall x_i, x_j \in U$, when x_i and x_j are in the positive domain at the same time, x_i is said to be better than x_j, if $EL_{\mathbb{P}}(x_i) < EL_{\mathbb{P}}(x_j)$. When both x_i and x_j are in the negative domain, x_i is inferior to x_j, if $EL_{\mathbb{N}}(x_i) < EL_{\mathbb{N}}(x_j)$. According to the above ranking rules, we can find the best and the worst in the U.

Example 8. According to the rules (P_1), (B_1), (N_1), and Table 9, we can obtain the classification result $POS(U) = \{x_4\}$, $BND = \{x_5, x_8\}$, $NEG = \{x_1, x_2, x_3, x_6, x_7, x_9\}$. In addition, we can determine the best object is x_4 and the worst object is x_7.

In order to more intuitively show the ideas and decision-making strategies of this paper, we summarize the method steps of the designed model as follows.

Step 1: From Definition 14, a complete ordered information system with confidence is obtained.

Step 2: According to Formulas (26) and (30), the distances between the objects in the universe of discourse and the positive ideal solution (PIS) as well as the negative ideal solution (NIS) are calculated, respectively.

Step 3: The dominance relations with respect to PIS and NIS are calculated by Formulas (27) and (31).

Step 4: Calculate the fuzzy pre-decision and relative loss function matrix of each object in the universe of discourse.

Step 5: The degree of each object in the universe of discourse belongs to the PIS class and the NIS class is calculated by the Formulas (34)-(39).

Step 6: According to the decision rules $(P_3) - (N_3)$, we can get classified results.

Step 7: The object with the minimum expected loss in the positive region is selected as the optimal one.

The method for the key steps is given as Algorithm 2.

Algorithm 2 A novel three-way decision model in incomplete ordered information systems.

```
input An incomplete ordered information system IOIS = (U, A, f, V ∪ {*}), positive ideal solution S+ = (u1+, u2+, ..., um+), negative ideal solution S− = (u1−, u2−, ..., um−).
output The classification of all objects and optimal object.
According to Definition 14, a complete ordered information system with confidence is obtained
for x_i in U do
        Calculate the distances d_i+ and d_i− between x_i and positive ideal solution S+, and between x_i and negative ideal solution S−, respectively. //by Definitions 15 and 17
end
for x_i in U do
    for x_j in U do
        Calculate the dominance relation between x_i and x_j relative to the PIS and the NIS.//by Definitions 16 and 18
    end
end
for x_i in U do
    Calculate the fuzzy pre-decision value of x_i, D(x_i) = (D_PIS(x_i), D_NIS(x_i)) and relative loss function matrix
end
for x_i in U do
Calculate the degree of x_i belong to the classes of positive and negative ideal solution.//by Formulas (34)-(39)
end
obtain three regions: POS(U), BND(U), NEG(U).
for x_i in U do
    Determine the in which each object is located by the rules (P3) − (N3)
    if BD_PIS(x_i) > α_i then x_i ∈ POS(U),
    if β_i ≤ BD_PIS(x_i) ≤ α_i then x_i ∈ BND(U),
    if BD_PIS(x_i) < β_i then x_i ∈ NEG(U).
end
for x in POS(U) do
    Calculate the expected loss EL_P(x)
end
x_best = arg min{EL_P(x)|x ∈ POS(U)}.
```

4. Dataset experimental analysis and discussion

In order to provide further illustration of the rationality and effectiveness of the model constructed in this paper, two sets of experiments are conducted in this section. The first set of experiments is designed to compare the classification error rate of the proposed method with those of existing algorithms within incomplete ordered information systems. Because there is an order relationship such as $POS > BND > NEG$ in the three-way decision, in order to sort the objects in the universe of discourse reasonably, the three-way decision algorithm must be able to classify the objects in these three regions more accurately. If an object belonging to a positive domain is classified into a boundary domain or a negative domain, its ranking position will be greatly affected. In the first set of experiments, we select the Gas Turbine CO and NOx Emission dataset from 2015 and the Iris dataset in the UCI database. The detailed information for the dataset is shown in Table 10. In the first set of experiments, we selected three models of three-way decision for comparative analysis, which are Yang et al.'s model ($\alpha = 0.61, \beta = 0$) [33], Peng et al.'s model ($L = 0.7, \vartheta = 0.9$) [16], Wang et al.'s model ($\beta = 0.1, \gamma = 0.3, \theta = 0.3, \delta = 0.4$) [29].

The second set of experiments is to measure the consistency between the ranking results obtained by the proposed method and the algorithm results in the existing literature through corresponding indicator in a complete ordered information system. In the second

Table 10
Datasets information.

Dataset	Data type	Number of objects	Number of attributes
Gas Turbine CO and NOx Emission	Real	7384	11
Iris	Real	150	4+1
Auto MPG	Real, Integer	398	7+1
Travel Reviews	Real	980	10
Wine Quality (red)	Real	1599	11+1
Energy Efficiency	Real, Integer	768	8+2
Concrete Compressive Strength	Real	1030	8+1

set of experiments, we select the Auto MPG dataset, the Travel Reviews dataset, the Wine Quality dataset, the Energy Efficiency dataset and the Concrete Compressive Strength dataset in the UCI database. In the second set of experiments, we also select three sorting algorithms for comparative analysis, which are the TOPSIS method, Qian et al.'s method [17], Wang et al.'s method [30].

4.1. Classification analysis of algorithm

Definition 19. Let $IOIS = (U, A, f, V \cup \{*\})$ be an incomplete ordered information system, $\forall x \in U$, $D(x)$ is the label corresponding to x in the data table. $F(x)$ is the label assigned to x according to the classification algorithm, and the classification error rate of the algorithm is defined as follows:

$$CE = \frac{|\{x \in U \,|\, F(x) \neq D(x)\}|}{|U|}. \tag{41}$$

In the first set of experiments, this paper first preprocesses the dataset. The dataset Gas Turbine CO and NOx Emission contains 11 characteristic attributes: Ambient temperature (AT), Ambient pressure (AP), Ambient humidity (AH), Air filter difference pressure (AFDP), Gas turbine exhaust pressure (GTEP). Turbine inlet temperature (TIT), Turbine after temperature (TAT), Compressor discharge pressure (CDP), Turbine energy yield (TEY), Carbon monoxide (CO), Nitrogen oxides (NOx). We select Turbine energy yield (TEY) as the classification attribute and discretize it according to the amount of power generation. The power generation in the $[100, 120]$ is defined as the first category, in the $(120, 150]$ is located in the second category, and in the $(150, 180]$ is the third category. AT, AP, AH, NOx do not have the characteristics of benefit or cost attributes, so they are removed. Suppose positive ideal solution $S^+ = (5.2395, 40.716, 1100, 516.04, 15.159, 0.2128)$, negative ideal solution $S^- = (2.3688, 17.698, 1016, 550.59, 9.8708, 41.097)$, $\sigma = 0.29$ in the loss function matrix. After the dataset Gas Turbine CO and NOx Emission is processed and given parameters, we randomly miss 40, 80, 160, and 320 values for each attribute of the dataset, that is, 240 values, 480 values, 960 values, and 1920 values as a whole. Under these four different numbers of missing values, we conduct 10 experiments respectively. The specific data are shown in Table 12.

The Iris dataset contains four feature attributes (sepal length, sepal width, petal length, petal width) and one classification attribute (class of iris plant). Let positive ideal solution $S^+ = (7.9, 2.9, 6.9, 2.5)$, negative ideal solution $S^- = (4.3, 4.4, 1, 0.1)$, $\sigma = 0.29$ in the loss function matrix. Additionally, 5, 15, 25, and 35 values are randomly removed for each feature attribute in the Iris dataset, resulting in a total of 20, 60, 100, and 140 missing values, respectively. Ten experiments are conducted with these four distinct missing value counts. Table 13 displays the specific data.

Figs. 1 and 2 are the average values of the classification error rate of the four algorithms in ten experiments on datasets Iris and Gas Turbine CO and NOx Emission, respectively. From Figs. 1 and 2, we can see that the classification error rate of the proposed algorithm on an incomplete ordered information system is lower than that of the other three algorithms.

4.2. Sorting analysis of algorithm

In the Auto MPG dataset, we delete objects that contain missing data and delete attributes: weight, model year, and origin. The positive ideal solution $S^+ = (46.6, 8, 455, 230, 24.8)$, the negative ideal solution $S^- = (9, 3, 68, 46, 8)$, and the parameter $\sigma = 0.7$ in the loss function matrix. In the Travel Reviews dataset, the positive ideal solution $S^+ = (3.22, 3.64, 3.62, 3.44, 3.3, 3.76, 3.21, 3.39, 3.17, 3.66)$, the negative ideal solution $S^- = (0.34, 0, 0.13, 0.15, 0.06, 0.14, 3.16, 2.42, 0.74, 2.14)$, and the parameter $\sigma = 0.7$ in the loss function matrix. In the Wine Quality dataset, the Energy Efficiency dataset and the Concrete Compressive Strength dataset, we delete classification attribute and set the parameter $\sigma = 0.7$ in the loss function matrix. In the Wine Quality dataset, the positive ideal solution $S^+ = \{15.9, 1.58, 1, 15.5, 0.611, 72, 289, 1, 4.01, 2, 14.9\}$ and the negative ideal solution $S^- = \{4.6, 0.12, 0, 0.9, 0.012, 1, 6, 0.99, 2.74, 0.33, 8.4\}$. In the Energy Efficiency dataset, the positive ideal solution $S^+ = \{0.98, 808.5, 416.5, 220.5, 7, 5, 0.4, 5\}$ and the negative ideal solution $S^- = \{0.62, 514.5, 245, 110.25, 3.5, 2, 0, 0\}$. In the Concrete Compressive Strength dataset, the positive ideal solution $S^+ = \{540, 359.4, 200.1, 247, 32.2, 1145, 992.6, 365\}$ and the negative ideal solution $S^- = \{102, 0, 0, 121.75, 0, 801, 594, 1\}$.

Table 11 illustrates the results of our method, the TOPSIS method, Qian et al.'s method [17], and Wang et al.'s method [30] in five datasets: the Auto MPG dataset, the Travel Reviews dataset, the Wine Quality dataset, the Energy Efficiency dataset and the Concrete Compressive Strength dataset.

To assess the degree of consistency among the ranking outcomes derived from various techniques, we utilize the concept of the Spearman's rank correlation coefficient (SRCC), which is the correlation between two ranking sequences [8],

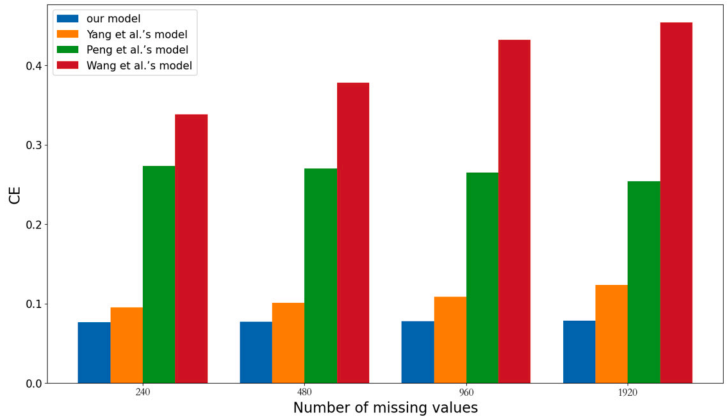

Fig. 1. CE (Gas Turbine CO and NOx Emission).

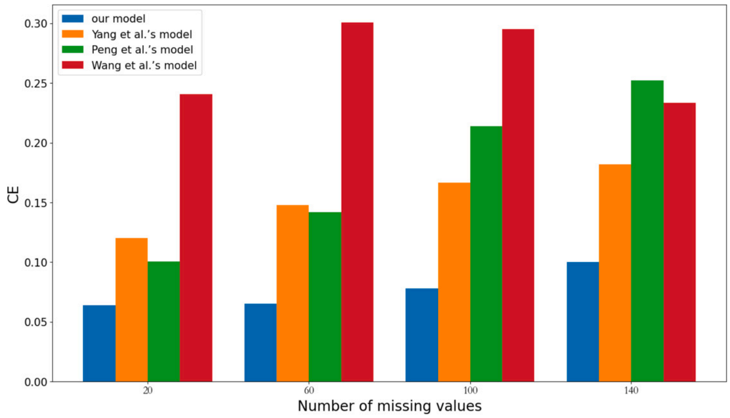

Fig. 2. CE (Iris).

Table 11
The best object.

	Our method	The TOPSIS method	Qian et al.'s method	Wang et al.'s method
Auto MPG	a_{360}	a_{360}	a_9	a_9
Travel Reviews	a_{667}	a_{667}	a_{667}	a_{667}
Wine Quality	a_{151}	a_{151}	a_{151}	a_{151}
Energy Efficiency	a_{19}	a_{19}	a_{19}	a_{19}
Concrete Compressive Strength	a_{66}	a_{66}	a_{42}	a_{66}

$$SRCC = 1 - \frac{6\sum_{i=1}^{n}(d_i)^2}{n^3 - n}, \tag{42}$$

where d_i denotes the difference between the ranking index values of object x_i in the two sequences, $x_i \in U(i = 1, 2, \cdots, n)$.

The SRCC results are generally analyzed from two aspects: (1) Based on the Spearman rank correlation threshold table, at the significance level of 0.05, when the ranking results of the two methods are greater than 0.648, the ranking performance of the two methods is highly consistent [28]. (2) In view of the excellent sorting performance of the classical decision method, if the SRCC between the first method and the classical method is greater than the SRCC between the second method and the classical method, the first method is superior to the second method in sorting performance [28]. Therefore, the calculation results in Figs. 3, 4, 5, 6 and 7 are analyzed below.

In light of Figs. 3 and 6, the values of SRCC between the our method and other counterparts, including the TOPSIS method, Qian et al.'s method [17], Wang et al.'s method [30], are all higher than 0.648, which indicates the ranking performance of these methods is highly consistent. In Figs. 4 and 5, the values of SRCC of our method and the TOPSIS method, Qian et al.'s method [17] are higher than 0.648. From Figs. 3, 4, 5, 6 and 7, we can also see that the SRCC of our method and the classical method TOPSIS is not less than 0.9997 on the datasets: Auto MPG, Travel Reviews, Wine Quality, Energy Efficiency and Concrete Compressive Strength. This indicates that the two ranking results are extremely similar.

Table 12
Classification error rate (Gas Turbine CO and NOx Emission).

240 missing values					480 missing values				
Times	Our model	Yang et al.'s model	Peng et al.'s model	Wang et al.'s model	Times	our model	Yang et al.'s model	Peng et al.'s model	Wang et al.'s model
	CE	CE	CE	CE		CE	CE	CE	CE
1	0.0768	0.0947	0.2738	0.3317	1	0.0776	0.1025	0.2690	0.3500
2	0.0771	0.0952	0.2752	0.3398	2	0.0775	0.1009	0.2730	0.3818
3	0.0769	0.0949	0.2734	0.3076	3	0.0772	0.1001	0.2709	0.3726
4	0.0769	0.0956	0.2751	0.3602	4	0.0761	0.1005	0.2700	0.3937
5	0.0768	0.0955	0.2732	0.3359	5	0.0769	0.1025	0.2688	0.3959
6	0.0767	0.0955	0.2733	0.3589	6	0.0769	0.1014	0.2719	0.3689
7	0.0771	0.0956	0.2730	0.3564	7	0.0775	0.1027	0.2684	0.3841
8	0.0768	0.0939	0.2733	0.3254	8	0.0773	0.0994	0.2702	0.3681
9	0.0768	0.0963	0.2746	0.3341	9	0.0768	0.1020	0.2704	0.3954
10	0.0768	0.0952	0.2728	0.3337	10	0.0772	0.1001	0.2709	0.3750
960 missing values					1920 missing values				
Times	Our model	Yang et al.'s model	Peng et al.'s model	Wang et al.'s model	Times	our model	Yang et al.'s model	Peng et al.'s model	Wang et al.'s model
	CE	CE	CE	CE		CE	CE	CE	CE
1	0.0773	0.1101	0.2650	0.4330	1	0.0780	0.1314	0.2546	0.4377
2	0.0784	0.1121	0.2648	0.4236	2	0.0776	0.1348	0.2574	0.4612
3	0.0776	0.1132	0.2629	0.4436	3	0.0788	0.1346	0.2501	0.4468
4	0.0780	0.1111	0.2631	0.4237	4	0.0773	0.1324	0.2558	0.4441
5	0.0771	0.1134	0.2668	0.4267	5	0.0798	0.1337	0.2623	0.4604
6	0.0772	0.0822	0.2660	0.4462	6	0.0775	0.1334	0.2519	0.4632
7	0.0775	0.1120	0.2667	0.4336	7	0.0780	0.1024	0.2500	0.4428
8	0.0776	0.1102	0.2669	0.4220	8	0.0781	0.0971	0.2571	0.4743
9	0.0777	0.1115	0.2668	0.4164	9	0.0785	0.1009	0.2541	0.4331
10	0.0776	0.1132	0.2628	0.4356	10	0.0796	0.1385	0.2523	0.4576

Table 13
Classification error rate (iris).

20 missing values					60 missing values				
Times	Our model	Yang et al.'s model	Peng et al.'s model	Wang et al.'s model	Times	our model	Yang et al.'s model	Peng et al.'s model	Wang et al.'s model
	CE	CE	CE	CE		CE	CE	CE	CE
1	0.0733	0.1067	0.0933	0.1933	1	0.0600	0.1400	0.1667	0.2867
2	0.0667	0.1133	0.0867	0.2867	2	0.0800	0.1467	0.1200	0.3267
3	0.0533	0.1133	0.1000	0.2067	3	0.0800	0.1200	0.1267	0.3067
4	0.0600	0.1200	0.1000	0.2667	4	0.0600	0.1667	0.1200	0.2867
5	0.0667	0.1267	0.1133	0.2000	5	0.0533	0.1400	0.1600	0.2733
6	0.0600	0.1067	0.0933	0.2000	6	0.0667	0.1800	0.1600	0.2533
7	0.0667	0.1333	0.1000	0.3333	7	0.0800	0.1133	0.1267	0.3267
8	0.0667	0.1200	0.1133	0.2000	8	0.0600	0.1733	0.1600	0.3333
9	0.0600	0.1400	0.0933	0.1867	9	0.0467	0.1467	0.1600	0.2600
10	0.0667	0.1200	0.1133	0.3333	10	0.0667	0.1533	0.1200	0.3533
100 missing values					140 missing values				
Times	Our model	Yang et al.'s model	Peng et al.'s model	Wang et al.'s model	Times	our model	Yang et al.'s model	Peng et al.'s model	Wang et al.'s model
	CE	CE	CE	CE		CE	CE	CE	CE
1	0.0800	0.1333	0.2267	0.2600	1	0.0800	0.1867	0.2600	0.3400
2	0.0667	0.1667	0.2267	0.3333	2	0.0733	0.2000	0.2467	0.1933
3	0.0600	0.1933	0.2000	0.3400	3	0.1000	0.1800	0.2467	0.1533
4	0.1000	0.1733	0.2400	0.2733	4	0.1000	0.1400	0.2667	0.2467
5	0.1000	0.1267	0.2333	0.2600	5	0.0667	0.2200	0.2333	0.1667
6	0.0867	0.1600	0.2067	0.2800	6	0.1067	0.1867	0.2600	0.3600
7	0.0733	0.2067	0.1867	0.4133	7	0.2200	0.1533	0.2467	0.1000
8	0.0600	0.1867	0.2267	0.3000	8	0.1067	0.2000	0.2200	0.2067
9	0.0667	0.1533	0.1867	0.3200	9	0.0733	0.1467	0.2600	0.1467
10	0.0867	0.1667	0.2067	0.1733	10	0.0733	0.2067	0.2800	0.4200

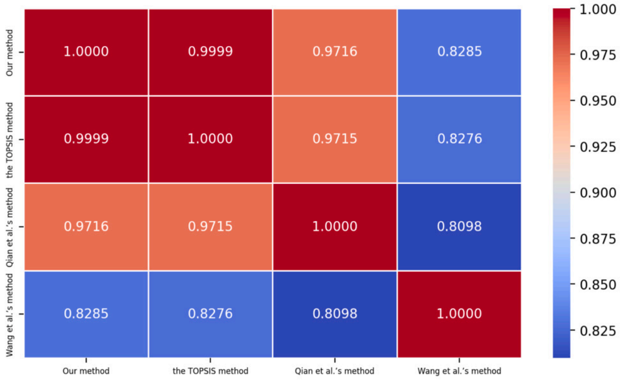

Fig. 3. SRCC (Auto MPG).

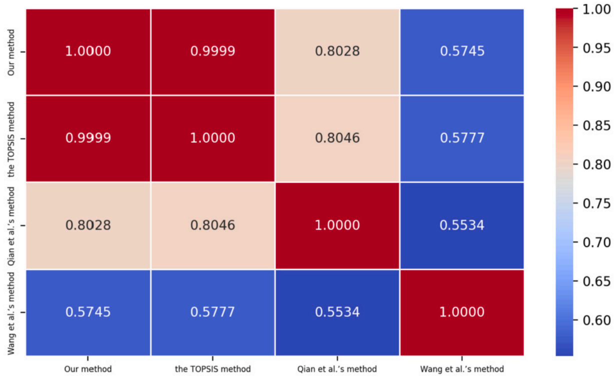

Fig. 4. SRCC (Travel Reviews).

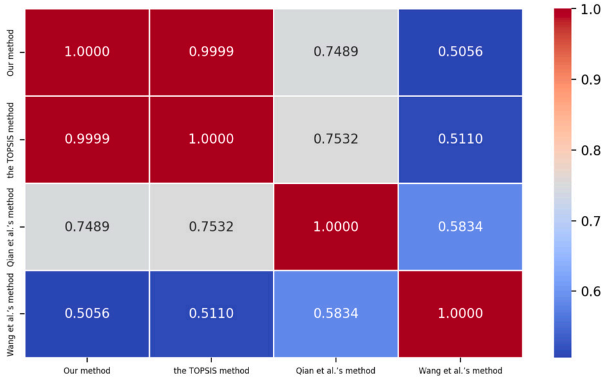

Fig. 5. SRCC (Wine Quality).

5. Conclusions

This paper presents a novel approach of three-way decision in incomplete ordered information systems. In contrast to previous research, the proposed method does not construct the binary relation directly. Instead, the filling value with confidence is obtained by Definition 14, resulting in a three-way decision model in a complete ordered information system with confidence. Similar to those models of three-way decision with decision-theoretic rough sets, this paper defines the relative loss function matrix and the dominance relation, and induces the corresponding three-way decision rules according to the minimum risk decision theory. Firstly, we use the ideas of TOPSIS method and regret theory to define a novel dominance relation between objects that reflects the advantages and disadvantages of objects and the degree of advantages and disadvantages and has good properties of reflexivity, symmetry, and transmission. Secondly, according to the relative distance in TOPSIS method, we introduce the fuzzy pre-decision and relative loss function matrix. Additionally, a method for calculating the membership degree between the object and the PIS and NIS classes is proposed. Ultimately, the decision rules analogous to the conventional three-way decision model are derived through the application

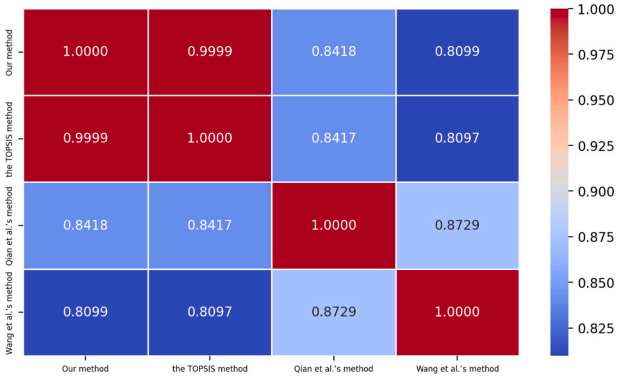

Fig. 6. SRCC (Energy Efficiency).

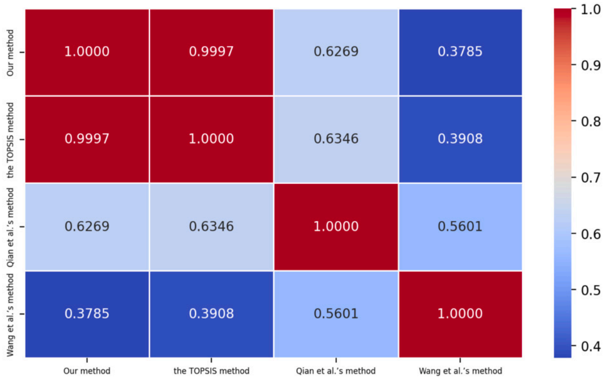

Fig. 7. SRCC (Concrete Compressive Strength).

of Bayesian minimum risk decision theory. In the final experimental section of the paper, we compare it with other models of three-way decision in incomplete ordered information systems. This paper is superior to the other three in terms of classification accuracy. Then, the three-way decision model of this paper is applied to the complete ordered information system, and the ranking results are highly consistent with the classical algorithm TOPSIS and also have good correlation with the other two algorithms.

The three-way decision model proposed in this paper is suitable for incomplete ordered information systems, and PIS and NIS need to be given. In our future work, we will consider extending the model to general incomplete information systems and propose an algorithm to find extreme values such as PIS and NIS.

CRediT authorship contribution statement

Zhao Li: Writing – original draft, Methodology, Conceptualization. **Ju Sheng Mi:** Writing – review & editing, Methodology, Conceptualization. **Lei Jun Li:** Writing – review & editing, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Nos. 62476078, 62076088, 61502144, 12101182); the Key Project of Natural Science Foundation of Hebei Province (No. F2023205006); the Key Development Fund of Hebei Normal University (Nos. L2024ZD06, L2024ZD11), the Special Science Foundation of Hebei Normal University (No. L2021T01).

Data availability

No data was used for the research described in the article.

References

[1] D.E. Bell, Regret in decision making under uncertainty, Oper. Res. 30 (1982) 961–981.
[2] F.E. Boran, S. Genç, M. Kurt, D. Akay, A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method, Expert Syst. Appl. 36 (2009) 11363–11368.
[3] C.T. Chen, Extensions of the TOPSIS for group decision-making under fuzzy environment, Fuzzy Sets Syst. 114 (2000) 1–9.
[4] J. Dai, B. Wei, X. Zhang, Q. Zhang, Uncertainty measurement for incomplete interval-valued information systems based on α-weak similarity, Knowl.-Based Syst. 136 (2017) 159–171.
[5] J. Dai, Q. Xu, Approximations and uncertainty measures in incomplete information systems, Inf. Sci. 198 (2012) 62–80.
[6] J. Deng, J. Zhan, E. Herrera-Viedma, F. Herrera, Regret theory-based three-way decision method on incomplete multiscale decision information systems with interval fuzzy numbers, IEEE Trans. Fuzzy Syst. 31 (2023) 982–996.
[7] W.S. Du, B.Q. Hu, Dominance-based rough set approach to incomplete ordered information systems, Inf. Sci. 346–347 (2016) 106–129.
[8] T. Gauthier, Detecting trends using Spearman's rank correlation coefficient, Environ. Forensics 2 (2001) 359–362.
[9] X. Han, J. Zhan, Y. Bao, B. Sun, Three-way group consensus with experts' attitudes based on probabilistic linguistic preference relations, Inf. Sci. 657 (2024) 119919.
[10] F. Jia, P. Liu, Multi-attribute three-way decisions based on ideal solutions under interval-valued intuitionistic fuzzy environment, Int. J. Approx. Reason. 138 (2021) 12–37.
[11] M. Kryszkiewicz, Rough set approach to incomplete information systems, Inf. Sci. 112 (1998) 39–49.
[12] J. Liang, Z. Shi, D. Li, M.J. Wierman, Information entropy, rough entropy and knowledge granulation in incomplete information systems, Int. J. Gen. Syst. 35 (2006) 641–654.
[13] B. Lin, W. Xu, Multi-granulation rough set for incomplete interval-valued decision information systems based on multi-threshold tolerance relation, Symmetry-Basel 10 (2018) 208.
[14] D. Liu, D. Liang, C. Wang, A novel three-way decision model based on incomplete information system, Knowl.-Based Syst. 91 (2016) 32–45.
[15] J. Luo, M. Hu, K. Qin, Three-way decision with incomplete information based on similarity and satisfiability, Int. J. Approx. Reason. 120 (2020) 151–183.
[16] L. Peng, X. Zhou, J. Zhao, Y. Sun, H. Li, Three-way multi-attribute decision making under incomplete mixed environments using probabilistic similarity, Inf. Sci. 614 (2022) 432–463.
[17] J. Qian, T. Wang, H. Jiang, Y. Yu, D. Miao, A TOPSIS method based on sequential three-way decision, Appl. Intell. 53 (2023) 30661–30676.
[18] H. Qin, Q. Peng, X. Ma, A novel interval-valued Fermatean fuzzy three-way decision making method with probability dominance relations, Expert Syst. Appl. 242 (2024) 122727.
[19] D. Selang, H. Zhang, Y. He, Three-way decision-making methods with multi-intuitionistic β-neighborhood-based multiattribute group decision-making problems, Inf. Sci. 659 (2024) 120063.
[20] M.W. Shao, W.X. Zhang, Dominance relation and rules in an incomplete ordered information system, Int. J. Intell. Syst. 20 (2005) 13–27.
[21] Z. Shi, L. Li, C. Bo, J. Zhan, Two novel three-way decision models based on fuzzy β -covering rough sets and prospect theory under q-rung orthopair fuzzy environments, Expert Syst. Appl. 244 (2024) 123050.
[22] X. Su, X. Yan, C.L. Tsai, Linear regression, WIREs: Comput. Stat. 4 (2012) 275–294.
[23] L. Sun, J. Xu, T. Xu, Information entropy and information granulation-based uncertainty measures in incomplete information systems, Appl. Math. Inf. Sci. 8 (2014) 2073–2083.
[24] A. Tan, W. Wu, J. Li, G. Lin, Evidence-theory-based numerical characterization of multigranulation rough sets in incomplete information systems, Fuzzy Sets Syst. 294 (2016) 18–35.
[25] J. Tu, S. Su, Method for three-way decisions using similarity in incomplete information systems, Int. J. Mach. Learn. Cybern. 14 (2023) 2053–2070.
[26] T. Wang, H. Li, Y. Qian, B. Huang, X. Zhou, A regret-based three-way decision model under interval type-2 fuzzy environment, IEEE Trans. Fuzzy Syst. 30 (2022) 175–189.
[27] W. Wang, J. Zhan, E. Herrera-Viedma, A three-way decision approach with a probability dominance relation based on prospect theory for incomplete information systems, Inf. Sci. 611 (2022) 199–224.
[28] W. Wang, J. Zhan, J. Mi, A three-way decision approach with probabilistic dominance relations under intuitionistic fuzzy information, Inf. Sci. 582 (2022) 114–145.
[29] W. Wang, J. Zhan, C. Zhang, E. Herrera-Viedma, G. Kou, A regret-theory-based three-way decision method with a priori probability tolerance dominance relation in fuzzy incomplete information systems, Inf. Fusion 89 (2023) 382–396.
[30] Y. Wang, P. Liu, Y. Yao, BMW-TOPSIS: a generalized TOPSIS model based on three-way decision, Inf. Sci. 607 (2022) 799–818.
[31] X.W. Xin, J.B. Sun, Z.A. Xue, J.H. Song, W.M. Peng, A novel intuitionistic fuzzy three-way decision model based on an intuitionistic fuzzy incomplete information system, Int. J. Mach. Learn. Cybern. 13 (2022) 907–927.
[32] H.L. Yang, S.Y. Xue, Y.H. She, General three-way decision models on incomplete information tables, Inf. Sci. 605 (2022) 136–158.
[33] X. Yang, T. Li, A. Tan, Three-way decisions in fuzzy incomplete information systems, Int. J. Mach. Learn. Cybern. 11 (2020) 667–674.
[34] X. Yang, Z. Lu, T.J. Li, Decision-theoretic rough sets in incomplete information system, Fundam. Inform. 126 (2013) 353–375.
[35] X. Yang, X. Zhou, B. Huang, H. Li, T. Wang, A three-way decision method on multi-scale single-valued neutrosophic decision systems, Artif. Intell. Rev. 57 (2024) 109.
[36] Y. Yao, Three-way decisions with probabilistic rough sets, Inf. Sci. 180 (2010) 341–353.
[37] J. Zhan, J. Ye, W. Ding, A novel three-way decision model based on utility theory in incomplete fuzzy decision systems, IEEE Trans. Fuzzy Syst. 30 (2022) 2210–2226.
[38] Q. Zhan, L. Jin, R.R. Yager, A modified topsis approach with three-way decision, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 31 (2023) 795–823.
[39] S. Zhang, J. Zhu, X. Liu, Y. Chen, Regret theory-based group decision-making with multidimensional preference and incomplete weight information, Inf. Fusion 31 (2016) 1–13.
[40] J. Zhu, X. Ma, L. Martínez, J. Zhan, A probabilistic linguistic three-way decision method with regret theory via fuzzy c-means clustering algorithm, IEEE Trans. Fuzzy Syst. 31 (2023) 2821–2835.