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The SMAA-TWD model: A novel stochastic multi-attribute three-way decision with interrelated attributes in triangular fuzzy information systems

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As a decision model to depict the human cognitive process, three-way decision (TWD) offers a reasonable semantic interpretation for solving practical multi-attribute decision-making (MADM) problems. Due to the complexity of the decision-making environment, uncertainties usually exist in multi-attribute three-way decision making problems. To stress these uncertainties simultaneously, a novel stochastic multi-attribute TWD model that incorporates TWD, ϵ-almost stochastic dominance, and stochastic multiobjective acceptability analysis (SMAA) is proposed for dealing with stochastic MADM problems with interrelated attributes in triangular fuzzy information systems. First, based on the ϵ-almost stochastic dominance, a novel ϵ-almost stochastic dominance degree is proposed for measuring the quantitative relationship of two triangular fuzzy numbers. Second, a novel stochastic TWD model is presented, in which the set of two states, conditional probability, and relative loss, can be obtained according to an information system. Third, the Choquet integral with respect to bi-capacity is utilized to aggregate the expected losses of three actions to strengthen their interpretability. Fourth, a SMAA-TWD model is proposed for multi-attribute TWDs with interrelated attributes in triangular fuzzy information systems. Finally, an application to medical diagnosis is given, and a comparative analysis is performed to verify the applicability and effectiveness of the proposed model.

1. Introduction

In real-world decision making processes, clear judgments (i.e., acceptance or rejection) can usually be made immediately when relevant decision conditions are fully grasped, while assessment tends to be postponed with partially known information. To adopt this primary cognitive mode of human beings, Yao [1] proposed a three-way decision (TWD) model based on decision-theoretic rough sets (DTRSs) in 2009. This decision model added three new semantic interpretations to the positive domain, negative domain, and boundary domain of the rough set, where the deferment decision corresponding to the bound-

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ary domain can largely avoid the loss caused by wrong choices [2,3]. Consequently, TWD has received considerable attention from researchers [4–6].

Two primary elements of the TWD model have been intensively studied, namely conditional probability and the loss function. Conditional probability is measured by equivalence class and state set, where the equivalence classes mainly include dominating and dominated relations [7], α-fuzzy neighborhood classes [8,9], probabilistic dominance relations [10,11], similarity degrees [12–18], fuzzy clustering algorithms [19], and outranking relations [20,21]. The state set can be determined in two ways. First, it is subjectively determined by a given decision attribute [13,7] or fuzzy set [20,12]. Second, it is objectively obtained by the information system. For example, Liang et al. [22] utilized the technique for order preference by similarity to ideal solution (TOPSIS) method to calculate the conditional probability. Similarly, Liang et al. [13] proposed a novel evaluation function for conditional probability under an interval type-2 fuzzy environment. Combining grey incidence analysis and the TOPSIS method, Du et al. [23] proposed a novel acceptance-rejection evaluation function of conditional probability. A state set based on Euclidean distance was defined by Wang et al. [21] and utilized to calculate conditional probability in hesitant fuzzy information systems. Liang et al. [24] utilized a power average operator and a fuzzy cross-entropy measure to obtain an overall conditional probability. Moreover, Wang et al. [11] defined a fuzzy set as a state set and estimated conditional probability through the proposed state set and dominance relation. The loss function is divided into two categories: subjective assessments by the experience of decision-makers(DMs) and objective judgments according to information systems. Generally, it is unrealistic for loss values to be directly given or to be the same for different actions in different states. Therefore, the objective method has become the primary research field. For example, Zhan et al. [12] proposed a relative utility function in an incomplete fuzzy decision system. Considering the DMs' risk attitude, Huang and Zhan [7] proposed a novel relative utility function combined with regret theory in the multi-scale decision information system. Liang et al. [13] proposed relative loss and benefit functions according to different kinds of aspirations of DMs. Wang et al. [25] proposed a regret-based utility function in an interval type-2 fuzzy information system. From the above analysis, the existing research on conditional probability and loss functions has shown both feasibility and superiority; however, there are two notable limitations:

(1) The existing research uses different fuzzy sets or fuzzy numbers to describe the uncertain environment [22,13,25,11,21,26,24,27,28], including triangular fuzzy numbers. According to the α-cut [29], the triangular fuzzy number can be translated into the confidence level interval. When α is a specific value chosen from $[0,1]$, it significantly reduces its superiority in dealing with uncertainty [30–32]. Therefore, a feasible method is needed when α is a stochastic variable estimated from $[0,1]$.
(2) Considering the multi-attribute TWD problem, both the conditional probability and loss function under each attribute need to be aggregated by additive measures [5,12]. The above procedures assume that no interrelations exist with the shortcoming of the hypothesis being inconsistent with reality [33]. To address this situation, the Choquet integral, one of the effective nonadditive measures, is utilized to describe the interactions between attributes [34]. For example, Liang et al. [13] utilized the Choquet integral to aggregate loss values involving interrelated attributes. However, in the method proposed by Liang et al. [13], the threshold nonadditive measures are subjectively given and different thresholds may cause different results. In addition, the decision results are likely to be affected by the scale of attribute measurements.

For the first problem, almost stochastic dominance [35–37] is a powerful tool to determine the qualitative relationship between two stochastic variables. As a flexible multi-attribute decision aid tool, stochastic multiobjective acceptability analysis (SMAA) can deal with the completely unknown or partially known input information in MADM [38,39]. Specifically, using the probability distribution to express uncertain information, SMAA calculates the additive utility function by randomly exploring the attribute weight space through Monte Carlo simulation, sorts objects by comparing their utility values, and calculates several indices to analyze the robust results. There are few studies on combining the SMAA and TWD model, especially when α is considered to be a stochastic variable that is estimated from $[0,1]$. For the second problem, there is no need for DMs to provide any further preference information on nonadditive measures if the SMAA is extended into the multi-attribute TWD model. Bi-capacity [40,41] is a generalization of fuzzy measures and can efficiently describe interrelated attributes in terms of bipolar scales. Subsequently, bi-capacity has attracted more widespread attention because of its general application significance. For example, Lin et al. [42] proposed a novel VIKOR (VIseKriterijumska Optimizacija I Kompromisno Resenje) (VlseKriterijumska Optimizacija IKompromisno Resenje), which combined the VIKOR and bi-capacity, for interaction attribute values with a bipolar scale. Zhang et al. [43] proposed a novel Bi-TOPSIS method for interrelated attributes with bipolar scale combined with bi-capacity. Thus, when the loss values corresponding to tree actions are measured by a bipolar scale, bi-capacity is suitable for a multi-attribute TWD model with interrelated attributes, but no attempt has been made to combine the two.

In summary, we propose a novel SMAA-TWD model for triangular fuzzy multi-attribute TWD problems involving interrelated attributes. Specifically, the main contributions of this study are summarized below: (1) Considering the interval uncertainty from randomness, the fuzzy interval value can be considered a stochastic variable estimated from the fuzzy interval value. Based on ϵ-almost stochastic dominance, a novel ϵ-almost stochastic dominance degree is defined to identify the quantitative relationship between two stochastic variables. (2) A novel TWD model combined with ϵ-almost stochastic dominance is proposed. In the model, a set of two states can be objectively determined. Based on ϵ-almost stochastic dominance, a stochastic dominance class of objects is proposed. The conditional probability can be obtained from the stochastic dominance class and degree. Inspired by the Hellinger distance [44], a novel relative loss function based on the distance measure is developed. (3) To enhance the interpretability of the decision results obtained by the novel TWD, it is reasonable to measure them using a bipolar scale. For a multi-attribute TWD with interrelated attributes, the Choquet integral with respect to bi-capacity is utilized to obtain the overall loss values of objects. Furthermore, the existing TWDs are inadequate for dealing with uncertain input information, for example, a parameter α that obeys a uniform distribution in the interval $[0,1]$. For this reason, a SMAA-TWD model is proposed by extending the SMAA into the novel proposed TWD to address uncertain input information. (4) Monte Carlo simulation of the proposed model is presented for obtaining three indices. By analyzing these indices, the classification results of multi-attribute TWDs are proven to be robust and reliable. A comparison and sensitivity analysis is also conducted to demonstrate the efficiency and practicability of our proposal.

The rest of this paper contains the following sections: Section 2 introduces some basic concepts of triangular fuzzy numbers, the Choquet integral with respect to bi-capacity, the decision-theoretic rough sets model and ϵ-almost stochastic dominance. A novel stochastic TWD model combined with ϵ-almost stochastic dominance is proposed in Section 3. In the proposed stochastic TWD model, the set of two states, conditional probability and loss function are derived by attribute assessments. Section 4 proposes a multi-attribute TWD involving interrelated attributes. Section 5 presents a SMAA-TWD model for multi-attribute TWDs involving interrelated attributes and Monte Carlo simulation to realize the proposed model. To verify the feasibility and practicability of the proposed model, an illustrative example of traditional Chinese medicine (TCM) diagnosis of insomnia is provided in Section 6. Furthermore, comparison, discussion and sensitivity analysis are also provided to demonstrate the efficiency and validity of the proposed model. Section 7 gives the conclusions of this paper and future research directions.

2. Preliminaries

In this section, we briefly review the concepts of triangular fuzzy numbers, Choquet integrals with respect to bi-capacity, the decision-theoretic rough sets model and ϵ-almost stochastic dominance.

2.1. Triangular fuzzy number

Definition 1 [45]. Let $T = (t^1, t^2, t^3)$ be a triangular fuzzy number with $0 < t^1 < t^2 < t^3 \in R$, its membership funcition is defined as follows:

$$\omega_T(x) = \begin{cases} \frac{x-t^1}{t^2-t^1}, & t^1 \leqslant x < t^2, \\ 1, & x = t^2, \\ \frac{t^3-x}{t^3-t^2}, & t^2 < x \leqslant t^3. \end{cases} \tag{1}$$

where t^1 and t^3 are the infimum and supremum of triangular fuzzy number, and t^2 is the median of triangular fuzzy number.

Definition 2 [29]. Let Q be a fuzzy set on discourse universe X, for $\alpha \in [0,1]$, α-cut set of Q is given as follows:

$$Q^\alpha = \{x \in X | \omega_Q(x) \geqslant \alpha\}. \tag{2}$$

For the triangular fuzzy number $T = (t^1, t^2, t^3)$ and $\alpha \in [0,1]$, the confidence level interval of T is given as follows:

$$T^\alpha = \left[t^1 + (t^2 - t^1)\alpha, t^3 - (t^3 - t^2)\alpha\right]. \tag{3}$$

2.2. Choquet integral with respect to bi-capacity

Let $M = \{1, 2, \ldots, m\}$ be a finite set, the power set $\mathscr{P}(M)$ is given by $\mathscr{P}(M) := \{S | S \subseteq M\}$.

Definition 3 (*[40,41]*). Let $\mathscr{Q}(M) = \{(M_1, M_2) \in \mathscr{P}(M) \times \mathscr{P}(M) | M_1 \cap M_2 = \varnothing\}$. A bi-capacity function $v : \mathscr{Q}(M) \to [-1, 1]$ requires the following conditions to be satisfied:

(1) $v(\varnothing, \varnothing) = 0$, $v(M, \cdot) = 1$, $v(\cdot, M) = -1$;
(2) For any disjoint subsets $M_1, M_2 \subseteq M$ and $M_1 \subseteq M_2$, we have $v(M_1, \cdot) \leqslant v(M_2, \cdot)$ and $v(\cdot, M_1) \geqslant v(\cdot, M_2)$.

For any $(M_1, M_2) \in \mathscr{Q}(M)$, the transformations between Möbius function ϑ and bi-capacity function v are given as follows [40,41]:

$$\vartheta(M_1, M_2) = \sum_{\substack{N_1 \subseteq M_1 \\ M_2 \subseteq N_2 \subseteq M_1^c}} (-1)^{|M_1 \setminus N_1| + |N_2 \setminus M_2|} v(N_1, N_2), \tag{4}$$

$$v(M_1, M_2) = \sum_{(N_1, N_2) \sqsubseteq (M_1, M_2)} \vartheta(N_1, N_2). \tag{5}$$

For any $h \in M$, the Shapley values $I_{h,\varnothing}$ and $I_{\varnothing,h}$ of ϑ are given as follows [40,41]:

$$I_{h,\varnothing} = \sum_{(h, h^c) \sqsubseteq (M_1, M_2)} \frac{1}{m - m_2} \vartheta(M_1, M_2), \tag{6}$$

$$I_{\varnothing,h} = \sum_{(\varnothing, h^c) \sqsubseteq (M_1, M_2), (M_1, M_2) \in \mathcal{Q}(M \setminus h)} \frac{1}{m - m_2} \vartheta(M_1, M_2), \tag{7}$$

where m and m_2 are the numbers of elements in sets M and M_2, respectively.

For any $(M_1, M_2) \in \mathcal{Q}(M)$, the interaction transform I_{M_1, M_2} of ϑ is given as follows:

$$I_{M_1, M_2} = \sum_{(M_1\prime, M_2\prime) \in [(M_1, M \setminus (M_1 \cup M_2)), (M \setminus M_2, \varnothing)]} \frac{\vartheta(M_1\prime, M_2\prime)}{m - m_1 - m_2 - m_2\prime + 1}, \tag{8}$$

where m and m_1 are the numbers of elements in the sets M and M_1, and m_2 and $m_2\prime$ are the numbers of elements in the sets M_2 and $M_2\prime$, respectively.

Given the Möbius function ϑ and a real-valued function f on M, the Choquet integral is given as follows [40,41]:

$$C_\vartheta(f) = \sum_{M_2 \subseteq M} \vartheta(\varnothing, M_2) \left(\bigwedge_{j \in M_2^c \cap M^-} f_j \right) + \sum_{\substack{(M_1, M_2) \in \mathcal{Q}(M) \\ M_1 \neq \varnothing}} \vartheta(M_1, M_2) \left[\left(\bigwedge_{j \in (M_1 \cup M_2)^c \cap M^-} f_j + \bigwedge_{j \in M_1} f_j \right) \vee 0 \right], \tag{9}$$

with the convention $\bigwedge_\varnothing f_j = 0$, where $\bigwedge$ represents the operation of minimum, $M^+ = \{ j \in M | f_j \geqslant 0 \}$ and $M^- = \{ j \in M | f_j < 0 \}$.

2.3. Decision-theoretic rough sets model

Using ideas from Bayesian decision theory [46,47], the DTRS model describes the process of TWD with two states and three actions. The set of two states is denoted $\Omega = \{C, \neg C\}$, indicating that an object either belongs to state C or does not belong to state C. The set of three actions is denoted $\Pi = \{\pi_P, \pi_B, \pi_N\}$, where π_P denotes accepting object o_i i.e., $o_i \in POS(C), \pi_B$ denotes needing further analysis for object o_i, i.e., $o_i \in BND(C)$, and π_N denotes rejecting object o_i, i.e., $o_i \in NEG(C)$. The DTRS model contains two key factors: (1) the loss function, and (2) the conditional probability. The loss functions for the cost of three actions in two states are shown in Table 1.

In Table 1, $\lambda_{PP}, \lambda_{BP}$ and λ_{NP} represent the loss resulting from taking three actions when the object belongs to state C, while $\lambda_{PN}, \lambda_{BN}$ and λ_{NN} represent the loss resulting from taking three actions when the object belongs to state $\neg C$. The loss functions satisfy the following requirements: $\lambda_{PP} \leqslant \lambda_{BP} < \lambda_{NP}$ and $\lambda_{NN} \leqslant \lambda_{BN} < \lambda_{PN}$. If $\Pr(C|[o_i])$ denotes the conditional probability that any object belongs to C under the condition of belonging to the equivalence class $[o_i]$, we have $\Pr(C|[o_i]) + \Pr(\neg C|[o_i]) = 1$, where $[o_i]$ is the equivalence class of the object o_i.

Based on Bayesian decision making process, the expected losses of three actions $\Phi(\pi_\Delta|[o_i])$ $(\Delta = P, B, N)$ are calculated as follows:

$\Phi(\pi_P|[o_i]) = \lambda_{PP}\Pr(C|[o_i]) + \lambda_{PN}\Pr(\neg C|[o_i])$;
$\Phi(\pi_B|[o_i]) = \lambda_{BP}\Pr(C|[o_i]) + \lambda_{BN}\Pr(\neg C|[o_i])$;
$\Phi(\pi_N|[o_i]) = \lambda_{NP}\Pr(C|[o_i]) + \lambda_{NN}\Pr(\neg C|[o_i])$.

Thus, through comparing the three expected losses, the decision rules (P)-(N) of TWD are given as follows:

(P) If $\Phi(\pi_P|[o_i]) \leqslant \Phi(\pi_B|[o_i])$ and $\Phi(\pi_P|[o_i]) \leqslant \Phi(\pi_N|[o_i])$, then $o_i \in POS(C)$;
(B) If $\Phi(\pi_B|[o_i]) \leqslant \Phi(\pi_P|[o_i])$ and $\Phi(\pi_B|[o_i]) \leqslant \Phi(\pi_N|[o_i])$, then $o_i \in BND(C)$;

Table 1
The loss functions about the cost of three actions in two states.

	$C(P)$	$\neg C(N)$
π_P	λ_{PP}	λ_{PN}
π_B	λ_{BP}	λ_{BN}
π_N	λ_{NP}	λ_{NN}

(N) If $\Phi(\pi_N|[o_i]) \leqslant \Phi(\pi_P|[o_i])$ and $\Phi(\pi_N|[o_i]) \leqslant \Phi(\pi_B|[o_i])$, then $o_i \in NEG(C)$.

2.4. ϵ-almost stochastic dominance

Leshno and Levy [35] first proposed the concept of almost stochastic dominance. Tzeng et al. [36] later proposed an alternative version of almost stochastic dominance that resolved some issues with Leshno and Levy's orginal theorem. Based on these two versions of almost stochastic dominance, Chang et al. [37] proposed the concept of ϵ-almost stochastic dominance, which is explained below.

Let U_1 be the set of all non-decreasing differentiable utility function, i.e., $U_1 = \{u|u\prime \geqslant 0\}$ and U_2 be the set of all non-decreasing and concave function, i.e., $U_2 = \{u|u\prime \geqslant 0\, and\, u\prime\prime \leq 0\}$. For $0 < \epsilon < 0.5$, with some restrictions, the subsets of U_1 and U_2 are given as below.

$$U_1^*(\epsilon) = \left\{u \in U_1 : u\prime(x) \leqslant inf[u\prime(x)]\left(\tfrac{1}{\epsilon} - 1\right), \forall x \in [\underline{a}, \bar{a}]\right\},$$
$$U_2^*(\epsilon) = \left\{u \in U_2 : -u\prime\prime(x) \leqslant inf[-u\prime\prime(x)]\left(\tfrac{1}{\epsilon} - 1\right), \forall x \in [\underline{a}, \bar{a}]\right\}.$$

Let F and G be two cumulative distribution functions of variables Y and Z, respectively, which are defined on a finite interval $[\underline{a}, \bar{a}]$ $(-\infty < \underline{a} < \bar{a} < +\infty)$.

Definition 4 [37]. For $0 < \epsilon < 0.5, F$ dominates G by ϵ-almost first stochastic dominance if and only if $\epsilon_1 \leqslant \epsilon$, with

$$\epsilon_1 = \frac{\int_{S_1(F,G)} (F(x) - G(x))dx}{\int_{S(F,G)} |F(x) - G(x)|dx}, \tag{10}$$

where $S(F,G) = \{x : x \in [\underline{a}, \bar{a}]\}$ and $S_1(F,G) = \{x \in S(F,G) : G(x) < F(x)\}$.

Theorem 1 [37]. F dominates G by ϵ-almost first stochastic dominance if and only if $E_F[u(Y)] \geqslant E_G[u(Z)]$ for all $u \in U_1^*(\epsilon)$.

Definition 5 [37]. For $0 < \epsilon < 0.5, F$ dominates G by ϵ-almost second stochastic dominance if and only if $\epsilon_2 \leqslant \epsilon$, with

$$\epsilon_2 = \frac{\int_{S_2} (F(x) - G(x))dx}{\int_{S_2} (F(x) - G(x))dx + \int_{\bar{S}_2} (G(x) - F(x))dx}, \tag{11}$$

where $S_2(F,G) = \left\{x \in S_1(F,G) : G^{(2)}(x) < F^{(2)}(x)\right\}, \bar{S}_2(F,G) = \left\{x \in S_1(F,G) : G^{(2)}(x) \geqslant F^{(2)}(x)\right\}$, $S_1(F,G) = \{x \in S(F,G) : G(x) < F(x)\}, S(F,G) = \{x : x \in [\underline{a}, \bar{a}]\}, F^{(2)}(x) = \int_{\underline{a}}^{x} F(y)dy$ and $G^{(2)}(x) = \int_{\underline{a}}^{x} G(y)dy$.

Theorem 2. [37] F dominates G by ϵ-almost second stochastic dominance if and only if $E_F[u(Y)] \geqslant E_G[u(Z)]$ for all $u \in U_2^*(\epsilon)$.

3. A novel TWD model combined with ϵ-almost stochastic dominance

In this section, we first define a novel ϵ-almost stochastic dominance. Second, a new set of two states by the ϵ-almost stochastic dominance degree is presented. Third, a novel class of objects under each attribute based on the ϵ-almost stochastic dominance can be defined. Finally, inspired by the Hellinger distance [44], a novel relative loss function combined with a distance measure is proposed. On the basis of the above study, a novel TWD model combined with a ϵ-almost stochastic dominance is presented.

3.1. A novel ϵ-almost stochastic dominance degree

Definition 6 [48]. An information system is given as a quadruple $\mathscr{S} = \{\mathscr{O}, \mathscr{A}, \mathscr{V}, \mathscr{F}\}, \mathscr{O} = \{o_1, o_2, \ldots, o_n\}$is a nonempty finite set of n objects, and $\mathscr{A} = \{c_1, c_2, \ldots, c_m\}$ is a nonempty finite set of m attributes and $\mathscr{V} = \bigcup_{c_j \in \mathscr{A}} \mathscr{V}_{c_j}$, where $\mathscr{V}_{c_j}$ is the domain of values for the attribute c_j, and $\mathscr{F} = \left\{\mathscr{F}_j : \mathscr{O} \to \mathscr{V}_{c_j}\right\}$ is a mapping set, such that $\mathscr{F}(o_i, c_j) = c_j(o_i) = t_{ij} \in \mathscr{V}_{c_j}$ for any $o_i \in \mathscr{O}$.

The information system is shown in Table 2.

According to Definition 2, a triangular fuzzy number can be considered a fuzzy interval number by using an α-cut. From Definitions 4 and 5, ϵ-almost first stochastic dominance and ϵ-almost second stochastic dominance are effective dealing with these uncertainties by introducing probability distributions. In what follows, we will define the qualitative and quantitative relationship of two triangular fuzzy numbers in the information system $\mathscr{S} = \{\mathscr{O}, \mathscr{A}, \mathscr{V}, \mathscr{F}\}$.

Table 2
An information system $\mathscr{S} = \{\mathscr{O}, \mathscr{A}, \mathscr{V}, \mathscr{F}\}$.

$\mathscr{O}/\mathscr{A}$	c_1	c_2	$\cdots$	c_m
o_1	t_{11}	t_{12}	$\cdots$	t_{1m}
o_2	t_{21}	t_{22}	$\cdots$	t_{2m}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
o_n	t_{n1}	t_{n2}	$\cdots$	t_{nm}

Definition 7. In the information system $\mathscr{S} = \{\mathscr{O}, \mathscr{A}, \mathscr{V}, \mathscr{F}\}$, let triangular fuzzy numbers $t_{ij} = \left(t_{ij}^1, t_{ij}^2, t_{ij}^3\right)$ and $t_{kj} = \left(t_{kj}^1, t_{kj}^2, t_{kj}^3\right)$ be the values of objects o_i and o_k with respect to the attribute c_j, i.e., $t_{ij}, t_{kj} \in \mathscr{V}_{c_j}$. Given a parameter $\alpha \in [0,1]$, confidence level intervals of t_{ij} and t_{kj} are $t_{ij}^\alpha = \left[t_{ij}^1 + \left(t_{ij}^2 - t_{ij}^1\right)\alpha, t_{ij}^3 - \left(t_{ij}^3 - t_{ij}^2\right)\alpha\right]$ and $t_{kj}^\alpha = \left[t_{kj}^1 + \left(t_{kj}^2 - t_{kj}^1\right)\alpha, t_{kj}^3 - \left(t_{kj}^3 - t_{kj}^2\right)\alpha\right]$, respectively. x_{ij} and x_{kj} are variables estimated from intervals t_{ij}^α and t_{kj}^α, respectively. Let F and G be the cumulative distribution functions of random variables x_{ij} and x_{kj}, respectively.

(1) For $0 < \epsilon < 0.5, o_i$ dominates o_k under the attribute c_j by ϵ-almost first stochastic dominance, denoted as $o_i \succeq^j_{\epsilon-AFSD} o_k$ if and only if $\epsilon_1 < \epsilon$, with

$$\epsilon_1 = \frac{\int_{S_1(F,G)} (F(t) - G(t))dt}{\int_{S(F,G)} |F(t) - G(t)|dt}, \tag{12}$$

where $S(F,G) = \{t \in [\underline{t}, \bar{t}]\}, S_1(F,G) = \{t \in S(F,G) : G(t) < F(t)\}, \underline{t} = \min\left\{t_{ij}^1 + \left(t_{ij}^2 - t_{ij}^1\right)\alpha, t_{kj}^1 + \left(t_{kj}^2 - t_{kj}^1\right)\alpha\right\}$ and $\bar{t} = \max\left\{t_{ij}^3 - \left(t_{ij}^3 - t_{ij}^2\right)\alpha, t_{kj}^3 - \left(t_{kj}^3 - t_{kj}^2\right)\alpha\right\}$.

(2) For $0 < \epsilon < 0.5, o_i$ dominates o_k under the attribute c_j by ϵ-almost second stochastic dominance, denoted as $o_i \succeq^j_{\epsilon-ASSD} o_k$ if and only if $\epsilon_2 < \epsilon$, with

$$\epsilon_2 = \frac{\int_{S_2(F,G)} (F(t) - G(t))dt}{\int_{S_2(F,G)} (F(t) - G(t))dt + \int_{\bar{S}_2(F,G)} (G(t) - F(t))dt}, \tag{13}$$

where $S_2(F,G) = \left\{t \in S_1(F,G) : G^{(2)}(t) < F^{(2)}(t)\right\}, \bar{S}_2(F,G) = \left\{t \in S_1(F,G) : G^{(2)}(t) \geqslant F^{(2)}(t)\right\}$, $S_1(F,G) = \{t \in S(F,G) : G(t) < F(t)\}, S(F,G) = \{t \in [\underline{t}, \bar{t}]\}, F^{(2)}(t) = \int_{\underline{t}}^t F(x)dx, G^{(2)}(t) = \int_{\underline{t}}^t G(x)dx, \underline{t} = \min\left\{t_{ij}^1 + \left(t_{ij}^2 - t_{ij}^1\right)\alpha, t_{kj}^1 + \left(t_{kj}^2 - t_{kj}^1\right)\alpha\right\}$ and $\bar{t} = \max\left\{t_{ij}^3 - \left(t_{ij}^3 - t_{ij}^2\right)\alpha, t_{kj}^3 - \left(t_{kj}^3 - t_{kj}^2\right)\alpha\right\}$.

Here, we assume that variables x_{ij} and x_{kj} obey normal distribution, i.e., $x_{ij} \sim N\left(\mu_{x_{ij}}, \sigma^2_{x_{ij}}\right)$ and $x_{kj} \sim N\left(\mu_{x_{kj}}, \sigma^2_{x_{kj}}\right)$. According to 3σ principle [49] in the knowledge of probability and statistics, x_{ij} and x_{kj} are covered by a probability of 99.73% in the intervals $t_{ij}^\alpha = \left[t_{ij}^1 + \left(t_{ij}^2 - t_{ij}^1\right)\alpha, t_{ij}^3 - \left(t_{ij}^3 - t_{ij}^2\right)\alpha\right]$ and $t_{kj}^\alpha = \left[t_{kj}^1 + \left(t_{kj}^2 - t_{kj}^1\right)\alpha, t_{kj}^3 - \left(t_{kj}^3 - t_{kj}^2\right)\alpha\right]$. Therefore, expected values and standard deviations of variables x_{ij} and x_{kj} are represented as follows:

$$\mu_{x_{ij}} = \frac{2t_{ij}^2\alpha + (1-\alpha)\left(t_{ij}^1 + t_{ij}^3\right)}{2} \text{ and } \sigma_{x_{ij}} = \frac{\left(t_{ij}^3 - t_{ij}^1\right)(1-\alpha)}{6};$$

$$\mu_{x_{kj}} = \frac{2t_{kj}^2\alpha + (1-\alpha)\left(t_{kj}^1 + t_{kj}^3\right)}{2} \text{ and } \sigma_{x_{kj}} = \frac{\left(t_{kj}^3 - t_{kj}^1\right)(1-\alpha)}{6}.$$

Example 1. Given an information system $\mathscr{S} = \{\mathscr{O}, \mathscr{A}, \mathscr{V}, \mathscr{F}\}$, let $\mathscr{O} = \{o_1, o_2, o_3, o_4, o_5\}$ represent the set of patients with sleep disorders and $\mathscr{A} = \{c_1, c_2, c_3, c_4\}$ represent the set of attributes, i.e., inspection(c_1), smelling(c_2), listening(c_3), and interrogation and palpation(c_4). For each attribute c_j, the values in domains $\mathscr{V}_{c_j}$ are expressed by triangular fuzzy numbers. Then, the information system is shown in Table 3.

Suppose $\alpha = 0.75$ and the confidence level intervals of triangular fuzzy numbers of object $o_i (i = 1,2,3,4,5)$ under attribute c_1 are $[0.188, 0.242]$, $[0.148, 0.342]$, $[0.100, 0.333]$, $[0.240, 0.368]$ and $[0.190, 0.258]$, respectively. Let $x_{11}, x_{21}, x_{31}, x_{41}$ and x_{51} be the estimated variables from previous intervals that obey the following normal distributions: $x_{11} \sim N\left(0.215, 0.009^2\right), x_{21} \sim N\left(0.245, 0.033^2\right), x_{31} \sim N\left(0.216, 0.039^2\right), x_{41} \sim N\left(0.304, 0.021^2\right)$ and $x_{51} \sim N\left(0.224, 0.011^2\right)$. By Definition 7, we can identify the ϵ-almost first stochastic dominance of the object o_2 and other

Table 3
An information system $\mathscr{S} = \{\mathscr{O}, \mathscr{A}, \mathscr{V}, \mathscr{F}\}$.

$\mathscr{O}/\mathscr{A}$	c_1	c_2	c_3	c_4
o_1	$(0.15, 0.20, 0.37)$	$(0.10, 0.60, 1.00)$	$(0.24, 0.46, 0.77)$	$(0.13, 0.17, 0.95)$
o_2	$(0.02, 0.19, 0.80)$	$(0.31, 0.55, 0.85)$	$(0.31, 0.41, 0.90)$	$(0.15, 0.68, 1.00)$
o_3	$(0.07, 0.11, 1.00)$	$(0.12, 0.67, 0.79)$	$(0.07, 0.14, 0.95)$	$(0.12, 0.41, 0.88)$
o_4	$(0.21, 0.25, 0.72)$	$(0.17, 0.58, 0.89)$	$(0.15, 0.29, 0.90)$	$(0.37, 0.61, 0.77)$
o_5	$(0.16, 0.20, 0.43)$	$(0.25, 0.61, 1.00)$	$(0.32, 0.53, 0.83)$	$(0.18, 0.62, 0.80)$

objects under the attribute c_1 with $\epsilon = 0.47$. We have $o_2 \succeq^1_{\epsilon-AFSD} o_1, o_2 \succeq^1_{\epsilon-AFSD} o_3, o_2 \succeq^1_{\epsilon-AFSD} o_5$ and $o_4 \succeq^1_{\epsilon-AFSD} o_2$, since $\epsilon_1^{21} = 0.036 < \epsilon, \epsilon_1^{23} = 0 < \epsilon, \epsilon_1^{25} = 0.073 < \epsilon$ and $\epsilon_1^{42} = 0 < \epsilon$. The ϵ-almost second stochastic dominance can be identified in a similar manner, which we omit here.

Proposition 1. For any three triangular fuzzy numbers $t_{ij}, t_{kj}, t_{lj} \in \mathscr{V}_{c_j}$,

(1) If $t_{ij} \succeq^j_{\epsilon-AFSD} t_{kj}$, then $t_{ij} \succeq^j_{\epsilon-ASSD} t_{kj}$;
(2) If $t_{ij} \succeq^j_{\epsilon-AFSD} t_{kj}$ and $t_{kj} \succeq^j_{\epsilon-AFSD} t_{lj}$, then $t_{ij} \succeq^j_{\epsilon-AFSD} t_{lj}$;
(3) If $t_{ij} \succeq^j_{\epsilon-ASSD} t_{kj}$ and $t_{kj} \succeq^j_{\epsilon-ASSD} t_{lj}$, then $t_{ij} \succeq^j_{\epsilon-ASSD} t_{lj}$.

Proof.

(1) From Definition 7, we have

$$\epsilon_1 = \frac{\int_{S_1(F,G)} (F(t) - G(t))dt}{\int_{S(F,G)} (F(t) - G(t))dt} = \frac{\int_{S_1(F,G)} (F(t) - G(t))dt}{\int_{S_1(F,G)} (F(t) - G(t))dt + \int_{\bar{S}_1(F,G)} (G(t) - F(t))dt}.$$

According to $\bar{S}_2(F,G) = \bar{S}_1(F,G) + [S_1(F,G) - S_2(F,G)]$, we have

$$\begin{aligned}\epsilon_2 &= \frac{\int_{S_2(F,G)} (F(t)-G(t))dt}{\int_{S_2(F,G)} (F(t)-G(t))dt + \int_{\bar{S}_2(F,G)} (G(t)-F(t))dt} \\ &= \frac{\int_{S_2(F,G)} (F(t)-G(t))dt}{\int_{S_2(F,G)} (F(t)-G(t))dt + \int_{\bar{S}_1(F,G)} (G(t)-F(t))dt + \int_{S_1(F,G)} (G(t)-F(t))dt - \int_{S_2(F,G)} (G(t)-F(t))dt}.\end{aligned}$$

Let $p_1 = \int_{S_1(F,G)} (F(t) - G(t))dt > 0, p_2 = \int_{\bar{S}_1(F,G)} (G(t) - F(t))dt > 0$ and $p_3 = \int_{S_2(F,G)} (F(t) - G(t))dt > 0$. The expressions of ϵ_1 and ϵ_2 can be rewritten as $\epsilon_1 = \frac{p_1}{p_1 + (p_2 - p_1) + p_1}$ and $\epsilon_2 = \frac{p_3}{p_3 + (p_2 - p_1) + p_3}$, respectively. If $t_{ij} \succeq^j_{\epsilon-AFSD} t_{kj}$, we have $\epsilon_1 < \epsilon < 0.5$. Since $S_2(F,G) \subset S_1(F,G)$, we have $p_1 \geqslant p_3$. At last, we have $\epsilon_2 < \epsilon_1 < \epsilon < 0.5$.

(2) Let $\breve{U}_1^*(\epsilon) = \{u \in U_1 : u\prime(t) \leqslant inf\{u\prime(t)\}(\frac{1}{\epsilon} - 1), \forall t \in [\underline{t}, \bar{t}]\}$, where $\underline{t} = \min\left\{t_{ij}^1 + \left(t_{ij}^2 - t_{ij}^1\right)\alpha, t_{kj}^1 + \left(t_{kj}^2 - t_{kj}^1\right)\alpha, t_{lj}^1 + \left(t_{lj}^2 - t_{lj}^1\right)\alpha\right\}$ and $\bar{t} = \max\left\{t_{ij}^3 - \left(t_{ij}^3 - t_{ij}^2\right)\alpha, t_{kj}^3 - \left(t_{kj}^3 - t_{kj}^2\right)\alpha, t_{lj}^3 - \left(t_{lj}^3 - t_{lj}^2\right)\alpha\right\}$. According to the sufficient condition of Theorem 1, if $t_{ij} \succeq^j_{\epsilon-AFSD} t_{kj}$ and $t_{kj} \succeq^j_{\epsilon-AFSD} t_{lj}$, we have $E_F[u(x_{ij})] \geqslant E_G[u(x_{kj})]$ and $E_G[u(x_{kj})] \geqslant E_H[u(x_{lj})]$ for all $u \in \breve{U}_1^*(\epsilon)$. Thus, $E_F[u(x_{ij})] \geqslant E_H[u(x_{lj})]$. According to the necessary condition of Theorem 1, we have $t_{ij} \succeq^j_{\epsilon-AFSD} t_{lj}$.

(3) Let $\breve{U}_2^*(\epsilon) = \{u \in U_2 : -u\prime\prime(t) \leqslant inf\{-u\prime\prime(t)\}(\frac{1}{\epsilon} - 1), \forall t \in [\underline{t}, \bar{t}]\}$, where $\underline{t} = \min\left\{t_{ij}^1 + \left(t_{ij}^2 - t_{ij}^1\right)\alpha, t_{kj}^1 + \left(t_{kj}^2 - t_{kj}^1\right)\alpha, t_{lj}^1 + \left(t_{lj}^2 - t_{lj}^1\right)\alpha\right\}$ and $\bar{t} = \max\left\{t_{ij}^3 - \left(t_{ij}^3 - t_{ij}^2\right)\alpha, t_{kj}^3 - \left(t_{kj}^3 - t_{kj}^2\right)\alpha, t_{lj}^3 - \left(t_{lj}^3 - t_{lj}^2\right)\alpha\right\}$. According to the sufficient condition of Theorem 2, if $t_{ij} \succeq^j_{\epsilon-ASSD} t_{kj}$ and $t_{kj} \succeq^j_{\epsilon-ASSD} t_{lj}$, we have $E_F[u(x_{ij})] \geqslant E_G[u(x_{kj})]$ and $E_G[u(x_{kj})] \geqslant E_H[u(x_{lj})]$ for all $u \in \breve{U}_2^*(\epsilon)$. Thus, we have the inequality $E_F[u(x_{ij})] \geqslant E_H[u(x_{lj})]$. According to the necessary condition of Theorem 2, we have $t_{ij} \succeq^j_{\epsilon-ASSD} t_{lj}$.

According to Proposition 1, for any two triangular fuzzy numbers with respect to the attribute c_j, if the ϵ-almost first stochastic dominance relationship ($\succeq^j_{\epsilon-AFSD}$) is not met, we can further discuss the ϵ-almost second stochastic dominance relationship ($\succeq^j_{\epsilon-ASSD}$). As long as one of the two relationships is satisfied, we consider the two triangular fuzzy numbers as satisfying the ϵ-almost stochastic dominance relationship, denoted as $\succeq^j_{\epsilon-ASD}$. Based on the ϵ-almost stochastic dominance relationship, a ϵ-almost stochastic dominance degree is given as follows.

Definition 8. Let $\mathscr{S} = \{\mathscr{O}, \mathscr{A}, \mathscr{V}, \mathscr{F}\}$ be an information system and the triangular fuzzy numbers $t_{ij} = \left(t_{ij}^1, t_{ij}^2, t_{ij}^3\right)$ and $t_{kj} = \left(t_{kj}^1, t_{kj}^2, t_{kj}^3\right)$ be the values of objects o_i and o_k with respect to the attribute c_j, i.e., $t_{ij}, t_{kj} \in \mathscr{V}_{c_j}$. Given a parameter $\alpha \in [0, 1]$, the confidence level intervals of t_{ij} and t_{kj} are $t_{ij}^{\alpha} = \left[t_{ij}^1 + \left(t_{ij}^2 - t_{ij}^1\right)\alpha, t_{ij}^3 - \left(t_{ij}^3 - t_{ij}^2\right)\alpha\right]$ and $t_{kj}^{\alpha} = \left[t_{kj}^1 + \left(t_{kj}^2 - t_{kj}^1\right)\alpha, t_{kj}^3 - \left(t_{kj}^3 - t_{kj}^2\right)\alpha\right]$, respectively. x_{ij} and x_{kj} are variables estimated from intervals t_{ij}^{α} and t_{kj}^{α}, respectively. For $0 < \epsilon < 0.5$, if $t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}$, then ϵ-almost stochastic dominance degree is defined as follows:

$$D\left(t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}\right) = (1-\epsilon)^{\frac{\left(\sigma_{x_{ij}} + \sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)}}, \tag{14}$$

where $\mu_{x_{ij}}$ and $\mu_{x_{kj}}$ are the expected values of x_{ij} and x_{kj}, respectively; $\sigma_{x_{ij}}$ and $\sigma_{x_{kj}}$ are the standard deviations of x_{ij} and x_{kj}, respectively.

Example 2. (Continue with **Example 1**) From **Example 1**, with $\epsilon = 0.47$ and $\alpha = 0.75$, o_2 dominates other objects under the attribute c_1 by ϵ-almost stochastic dominance can be identified, i.e., $t_{21} \succeq^1_{\epsilon-ASD} t_{11}, t_{21} \succeq^1_{\epsilon-ASD} t_{31}$ and $t_{21} \succeq^1_{\epsilon-ASD} t_{51}$. Then, the corresponding ϵ-almost stochastic dominance degree under the attribute c_1 are given as follows: $D(t_{21} \succeq^1_{\epsilon-ASD} t_{11}) = 0.414$, $D(t_{21} \succeq^1_{\epsilon-ASD} t_{31}) = 0.207$, and $D(t_{21} \succeq^1_{\epsilon-ASD} t_{51}) = 0.271$.

Note 1. According to the propositions of ϵ-almost stochastic dominance [37], if $t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}$, then $\mu_{x_{ij}} > \mu_{x_{kj}}$. Therefore, the novel ϵ-almost stochastic dominance degree is nonnegative and finite. Below, we will give more specific properties of the ϵ-almost stochastic dominance degree.

Proposition 2. For any two triangular fuzzy numbers $t_{ij}, t_{kj} \in \mathscr{V}_{c_j}$, if $t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}$, the ϵ-almost stochastic dominance degree has the following properties:

(1) $0 < D\left(t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}\right) \leqslant 1$;

(2) When the value of $\frac{\left(\sigma_{x_{ij}} + \sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)}$ is fixed, $D\left(t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}\right)$ is monotonically increasing with the decreasing of ϵ;

(3) When the values of ϵ and $\left(\sigma_{x_{ij}} + \sigma_{x_{kj}}\right)$ are fixed, $D\left(t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}\right)$ is monotonically increasing with the increasing of $\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)$;

(4) When the values of ϵ and $\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)$ are fixed, $D\left(t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}\right)$ is monotonically increasing with the decreasing of $\left(\sigma_{x_{ij}} + \sigma_{x_{kj}}\right)$.

Proof.

(1) If $t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}$, then we have $\mu_{x_{ij}} > \mu_{x_{kj}}$. According to $0 < \epsilon < 0.5$, we have $0.5 < 1 - \epsilon < 1$. Based on the properties of the power function, when $\frac{\left(\sigma_{x_{ij}} + \sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)} > 0, 0 < (1-\epsilon)^{\frac{\left(\sigma_{x_{ij}} + \sigma_{x_{kj}} s\right)}{\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)}} \leqslant 1$. Especially, $(1-\epsilon)^{\frac{\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)}{\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)}} = 1$ if and only if $\left(\sigma_{x_{ij}} + \sigma_{x_{kj}}\right) = 0$.

(2) When the value of $\frac{\left(\sigma_{x_{ij}} + \sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)}$ is fixed, $D\left(t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}\right)$ is a power function about ϵ. The first derivative of $D\left(t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}\right)$ about $\epsilon$is $D'\left(t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}\right) = -\frac{\left(\sigma_{x_{ij}} + \sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)}(1-\epsilon)^{\left(\frac{\left(\sigma_{x_{ij}} + \sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)} - 1\right)}$. From $\frac{\left(\sigma_{x_{ij}} + \sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)} > 0$ and $(1-\epsilon)^{\left(\frac{\left(\sigma_{x_{ij}} + \sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}} - \mu_{x_{kj}}\right)} - 1\right)} > 0$, we have $D'\left(t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}\right) < 0$. Therefore, $D\left(t_{ij} \succeq^j_{\epsilon-ASD} t_{kj}\right)$ is monotonically increasing with the decreasing of ϵ.

(3) When the values of ϵ and $\left(\sigma_{x_{ij}}+\sigma_{x_{kj}}\right)$ are fixed, $D\left(t_{ij}\succeq^{j}_{\epsilon-ASD}t_{kj}\right)$ is a exponential function about $\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)$. The first derivative of $D\left(t_{ij}\succeq^{j}_{\epsilon-ASD}t_{kj}\right)$ about $\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)$ is $D'\left(t_{ij}\succeq^{j}_{\epsilon-ASD}t_{kj}\right)=(1-\epsilon)^{\frac{\left(\sigma_{x_{ij}}+\sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)}}\times\ln(1-\epsilon)\times\frac{-\left(\sigma_{x_{ij}}+\sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)^{2}}$. From $(1-\epsilon)^{\frac{\left(\sigma_{x_{ij}}+\sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)}}>0,\ln(1-\epsilon)<0$ with $0.5<1-\epsilon<1$ and $\frac{-\left(\sigma_{x_{ij}}+\sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)^{2}}<0$, we have $D'\left(t_{ij}\succeq^{j}_{\epsilon-ASD}t_{kj}\right)>0$. Therefore, $D\left(t_{ij}\succeq^{j}_{\epsilon-ASD}t_{kj}\right)$ is monotonically increasing with the increasing of $\frac{\left(\sigma_{x_{ij}}+\sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)}$.

(4) When the values of ϵ and $\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)$ are fixed, $D\left(t_{ij}\succeq^{j}_{\epsilon-ASD}t_{kj}\right)$ is a exponential function about $\left(\sigma_{x_{ij}}+\sigma_{x_{kj}}\right)$. The first derivative of $D\left(t_{ij}\succeq^{j}_{\epsilon-ASD}t_{kj}\right)$ about $\left(\sigma_{x_{ij}}+\sigma_{x_{kj}}\right)$ is $D'\left(t_{ij}\succeq^{j}_{\epsilon-ASD}t_{kj}\right)=(1-\epsilon)^{\frac{\left(\sigma_{x_{ij}}+\sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)}}\times\ln(1-\epsilon)^{\frac{1}{\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)}}$. From $(1-\epsilon)^{\frac{\left(\sigma_{x_{ij}}+\sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)}}>0$and $\ln(1-\epsilon)^{\frac{1}{\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)}}<0$ with $0.5<1-\epsilon<1$, we have $D'\left(t_{ij}\succeq_{\epsilon-ASD}t_{kj}\right)<0$. Therefore, $D\left(t_{ij}\succeq^{j}_{\epsilon-ASD}t_{kj}\right)$ is monotonically increasing with the decreasing of $\frac{\left(\sigma_{x_{ij}}+\sigma_{x_{kj}}\right)}{\left(\mu_{x_{ij}}-\mu_{x_{kj}}\right)}$.

3.2. A new set of two states based on ϵ-almost stochastic dominance

In this section, the ϵ-almost stochastic dominance degree is utilized to determine the set of two states $\Omega=\{C,\neg C\}$ in the TWD model. The specific process can be divided into the following two steps: First, identify the relationship between any two objects under each attribute by ϵ-almost stochastic dominance. Second, calculate the ϵ-almost stochastic dominance degree and build the ϵ-almost stochastic dominance degree matrix.

Definition 9. Let $\mathscr{S}=\{\mathscr{O},\mathscr{A},\mathscr{V},\mathscr{F}\}$ be an information system and the triangular fuzzy numbers $t_{ij}=\left(t^{1}_{ij},t^{2}_{ij},t^{3}_{ij}\right)$ and $t_{kj}=\left(t^{1}_{kj},t^{2}_{kj},t^{3}_{kj}\right)$ be the values of objects o_i and o_k with respect to the attribute c_j, i.e., $t_{ij},t_{kj}\in\mathscr{V}_{c_j}$. For $0<\epsilon<0.5$, if $t_{ij}\succeq^{j}_{\epsilon-ASD}t_{kj}$, it means that the object o_i dominates the object o_k under the attribute c_j by ϵ-almost stochastic dominance, denoted as $o_i\succeq^{j}_{\epsilon-ASD}o_k$. Let $\psi(o_i)^{+}$ be the dominance degree, which measures how the object o_i dominats the other objects, and $\psi(o_i)^{-}$ be the non-dominant degree, which measures how the other objects dominant the object o_i. Here, $\psi(o_i)^{+}$ and $\psi(o_i)^{-}$ can be calculated by the following formulas.

$$\psi(o_i)^{+}=\frac{\sum_{k=1}^{n}\sum_{j=1}^{m}D\left(o_i\succeq^{j}_{\epsilon-ASD}o_k\right)}{\sum_{k=1}^{n}\sum_{j=1}^{m}\Gamma^{j}_{ik}},i=1,2,\ldots,n, \tag{15}$$

$$\psi(o_i)^{-}=\frac{\sum_{k=1}^{n}\sum_{j=1}^{m}\left(1-D\left(o_i\succeq^{j}_{\epsilon-ASD}o_k\right)\right)}{\sum_{k=1}^{n}\sum_{j=1}^{m}\Gamma^{j}_{ik}},i=1,2,\ldots,n, \tag{16}$$

where $D\left(o_i\succeq^{j}_{\epsilon-ASD}o_k\right)$ is the ϵ-almost stochastic dominance degree and $\Gamma^{j}_{ik}=\begin{cases}1, & if\ o_i\succeq^{j}_{\epsilon-ASD}o_k,\\ 0, & otherwise.\end{cases}$

Definition 10. Let $\mathscr{O}=\{o_1,o_2,\ldots,o_n\}$be a nonempty finite set of n objects, with the aid of the dominance degrees $\psi(o_i)^{+}$ and $\psi(o_i)^{-}$, the fuzzy sets C and $\neg C$ in $\Omega=\{C,\neg C\}$ can be determined as follows:

$$C = \frac{\psi(o_1)^+}{o_1} + \frac{\psi(o_2)^+}{o_2} + \cdots + \frac{\psi(o_n)^+}{o_n}, \tag{17}$$

$$\neg C = \frac{\psi(o_1)^-}{o_1} + \frac{\psi(o_2)^-}{o_2} + \cdots + \frac{\psi(o_n)^-}{o_n}. \tag{18}$$

Example 3. (Continue with **Example 2**) From **Example 2**, with $\epsilon = 0.47$ and $\alpha = 0.75$, all the ϵ-almost stochastic dominance degree matrices of the object $o_i (i = 1,2,3,4,5)$ under each attribute $c_j (j = 1,2,3,4)$ are obtained as follows:

$$D(o_i \succ^1_{\varepsilon-ASD} o_k) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.414 & 0 & 0.207 & 0 & 0.271 \\ 0 & 0 & 0 & 0 & 0 \\ 0.804 & 0.559 & 0.647 & 0 & 0.773 \\ 0.227 & 0 & 0.015 & 0 & 0 \end{pmatrix},$$

$$D(o_i \succ^2_{\varepsilon-ASD} o_k) = \begin{pmatrix} 0 & 0.281 & 0 & 0.117 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.236 & 0.580 & 0 & 0.470 & 0 \\ 0 & 0.036 & 0 & 0 & 0 \\ 0.190 & 0.545 & 0 & 0.431 & 0 \end{pmatrix},$$

$$D(o_i \succ^3_{\varepsilon-ASD} o_k) = \begin{pmatrix} 0 & 0.093 & 0.855 & 0.759 & 0 \\ 0 & 0 & 0.842 & 0.725 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.690 & 0 & 0 \\ 0.675 & 0.703 & 0.888 & 0.841 & 0 \end{pmatrix},$$

$$D(o_i \succ^4_{\varepsilon-ASD} t_k) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.893 & 0 & 0.824 & 0.541 & 0.556 \\ 0.782 & 0 & 0 & 0 & 0 \\ 0.909 & 0 & 0.833 & 0 & 0.115 \\ 0.889 & 0 & 0.790 & 0 & 0 \end{pmatrix}.$$

According to Eqs. (17) and (18), we have

$$C = \frac{0.421}{o_1} + \frac{0.586}{o_2} + \frac{0.414}{o_3} + \frac{0.596}{o_4} + \frac{0.563}{o_5},$$

$$\neg C = \frac{0.579}{o_1} + \frac{0.414}{o_2} + \frac{0.586}{o_3} + \frac{0.404}{o_4} + \frac{0.437}{o_5}.$$

3.3. A novel conditional probability based on ϵ-almost stochastic dominance

Definition 11. Let $\mathscr{S} = \{\mathscr{O}, \mathscr{A}, \mathscr{V}, \mathscr{F}\}$ be an information system, combined with the ϵ-almost stochastic dominance relation $\succeq^j_{\epsilon-ASD}$, a class of the object $o_i \in \mathscr{O}$ with respect to the attribute c_j can be defined as follows:

$$[o_i]_{\succeq^j_{\epsilon-ASD}} = \left\{ o_k | o_i \succeq^j_{\epsilon-ASD} o_k \wedge o_k \in \mathscr{O} \right\} \bigcup \{o_i\}. \tag{19}$$

If $o_k \in [o_i]_{\succeq^j_{\epsilon-ASD}}$, we have $o_i \succeq^j_{\epsilon-ASD} o_k$. Let $\mathscr{O}/\succeq^j_{\epsilon-ASD} = \left\{ [o_i]_{\succeq^j_{\epsilon-ASD}} | o_i \in \mathscr{O} \right\}$ denote a possibly overlapping classification of all objects with respect to the attribute c_j. In general, $\mathscr{O}/\succeq^j_{\epsilon-ASD} = \left\{ [o_i]_{\succeq^j_{\epsilon-ASD}} | o_i \in \mathscr{O} \right\}$ is a covering of $\mathscr{O}$ with respect to the attribute c_j.

Note 2. Two extreme cases may occur for the object o_i: (1) the object o_i does not dominate any other objects, but some other objects dominate the object o_i by ϵ-almost stochastic dominance or (2) the object o_i does not dominate any other object and any other object does not dominate the object o_i by ϵ-almost stochastic dominance. Although case (2) is rare, it does happen in practice. For these two cases, the dominant set of the object o_i is empty, i.e., $\left\{ o_k | o_i \succeq^j_{\epsilon-ASD} o_k \wedge o_k \in \mathscr{O} \right\} = \varnothing$. To satisfy the coverage requirement, we add the object o_i to its own dominant set as a class, i.e., $\left\{ o_k | o_i \succeq^j_{\epsilon-ASD} o_k \wedge o_k \in \mathscr{O} \right\} \bigcup \{o_i\}$.

Example 4. (Continue with **Example 1**) With $\epsilon = 0.47$ and $\alpha = 0.75$, according to Eq. (19), the class of the object $o_i (i = 1,2,3,4,5)$ under the attribute c_1 is given as follows:$[o_1]_{\succeq^1_{\epsilon-ASD}} = \{o_1\}$, $[o_2]_{\succeq^1_{\epsilon-ASD}} = \{o_1, o_2, o_3, o_5\}$, $[o_3]_{\succeq^1_{\epsilon-ASD}} = \{o_3\}$, $[o_4]_{\succeq^1_{\epsilon-ASD}} = \{o_1, o_2, o_3, o_4, o_5\}$, and $[o_5]_{\succeq^1_{\epsilon-ASD}} = \{o_1, o_3, o_5\}$.

Proposition 3. Given an information system $\mathscr{S} = \{\mathscr{O}, \mathscr{A}, \mathscr{V}, \mathscr{F}\}$, the class $[o_i]_{\succeq^j_{\epsilon-ASD}}$ of the object $o_i \in \mathscr{O}$ with respect to the attribute c_j has the following properties.

(1) For any three objects $o_i, o_k, o_l \in \mathcal{O}$, if $o_k \in [o_i]_{\succeq^j_{\epsilon-ASD}}$ and $o_l \in [o_k]_{\succeq^j_{\epsilon-ASD}}$, then $o_l \in [o_i]_{\succeq^j_{\epsilon-ASD}}$;
(2) For any two objects $o_i, o_k \in \mathcal{O}$, if $\varrho < \epsilon$ and $o_k \in [o_i]_{\succeq^j_{\varrho-ASD}}$, then $o_k \in [o_i]_{\succeq^j_{\epsilon-ASD}}$.

Proof.

(1) According to Definition 7, if $o_k \in [o_i]_{\succeq^j_{\epsilon-ASD}}$ and $o_l \in [o_k]_{\succeq^j_{\epsilon-ASD}}$, we have $o_i \succeq^j_{\epsilon-ASD} o_k$ and $o_k \succeq^j_{\epsilon-ASD} o_l$. According to Proposition 1, we have $o_i \succeq^j_{\epsilon-ASD} o_l$. Thus, $o_l \in [o_i]_{\succeq^j_{\epsilon-ASD}}$.
(2) According to Definition 7, the conclusion is obvious, and the proof process is omitted.

Definition 12. Let $\mathcal{S} = \{\mathcal{O}, \mathcal{A}, \mathcal{V}, \mathcal{F}\}$ be an information system, the conditional probability of the object $o_i \in \mathcal{O}$ under the attribute $c_j \in \mathcal{A}$ is determined as follows:

$$\Pr\left(C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right) = \frac{\sum_{[o_i]_{\succeq^j_{\epsilon-ASD}}} \psi(o_i)^+}{\left|[o_i]_{\succeq^j_{\epsilon-ASD}}\right|}, \tag{20}$$

where $\Pr\left(C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right)$ is the probability that the object o_i in $[o_i]_{\succeq^j_{\epsilon-ASD}}$ belongs to the state C under the attribute c_j and $|\cdot|$ represents the number of granularity.

Example 5. Combined with the results of **Examples** 3 and **4**, according to Eq. (20), all the conditional probabilities of the object $o_i \in \mathcal{O}$ under the attribute c_1 are given as follows:$\Pr\left(C|[o_1]_{\succeq^1_{\epsilon-ASD}}\right) = 0.421, \Pr\left(C|[o_2]_{\succeq^1_{\epsilon-ASD}}\right) = 0.496, \Pr\left(C|[o_3]_{\succeq^1_{\epsilon-ASD}}\right) = 0.414, \Pr\left(C|[o_4]_{\succeq^1_{\epsilon-ASD}}\right) = 0.516$ and $\Pr\left(C|[o_5]_{\succeq^1_{\epsilon-ASD}}\right) = 0.466$.

Proposition 4. Let $\Omega = \{C, \neg C\}$ be a set of states in the TWD model, $\Pr\left(C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right)$ is the probability that the object o_i in $[o_i]_{\succeq^j_{\epsilon-ASD}}$ belongs to the state C under the attribute c_j, then it satisfies the following property.

$$\Pr\left(C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right) + \Pr\left(\neg C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right) = 1.$$

Proof. For any object $o_i \in \mathcal{O}$, we have $\neg C(o_i) = 1 - C(o_i)$ because $\neg C$ is the complement of C. Therefore, we have $\Pr\left(C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right) + \Pr\left(\neg C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right) = \frac{\sum_{[o_i]_{\succeq^j_{\epsilon-ASD}}} \psi(o_i)^+}{\left|[o_i]_{\succeq^j_{\epsilon-ASD}}\right|} + \frac{\sum_{[o_i]_{\succeq^j_{\epsilon-ASD}}} \psi(o_i)^-}{\left|[o_i]_{\succeq^j_{\epsilon-ASD}}\right|} = 1.$

3.4. A novel relative loss function based on Hellinger distance

Combined with the Hellinger distance [44], a novel distance for measuring any two triangular fuzzy numbers is first proposed. Then, a relative loss function combined with the novel distance is given.

Definition 13. Let $\mathcal{S} = \{\mathcal{O}, \mathcal{A}, \mathcal{V}, \mathcal{F}\}$ be an information system and the triangular fuzzy numbers $t_{ij} = \left(t^1_{ij}, t^2_{ij}, t^3_{ij}\right)$ and $t_{kj} = \left(t^1_{kj}, t^2_{kj}, t^3_{kj}\right)$ be the values of objects o_i and o_k with respect to the attribute c_j, i.e., $t_{ij}, t_{kj} \in \mathcal{V}_{c_j}$. Given a parameter $\alpha \in [0, 1]$, the confidence level intervals of t_{ij} and t_{kj} are $t^\alpha_{ij} = \left[t^1_{ij} + \left(t^2_{ij} - t^1_{ij}\right)\alpha, t^3_{ij} - \left(t^3_{ij} - t^2_{ij}\right)\alpha\right]$ and $t^\alpha_{kj} = \left[t^1_{kj} + \left(t^2_{kj} - t^1_{kj}\right)\alpha, t^3_{kj} - \left(t^3_{kj} - t^2_{kj}\right)\alpha\right]$, respectively. x_{ij} and x_{kj} are variables estimated from intervals t^α_{ij} and t^α_{kj}, respectively. Then, Hellinger distance between t_{ij} and t_{kj} is calculated by

$$H(t_{ij}, t_{kj}) = \sqrt{\frac{1}{2}\int \left(\sqrt{f_{x_{ij}}(x)} - \sqrt{f_{x_{kj}}(x)}\right)^2 dx}, \tag{21}$$

where $f_{x_{ij}}(x)$ and $f_{x_{kj}}(x)$ are the probability density functions of x_{ij} and x_{kj}, respectively.

Proposition 5. $\forall t_{ij}, t_{kj}, t_{lj} \in \mathscr{V}_{c_j}$, the Hellinger distance $H(\cdot)$ has the following properties.

(1) Bounded: $\forall t_{ij}, t_{kj} \in \mathscr{V}_{c_j}, 0 \leqslant H(t_{ij}, t_{kj}) \leqslant 1$;
(2) Symmetry: $\forall t_{ij}, t_{kj} \in \mathscr{V}_{c_j}, H(t_{ij}, t_{kj}) = H(t_{kj}, t_{ij})$;
(3) Triangle Inequality: $\forall t_{ij}, t_{kj}, t_{kj} \in \mathscr{V}_{c_j}, H(t_{kj}, t_{lj}) \leqslant H(t_{ij}, t_{kj}) + H(t_{ij}, t_{lj})$.

Proof.

(1) Given the probability density functions $f_{x_{ij}}(x)$ and $f_{x_{kj}}(x)$ of variables x_{ij} and x_{kj}, respectively. According to the Hellinger distance, it satisfies nonnegative, i.e., $H(t_{ij}, t_{kj}) \geqslant 0$. Depending on the properties of probability function, we have $0 \leqslant f_{x_{ij}}(x), f_{x_{kj}}(x) \leqslant 1, \int f_{x_{ij}}(x)dx = 1$ and $\int f_{x_{kj}}(x)dx = 1$. By Cauchy–Schwarz inequality, we have $H(t_{ij}, t_{kj}) = \frac{1}{\sqrt{2}}\sqrt{\int\left(\sqrt{f_{x_{ij}}(x)} - \sqrt{f_{x_{kj}}(x)}\right)^2 dx} = \frac{1}{\sqrt{2}}\sqrt{\int f_{x_{ij}}(x) - 2\sqrt{f_{x_{ij}}(x) f_{x_{kj}}(x)} + f_{x_{kj}}(x)dx} \leqslant \frac{1}{\sqrt{2}}\sqrt{\int\left(f_{x_{ij}}(x) + f_{x_{kj}}(x)\right)dx} \leqslant 1$.
Therefore, the boundedness of the Hellinger distance about triangular fuzzy numbers has been proven.
(2) Given the probability density functions $f_{x_{ij}}(x)$ and $f_{x_{kj}}(x)$ of x_{ij} and x_{kj}, respectively. The Hellinger distance of t_{ij} and t_{kj} is $H(t_{ij}, t_{kj}) = \frac{1}{\sqrt{2}}\sqrt{\int\left(\sqrt{f_{x_{ij}}(x)} - \sqrt{f_{x_{kj}}(x)}\right)^2 dx}$. Similarly, the Hellinger distance of t_{kj} and t_{ij} is $H(t_{kj}, t_{ij}) = \frac{1}{\sqrt{2}}\sqrt{\int\left(\sqrt{f_{x_{kj}}(x)} - \sqrt{f_{x_{ij}}(x)}\right)^2 dx}$. Obviously, we have $H(t_{ij}, t_{kj}) = H(t_{kj}, t_{ij})$. Therefore, the symmetry of the Hellinger distance about triangular fuzzy numbers has been proven.
(3) Given the probability density functions $f_{x_{ij}}(x), f_{x_{kj}}(x)$ and $f_{x_{lj}}(x)$ of x_{ij}, x_{kj} and x_{lj}, respectively. According to Cauchy–Schwarz inequality, we have $\left(\int\left|\sqrt{f_{x_{ij}}(x)} - \sqrt{f_{x_{kj}}(x)}\right| \cdot \left|\sqrt{f_{x_{kj}}(x)} - \sqrt{f_{x_{lj}}(x)}\right| dx\right)^2 \leqslant \int\left(\left|\sqrt{f_{x_{ij}}(x)} - \sqrt{f_{x_{kj}}(x)}\right|\right)^2 dx \cdot \int\left(\left|\sqrt{f_{x_{kj}}(x)} - \sqrt{f_{x_{lj}}(x)}\right|\right)^2 dx$. Combined with $\int\left|\sqrt{f_{x_{ij}}(x)} - \sqrt{f_{x_{kj}}(x)}\right| + \left|\sqrt{f_{x_{kj}}(x)} - \sqrt{f_{x_{lj}}(x)}\right| dx \leqslant \int\left|\sqrt{f_{x_{ij}}(x)} - \sqrt{f_{x_{kj}}(x)}\right| dx + \int\left|\sqrt{f_{x_{kj}}(x)} - \sqrt{f_{x_{lj}}(x)}\right| dx$, we have

$$\begin{aligned}
H(t_{ij}, t_{lj}) &= \frac{1}{\sqrt{2}}\sqrt{\int\left(\sqrt{f_{x_{ij}}(x)} - \sqrt{f_{x_{lj}}(x)}\right)^2 dx} \\
&= \frac{1}{\sqrt{2}}\sqrt{\int\left(\sqrt{f_{x_{ij}}(x)} - \sqrt{f_{x_{kj}}(x)} + \sqrt{f_{x_{kj}}(x)} - \sqrt{f_{x_{lj}}(x)}\right)^2 dx} \\
&\leqslant \frac{1}{\sqrt{2}}\sqrt{\int\left(\left|\sqrt{f_{x_{ij}}(x)} - \sqrt{f_{x_{kj}}(x)}\right| + \left|\sqrt{f_{x_{kj}}(x)} - \sqrt{f_{x_{lj}}(x)}\right|\right)^2 dx} \\
&\leqslant \frac{1}{\sqrt{2}}\left(\sqrt{\int\left(\left|\sqrt{f_{x_{ij}}(x)} - \sqrt{f_{x_{kj}}(x)}\right|\right)^2 dx} + \sqrt{\int\left(\left|\sqrt{f_{x_{kj}}(x)} - \sqrt{f_{x_{lj}}(x)}\right|\right)^2 dx}\right) \\
&\leqslant \frac{1}{\sqrt{2}}\sqrt{\int\left(\sqrt{f_{x_{ij}}(x)} - \sqrt{f_{x_{kj}}(x)}\right)^2 dx} + \frac{1}{\sqrt{2}}\sqrt{\int\left(\sqrt{f_{x_{kj}}(x)} - \sqrt{f_{x_{lj}}(x)}\right)^2 dx} \\
&= H(t_{ij}, t_{kj}) + H(t_{kj}, t_{lj}).
\end{aligned}$$

Therefore, the triangle inequality of the Hellinger distance about triangular fuzzy numbers has been proven.

Inspired by [5], the detailed computation process of relative loss functions combined with the Hellinger distance is given as follows:

Let C_j and $\neg C_j$ be two states corresponding to the attribute c_j. With triangular fuzzy number $t_{ij} = \left(t_{ij}^1, t_{ij}^2, t_{ij}^3\right)$ of the object o_i under the attribute c_j, the relative loss functions are given as follows:

$$\lambda_{NP}^j = H\left(t_{ij}, t_j^{\min}\right), \tag{22}$$

$$\lambda_{PN}^j = H\left(t_{ij}, t_j^{\max}\right), \tag{23}$$

where $t_j^{\min} = \left(\min_i\left\{t_{ij}^1\right\}, \min_i\left\{t_{ij}^2\right\}, \min_i\left\{t_{ij}^3\right\}\right)$ and $t_j^{\max} = \left(\max_i\left\{t_{ij}^1\right\}, \max_i\left\{t_{ij}^2\right\}, \max_i\left\{t_{ij}^3\right\}\right)$.

Table 4
The relative loss functions from the attribute assessment t_{ij}.

	C_j	$\neg C_j$
π_P	0	$H\left(t_{ij}, t_j^{\max}\right)$
π_B	$\theta H\left(t_{ij}, t_j^{\min}\right)$	$\theta H\left(t_{ij}, t_j^{\max}\right)$
π_N	$H\left(t_{ij}, t_j^{\min}\right)$	0

Table 5
The relative loss functions of all objects under the attribute c_1.

c_1	λ_{PP}^1	λ_{BP}^1	λ_{NP}^1	λ_{PN}^1	λ_{BN}^1	λ_{NN}^1
o_1	0	0.449	0.999	0.986	0.444	0
o_2	0	0.435	0.966	0.801	0.361	0
o_3	0	0.381	0.847	0.876	0.394	0
o_4	0	0.450	1.000	0.467	0.210	0
o_5	0	0.449	0.999	0.974	0.438	0

Table 6
The three actions' expected lossss of all objects under the attribute c_1.

c_1	o_1	o_2	o_3	o_4	o_5
$\Phi_1(\pi_P\|[o_i])$	0.571	0.404	0.514	0.226	0.520
$\Phi_1(\pi_B\|[o_i])$	0.446	0.397	0.389	0.334	0.444
$\Phi_1(\pi_N\|[o_i])$	0.421	0.479	0.350	0.516	0.466
decision rules	(N1)	(B1)	(N1)	(P1)	(B1)

With the risk avoidance coefficient $\theta \in (0, 0.5)$, $\lambda_{BP}^j = \theta\lambda_{NP}^j$ and $\lambda_{BN}^j = \theta\lambda_{PN}^j$ which satisfy $\lambda_{PP}^j \leqslant \lambda_{BP}^j \leqslant \lambda_{NP}^j$ and $\lambda_{NN}^j \leqslant \lambda_{BN}^j \leqslant \lambda_{PN}^j$. To make it clearer, all the relative loss functions are shown in Table 4.

Example 6. (For the information system $\mathscr{S} = \{\mathscr{O}, \mathscr{A}, \mathscr{V}, \mathscr{F}\}$ in the **Example** 1) we assume that $\theta = 0.45$, according to Eqs. (22) and (23), the relative loss functions of all objects under the attribute c_1 are shown in Table 5.

Based on Bayesian decision making process, three actions' expected losses $\Phi_j(\pi_\Delta|[o_i])$ $(\Delta = P, B, N)$ with respect to the attribute c_j are calculated as follows:

$$\Phi_j(\pi_P|[o_i]) = \lambda_{PP}^j \Pr\left(C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right) + \lambda_{PN}^j \Pr\left(\neg C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right);$$
$$\Phi_j(\pi_B|[o_i]) = \lambda_{BP}^j \Pr\left(C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right) + \lambda_{BN}^j \Pr\left(\neg C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right);$$
$$\Phi_j(\pi_N|[o_i]) = \lambda_{NP}^j \Pr\left(C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right) + \lambda_{NN}^j \Pr\left(\neg C|[o_i]_{\succeq^j_{\epsilon-ASD}}\right).$$

Hence, the decision rules (P1)-(N1) of TWD obtained by comparing three expected losses are shown below:(P1) If $\Phi_j(\pi_P|[o_i]) \leqslant \Phi_j(\pi_B|[o_i])$ and $\Phi_j(\pi_P|[o_i]) \leqslant \Phi_j(\pi_N|[o_i])$, then $o_i \in POS(C)$;(B1) If $\Phi_j(\pi_B|[o_i]) \leqslant \Phi_j(\pi_P|[o_i])$ and $\Phi_j(\pi_B|[o_i]) \leqslant \Phi_j(\pi_N|[o_i])$, then $o_i \in BND(C)$;(N1) If $\Phi_j(\pi_N|[o_i]) \leqslant \Phi_j(\pi_P|[o_i])$ and $\Phi_j(\pi_N|[o_i]) \leqslant \Phi_j(\pi_B|[o_i])$, then $o_i \in NEG(C)$.

Here, $POS(C)$ represents the acceptance domain, $BND(C)$ represents the deferment domain and $NEG(C)$ represents the rejection domain.

Example 7. (For the information system $\mathscr{S} = \{\mathscr{O}, \mathscr{A}, \mathscr{V}, \mathscr{F}\}$ in **Example** 1) With $\alpha = 0.75$ and $\epsilon = 0.47$, combined with conditional probability and relative losses of all objects under the attribute c_1, the three actions' expected loss and decision rules of all objects under the attribute c_1 are shown in Table 6.

For clarity, we summarize the complete implementation procedure of the novel TWD model combined with ϵ-almost stochastic dominance in Algorithm 1.

Algorithm 1: The algorithm of a novel TWD combined with ϵ-almost stochastic dominance.

Input : An information system $\mathcal{S} = \{\mathcal{O}, \mathcal{A}, \mathcal{V}, \mathcal{F}\}$ and parameter ϵ.
Output : The classification of all objects.

Begin
  Given a parameter $\epsilon \in (0, 0.5)$;
  for $j = 1$ **to** m *and* $i = 1$ **to** n **do**
    identify ϵ-almost stochastic dominance between objects o_i and o_k with respect to the attribute c_j, i.e., $o_i \succeq^j_{\epsilon-ASD} o_k$. // By Eqs.(12) and (13)
  end
  for $j = 1$ **to** m *and* $i = 1$ **to** n **do**
    calculate ϵ-almost stochastic dominance degree of the object o_i and o_k with respect to the attribute c_j, i.e., $D\left(o_i \succeq^j_{\epsilon-ASD} o_k\right)$. // By Eq.(14)
  end
  for $j = 1$ **to** m *and* $i = 1$ **to** n **do**
    calculate the dominance degrees $\psi(o_i)^+$ and non-dominant degree $\psi(o_i)^-$ of the object o_i. // By Eqs.(15) and (16)
  end
  for $j = 1$ **to** m *and* $i = 1$ **to** n **do**
    determine the set of two states $\Omega = \{C, \neg C\}$ // By Eqs.(17) and (18)
  end
  for $j = 1$ **to** m *and* $i = 1$ **to** n **do**
    determine the class of the object o_i with respect to the attribute c_j by ϵ-almost stochastic dominance, i.e., $[o_i]_{\succeq^j_{\epsilon-ASD}}$. // By Eq. (19)
  end
  for $j = 1$ **to** m **do**
    calculate the conditional probability that the object o_i with respect to the attribute c_j in $[o_i]_{\succeq^j_{\epsilon-ASD}}$ belongs to the state C, i.e., $\Pr\left(C|\,[o_i]_{\succeq^j_{\epsilon-ASD}}\right)$. // By Eq.(20)
  end
  Given a parameter $\theta \in (0, 0.5)$;
  for $j = 1$ **to** m *and* $i = 1$ **to** n **do**
    calculate the relative loss functions of the object o_i with respect to the attribute c_j, i.e., λ^j_{BP}, λ^j_{NP}, λ^j_{PN} and λ^j_{BN}. // By Eqs.(21) and (23)
    $\lambda^j_{PP} = 0$; $\lambda^j_{BP} = \theta H(c_{ij}, \min c_{ij})$; $\lambda^j_{NP} = H(c_{ij}, \min c_{ij})$;
    $\lambda^j_{PN} = H(c_{ij}, \max c_{ij})$; $\lambda^j_{BN} = \theta H(c_{ij}, \max c_{ij})$; $\lambda^j_{NN} = 0$.
  end
  for $j = 1$ **to** m *and* $i = 1$ **to** n **do**
    compute three actions' expected lossss of the object o_i under the attribute c_j: $\Phi_j(\pi_\Delta|\,[o_i]), (\Delta = P, B, N)$.
    $\Phi_j(\pi_P|\,[o_i]) = \lambda^j_{PP} \Pr\left(C|\,[o_i]_{\succeq^j_{\epsilon-ASD}}\right) + \lambda^j_{PN} \Pr\left(\neg C|\,[o_i]_{\succeq^j_{\epsilon-ASD}}\right)$;
    $\Phi_j(\pi_B|\,[o_i]) = \lambda^j_{BP} \Pr\left(C|\,[o_i]_{\succeq^j_{\epsilon-ASD}}\right) + \lambda^j_{BN} \Pr\left(\neg C|\,[o_i]_{\succeq^j_{\epsilon-ASD}}\right)$;
    $\Phi_j(\pi_N|\,[o_i]) = \lambda^j_{NP} \Pr\left(C|\,[o_i]_{\succeq^j_{\epsilon-ASD}}\right) + \lambda^j_{NN} \Pr\left(\neg C|\,[o_i]_{\succeq^j_{\epsilon-ASD}}\right)$.
  end
  for $j = 1$ **to** m *and* $i = 1$ **to** n **do**
    determine the decision rules (P1)-(N1) and three domains $POS(C)$, $BND(C)$ and $NEG(C)$:
    (P1) If $\Phi_j(\pi_P|\,[o_i]) \le \Phi_j(\pi_B|\,[o_i])$ and $\Phi_j(\pi_P|\,[o_i]) \le \Phi_j(\pi_N|\,[o_i])$, then $o_i \in POS(C)$;
    (B1) If $\Phi_j(\pi_B|\,[o_i]) \le \Phi_j(\pi_P|\,[o_i])$ and $\Phi_j(\pi_B|\,[o_i]) \le \Phi_j(\pi_N|\,[o_i])$, then $o_i \in BND(C)$;
    (N1) If $\Phi_j(\pi_N|\,[o_i]) \le \Phi_j(\pi_P|\,[o_i])$ and $\Phi_j(\pi_N|\,[o_i]) \le \Phi_j(\pi_B|\,[o_i])$, then $o_i \in NEG(C)$.
  end
  return $POS(C)$, $BND(C)$ *and* $NEG(C)$.
End

4. Multi-attribute TWD involving interrelated attributes measured by bipolar scale

As an essential branch of decision analysis, MADM addresses decision problems with a finite set of objects and multiple attributes. It is one of the essential techniques that has been successfully applied to real-life medical diagnoses, engineering, finances, market analysis, management, etc [4,8,10,11,13]. To describe the problem of multi-attribute TWD, we use the medical diagnosis as an example. In medical diagnosis, through the four diagnostics (i.e., four attributes), "inspection", "smelling", "listening", and "interrogation and palpation", information about the body is obtained. Under each diagnostic, explicit judgments, either treatment needed or no need for treatment, can usually be made immediately when the obtained information is fully grasped, whereas assessment tends to be postponed with complex information. Some examinations even need to be diagnosed further. Given these three situations (treatment needed, further observation needed and no need for treatment), we extend the proposed TWD model in Section 3 to multi-attribute to grasp the human system as a whole and strengthen the explanation of human cognition. Moreover, the independence assumption among attributes is unreasonable in practice. Therefore, the Choquet integral with respect to bi-capacity is utilized for aggregating the interrelated attributes with a bipolar scale in the TWD model. First, the bipolar scale $\Psi_{ij}(\cdot)$ corresponding to three actions of the object $o_i(i=1,2,\dots,n)$ under the attribute $c_i(j=1,2,\dots,m)$ is defined as follows:

$$\Psi_{ij}(\Delta)=\begin{cases}-1, & \Phi_j(\pi_\Delta|[o_i])=\min\left\{\Phi_j(\pi_P|[o_i]),\Phi_j(\pi_B|[o_i]),\Phi_j(\pi_N|[o_i])\right\},\\ 1, & \Phi_j(\pi_\Delta|[o_i])=\max\left\{\Phi_j(\pi_P|[o_i]),\Phi_j(\pi_B|[o_i]),\Phi_j(\pi_N|[o_i])\right\},\\ 0, & \textit{otherwise}.\end{cases}$$

where $\Delta\in\{P,B,N\}$.

Then, we utilize Choquet integral with respect to bi-capacity in terms of Möbius function ϑ to obtain the overall expected losses of the object o_i taking three actions, i.e.,

$$\mathbb{C}_\vartheta\left(o_i^\Delta\right)=\sum_{M_2\subseteq M}\vartheta(\varnothing,M_2)\left(\bigwedge_{j\in M_2^c\cap M^-}\Psi_{ij}(\Delta)\right)+\sum_{\substack{(M_1,M_2)\in\mathcal{Q}(M)\\ M_1\neq\varnothing}}\vartheta(M_1,M_2)\left[\left(\bigwedge_{j\in(M_1\cup M_2)^c\cap M^-}\Psi_{ij}(\Delta)+\bigwedge_{j\in M_1}\Psi_{ij}(\Delta)\right)\vee 0\right],$$

where $M=\{1,2,\dots,m\},M^-=\left\{j\in M|\Psi_{ij}(\Delta)<0,i=1,2,\dots,n,\Delta=P,B,N\right\}$.

For any subset (M_1,M_2) of $\mathcal{Q}(M)$, the bi-capacity $v(M_1,M_2)$ should be determined in advance. Because $v(\varnothing,\varnothing)=0$, $v(M,\varnothing)=1$ and $v(\varnothing,M)=-1$, there are 3^m-3 parameters that need to be determined. Usually, it is difficult for DMs to deduce all these parameters. Thus, k-additive bi-capacity has been proposed in [40,41] based on its corresponding Möbius representation. In particular, bi-capacity is said to be k-additive if its Möbius representation ϑ is nonzero for all elements (M_1,M_2) of $\mathcal{Q}(M)$ such that $|M_2|\geqslant m-k$. When $k=2$, the bi-capacity is 2-additive, it is sufficient for measuring the interactions among attributes in practical applications. First, it can indicate the importance of a single attribute and the interactions between any two attributes in practical MADM problems. Furthermore, for more attributes, their interactions cannot be provided directly by DMs. Second, the fewer the number of parameters, the more straightforward the calculation; here, only $2m^2+1$ parameters need to be calculated. For any $h,j\in M$, the elements of the 2-additive bi-capacity in terms of Möbius measures are expressed by $\left(\varnothing,h^c\right),\left(\varnothing,\{h,j\}^c\right),\left(h,h^c\right)$ and $\left(\{h,j\},\{h,j\}^c\right)$.

The standardized and monotonic conditions of 2-additive bi-capacity in terms of Möbius measures ϑ are as follows:

(1) $\sum_{(M_1,M_2)\in\mathcal{Q}(M)}\vartheta(M_1,M_2)=1,\sum_{M_2\subsetneq M}\vartheta(\varnothing,M_2)=1$ and $\vartheta(\varnothing,M)=-1$.
(2) For all $h\in M$, for all $(M_1,M_2)\in\mathcal{Q}(M\setminus h)$, $\vartheta\left(h,h^c\right)+\sum_{j\in(M_2\cup h)^c}\vartheta\left(h,\{h,j\}^c\right)+\sum_{j\in M_1}\vartheta\left(\{h,j\},\{h,j\}^c\right)\geqslant 0$, $\vartheta\left(\varnothing,h^c\right)+\sum_{j\in(M_2\cup h)^c}\vartheta\left(\varnothing,\{h,j\}^c\right)+\sum_{j\in M_1}\vartheta\left(j,\{h,j\}^c\right)\geqslant 0$.

The Choquet integral of o_i with respect to 2-additive bi-capacity in terms of Möbius measures ϑ can be expressed by

$$\begin{aligned}\mathbb{C}_\vartheta\left(o_i^\Delta\right)=&\sum_{h\in M^-}\vartheta\left(\varnothing,h^c\right)\Psi_{ih}(\Delta)+\sum_{\{h,j\}\subseteq M^-}\vartheta\left(\varnothing,\{h,j\}^c\right)\left(\Psi_{ih}(\Delta)\bigwedge\Psi_{ij}(\Delta)\right)+\sum_{\substack{h\in M^-\\ j\in M^+}}\vartheta\left(\varnothing,\{h,j\}^c\right)\Psi_{ih}(\Delta)\\&+\sum_{\substack{h\in M^-\\ j\in M^+}}\vartheta\left(h,\{h,j\}^c\right)\left[\left(\Psi_{ih}(\Delta)+\Psi_{ij}(\Delta)\right)\bigvee 0\right]+\sum_{\{h,j\}\subseteq M^+}\vartheta\left(h,\{h,j\}^c\right)\Psi_{ih}(\Delta)\\&+\sum_{\{h,j\}\subseteq M^+}\vartheta\left(\{h,j\},\{h,j\}^c\right)\left(\Psi_{ih}(\Delta)\bigwedge\Psi_{ij}(\Delta)\right)+\sum_{h\in M^+}\vartheta\left(h,(h)^c\right)\Psi_{ih}(\Delta),\end{aligned}$$

where $M=\{1,2,\dots,m\},M^-=\left\{j\in M|\mathbb{C}_\vartheta\left(o_i^\Delta\right)<0,i=1,2,\dots,n,\Delta=P,B,N\right\}$.

The importance of attributes and their interactions in terms of Möbius measures are given as follows:

(1) $I_{h,\varnothing}=\vartheta\left(h,h^c\right)+\sum_{h\neq j}\frac{1}{2}\left[\vartheta\left(h,\{h,j\}^c\right)+\vartheta\left(\{h,j\},\{h,j\}^c\right)\right]$;
(2) $I_{\varnothing,h}=\vartheta\left(\varnothing,h^c\right)+\sum_{h\neq j}\frac{1}{2}\left[\vartheta\left(h,\{h,j\}^c\right)+\vartheta\left(\varnothing,\{h,j\}^c\right)\right]$;
(3) $I_{\{h,j\},\varnothing}=\vartheta\left(\{h,j\},\{h,j\}^c\right)$;
(4) $I_{\varnothing,\{h,j\}}=\vartheta\left(\varnothing,\{h,j\}^c\right)$;
(5) $I_{h,j}=\vartheta\left(h,\{h,j\}^c\right)$.

The expected losses of three actions with respect to interrelated attributes with synergy and redundancy relationships can be obtained. According to the overall losses, the decision rules (P2)-(N2) of the object $o_i (i = 1, 2, \ldots, n)$ are given as follows:

(P2) If $\mathbb{C}_\vartheta(o_i^P) \leqslant \mathbb{C}_\vartheta(o_i^N)$ and $\mathbb{C}_\vartheta(o_i^P) \leqslant \mathbb{C}_\vartheta(o_i^B)$, then $o_i \in POS(C)$;

(B2) If $\mathbb{C}_\vartheta(o_i^B) \leqslant \mathbb{C}_\vartheta(o_i^N)$ and $\mathbb{C}_\vartheta(o_i^B) \leqslant \mathbb{C}_\vartheta(o_i^P)$, then $o_i \in BND(C)$;

(N2) If $\mathbb{C}_\vartheta(o_i^N) \leqslant \mathbb{C}_\vartheta(o_i^P)$ and $\mathbb{C}_\vartheta(o_i^N) \leqslant \mathbb{C}_\vartheta(o_i^B)$, then $o_i \in NEG(C)$.

5. The SMAA-TWD model for multi-attribute TWD problems involving interrelated attributes

In this section, we propose a novel SMAA-TWD model for multi-attribute TWDs with using a Choquet integral with respect to bi-capacity. The corresponding Monte Carlo simulation is conducted for determining three indices to analyze the decision results.

5.1. The SMAA-TWD model for multi-attribute TWD problems with the Choquet integral with respect to bi-capacity

The total loss $\mathbb{C}_\vartheta(o_i^\Delta)$ of each object o_i taking action $\Delta(= P, B, N)$ is considered to be the ranking function $U(o_i^\Delta, \boldsymbol{X}^\alpha, \vartheta)$ in SMAA, which is composed of the Choquet integral with respect to bi-capacity, i.e., $U(o_i^\Delta, \boldsymbol{X}^\alpha, \vartheta) = \mathbb{C}_\vartheta(o_i^\Delta)$. For each object o_i, the overall expected losses of the object o_i take three action maps to three values, which are combined with α and the Möbius measure vector ϑ.

The triangular fuzzy number t_{ij} represents the evaluation information of object $o_i (i = 1, 2, \ldots, n)$ with respect to attribute $c_j (j = 1, 2, \ldots, m)$. For a given $\alpha \in [0, 1]$, the triangular fuzzy numbers can be transformed into confidence level intervals $t_{ij}^\alpha = \left[t_{ij}^1 + \left(t_{ij}^2 - t_{ij}^1\right)\alpha, t_{ij}^3 - \left(t_{ij}^3 - t_{ij}^2\right)\alpha\right]$. Suppose variable x_{ij} obeys a normal distribution, which is estimated from the confidence level interval and its probability density function is denoted $f(x_{ij})$. In this way, the elements in the evaluation matrix are changed from triangular fuzzy number t_{ij} to random variable x_{ij}. For each $\alpha \in [0, 1]$, an evaluation matrix of random variables estimated from the interval values can be generated. Therefore, the joint probability density function of variables corresponding to elements in the evaluation matrices $\mathbb{F}_\aleph(\boldsymbol{X}^\alpha)$ is defined in the evaluation information space $\aleph$.

The completely unknown preferences on the importance of attributes and interactions between two attributes, which are measured by 2-additive bi-capacity in terms of the Möbius measure vector ϑ, are also considered stochastic variables, and their joint probability distribution $\mathbb{F}_\Theta(\vartheta)$ is defined in the Möbius measure space Θ. For completely unknown preferences, the Möbius measure space Θ depends on the standardized and monotonic conditions, i.e.,

$$\Theta\begin{cases} \sum_{(M_1,M_2)\in Q(M)}\vartheta(M_1,M_2) = 1, \sum_{M_2\subseteq M}\vartheta(\varnothing,M_2) = 1\, and\, \vartheta(\varnothing,M) = -1, \\ \qquad\qquad \forall h \in M, \forall (M_1,M_2) \in \mathcal{Q}(M\setminus h) \\ \vartheta(h,h^c) + \sum_{j\in(M_2\cup h)^c}\vartheta(h,\{h,j\}^c) + \sum_{j\in M_1}\vartheta(\{h,j\},\{h,j\}^c) \geqslant 0, \\ \vartheta(\varnothing,h^c) + \sum_{j\in(M_2\cup h)^c}\vartheta(\varnothing,\{h,j\}^c) + \sum_{j\in M_1}\vartheta(j,\{h,j\}^c) \geqslant 0. \end{cases}$$

In the SMAA-TWD model, for $\boldsymbol{X}^\alpha \in \aleph, \vartheta \in \Theta$, the rank function of the object o_i taking action $\Delta(= P, B, N)$ is given as follows:

$$rank(o_i^\Delta, \boldsymbol{X}^\alpha, \vartheta) = 1 + \sum_{\nabla\neq\triangle}\rho\left(U(o_i^\Delta, \boldsymbol{X}^\alpha, \vartheta) > U(o_i^\nabla, \boldsymbol{X}^\alpha, \vartheta)\right),$$

where $\rho(false) = 0$ and $\rho(true) = 1$.

Obviously, the rank function depends on the compatible Möbius measure vector ϑ and each evaluation information matrix $\boldsymbol{X}^\alpha \in \aleph$. Therefore, the favorable rank preferences on the Möbius measure vector ϑ can ensure that the object o_i taking action $\triangle(= P, B, N)$ ranks $r(= 1, 2, 3)$, which can be calculated as follows:

$$\Theta_\triangle^r(\boldsymbol{X}^\alpha) = \left\{\vartheta \in \Theta : rank(o_i^\Delta, \boldsymbol{X}^\alpha, \vartheta) = r\right\}.$$

(1) The rank acceptability indices (RAIs) are the ratio of the expected volume of $\Theta_\triangle^r(\boldsymbol{X}^\alpha)$ to the volume of the feasible Möbius measure space Θ, which can be calculated by

$$b_{o_i^\Delta}^r = \int_{\boldsymbol{X}^\alpha\in\aleph}\mathbb{F}_\aleph(\boldsymbol{X}^\alpha)\int_{\vartheta\in\Theta_\triangle^r(\boldsymbol{X}^\alpha)}\mathbb{F}_\Theta(\vartheta)d\vartheta d\boldsymbol{X}^\alpha,$$

where $b_{o_i^\Delta}^r \in [0, 1]$. When $b_{o_i^\Delta}^r = 0$, it means that object o_i taking action $\triangle$ never ranks r with the current compatible preference information. When $b_{o_i^\Delta}^r = 1$, object o_i taking action $\triangle$ always ranks r with the current compatible preference information.

(2) The central Möbius measure vectors (CMMVs) are defined as the expected center of gravity of the favorable Möbius measure vector which can ensure that the object o_i taking action $\triangle$ ranks first, which can be calculated by

$$\vartheta^c_{o_i^\Delta} = \frac{1}{b^1_{o_i^\Delta}} \int_{\mathbf{X}^\alpha \in \aleph} \mathbb{F}_\aleph(\mathbf{X}^\alpha) \int_{\vartheta \in \Theta^r_\triangle(\mathbf{X}^\alpha)} \mathbb{F}_\Theta(\vartheta)\vartheta d\vartheta d\mathbf{X}^\alpha.$$

(3) The pairwise winning indices (PWIs) are defined as the probabilities that an object o_i taking action $\triangle$ is more preferred than the object o_i taking action ∇, which can be computed by

$$p_{o_i^\triangle o_i^\nabla} = \int_{\vartheta \in \Theta : rank\left(o_i^\Delta, \mathbf{X}^\alpha, \vartheta\right) < rank\left(o_i^\nabla, \mathbf{X}^\alpha, \vartheta\right)} \mathbb{F}_\Theta(\vartheta) \int_{\mathbf{X}^\alpha \in \aleph} \mathbb{F}_\aleph(\mathbf{X}^\alpha) d\mathbf{X}^\alpha d\vartheta.$$

5.2. Monte Carlo simulation of the proposed SMAA-TWD model

Monte Carlo simulation of the proposed SMAA-TWD model is presented in the following steps. In addition, a flowchart for the proposed SMAA-TWD model is depicted in Fig. 1.

Step 1: Given a multi-attribute TWD problem, identify the evaluation space $\aleph$.
Step 2: Identify the Möbius measure space Θ through the standardized and monotonic conditions.
Step 3: Set a maximum number of iterations L, and for each iteration execute Step 4.
Step 4: For the l^{th} iteration.
 (1) Sample a decision matrix $\mathbf{X}^{\alpha^l}$ from the evaluation space $\aleph$ by runif in R software.
 (2) Sample a Möbius measure vector ϑ^l from the Möbius measure space Θ by Hit-and-Run sampling [50].
 (3) Obtain the classification of all objects.
Step 5: If $l < L$, then return to Step 4; if $l = L$, then proceed to Step 6.
Step 6: Update the three descriptive indices.
 (1) $b^r_{o_i^\Delta} \approx R_{o_i^\Delta r}/L, \forall \triangle, r \in \{1,2,3\}$, where $R_{o_i^\Delta r} = \sum_{l \in \{1,2,\ldots,L\}} \ell^l_1\left(rank\left(o_i^\triangle, \mathbf{X}^{\alpha^l}, \vartheta^l\right) = r\right)$, such that $\ell_1(true) = 1$ and $\ell^l_1(false) = 0$.
 (2) $\vartheta^c_{o_i^\Delta} \approx \Upsilon_{o_i^\Delta}/R_{o_i^\Delta 1}, \forall \triangle \in \{1,2,3\}$, where $\Upsilon_{o_i^\Delta} = \sum_{l \in \{1,2,\ldots,L\}} \ell^l_2\left(rank\left(o_i^\triangle, \mathbf{X}^{\alpha^l}, \vartheta^l\right) = 1\right)$, such that $\ell^l_2(true) = \vartheta^l_{o_i^\Delta}$ and $\ell^l_2(false) = \mathbf{0}$.
 (3) $p_{o_i^\triangle o_i^\nabla} \approx J_{o_i^\triangle o_i^\nabla}/L, \forall \triangle, \nabla \in \{1,2,3\}$, where $J_{o_i^\triangle o_i^\nabla} = \sum_{l \in \{1,2,\ldots,L\}} \left(\ell^l_1\left(rank\left(o_i^\triangle, \mathbf{X}^{\alpha^l}, \vartheta^l\right) < rank\left(o_i^\nabla, \mathbf{X}^{\alpha^l}, \vartheta^l\right)\right)\right)$.

6. Numerical analysis

In this section, we apply the proposed SMAA-TWD model to address the decision problem regarding the TCM diagnosis of insomnia. Then, a comparative analysis of the proposed model with several models is performed to display its excellent performance. Finally, sensitivity analysis is conducted to discuss the effect on the results with the change in parameters.

6.1. Case description

Insomnia, characterized by frequent awakening, dreaminess, early awakening, difficulty falling asleep, etc., is a prevalent disease among people in modern society. It not only impairs physical health conditions but also deteriorates mental well-being in the long term. According to the China Sleep Research Association, more than 300 million people suffered from sleep disorders in 2021, and the incidence of insomnia in adults was as high as 38.2%. Young people, including those born after 1990, comprise the largest segment of sleep-deprived individuals. The physical examination methods of traditional Chinese medicine (TCM) diagnosis is generally standardized for insomniacs, and consist of inspection, smelling and listening, interrogation and palpation.

To illustrate the performance of the proposed method, we utilize the example of TCM diagnosis on insomnia. Let $\Omega = \{C, \neg C\}$ be the set of two states of a clinic attendee corresponding to insomnia and no insomnia, respectively. The set of actions for the clinic attendee is denoted $\Pi = \{\pi_P, \pi_B, \pi_N\}$. The action π_P represents the case where clinic attendee merits treatment, while the action π_N represents the case where clinic attendee is not to be treated. The action π_B represents the case where doctor cannot immediately reach a conclusion and needs to make further observations. Twenty clinic attendees were selected for insomnia examination, denoted as $\mathcal{O} = \{o_1, o_2, \ldots, o_{20}\}$ and were observed by a doctor. Let $\mathcal{A} = \{c_1, c_2, c_3, c_4\}$ represent the set of attributes: inspection(c_1), smelling(c_2), listening(c_3), and interrogation and palpation(c_4). Common sense suggests that synergy and redundancy interactions among attributes exist and need to be considered. The initial evaluation of the disease can be expressed by linguistic variables $S = \{s_0 = Very\,bad; s_1 = Bad; s_2 = Medium; s_3 = Good; s_4 = Very\,good\}$. The corresponding linguistic evaluation matrices of the twenty clinic attendees with respect to four attributes can be given by doctors. From the corresponding relationships between triangular fuzzy numbers and linguistic variables, as shown in Table 7, these linguistic matrices can be transformed into triangular fuzzy number evaluation matrices. After considerable discussion, the doctors utilized a weighted processing method to reach group opinions.

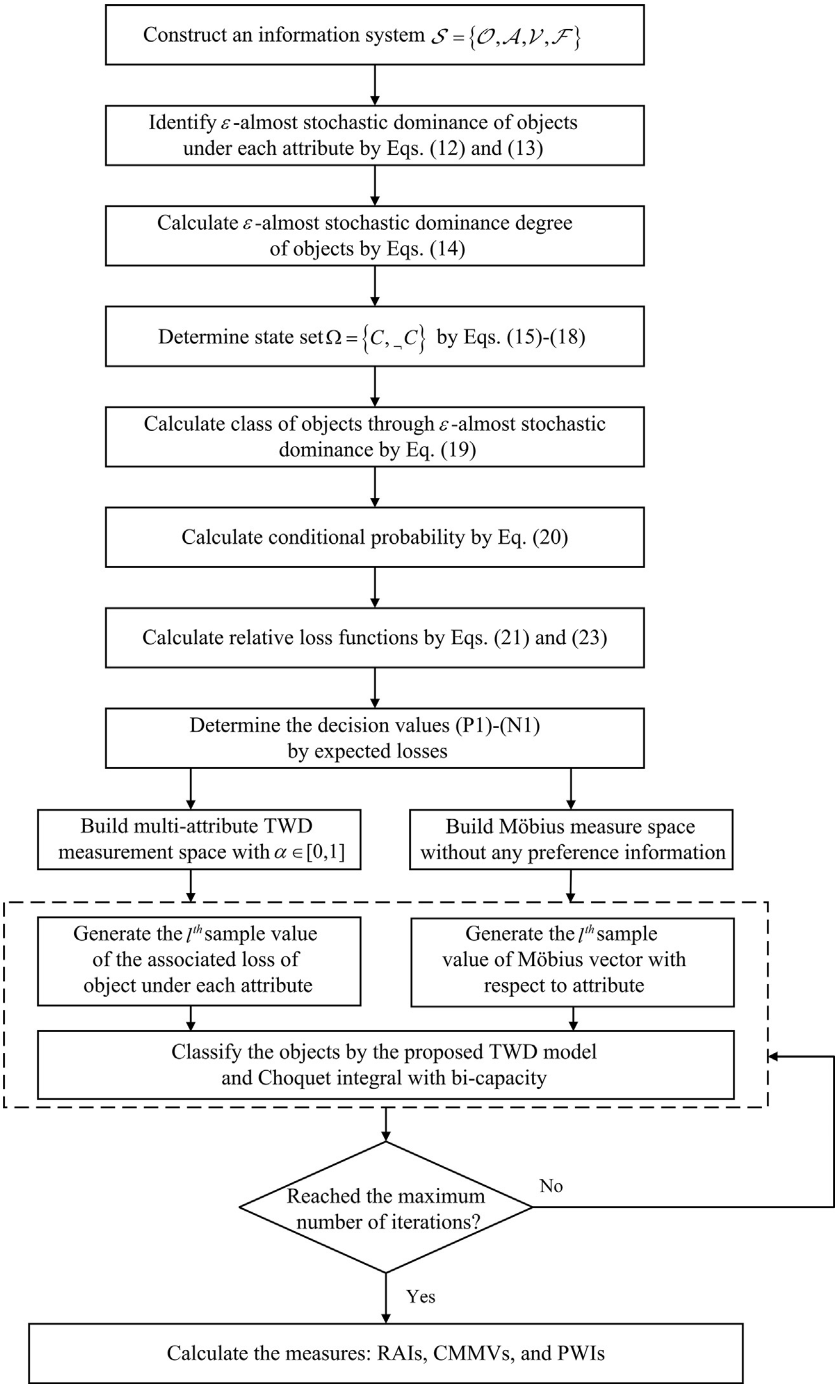

Fig. 1. Flow chart of the proposed SMAA-TWD model.

Table 7
The relationships between triangular fuzzy numbers and linguistic variables.

Linguistic variable	Triangular fuzzy number
Very bad	(0, 0, 0.3)
Bad	(0.1, 0.3, 0.5)
Medium	(0.3, 0.5, 0.7)
Good	(0.5, 0.7, 0.9)
Very good	(0.7, 0.99, 0.99)

Table 8
The collected triangular fuzzy number decision matrix of the twenty objects.

	c_1	c_2	c_3	c_4
o_1	(0.10,0.30,0.67)	(0.10,0.12,0.60)	(0.14,0.56,0.87)	(0.13,0.67,0.95)
o_2	(0.13,0.29,0.95)	(0.14,0.25,0.75)	(0.15,0.76,0.98)	(0.15,0.78,0.95)
o_3	(0.07,0.20,1.00)	(0.12,0.67,0.99)	(0.37,0.44,0.95)	(0.12,0.68,0.98)
o_4	(0.09,0.15,0.72)	(0.17,0.68,0.89)	(0.15,0.69,0.70)	(0.10,0.50,0.77)
o_5	(0.07,0.19,0.83)	(0.15,0.61,1.00)	(0.12,0.38,0.83)	(0.18,0.29,0.50)
o_6	(0.12,0.24,0.43)	(0.13,0.72,0.98)	(0.12,0.39,0.68)	(0.15,0.39,0.78)
o_7	(0.11,0.25,0.95)	(0.14,0.51,0.93)	(0.13,0.35,0.97)	(0.11,0.20,0.77)
o_8	(0.12,0.21,0.98)	(0.18,0.79,1.00)	(0.15,0.65,0.95)	(0.16,0.22,0.70)
o_9	(0.15,0.10,0.52)	(0.12,0.52,0.82)	(0.17,0.52,0.73)	(0.15,0.65,0.98)
o_{10}	(0.18,0.27,0.60)	(0.11,0.63,0.87)	(0.12,0.58,0.74)	(0.13,0.48,0.97)
o_{11}	(0.19,0.23,0.45)	(0.13,0.55,0.86)	(0.38,0.43,0.90)	(0.12,0.28,0.57)
o_{12}	(0.12,0.14,0.95)	(0.10,0.41,0.83)	(0.12,0.68,0.96)	(0.11,0.28,0.97)
o_{13}	(0.10,0.21,0.89)	(0.16,0.70,0.86)	(0.15,0.35,0.97)	(0.14,0.17,0.93)
o_{14}	(0.13,0.32,0.83)	(0.11,0.64,0.87)	(0.14,0.79,0.99)	(0.12,0.22,1.00)
o_{15}	(0.16,0.32,0.55)	(0.11,0.83,0.88)	(0.13,0.39,0.75)	(0.15,0.20,0.79)
o_{16}	(0.14,0.20,0.72)	(0.17,0.68,1.00)	(0.23,0.29,0.77)	(0.21,0.27,0.72)
o_{17}	(0.10,0.32,0.73)	(0.12,0.31,0.71)	(0.13,0.67,0.73)	(0.20,0.31,0.75)
o_{18}	(0.11,0.22,0.42)	(0.12,0.73,0.85)	(0.17,0.70,0.80)	(0.14,0.26,0.48)
o_{19}	(0.11,0.15,0.78)	(0.10,0.41,1.00)	(0.12,0.36,0.75)	(0.11,0.18,0.90)
o_{20}	(0.13,0.15,0.97)	(0.56,0.90,1.00)	(0.15,0.70,1.00)	(0.10,0.46,0.85)

Our main objective is to classify each without the group decision and aggregation method. Therefore, we adopt the simulated data matrix as the collected triangular fuzzy number decision matrix of the twenty objects, as shown in Table 8.

6.2. The decision processes and results of the proposed model

Given $\epsilon = 0.320$, we can first identify the ϵ-almost stochastic dominance relation of any two objects under each attribute. Second, combined with the ϵ-almost stochastic dominance degree of objects under each attribute, we can calculate the conditional probability. Thirdly, with the risk avoidance coefficient $\theta = 0.415$, the loss function can be obtained by using the triangular fuzzy Hellinger distance. Full details are shown in Algorithm 1. Without any preference information on attributes, Monte Carlo simulation of SMAA-TWD model is implemented by exploring the parameter space $\alpha \in [0,1]$. Assuming that the number of iterations is $L = 10000$, RAIs, CMMVs and PWIs of clinic attendees are obtained after the end of the simulation. We only show the RAIs, CMMVs and PWIs of the clinic attendee o_1 due to space considerations. The RAIs and PWIs of the clinic attendee o_1 are shown in Tables 8 and 9, respectively. The CMMVs of the clinic attendee o_1 are shown in Fig. 2. In Fig. 2, the horizontal coordinate-axis (content coordinate) consists of Möbius measures according to the set sequence. For example, scale "1" corresponds to Möbius measure $\vartheta(\varnothing,\{2,3,4\})$ and $\vartheta(\varnothing,\{2,3,4\}) = 0.216$. This means that when the positive scale is empty and the negative scale is the set of attributes $\{c_2,c_3,c_4\}$, the Möbius measure is 0.216. Scale "2" corresponds to Möbius measure $\vartheta(\varnothing,\{1,3,4\})$ and $\vartheta(\varnothing,\{1,3,4\}) = 0.496$. This means that when the positive scale is empty and the negative scale is the set of attributes $\{c_1,c_3,c_4\}$, the Möbius measure is 0.496. The rest of the scales (i.e.,3, 4, .., 32) correspond to the following Möbius measures, i.e., $\vartheta(\varnothing,\{1,2,4\}), \vartheta(\varnothing,\{1,2,3\}), \vartheta(1,\{2,3,4\}), \vartheta(2,\{1,3,4\}), \vartheta(3,\{1,2,4\}), \vartheta(4,\{1,2,3\}), \vartheta(\varnothing,\{3,4\}), \vartheta(\varnothing,\{2,4\}), \vartheta(\varnothing,\{2,3\}), \vartheta(\varnothing,\{1,4\}), \vartheta(\varnothing,\{1,3\}), \vartheta(\varnothing,\{1,2\}), \vartheta(\{1,2\},\{3,4\}), \vartheta(\{1,3\},\{2,4\}), \vartheta(\{1,4\},\{2,3\}), \vartheta(\{2,3\},\{1,4\}), \vartheta(\{2,4\},\{1,3\}), \vartheta(\{3,4\},\{1,2\}), \vartheta(1,\{2,3\}), \vartheta(1,\{2,4\}), \vartheta(1,\{3,4\}), \vartheta(2,\{1,3\}), \vartheta(2,\{1,4\}), \vartheta(2,\{3,4\})$, $\vartheta(3,\{1,2\})$,$\vartheta(3,\{1,4\})$,$\vartheta(3,\{2,4\})$,$\vartheta(4,\{1,2\})$,$\vartheta(4,\{1,3\})$ and $\vartheta(4,\{2,3\})$.

From Fig. 2, one can see that the CMMV ϑ is nonzero. This means that the synergistic and redundant interactions among attributes exist and need to be considered. In addition, the CMMV ϑ is the preference basis for DMs. In Table 9, according to the principle of minimizing total loss, we only consider the RAIs of the clinic attendee o_1 taking three actions for the third rank. $b^3_{o_1^P}$ represents that the probability of clinic attendee o_1 receiving an acceptance decision is 31.0%. Similarly, $b^3_{o_1^B}$ represents that the probability of clinic attendee o_1 receiving a deferment decision is 35.9%, and $b^3_{o_1^N}$ represents that the probability of the clinic attendee o_1 receiving a rejection decision is 33.1%. One can see that the clinic attendee o_1 is most likely to take the deferment decision because the probability of the clinic attendee o_1 making the deferment decision is maximal

Table 9
The matrix of RAIs(%) for the clinic attendee o_1.

	$b^1_{o_1^\triangle}$	$b^2_{o_1^\triangle}$	$b^3_{o_1^\triangle}$
o_1^P	47.8	21.2	31.0
o_1^B	0	64.1	35.9
o_1^N	52.2	14.7	33.1

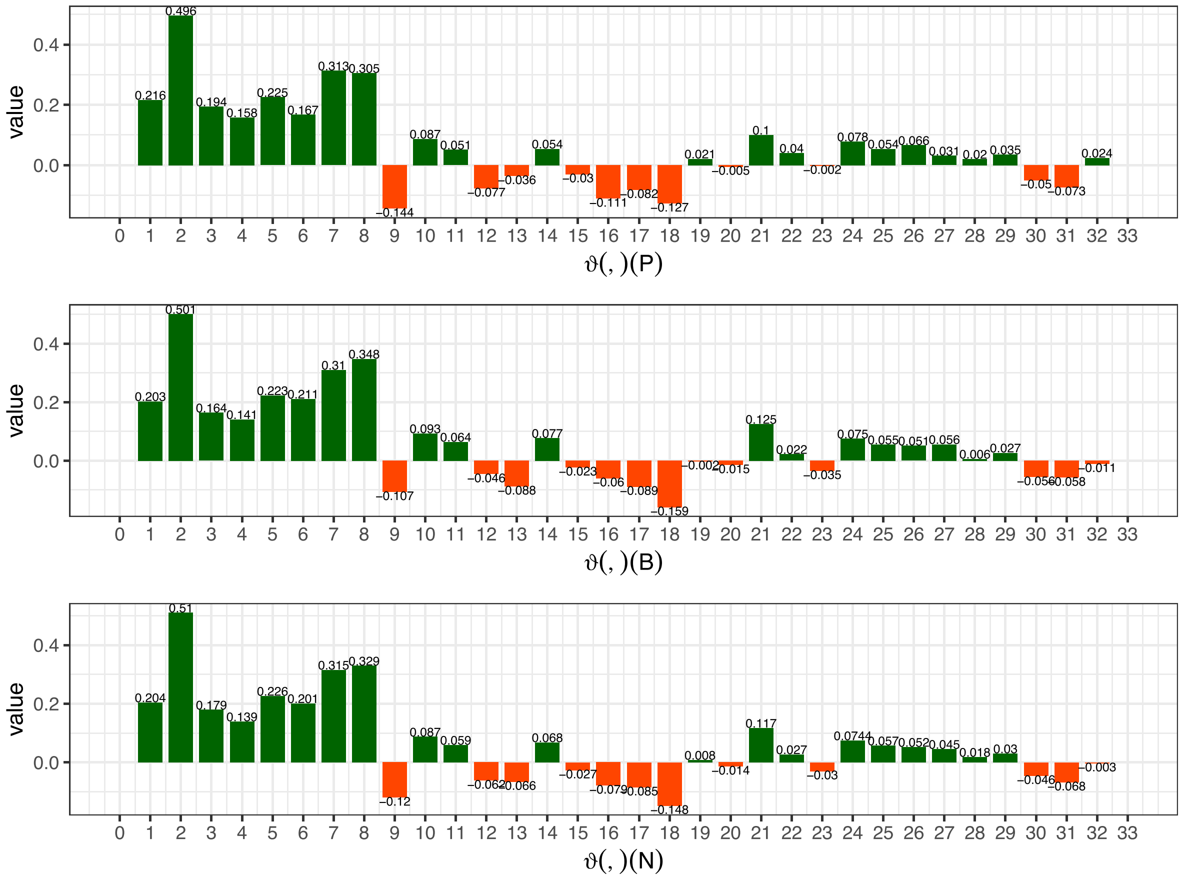

Fig. 2. The CMMVs of three domains for the clinic attendee o_1.

among these three actions for the third rank, i.e., $b^3_{o^B_1} = max\left\{b^3_{o^P_1}, b^3_{o^B_1}, b^3_{o^N_1}\right\} = 35.9\%$. However the difference among these three values is slight. In this case, we need to analyze the PWIs in Table 10. From Table 10, one can see that $p_{PB} = 69.0\% > p_{BP} = 31.0\%$, which means that the clinic attendee o_1 receiving an acceptance decision is preferred over a deferment decision. Similarly, the clinic attendee o_1 receiving a rejection decision is preferred over a deferment decision because $p_{NB} = 66.9\% > p_{BN} = 33.1\%$. Thus, it is more reliable to confirm that the clinic attendee o_1 belongs to the deferment domain. By analyzing the RAIs of all clinic attendees for three actions, the probabilities of all clinic attendees belonging to the three domains can be calculated and are shown in Fig. 3. According to the above analysis, the decision results of all clinic attendees can be obtained, as shown in Table 11. From Table 11, one can see that clinic attendees o_2, o_3, o_8 and o_{20} are diagnosed with insomnia and need to be admitted to the hospital for treatment; clinic attendees $o_1, o_5, o_{11}, o_{12}, o_{13}, o_{14}, o_{16}$ and o_{18} still need further observation, and the remaining clinic attendees do not require treatment.

6.3. Comparison and discussion

In this section, to demonstrate the effectiveness of the proposed method, we compare it with the other two TWD methods [5,13] for the problem of TCM diagnosis of insomnia. The classification results of clinic attendees derived from different TWD methods are presented in Table 12 and Fig. 4.

Table 10
The matrix of PWIs(%) for the clinic attendee o_1.

	o^P_1	o^B_1	o^N_1
o^P_1	0	69.0	47.8
o^B_1	31.0	0	33.1
o^N_1	52.2	66.9	0

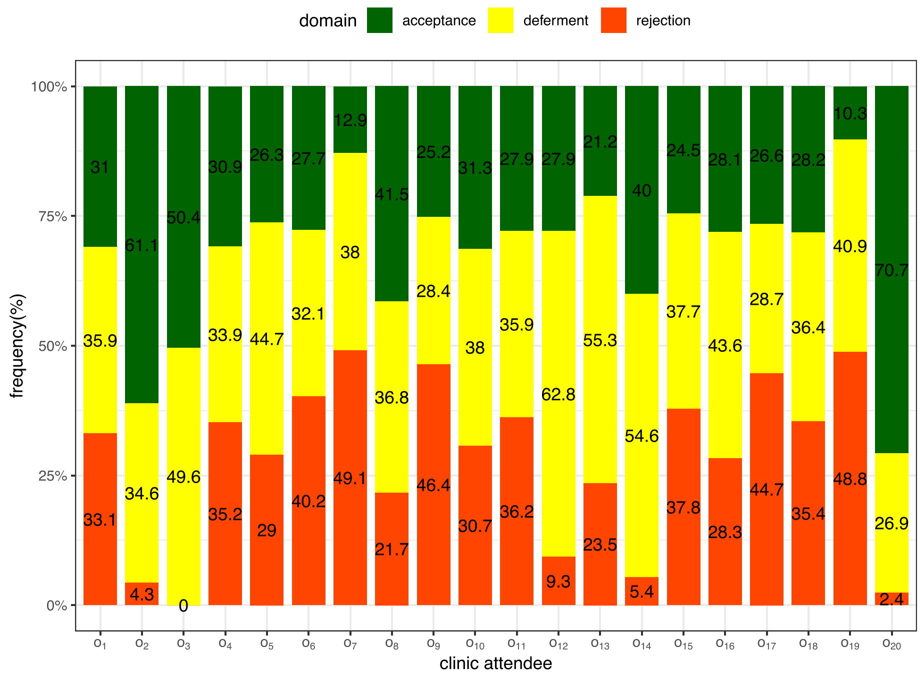

Fig. 3. The frequency of clinic attendee belonging to three domains.

Table 11
The decisions results of twenty clinic attendees.

Domains	POS(C)	BND(C)	NEG(C)
Clinic attendees	o_2, o_3, o_8, o_{20}	$o_1, o_5, o_{10}, o_{12}, o_{13}, o_{14}, o_{16}, o_{18}$	$o_4, o_6, o_7, o_9, o_{11}, o_{15}, o_{17}, o_{19}$

Table 12
The classification results of different methods.

classification	POS(C)	BND(C)	NEG(C)
Our proposed method	o_2, o_3, o_8, o_{20}	$o_1, o_5, o_{10}, o_{12}, o_{13}, o_{14}, o_{16}, o_{18}$	$o_4, o_6, o_7, o_9, o_{11},$ o_{15}, o_{17}, o_{19}
Liang et al.' method [13]	o_2, o_3, o_{20}	$o_1, o_4, o_5, o_8, o_9, o_{10}, o_{12}, o_{13}, o_{14}$	$o_6, o_7, o_{11}, o_{15}, o_{16}$ o_{17}, o_{18}, o_{19}
Fan and Liu' method [5]	$o_2, o_3, o_8, o_{14}, o_{20}$	$o_{10}, o_{11}, o_{12}, o_{13}$	$o_1, o_4, o_5, o_6, o_7, o_9$ $o_{15}, o_{16}, o_{17}, o_{18}, o_{19}$

From Table 12, one can see that although the classification of twenty clinic attendees obtained by the three different methods is not identical, the acceptance domain $POS(C)$ has overlapped clinic attendees, o_2, o_3 and o_{20}. This result is consistent with the decision results determined by the proposed method based on three actions, which provides strong support for the validity of our proposed model. In addition to the overlapped clinic attendees, the number of clinic attendees in the acceptance domain $POS(C)$ obtained by the method [13] is less than the number obtained by our proposed method. Specifically, the method [13] failed to judge the treatment of clinic attendee o_8, and determined that further observation was needed. Therefore, the clinic attendee o_8 is assigned to the deferment domain $BND(C)$. The number of clinic attendees in the acceptance domain $POS(C)$ obtained by the method [5] is more than the number obtained by our proposed method, and includes o_{14}. However, the number of clinic attendees in the deferment domain $BND(C)$ is less than the number obtained by our method. By comparing the three deferment domains $BND(C)$ obtained by these methods, the method [13] confirms the conclusions of our method, i.e., o_{14} needs further observation. Aside from overlapping clinic attendees, the deferment domain $BND(C)$ obtained by our method has more clinic attendees than two other methods. In addition, one can see that the method [5] has a larger rejection domain $NEG(C)$ than the two other methods. These differences might be attributed to the bipolar scale and Choquet integral with respect to bi-capacity, which distinguishes the three actions and measures the synergy and redundancy interactions among attributes with the bipolar scale. In contrast, for the conservative principle

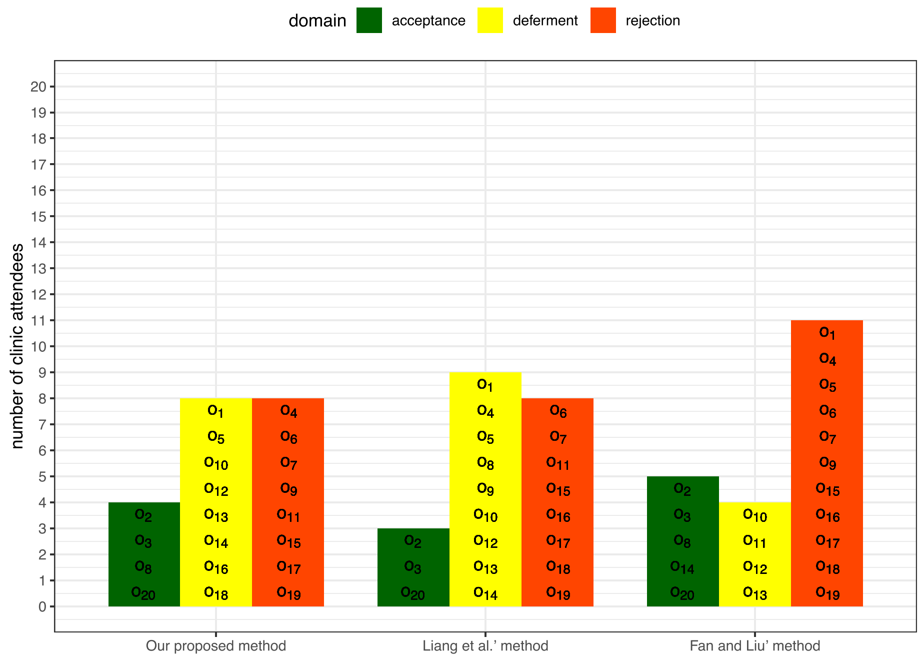

Fig. 4. The classification results of different methods.

of risk, the proposed method is more effective in coping with multi-attribute TWD problems than those of Liang et al. [13] and Fan and Liu [5].

Based on the above comparison and discussion, the proposed model is verified to be as effective as previous TWD models. Compared with previous TWD models, there are some highly expedient advantages to combining the TWD and stochastic MCDM successfully, which we summarize below:(1) One of the main advantages is to offer a stochastic TWD, which is suitable for the TWD under an uncertain environment. In the stochastic TWD model, we utilize ϵ-almost stochastic dominance to measure qualitative relationships of interval uncertainty and propose the ϵ-almost stochastic dominance degree to measure their quantitative relationships. Combined with ϵ-almost stochastic dominance and ϵ-almost stochastic dominance degree, a novel objective method combined with ϵ-almost stochastic dominance degree is proposed for determining the set of two states. Then, the conditional probability is calculated to avoid subjective bias. A novel calculation method of the relative loss function combined with the Hellinger distance is proposed, ensuring differences in relative losses under each attribute. Therefore, a novel TWD combined with ϵ-almost stochastic dominance is proposed. (2) To describe the interrelated attributes in terms of bipolar scale, we utilized the bi-capacity to distinguish three kinds of decision results under each attribute obtained by the novel TWD. The Choquet integral with respect to bi-capacity is utilized to aggregate the overall expected loss values corresponding to the decision actions. (3) The proposed model can reasonably deal with uncertainty. When considering α to be a random variable, SMAA is extended into the novel TWD to deal with input uncertainty information, i.e., attribute values and Möbius measures. Furthermore, Monte Carlo simulation of the novel model is proposed for calculating three indices. A more robust result can be achieved by analyzing indices.

6.4. Sensitivity analysis with parameters θ and ϵ

A sensitivity analysis is implemented to further discuss the parameters θ (in the relative loss function) and ϵ (in the ϵ-almost stochastic dominance) on the TWD results. The decision-making results with different values of θ are displayed in Fig. 5. Similarly, the decision-making results with different values of ϵ are displayed in Fig. 6.

According to Fig. 5, when $0.250 \leqslant \theta \leqslant 0.275$, there are only two domains, $BND(C)$ and $NEG(C)$, which indicates that all clinic attendees are divided into two domains: further observation needed and no need for treatment. One clinic attendee has no need for treatment, and the remaining clinic attendees need to be further observed. When $0.300 \leqslant \theta \leqslant 0.450$, there are three domains, i.e., $POS(C), BND(C)$ and $NEG(C)$, which indicate that all clinic attendees are divided into three domains: treatment needed, further observation needed and no need for treatment. Moreover, when $0.300 \leqslant \theta \leqslant 0.375, POS(C)$ is unchanged, and $BND(C)$ and $NEG(C)$ indicate one falling and the other rising. This means that when the parameter θ is smaller, the number of clinic attendees needing treatment is less affected by θ. When $\theta = 0.475$, there are only two domains, $POS(C)$ and $NEG(C)$, indicating that all clinic attendees are divided between the domains of treatment needed and no need for treatment, and no clinic attendee needs to be further observed. Furthermore, one can see that the number of clinic atten-

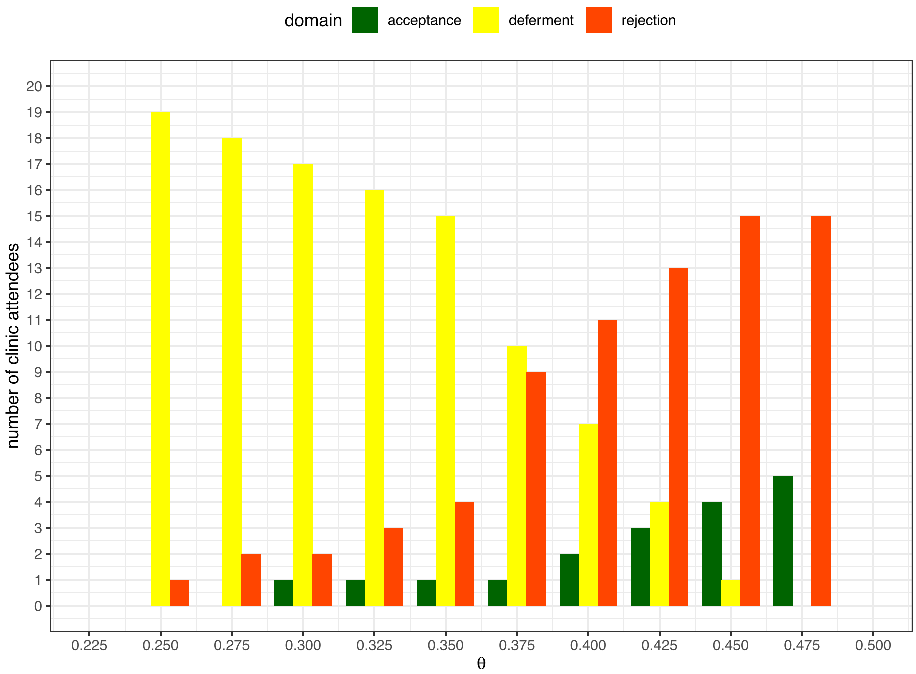

Fig. 5. The sensitivity analysis of θ.

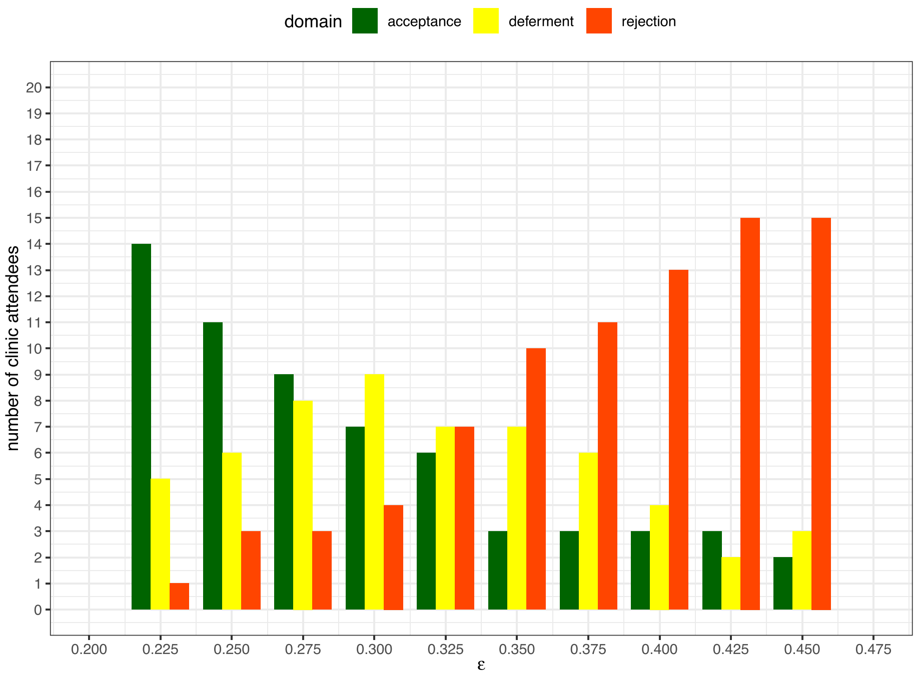

Fig. 6. The sensitivity analysis of ϵ.

dees placed into $NEG(C)$ increases with increasing θ. The number of clinic attendees placed into $BND(C)$ increases with decreasing θ. According to Fig. 6, when $0.225 \leqslant \epsilon \leqslant 0.450$, there are three domains, i.e., $POS(C), BND(C)$ and $NEG(C)$, which indicate that all clinic attendees are divided into three domains: need for treatment, need further observation and no need for treatment. In addition, when $\epsilon = 0.225$, only one clinic attendee is assigned to $NEG(C)$, fourteen clinic attendees are assigned to $POS(C)$ and five clinic attendees are assigned to $BND(C)$. This means that when the parameter ϵ is smaller, the number of clinic attendees needing further observation is less affected by ϵ. When $0.225 \leqslant \epsilon \leqslant 0.300$, the number of attendees assigned to $BND(C)$ increases with increasing ϵ, which indicates that more clinic attendees need to be further observed. When $0.300 \leqslant \epsilon \leqslant 0.450$, the number of attendees assigned to $BND(C)$ decreases with increasing ϵ, which indicates that fewer clinic attendees need to be further observed. Furthermore, one can see that the number of clinic attendees placed into $NEG(C)$ increases with increasing ϵ. The number of clinic attendees placed into $POS(C)$ increases with decreasing ϵ.

7. Conclusions

To successfully integrate TWD and stochastic MADM, a SMAA-TWD model is proposed for multi-attribute TWDs with interrelated attributes in a triangular fuzzy environment. To distinguish the TWDs, we utilize a bipolar scale to measure the TWDs and the Choquet integral with respect to bi-capacity to aggregate the expected loss of objects under all attributes. Monte Carlo simulation is executed to implementat the proposed model by randomly generating Möbius measures and observing the resultant preference orderings of objects. Through experimental studies, comparison and sensitivity analysis, the validity and stability of the proposed model are demonstrated. The main findings of this study are summarized as follows: (1) To show the advantage of being objective and accurate, the conditional probability and loss function in the novel TWD model can be determined according to the information system. (2) By adjusting the parameters ϵ (in the ϵ-almost stochastic dominance) and θ (in the loss function), DMs can make different decisions in the novel TWD model according to different needs of DMs. (3) The novel TWD model is more powerful than the traditional TWD because it can deal with complex and uncertain input information (including attribute assessments of objects, fuzzy measures, different kinds of parameters, etc). (4) The RAIs, CMMVs, and PWIs are complementary, and the robustness of decision results is strengthened by analyzing these indices when deciding the resultant preference orderings of objects. Hence, the effectiveness and feasibility of the proposed model is clear.

From the perspective of the structure of attributes, we only emphasize the typical single structure of attributes and ignore the hierarchical structure of attributes. We will further generalize the novel TWD to handle the MADM problem with the hierarchical structure of attributes. From the bounded rationality perspective of DMs, we do not consider behavior theory in our model. In the future, enriching the behavioral TWD model into the multi-attribute group decision-making problem is another critical research direction.

CRediT authorship contribution statement

Qian Zhao: Conceptualization, Methodology, Writing - original draft, Writing - review & editing. **Yanbing Ju:** Conceptualization, Methodology, Data curation, Writing - review & editing. **Luis Martínez:** Conceptualization, Methodology, Data curation, Writing - review & editing. **Peiwu Dong:** Methodology, Data curation, Writing - review & editing, Supervision. **Jingfeng Shan:** Writing - review & editing, Supervision.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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