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Multi-attribute predictive analysis based on attribute-oriented fuzzy rough sets in fuzzy information systems

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ABSTRACT

Multi-attribute decision making (MADM) plays an important role in decision analysis. Based on a new fuzzy rough set theory (FRS), this paper presents a multi-attribute prediction analysis model in fuzzy information systems (FISs). In this prediction model, the proposed FRS is based on an attribute-oriented fuzzy similarity relation; the cosine deviation of a candidate to optimistic decision and pessimistic decision is proposed to construct a trend function, for further predicting the development trend of the candidate. It deals with MADM problems from the perspective of multi-attribute predictive analysis, and is a valuable continuation of fuzzy decision making research. To evaluate the validity and feasibility of the proposed prediction model, experiments with real data sets and comparative experimental analysis are conducted. The experimental results prove that the prediction performance of the proposed model is well-pleasing in terms of both prediction accuracy and AUC results.

1. Introduction

Decision analysis is closely related to psychology, systems science, economics, mathematics and management [1,2]. Multi-attribute decision making (MADM) is an important part of decision analysis. It is widely applied in economy [3], medicine [4][5], industry [6], management [7,8] and military and many other fields [9–12], such as, risk evaluation, investment decision, project evaluation, plant site selection, bidding and tendering, industrial sector development ranking, automotive engineering, comprehensive evaluation of economic benefits [13–15]. Some classical methods have been widely used in the research of various MADM problems, such as ELECTRE method [16] and PROMETHEE method [17], TOPSIS method [18] and EDAS method [19].

However, with the development of MADM research, people are faced with solving decision problems with fuzzy information. To improve the decision quality of this kind of problems, it is necessary to enhance the ability to deal with inaccurate information. Therefore, scholars introduce fuzzy set theory (FST) [20] and put forward many fuzzy decision methods by taking advantage of FST to deal with uncertain problems. Most of them are the extensions of traditional MADM methods to

Abbreviations: FIS, AFRS; MADM, MAPA.

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fuzzy environment. Such as the fuzzy ELECTRE method [21,22], fuzzy PROMETHEE method [23](24), fuzzy TOPSIS method [25,26], Extended EDAS Method [27,28] and other decision making methods under fuzzy environment [29–32].

Fuzzy rough set (FRS) [33] and extended or modified FRS theory [34–36] integrate the advantages of FST and rough set theory [37] in dealing with fuzzy and imprecise information. With the increasing complexity of fuzzy decision making environment, the combination of FRS theory and classical MADM method can make up for fuzzy MADM model. The FRS-based MADM approaches provide a new strategy for dealing with MADM problems with fuzzy information [38]. In recent years, it has attracted many researchers to study related problems. Hereinto, the research team of Zhan and Jiang et al. has done extensive researches in this area. Such as, covering-based FRS models and covering-based generalized fuzzy rough sets have been proposed for handling MADA problems with fuzzy information [39] [40] or intuitionistic fuzzy information [41]. In addition, Zhang and Ye et al. presented FRS models to deal with MADA problems by incorporating with PROMETHEE method [42,43], TOPSIS method [44,45] and fuzzy information evaluation method [46]. Further, Ye and Zhang et al. [47] designed new FRS models based on a fuzzy neighborhood operator, then coped with MADM problems in fuzzy environment by using the idea of the PROMETHEE II methods.

It's worth noting that the mentioned researches above of MADM are using existing multi-attribute decision information to sort and select a group of (limited) alternatives in a certain way. But we would be confronted with such problems, for example, we want to judge whether a company will go bankrupt in the future, we can process the known information of the bankrupt and not bankrupt companies, so as to further decide the bankruptcy trend of the company. That is to say, it is according to the known knowledge to make trend decisions of an object. Obviously, the existing MADM method cannot reach this target. A different strategy should be found, therefore, we intend to design a new approach for dealing with such MADM problems based on FRS theory. Three research motivations of this paper as follows:

- The existing MADM cannot deal with multi-attribute trend decision making.
- Research the influence of correlation between attributes on decision making.
- FRS-based MADM approaches can make up for fuzzy MADM methods. The FRS-based models of above MADM methods are designed on the basis of object-oriented. However, object-oriented FRS models should not be the unique model to handle fuzzy decision making problems.
- Multi-attribute trend prediction analysis can be helpful for uncertain decision problem.

Motivated by those opinions, we have an ideal with designing a multi-attribute predictive analysis method based on attribute-oriented fuzzy rough sets in fuzzy information systems (FISs). In pursuing this mission, the main contributions of this paper are specifically offered in the following:

- In general, fuzzy similarity relation is a relation generated based on attributes, which describes the similarity relation between objects. To explore the correlation relation between attributes, this paper constructs a fresh fuzzy similarity relation to measure the correlation relationship between attributes, which generated based on objects.
- As stated above, FRS theory has unique advantages in dealing with uncertain decisions. Therefore, a novel attribute-oriented FRS model of FIS is established on the basis of the proposed fuzzy similarity relation. Related properties and theorems are investigated.
- From the perspective of decision strategy, the lower and upper approximation sets of the proposed FRS model are considered as pessimistic decision scheme and optimistic decision scheme. They are applied to multi-directional predictive modeling.
- A prediction estimation method by least deviation to optimistic and pessimistic predictions is proposed. By analyzing the cosine distances with a candidate to the optimistic prediction and pessimistic prediction, the development trend of the candidate can be predict with a trend function.
- Our work deals with MADM problems from the perspective of multi-attribute predictive analysis. Decisions are made by predicting the development trend of the object, it is a valuable continuation of fuzzy decision making research.

The outline of this paper is listed as follows: we review some basic definitions and notions in Section 2, such as FIS and FRS. In Section 3, we introduce a novel FRS model of FIS based on the proposed (γ, θ)-fuzzy similarity relation. Meanwhile, the fundamental properties of the novel FRS are investigated, and some examples are shown for further understanding the mechanism of this model. In Section 4, we put forward a FRS-based multi-attribute predictive analysis model, and present experiments with UCI data sets to verify the novel prediction model. Section 5 concludes the paper with a brief discussion about further research.

2. Preliminaries

For the convenience of the subsequent discussions, some theoretical preliminaries about FIS and FRS are shown as below. A fuzzy information system *FIS* can be described as a 4-tuple $\langle U, AT, V, f \rangle$, where.

- U called the universe of discourse, is a non-empty finite objects set;

- A is a non-empty finite attributes set;
- $V = \cup_{a \in A} V_a$, and $V = [0,1]$, where V_a is the set of values of attribute a;
- $f : U \times A \to V_a$, is a map such that $f(x,a) \in V_a$for every $x \in U$ and $a \in AT$.

In a *FIS*, suppose $A = \{a_1, a_2, \cdots, a_n\}$, then fuzzy set X in A can be expressed as $X = f(x,a_1)/a_1 + f(x,a_2)/a_2 + \cdots + f(x,a_n)/a_n, f(x,a)$ is the membership degree of $x \in U$ with $a \in A$. This expression of fuzzy set X shortly denotes as $X(a)$ with $a \in A$. All fuzzy sets (*FSs*) in A are expressed as $\widetilde{F}(A)$.

Theorem 2.1. Let $FIS = \langle U, A, V, f \rangle$, and $X, Y \in \widetilde{F}(A)$. For any $a \in A$, we have

(1) $X \subseteq Y \iff \forall a \in A, f_X(x,a) \leqslant f_Y(x,a)$;
In particular, $X = Y \iff \forall a \in A, f_X(x,a) = f_Y(x,a)$;
(2) $f_{(X \cap Y)}(x,a) = f_X(x,a) \wedge f_Y(x,a)$;
(3) $f_{(X \cup Y)}(x,a) = f_X(x,a) \vee f_Y(x,a)$;
(4) $f_{X^c}(x,a) = 1 - f_X(x,a)$.

$f_X(x,a)$ is the membership degree of $x \in U$ with $a \in A$ on fuzzy set X. Particularly, for each $X \in \widetilde{F}(A), X^1$ expresses for any $a \in A, X(a) = 1, X^0$ expresses for any $a \in A, X(a) = 0$. More details about *FSs* can be referred to ([20]).

Definition 2.1 [33]. Let $FIS = \langle U, A, V, f \rangle, R$ be an equivalence relation, and (U,R) be a fuzzy approximate space, for any $X \in \widetilde{F}(U)$. Then fuzzy lower and upper approximations of μ in (U,R) are respectively defined as $\underline{R}X$ and $\overline{R}X$,

$$\begin{aligned} \underline{R}X(x) &= \inf_{y \in U}\{\max(\mu(y), (1 - R(x,y)))\}, x \in U \\ \overline{R}X(x) &= \sup_{y \in U}\{\min(\mu(y), R(x,y))\}, x \in U. \end{aligned} \tag{1}$$

then the pair $[\underline{R}X, \overline{R}X]$ is a fuzzy rough set (FRS).

3. A novel fuzzy rough set model in fuzzy information system

In general, all objects of one data set can describe the relationships between attributes. Furthermore, the more objects there are, the stronger the relationships between attributes can be identified. For some problem, the correlation between attributes will affect the decision result. So, in the process of decision-making, we can make decisions on objects according to the relationship between attributes. In other words, this kind of problem can be handled by analyzing the correlations between data attributes. Aiming at this kind of problem, the similarity measure between attributes is introduced in this paper to explore the correlation between attributes. Though Yu et al. defined a fuzzy γ-similarity relation ([48]) by extending the notion of IND_λ ([49]) to explore the correlation between attributes with a *FIS*. The fuzzy γ-similarity relation is defined on the basis of excluding the objects with mutually exclusive attributes. However, the mutual exclusion case is based on that the distance between attributes is equal to 1.0, which is too strict in handling fuzzy problems. To deal effectively uncertain decision making problems, we need to improve this situation and make it applicable to different practical situations. In this regard, an attribute-oriented (γ, θ)-fuzzy similarity relation is defined on *FIS* in this paper.

Definition 3.1. Let $FIS = \langle U, A, V, f \rangle$. For any $(a_i, a_j) \in A \times A$, the (γ, θ)-fuzzy similarity relation R_γ^θ on A can be defined as:

$$R_\gamma^\theta(a_i, a_j) = r_{ij} = \frac{|J_\gamma(a_i, a_j)|}{|U| - |J^\theta(a_i, a_j)|} \tag{2}$$

where

$$J_\gamma(a_i, a_j) = \{x \in U \,|\, |f(x,a_i) - f(x,a_j)| \leqslant \gamma\}, \gamma \in [0, 0.5), \tag{3}$$

$$J^\theta(a_i, a_j) = \{x \in U \,|\, |f(x,a_i) - f(x,a_j)| \geqslant \theta\}, \theta \in [0.5, 1.0], \tag{4}$$

In Definition 3.1, R_γ^θ measures the correlation between attributes, it is defined from the point of view of objects. J_γ represents objects with certain similarity; whereas J^θ represents objects with certain difference. While $\theta = 1.0, J^1$ indicates that the objects with the greatest difference. Parameter θ guarantees that the proposed fuzzy similarity relation can apply to more uncertain systems.

R_γ^θ is usually characterized as a similarity matrix $[r_{ij}]_{|A| \times |A|}$, where r_{ij} is given in Definition 3.1. Actually, R_γ^θ is reflexive and symmetric but not necessarily transitive.

Theorem 3.1. Let $FIS = \langle U, A, V, f \rangle, \gamma_1, \gamma_2 \in [0, 0.5)$ and $\theta_1, \theta_2 \in [0.5, 1]$. Then (γ, θ)-fuzzy similarity relation has the following properties:

(1) If $\gamma_1 \leqslant \gamma_2$, then $R^{\theta}_{\gamma_1} \subseteq R^{\theta}_{\gamma_2}$;
(2) If $\theta_1 \leqslant \theta_2$, then $R^{\theta_1}_{\gamma} \supseteq R^{\theta_2}_{\gamma}$;
(3) If $\gamma_1 \leqslant \gamma_2$ and $\theta_1 \leqslant \theta_2$, then $R^{\theta_2}_{\gamma_1} \subseteq R^{\theta_1}_{\gamma_2}$.

Proof. (1) Since $0 \leqslant \gamma_1 \leqslant \gamma_2 < 0.5$, for any $a_i, a_j \in A$, by Definition 3.1, we have $|J_{\gamma_1}(a_i, a_j)| \leqslant |J_{\gamma_2}(a_i, a_j)|$, then $R^{\theta}_{\gamma_1}(a_i, a_j) \leqslant R^{\theta}_{\gamma_2}(a_i, a_j)$ holds, and it makes fuzzy similarity relation matrix $R^{\theta}_{\gamma_1} \subseteq R^{\theta}_{\gamma_2}$.

(2) Since $0.5 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, for any $a_i, a_j \in A$, by Definition 3.1, we have $|J^{\theta_1}(a_i, a_j)| \geqslant |J^{\theta_2}(a_i, a_j)|$, then $R^{\theta_1}_{\gamma}(a_i, a_j) \geqslant R^{\theta_2}_{\gamma}(a_i, a_j)$ holds, and it makes fuzzy similarity relation matrix $R^{\theta_1}_{\gamma} \supseteq R^{\theta_2}_{\gamma}$.

(3) Since $0 \leqslant \gamma_1 \leqslant \gamma_2 < 0.5$and $0.5 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, by property (2), we have $R^{\theta_2}_{\gamma_1} \subseteq R^{\theta_1}_{\gamma_1}$. Meanwhile, by property (1), we have $R^{\theta_1}_{\gamma_1} \subseteq R^{\theta_1}_{\gamma_2}$. Then we can obtain that, $R^{\theta_2}_{\gamma_1} \subseteq R^{\theta_1}_{\gamma_2}$. □

Example 1. Table 1 is a *FIS*. In which $U = \{x_1, x_2, x_3, x_4, x_5\}, A = \{a_1, a_2, a_3, a_4, a_5\}$. If $\gamma = 0.15, \theta = 0.85$, according to the (γ, θ)-fuzzy similarity relation of Definition 3.1, we can obtain a (γ, θ)-fuzzy similarity matrix R^{θ}_{γ} as in Table 2. In Table 2, for example, r_{12} is induced by $\{|f(x, a_1) - f(x, a_2)||x \in U\} = \{0.2651, 0.1205, 0.0298, 0.7314, 0.3023\}$, then $r_{12} = \frac{2}{5-0} = 0.4$.

On the basis of Definition 3.1, a novel attribute-oriented FRS model of *FISs* is proposed as below, which can be used for follow-up predictive analysis modeling. Moreover, the fundamental properties of the novel FRS model are investigated and examined in a *FIS*.

Definition 3.2. Suppose $FIS = \langle U, A, V, f \rangle, R^{\theta}_{\gamma}$ be a fuzzy similarity relation on A, $\left(A, R^{\theta}_{\gamma}\right)$ be a fuzzy approximation space, $\gamma \in [0, 0.5)$ and $\theta \in [0.5, 1], \forall X \in \widetilde{F}(A)$. Then the fuzzy lower and upper approximations of fuzzy set X are defined respectively as

$$\begin{aligned} \underline{\mathbf{R}^{\theta}_{\gamma}}X(a) &= \inf_{b \in A}\left\{\sup\left(X(b), \left(1 - R^{\theta}_{\gamma}(a, b)\right)\right)\right\}, a \in A \\ \overline{\mathbf{R}^{\theta}_{\gamma}}X(a) &= \sup_{b \in A}\left\{\inf\left(X(b), R^{\theta}_{\gamma}(a, b)\right)\right\}, a \in A. \end{aligned} \tag{5}$$

Then the pair $\left[\underline{\mathbf{R}^{\theta}_{\gamma}}X, \overline{\mathbf{R}^{\theta}_{\gamma}}X\right]$ is an attribute-oriented fuzzy rough set. Frankly, this model is a fuzzy model of (γ, δ)-rough fuzzy set model of Yu et al. proposed ([48]).

Table 1
An FIS

U/A	a_1	a_2	a_3	a_4	a_5
x_1	0.3998	0.1347	0.2396	0.8179	0.3908
x_2	0.8198	0.6993	0.2131	0.2136	0.8917
x_3	0.5209	0.4911	0.6189	0.6923	0.4912
x_4	0.2001	0.9315	0.2745	0.2802	0.3086
x_5	0.7035	0.4012	0.2219	0.7995	0.1895

Table 2
Fuzzy similarity matrix

$R^{0.85}_{0.15}$	a_1	a_2	a_3	a_4	a_5
a_1	1.0000	0.4000	0.4000	0.4000	0.8000
a_2	0.4000	1.0000	0.4000	0.0000	0.2000
a_3	0.4000	0.4000	1.0000	0.6000	0.6000
a_4	0.4000	0.0000	0.6000	1.0000	0.2000
a_5	0.8000	0.2000	0.6000	0.2000	1.0000

Example 2 (*Continued with Example 1*). Let $X = 0.7/a_1 + 0.5/a_2 + 0.6/a_3 + 0.9/a_4 + 0.3/a_5$. Then by Definition 3.2, we have $\underline{\mathbf{R}}^{0.85}_{0.15}X(a_1) = \inf(0.7 \vee (1-1), 0.5 \vee (1-0.4), 0.6 \vee (1-0.4), 0.9 \vee (1-0.4), 0.3 \vee (1-0.8)) = 0.3$.

Similarly, we have $\underline{\mathbf{R}}^{0.85}_{0.15}X = 0.3/a_1 + 0.5/a_2 + 0.4/a_3 + 0.6/a_4 + 0.3/a_5$.

$\overline{\mathbf{R}}^{0.85}_{0.15}X(a_1) = \sup(0.7 \wedge 1, 0.5 \wedge 0.4, 0.6 \wedge 0.4, 0.9 \wedge 0.4, 0.3 \wedge 0.8) = 0.7$.

Similarly, we have $\overline{\mathbf{R}}^{0.8}_{0.2}X = 0.7/a_1 + 0.5/a_2 + 0.6/a_3 + 0.9/a_4 + 0.7/a_5$.

Theorem 3.2. Let $FIS = \langle U, A, V, f \rangle, \gamma \in [0, 0.5)$ and $\theta \in [0.5, 1]$. For any $X, Y \in \widetilde{F}(A), a \in A$, we have

(1) $\underline{\mathbf{R}}^{\theta}_{\gamma}X \subseteq X \subseteq \overline{\mathbf{R}}^{\theta}_{\gamma}X$;

(2) $\underline{\mathbf{R}}^{\theta}_{\gamma}X^0 = X^0 = \overline{\mathbf{R}}^{\theta}_{\gamma}X^0$, $\underline{\mathbf{R}}^{\theta}_{\gamma}X^1 = X^1 = \overline{\mathbf{R}}^{\theta}_{\gamma}X^1$;

(3) $X \subseteq Y \Rightarrow \overline{\mathbf{R}}^{\theta}_{\gamma}X \subseteq \overline{\mathbf{R}}^{\theta}_{\gamma}Y, \underline{\mathbf{R}}^{\theta}_{\gamma}X \subseteq \underline{\mathbf{R}}^{\theta}_{\gamma}Y$;

(4) $\underline{\mathbf{R}}^{\theta}_{\gamma}(X \cup Y) \supseteq \underline{\mathbf{R}}^{\theta}_{\gamma}X \cup \underline{\mathbf{R}}^{\theta}_{\gamma}Y, \underline{\mathbf{R}}^{\theta}_{\gamma}(X \cap Y) = \underline{\mathbf{R}}^{\theta}_{\gamma}X \cap \underline{\mathbf{R}}^{\theta}_{\gamma}Y$;

(5) $\overline{\mathbf{R}}^{\theta}_{\gamma}(X \cup Y) = \overline{\mathbf{R}}^{\theta}_{\gamma}X \cup \overline{\mathbf{R}}^{\theta}_{\gamma}Y, \overline{\mathbf{R}}^{\theta}_{\gamma}(X \cap Y) \subseteq \overline{\mathbf{R}}^{\theta}_{\gamma}X \cap \overline{\mathbf{R}}^{\theta}_{\gamma}Y$;

(6) $\left(\overline{\mathbf{R}}^{\theta}_{\gamma}X\right)^c = \underline{\mathbf{R}}^{\theta}_{\gamma}(X^c), \left(\underline{\mathbf{R}}^{\theta}_{\gamma}X\right)^c = \overline{\mathbf{R}}^{\theta}_{\gamma}(X^c)$.

Proof. The proof of (4) is shown as follows. Other proofs are similar.

(4) i) Suppose that $X, Y \in \widetilde{F}(A)$. By Definition 3.2, $\forall a \in A$, we have

$$\begin{aligned}
&\left(\underline{\mathbf{R}}^{\theta}_{\gamma}X \cup \underline{\mathbf{R}}^{\theta}_{\gamma}Y\right)(a) \leqslant \left(\underline{\mathbf{R}}^{\theta}_{\gamma}X(a)\right) \vee \left(\underline{\mathbf{R}}^{\theta}_{\gamma}Y(a)\right) \\
&= \inf_{b \in A}\left\{\sup\left(X(b), \left(1 - R^{\theta}_{\gamma}(a,b)\right)\right)\right\} \vee \inf_{b \in A}\left\{\sup\left(Y(b), \left(1 - R^{\theta}_{\gamma}(a,b)\right)\right)\right\} \\
&= \inf_{b \in A}\left\{\sup\left(X(b), \left(1 - R^{\theta}_{\gamma}(a,b)\right)\right) \vee \sup\left(Y(b), \left(1 - R^{\theta}_{\gamma}(a,b)\right)\right)\right\} \\
&= \inf_{b \in A}\left\{\sup\left((X(b) \vee Y(b)), \left(1 - R^{\theta}_{\gamma}(a,b)\right)\right)\right\} \\
&= \underline{\mathbf{R}}^{\theta}_{\gamma}(X \cup Y)(a).
\end{aligned}$$

Therefore, $\underline{\mathbf{R}}^{\theta}_{\gamma}(X \cup Y) \supseteq \underline{\mathbf{R}}^{\theta}_{\gamma}X \cup \underline{\mathbf{R}}^{\theta}_{\gamma}Y$.

ii) For any $X, Y \in \widetilde{F}(A)$. By Definition 3.2, $\forall a \in A$, we have

$$\begin{aligned}
&\left(\underline{\mathbf{R}}^{\theta}_{\gamma}X \cap \underline{\mathbf{R}}^{\theta}_{\gamma}Y\right)(a) = \left(\underline{\mathbf{R}}^{\theta}_{\gamma}X(a)\right) \wedge \left(\underline{\mathbf{R}}^{\theta}_{\gamma}Y(a)\right) \\
&= \inf_{b \in A}\left\{\sup\left(X(b), \left(1 - R^{\theta}_{\gamma}(a,b)\right)\right)\right\} \wedge \inf_{b \in A}\left\{\sup\left(Y(b), \left(1 - R^{\theta}_{\gamma}(a,b)\right)\right)\right\} \\
&= \inf_{b \in A}\left\{\sup\left(X(b), \left(1 - R^{\theta}_{\gamma}(a,b)\right)\right) \wedge \sup\left(Y(b), \left(1 - R^{\theta}_{\gamma}(a,b)\right)\right)\right\} \\
&= \inf_{b \in A}\left\{\sup\left((X(b) \wedge Y(b)), \left(1 - R^{\theta}_{\gamma}(a,b)\right)\right)\right\} \\
&= \underline{\mathbf{R}}^{\theta}_{\gamma}(X \cap Y)(a).
\end{aligned}$$

that means $\left(\underline{\mathbf{R}}^{\theta}_{\gamma}X \cap \underline{\mathbf{R}}^{\theta}_{\gamma}Y\right) \subseteq \underline{\mathbf{R}}^{\theta}_{\gamma}(X \cap Y, \delta)$, since $X \cap Y \subseteq X, Y$, by property (3), we have $\underline{\mathbf{R}}^{\theta}_{\gamma}(X \cap Y) \subseteq \left(\underline{\mathbf{R}}^{\theta}_{\gamma}X \cap \underline{\mathbf{R}}^{\theta}_{\gamma}Y\right)$. Hence, $\underline{\mathbf{R}}^{\theta}_{\gamma}(X \cap Y) = \underline{\mathbf{R}}^{\theta}_{\gamma}X \cap \underline{\mathbf{R}}^{\theta}_{\gamma}Y$.

□

In the following, an example is exhibited to understand Theorem 3.2.

Example 3. : (Continued with **Example** 2). Let $Y = 0.7/a_1 + 0.6/a_2 + 0.3/a_3 + 0.6/a_4 + 0.9/a_5$. Then by Definition 3.2, we have $\underline{\mathbf{R}}^{0.85}_{0.15}Y = 0.6/a_1 + 0.6/a_2 + 0.3/a_3 + 0.4/a_4 + 0.4/a_5$.

Similarly, we have $\overline{\mathbf{R}}^{0.85}_{0.15}Y = 0.8/a_1 + 0.6/a_2 + 0.6/a_3 + 0.6/a_4 + 0.9/a_5$.
On the other hand, $X \cap Y = 0.7/a_1 + 0.5/a_2 + 0.3/a_3 + 0.6/a_4 + 0.3/a_5$,
$X \cup Y = 0.7/a_1 + 0.6/a_2 + 0.6/a_3 + 0.9/a_4 + 0.9/a_5$.
Meanwhile, $\underline{\mathbf{R}}^{0.85}_{0.15}(X \cap Y) = 0.3/a_1 + 0.5/a_2 + 0.3/a_3 + 0.4/a_4 + 0.3/a_5$,
$\overline{\mathbf{R}}^{0.85}_{0.15}(X \cap Y) = 0.7/a_1 + 0.5/a_2 + 0.6/a_3 + 0.6/a_4 + 0.7/a_5$.
And, $\underline{\mathbf{R}}^{0.85}_{0.15}(X \cup Y) = 0.6/a_1 + 0.6/a_2 + 0.6/a_3 + 0.6/a_4 + 0.6/a_5$,

$\overline{\mathbf{R}_{0.15}^{0.85}}(X \cup Y) = 0.8/a_1 + 0.6/a_2 + 0.6/a_3 + 0.9/a_4 + 0.9/a_5$.
$\underline{\mathbf{R}_{0.15}^{0.85}}X \cap \underline{\mathbf{R}_{0.15}^{0.85}}Y = 0.3/a_1 + 0.5/a_2 + 0.3/a_3 + 0.4/a_4 + 0.3/a_5$,
$\overline{\mathbf{R}_{0.15}^{0.85}}X \cap \overline{\mathbf{R}_{0.15}^{0.85}}Y = 0.7/a_1 + 0.5/a_2 + 0.6/a_3 + 0.6/a_4 + 0.7/a_5$.
$\underline{\mathbf{R}_{0.15}^{0.85}}X \cup \underline{\mathbf{R}_{0.15}^{0.85}}Y = 0.6/a_1 + 0.6/a_2 + 0.4/a_3 + 0.6/a_4 + 0.4/a_5$,
$\overline{\mathbf{R}_{0.15}^{0.85}}X \cup \overline{\mathbf{R}_{0.15}^{0.85}}Y = 0.8/a_1 + 0.6/a_2 + 0.6/a_3 + 0.9/a_4 + 0.9/a_5$.
Therefore, $\underline{\mathbf{R}_{\gamma}^{\theta}}(X \cap Y) = \underline{\mathbf{R}_{\gamma}^{\theta}}X \cap \underline{\mathbf{R}_{\gamma}^{\theta}}Y$; $\underline{\mathbf{R}_{\gamma}^{\theta}}(X \cup Y) \supseteq \underline{\mathbf{R}_{\gamma}^{\theta}}X \cup \underline{\mathbf{R}_{\gamma}^{\theta}}Y$; $\overline{\mathbf{R}_{\gamma}^{\theta}}(X \cap Y) \subseteq \overline{\mathbf{R}_{\gamma}^{\theta}}X \cap \overline{\mathbf{R}_{\gamma}^{\theta}}Y$; $\overline{\mathbf{R}_{\gamma}^{\theta}}(X \cup Y) = \overline{\mathbf{R}_{\gamma}^{\theta}}X \cup \overline{\mathbf{R}_{\gamma}^{\theta}}Y$.

Lemma 3.1. Let $FIS = \langle U, A, V, f\rangle, \gamma \in [0, 0.5), 0.5 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, and $X \in \widetilde{F}(A)$. Then $\underline{\mathbf{R}_{\gamma}^{\theta_1}}X \subseteq \underline{\mathbf{R}_{\gamma}^{\theta_2}}X$ and $\overline{\mathbf{R}_{\gamma}^{\theta_1}}X \supseteq \overline{\mathbf{R}_{\gamma}^{\theta_2}}X$.

Proof. Since $0.5 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, by Theorem 3.1, we have $R_{\gamma}^{\theta_1} \supseteq R_{\gamma}^{\theta_2}$, then $\left(R_{\gamma}^{\theta_1}\right)^c \subseteq \left(R_{\gamma}^{\theta_2}\right)^c$. By Definition 3.2, $\forall a \in A, \underline{\mathbf{R}_{\gamma}^{\theta_1}}X(a) = \inf_{b \in A}\left\{\sup\left(X(b), \left(1 - R_{\gamma}^{\theta_1}(a,b)\right)\right)\right\} \leqslant \inf_{b \in A}\left\{\sup\left(X(b), \left(1 - R_{\gamma}^{\theta_2}(a,b)\right)\right)\right\} = \underline{\mathbf{R}_{\gamma}^{\theta_2}}X(a)$ holds. Then we can conclude that, $\underline{\mathbf{R}_{\gamma}^{\theta_1}}X \subseteq \underline{\mathbf{R}_{\gamma}^{\theta_2}}X$.

Similarly, $\forall a \in A$, by Definition 3.2, $\overline{\mathbf{R}_{\gamma}^{\theta_1}}X(a) = \sup_{b \in A}\left\{\inf\left(X(b), R_{\gamma}^{\theta_1}(a,b)\right)\right\} \geqslant \sup_{b \in A}\left\{\inf\left(X(b), R_{\gamma}^{\theta_2}(a,b)\right)\right\} = \overline{\mathbf{R}_{\gamma}^{\theta_2}}X(a)$ holds. Then we can conclude that, $\overline{\mathbf{R}_{\gamma}^{\theta_1}}X \supseteq \overline{\mathbf{R}_{\gamma}^{\theta_2}}X$.

□

Lemma 3.2. Let $FIS = \langle U, A, V, f\rangle, 0 \leqslant \gamma_1 \leqslant \gamma_2 < 0.5, \theta \in [0.5, 1]$, and $X \in \widetilde{F}(A)$. Then $\underline{\mathbf{R}_{\gamma_1}^{\theta}}X \supseteq \underline{\mathbf{R}_{\gamma_2}^{\theta}}X$ and $\overline{\mathbf{R}_{\gamma_1}^{\theta}}X \subseteq \overline{\mathbf{R}_{\gamma_2}^{\theta}}X$.

Proof. Since $0 \leqslant \gamma_1 \leqslant \gamma_2 < 0.5$, by Theorem 3.1, we have $R_{\gamma_1}^{\theta} \subseteq R_{\gamma_2}^{\theta}$. By Definition 3.2, $\forall a \in A, \underline{\mathbf{R}_{\gamma_1}^{\theta}}X(a) = \inf_{b \in A}\left\{\sup\left(X(b), \left(1 - R_{\gamma_1}^{\theta}(a,b)\right)\right)\right\} \geqslant \inf_{b \in A}\left\{\sup\left(X(b), \left(1 - R_{\gamma_2}^{\theta}(a,b)\right)\right)\right\} = \underline{\mathbf{R}_{\gamma_2}^{\theta}}X(a)$ holds. Then we can conclude that, $\underline{\mathbf{R}_{\gamma_1}^{\theta}}X \supseteq \underline{\mathbf{R}_{\gamma_2}^{\theta}}X$.

Similarly, $\forall a \in A$, by Definition 3.2, $\overline{\mathbf{R}_{\gamma_1}^{\theta}}X(a) = \sup_{b \in A}\left\{\inf\left(X(b), R_{\gamma_1}^{\theta}(a,b)\right)\right\} \leqslant \sup_{b \in A}\left\{\inf\left(X(b), R_{\gamma_2}^{\theta}(a,b)\right)\right\} = \overline{\mathbf{R}_{\gamma}^{\theta}}X(a)$ holds. Then we can conclude that, $\overline{\mathbf{R}_{\gamma_1}^{\theta}}X \subseteq \overline{\mathbf{R}_{\gamma_2}^{\theta}}X$.

□

Theorem 3.3. Let $FIS = \langle U, A, V, f\rangle, \gamma \in [0, 0.5), \theta_1, \theta_2 \in [0.5, 1]$, and $X \in \widetilde{F}(A)$. Then we have
(1) $\underline{\mathbf{R}_{\gamma}^{\theta_1 \vee \theta_2}}X = \underline{\mathbf{R}_{\gamma}^{\theta_1}}X \cup \underline{\mathbf{R}_{\gamma}^{\theta_2}}X$;
(2) $\overline{\mathbf{R}_{\gamma}^{\theta_1 \vee \theta_2}}X = \overline{\mathbf{R}_{\gamma}^{\theta_1}}X \cap \overline{\mathbf{R}_{\gamma}^{\theta_2}}X$;
(3) $\underline{\mathbf{R}_{\gamma}^{\theta_1 \wedge \theta_2}}X = \underline{\mathbf{R}_{\gamma}^{\theta_1}}X \cap \underline{\mathbf{R}_{\gamma}^{\theta_2}}X$;
(4) $\overline{\mathbf{R}_{\gamma}^{\theta_1 \wedge \theta_2}}X = \overline{\mathbf{R}_{\gamma}^{\theta_1}}X \cup \overline{\mathbf{R}_{\gamma}^{\theta_2}}X$;.

Proof. The proof of (1) is shown as follows. Others can be proved similarly.

(1). Since $\theta_1 \vee \theta_2 \geqslant \theta_1, \theta_2$, by Lemma 3.1, we have $\underline{\mathbf{R}_{\gamma}^{\theta_1 \vee \theta_2}}X \supseteq \underline{\mathbf{R}_{\gamma}^{\theta_1}}X$and $\underline{\mathbf{R}_{\gamma}^{\theta_1 \vee \theta_2}}X \supseteq \underline{\mathbf{R}_{\gamma}^{\theta_2}}X$, then we can conclude that $\underline{\mathbf{R}_{\gamma}^{\theta_1 \vee \theta_2}}X \supseteq \underline{\mathbf{R}_{\gamma}^{\theta_1}}X \cup \underline{\mathbf{R}_{\gamma}^{\theta_2}}X$;

Moreover, suppose $\theta_1 \leqslant \theta_2$, so that $\theta_1 \vee \theta_2 = \theta_2$, by Lemma 3.1, we have $\underline{\mathbf{R}_{\gamma}^{\theta_1}}X \subseteq \underline{\mathbf{R}_{\gamma}^{\theta_2}}X$, then $\underline{\mathbf{R}_{\gamma}^{\theta_1 \vee \theta_2}}X = \underline{\mathbf{R}_{\gamma}^{\theta_2}}X$, and $\underline{\mathbf{R}_{\gamma}^{\theta_1}}X \cup \underline{\mathbf{R}_{\gamma}^{\theta_2}}X = \underline{\mathbf{R}_{\gamma}^{\theta_2}}X$, Thus, $\underline{\mathbf{R}_{\gamma}^{\theta_1 \vee \theta_2}}X = \underline{\mathbf{R}_{\gamma}^{\theta_1}}X \cup \underline{\mathbf{R}_{\gamma}^{\theta_2}}X$.

□

Theorem 3.4. Let $FIS = \langle U, A, V, f\rangle, \gamma_1, \gamma_2 \in [0, 0.5), \theta \in [0.5, 1]$, and $X \in \widetilde{F}(A)$. Then we have
(1) $\underline{\mathbf{R}_{\gamma_1 \vee \gamma_2}^{\theta}}X \subseteq \underline{\mathbf{R}_{\gamma_1}^{\theta}}X \cup \underline{\mathbf{R}_{\gamma_2}^{\theta}}X$;
(2) $\overline{\mathbf{R}_{\gamma_1 \vee \gamma_2}^{\theta}}X \supseteq \overline{\mathbf{R}_{\gamma_1}^{\theta}}X \cap \overline{\mathbf{R}_{\gamma_2}^{\theta}}X$;
(3) $\underline{\mathbf{R}_{\gamma_1 \wedge \gamma_2}^{\theta}}X \supseteq \underline{\mathbf{R}_{\gamma_1}^{\theta}}X \cap \underline{\mathbf{R}_{\gamma_2}^{\theta}}X$;
(4) $\overline{\mathbf{R}_{\gamma_1 \wedge \gamma_2}^{\theta}}X \subseteq \overline{\mathbf{R}_{\gamma_1}^{\theta}}X \cup \overline{\mathbf{R}_{\gamma_2}^{\theta}}X$.

Proof. The proof of (1) is shown as follows. Others can be proved similarly.

(1). Since $\gamma_1 \vee \gamma_2 \geqslant \gamma_1, \gamma_2$, by Lemma 3.2, we have $\underline{\mathbf{R}^{\theta}_{\gamma_1 \vee \gamma_2}} X \subseteq \underline{\mathbf{R}^{\theta}_{\gamma_1}} X$and $\underline{\mathbf{R}^{\theta}_{\gamma_1 \vee \gamma_2}} X \subseteq \underline{\mathbf{R}^{\theta}_{\gamma_2}} X$, then we can conclude that $\underline{\mathbf{R}^{\theta}_{\gamma_1 \vee \gamma_2}} X \subseteq \underline{\mathbf{R}^{\theta}_{\gamma_1}} X \cup \underline{\mathbf{R}^{\theta}_{\gamma_2}} X$;

Moreover, suppose $\gamma_1 \leqslant \gamma_2$, so that $\gamma_1 \vee \gamma_2 = \gamma_2$, by Lemma 3.2, we have $\underline{\mathbf{R}^{\theta}_{\gamma_1}} X \supseteq \underline{\mathbf{R}^{\theta}_{\gamma_2}} X$, then $\underline{\mathbf{R}^{\theta}_{\gamma_1 \vee \gamma_2}} X = \underline{\mathbf{R}^{\theta}_{\gamma_2}} X \subseteq \underline{\mathbf{R}^{\theta}_{\gamma_1}} X \cup \underline{\mathbf{R}^{\theta}_{\gamma_2}} X$. Thus, $\underline{\mathbf{R}^{\theta}_{\gamma_1 \vee \gamma_2}} X \subseteq \underline{\mathbf{R}^{\theta}_{\gamma_1}} X \cup \underline{\mathbf{R}^{\theta}_{\gamma_2}} X$.

□

4. A FRS-based multi-attribute predictive analysis model in FISs

4.1. A multi-attribute predictive analysis model based on FRS

As we all know, fuzzy rough sets divide the universe of discourse into three regions, namely positive domain, negative domain and boundary domain. The positive domain is the lower approximation set of FRS, and the positive domain and boundary domain together form the upper approximation set. From the perspective of decision strategy, positive domain represents accepting the decision, negative domain represents rejecting the decision, and boundary domain represents delaying the decision. Decisions that are acceptable only if they are in the lower approximation set are considered as pessimistic decision, while decisions that are acceptable when they are in the upper approximation set are considered as optimistic decision.

Suppose that the upper and lower approximation sets of the target set of FRS are respectively considered as the optimistic and pessimistic decision schemes of the target set from the perspective of decision strategy, then the distances between the target object and the pessimistic and optimistic directions can be calculated to build a trend prediction model. Based on this framework, a multi-attribute predictive analysis model based on FRS was proposed in this paper. The multi-attribute predictive analysis modeling route is as follows:

1. Build a relationship that acts as a bridge in a *FIS* to measure the correlation between attributes, then a (γ, θ)-fuzzy similarity relation R^{θ}_{γ} is constructed in this paper.
2. Establish a FRS model on the basis of attribute-oriented, then the upper and lower approximation sets of the target set can be regards as optimistic and pessimistic predictions in a *FIS*.
3. The cosine estimation method is employed for calculating the cosine deviation of the candidate to the two directions (optimistic and pessimistic directions). In cosine, the smaller the included angle of candidate and optimistic (pessimistic) scheme, the greater the cosine; conversely, the smaller the cosine is. Based on that, an attribute-oriented prediction model by least deviation to optimistic and pessimistic predictions can be conducted.
4. The prediction model forecasts the development trend on the basis of the cosine distances between the candidate and the optimistic scheme and the pessimistic scheme. While cosine distances between candidate and the pessimistic scheme is greater than or equal to the cosine distances between candidate and the optimistic scheme, then the candidate will show a favorable development trend; conversely, it will show an undesirable development trend.

According to the modeling route, we give the corresponding algorithm as follows.

Algorithm 1: Multi-attribute Predictive Analysis with Fuzzy Rough Set (MAPA-FRS)

Input: A $FIS = \langle U, A, V, f \rangle$ with $U = \{x_1, x_2, \cdots, x_m\}$ and $A = \{a_1, a_2, \cdots, a_n\}$, a fuzzy set $X \in \widetilde{F}(A)$, parameter γ and θ
Output: Predicted trend.
Step 1: Normalized data with Min–Max Normalization, then a normalized *FIS* can be obtained;
Step 2: $\forall a_i, a_j \in A$, calculate the fuzzy similarity relation $R^{\theta}_{\gamma}(a_i, a_j)$of the *FIS* by Definition 3.1, where $i,j = 1, 2, \cdots, n$, then a fuzzy similarity matrix $[r_{ij}]_{n \times n}$ can be obtained;
Step 3: For each $a \in A$, compute $\underline{\mathbf{R}^{\theta}_{\gamma}} X(a)$ and $\overline{\mathbf{R}^{\theta}_{\gamma}} X(a)$ by Definition 3.2, then the pessimistic prediction scheme (PPS=$\underline{\mathbf{R}^{\theta}_{\gamma}} X$) and the optimistic prediction scheme (OPS=$\overline{\mathbf{R}^{\theta}_{\gamma}} X$) of X can be obtained;
Step 4: Calculate the cosine deviation from the candidate x to PPS and OPS, respectively, by
$$\underline{d} = \frac{\sum_{i=1}^{|A|} (X(a_i) \times PPS(a_i))}{\sqrt{\sum_{i=1}^{|A|} (X(a_i))^2} \times \sqrt{\sum_{i=1}^{|A|} (PPS(a_i))^2}}, \overline{d} = \frac{\sum_{i=1}^{|A|} (X(a_i) \times OPS(a_i))}{\sqrt{\sum_{i=1}^{|A|} (X(a_i))^2} \times \sqrt{\sum_{i=1}^{|A|} (OPS(a_i))^2}};$$
Step 5: Predict the trend of x by trend function $G(x)$, we define $G(x)$ as $G(x) = \begin{cases} 1, & \underline{d} - \overline{d} \geqslant t; \\ 0, & \underline{d} - \overline{d} < t; \end{cases}$
Step 6: Output the predicted trend.

NOTE:

1. While $G = 1$ indicates that the fuzzy object X will show an upward trend (favorable development trend); $G = 0$ indicates that the fuzzy object X will show a downward trend (undesirable development trend).

2. Theoretically, γ and θ can be set to $[0,1]$ with $\gamma \leqslant \theta$. Generally, for a uniform distribution, the value of θ is 0.5; For a left-skewed distribution, $\theta \in [0,0.5)$; For a right-skewed distribution, $\theta \in (0.5,1]$. However, it is generally recognized that the relationship between the two is not less than 0.5, which indicates that they have certain differences. If the relation between the two is less than 0.5, it indicates that they have certain similarities. So this paper suggests that $\gamma \in [0,0.5)$ and $\theta \in [0.5,1]$. Different data sets have different values of θ, which can be obtained through training.

3.Threshold t is a hyper-parameter. For different real data sets, the datum value of development trend may be different, so, the use of threshold t can enhance the applicability of the algorithm. The trend function G can be revised by adjusting threshold t in the application of different data sets.

4. In order to determine the optimal solution of the threshold t of this model, we give the parameter tuning method as follows.

The debugging of the optimal solution of threshold t is based on the optimal prediction accuracy obtained after traversing parameters γ and θ. We search it from the positive and negative directions with 0 as the cut-off point, and set the original step size as 0.1.

1). If the results present normal distribution, the threshold t are adjusted to the middle with the step size of 0.05/0.025/0.01/0.005 until the results show monotonicity increasing (decreasing), and then the range of the threshold t can be obtained. For example, when the prediction results obtained by threshold t with -0.1, 0 and 0.1 are normally distributed, we calculate the prediction results of threshold t with -0.05, 0 and 0.05. If the prediction results still show normal distribution, we compute the prediction results of threshold t with -0.025 and 0.025. If the results show monotonically increasing (decreasing), the range of the threshold t can be judged to be on the right (left), and continue to debug with 0.025 (-0.025) as the cut-off point at the step of 0.01. The value range of the threshold t can be obtained until the result becomes stable.

2). If the obtained results show a monotonically increasing (or decreasing) trend, continue to debug in the positive (or negative) direction with the step size of 0.1. When the results show a normal distribution, use the step size of 0.05/0.025/0.01/0.005 for further debugging until stable results are obtained, then the value range of the threshold t can be achieved.

Finally, the optimal solution of the threshold t is evaluated based on the corresponding value range.

Example 4. (Continued with **Example** 3). Let $X = 0.7/a_1 + 0.5/a_2 + 0.6/a_3 + 0.9/a_4 + 0.3/a_5$; $Y = 0.7/a_1 + 0.6/a_2 + 0.3/a_3 + 0.6/a_4 + 0.9/a_5$. $Z_1 = X \cap Y = 0.7/a_1 + 0.5/a_2 + 0.3/a_3 + 0.6/a_4 + 0.3/a_5$, $Z_2 = X \cup Y = 0.7/a_1 + 0.6/a_2 + 0.6/a_3 + 0.9/a_4 + 0.9/a_5$.

Then by **Step 3**, we can have.

$\underline{d_X} = 1.9298$, $\overline{d_X} = 1.9353$, then $\underline{d_X} < \overline{d_X}$;

$\underline{d_X} = 2.0087$, $\overline{d_X} = 2.0730$, then $\underline{d_Y} < \overline{d_Y}$;

$\underline{d_X} = 1.2074$, $\overline{d_X} = 1.2072$, then $\underline{d_{Z_1}} > \overline{d_{Z_1}}$;

$\underline{d_X} = 2.7836$, $\overline{d_X} = 2.8261$, then $\underline{d_{Z_2}} < \overline{d_{Z_2}}$;

Hence, we can obtain that the fuzzy objects X, Y and Z_2 will show a downward trend (undesirable development trend) while the fuzzy object Z_1 will show an upward trend (favorable development trend), with respect to our proposed MAPA-FRS algorithm.

4.2. Experimental evaluation and discussion

In this subsection, to investigate the effectiveness of the proposed attribute-oriented MAPA-FRS model, experimental analyses are conducted. The experiments are set up as follows:

(1) Data set downloaded from UCI Machine Learning Repository (https://archive.ics.uci.edu/ml/datasets/Audit + Data#) is a real data set. It is a one year (the year 2015 to 2016) non-confidential data of firms collected from the Auditor Office of India to establish a predictor for the classification of suspicious firms. Before the experiments, the firms with the missing information are deleted. Then, the experimental data set consists of 772 subjects. The data set description is shown in Table 3. The purpose of our experiment is to predict whether the firm is risky or suspicious. Label "0" represents no risk and "1" represents risk.

(2) To illustrate the stability of the prediction model, we perform three types of random data training experiments: 30%, 50% and 80%. The random data generates by normal distribution, and randomly selected for 1000 times for training experiments. The presentation of process data is generated based on 80% random data experiments.

(3) Comparative study is presented. FCM clustering, K-means clustering and RFS prediction method (Reference ([48])) are employed for contrastive analysis. The parameters γ and δ of RFS prediction method are suggested with $[0,0.5)$ and $[0.5,1.0]$, respectively.

Table 3
Data sets description

Data Set Characteristics:	Multivariate	Number of Instances:	777	Area:	N/A
Attribute Characteristics:	Real	Number of Attributes:	17	Class	2
Associated Tasks:	Classification	Missing Values?/Number	Yes/5	Class Label:	0,1

(4) Two indices, i.e., Prediction accuracy and AUC (Area Under ROC Curve) are applied to contrastive analysis. The optimal results among different methods in the following tables are emphasized with bold font.

The detailed experimental processes with MAPA-FRS are as follows.

Data preparation: An Audit_risk data set with no Missing Values, in which No risk samples are 286, and Risk samples are 486.

Step 1: Normalized data with Min–Max Normalization, then a $FIS = \langle U, A, V, f \rangle$ is given, $X \in \widetilde{F}(A)$;

Step 2: Randomly select training data with equal proportions of the two categories (No risk and Risk), the rest samples are marked as the test data X. 30%, 50% and 80% random data are tested in the following;

Step 3: Compute the fuzzy similarity matrices of No risk data and Risk data of training data set by Definition 3.1, which denote as T1 and T2 respectively, then calculate the fuzzy relation synthesis operation of them, i.e., $R1 = T1 \circ T2$, then the fuzzy similarity matrix R of training data set can be obtained by self-synthesis of $R1$;

Step 4: Compute $\overline{d}$ and $\underline{d}$ of alternative $x(x \in X)$by Step 2–5 of Algorithm 1;

Step 5: Predict the trend of x by trend function G;

Step 6: Output the predicted trend.

NOTE: Some real data sets are imbalanced in practical application, so fuzzy synthesis operation in **Step3** is needed. Fuzzy synthesis operator $M(\wedge, \vee)$ is applied to fuzzy synthesis operation.

In the following, some results of experiments with MAPA-FRS are shown as tables and figures for subsequent experimental analysis.

Fig. 1 displays the distance between two fuzzy similarity matrices T1 and T2. "Max($|T1 - T2|$)" indicates the maximum value of each column of $|T1 - T2|$, which caused by the imbalance of raw data set.

In order to verify the effectiveness of parameters and our proposed prediction model, we traverse parameters γ and θ according to MAPA-FRS model based on 80% random training set, and the prediction accuracies obtained is shown in the Fig. 2. Fig. 2 shows that the average prediction accuracy of MAPA-FRS prediction model obtained with all valid parameters is 80.78%, the maximal average prediction accuracy is 86.10% with $\gamma = 0.15$ and $\theta = 0.5$.

The parameters of RFS prediction model are also traversed, the result is displayed in Fig. 3. The average prediction accuracy of RFS prediction model obtained with all valid parameters is 77.46% which is lower than MAPA-FRS model by a margin of 3.32%, and the maximal average prediction accuracy is 77.81%. Those results indicate that our proposed prediction model outdoes RFS prediction model with the whole situation.

Table 4 records the prediction accuracy with different models under different random training data sets, which are depicted in Figs. 4–6.

In Table 4, compared with K-Means clustering model, the prediction accuracy of the proposed prediction model is lower than it in the case of No risk with 30% and 50% random training data, whereas much higher than it at least by a margin of 27.7% under Risk prediction situation. Compared with FCM clustering model, though the prediction accuracy of MAPA-FRS model is not as much as it in the case of No risk prediction situation, the prediction accuracy of MAPA-FRS model outdoes it at least by a margin of 26.2% under Risk prediction situation. It is worth emphasizing that, in the Risk prediction situation, the prediction accuracy of MAPA-FRS model and RFS model is neck and neck, but in the No risk prediction situation, MAPA-FRS model is at least 18% higher than RFS model. To my great delight, the average prediction accuracy is more than 84%, which outstrips the compared models.

ROC Curve (Receiver Operating Characteristic Curve), also is known as sensitivity curve. The curve is mapped with two variables: Specificity = FPR (False positive rate) and Sensitivity = TPR (True positive rate). Specificity and Sensitivity reflect

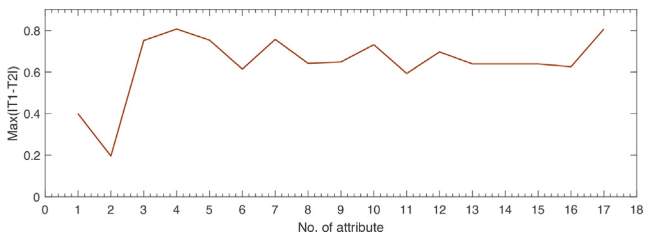

Fig. 1. The discrepancy of fuzzy similarity matrices T1 and T2.

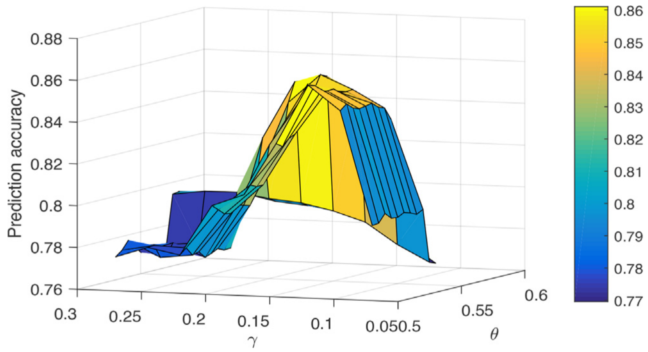

Fig. 2. Average prediction accuracy of MAPA-FRS model with different parameters.

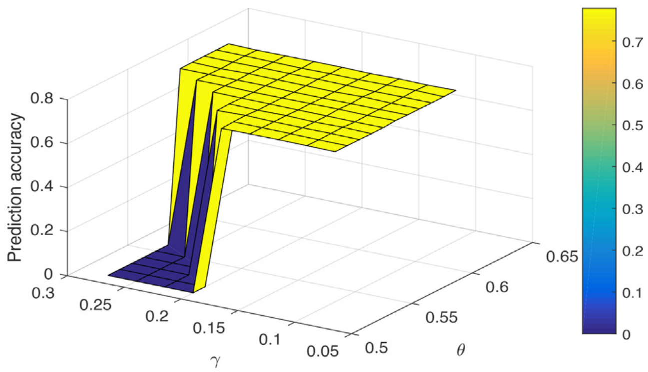

Fig. 3. Average prediction accuracy of RFS prediction model with different parameters.

Table 4
Prediction accuracy with different models under different random training data sets

Proportion	Case	K-Means	FCM	RFS ($\gamma = 0.15, \delta = 0.75$)	MAPA-FRS ($t = 0, \gamma = 0.15, \theta = 0.5$)
30%	No risk	98.49%	**100.00%**	76.68%	94.75%
	Risk	50.08%	50.27%	**77.97%**	77.91%
	Average	68.03%	68.71%	77.49%	**84.16%**
50%	No risk	98.00%	**100.00%**	76.62%	96.29%
	Risk	50.23%	50.50%	77.94%	**78.01%**
	Average	67.93%	68.84%	77.45%	**84.78%**
80%	No risk	97.80%	**100.00%**	77.41%	99.36%
	Risk	49.74%	51.82%	77.99%	**78.11%**
	Average	67.61%	69.74%	77.41%	**86.01%**

Note: The prediction accuracy is the average hit ratio of the two prediction trends. The values of the parameters are obtained by traversal.

the cost and benefit, respectively. Therefore, the ROC Curve can be used to evaluate forecasting performance. The smaller the specificity and the greater the sensitivity, then the better the forecasting performance is. Fig. 7 presents the ROC Curves with different percentages of random training data of prediction models. It shows that the ROC Curve of MAPA-FRS model is at the top among the compared models.

AUC is the area under ROC curve. The value range of AUC is [0.5,1.0]. When AUC comes closer to 1.0, the higher the authenticity of prediction method is. When the value of AUC is equal to 0.5, the authenticity is the lowest and the prediction method is ineffective. Table 5 records the AUC with different percentages of random training data of prediction models. It indicates that MAPA-FRS model archives the maximum AUC value among the compared models.

Fig. 7 and Table 5 indicate that the proposed MAPA-FRS model surpasses the compared popular, state-of-the-art prediction models in stability and prediction performance.

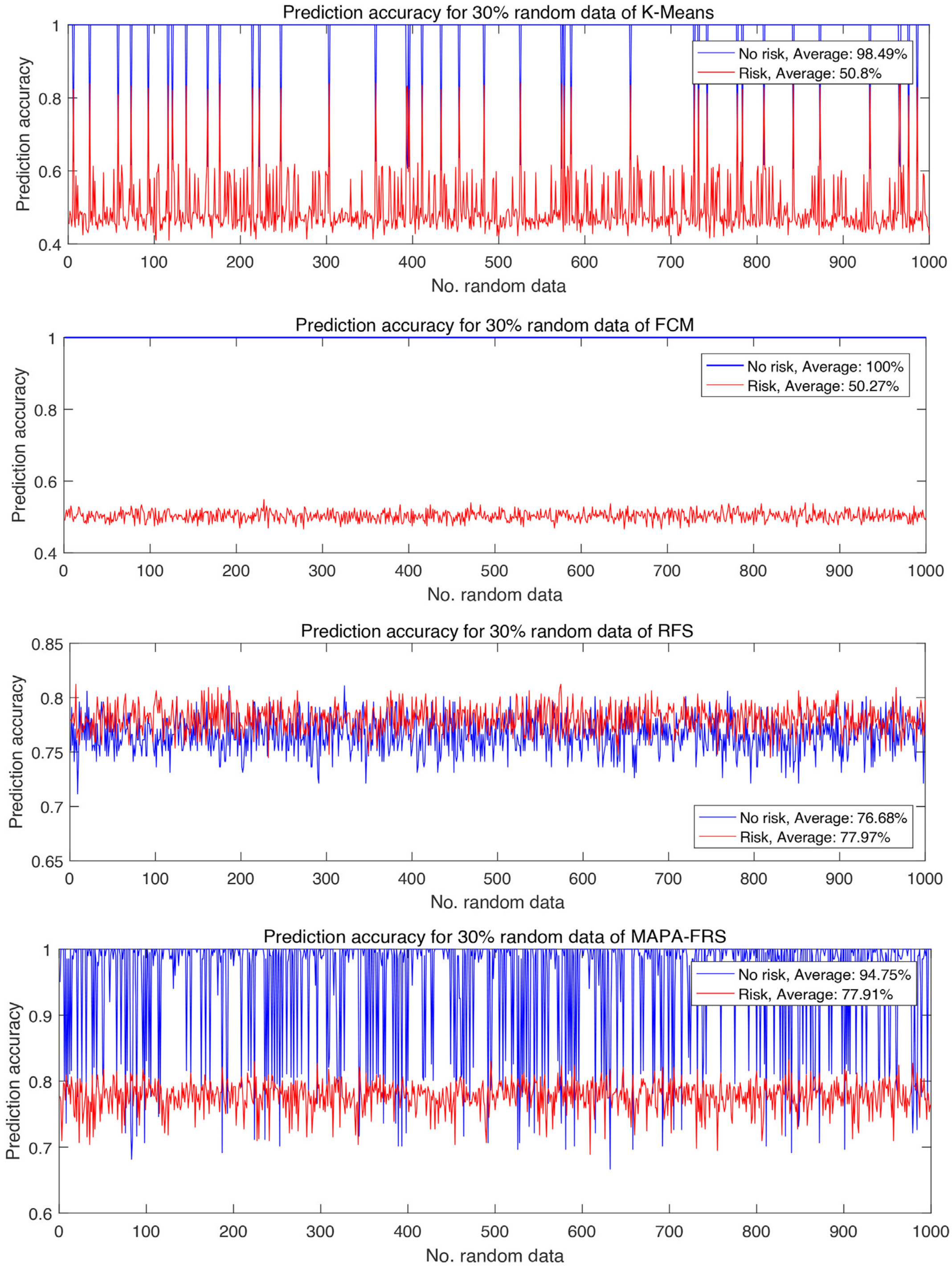

Fig. 4. Prediction Accuracy with compared models under 30% random training data.

4.3. Comparative analysis of numerical experiments

To further evaluate the trend forecasting performance of the proposed MAPA-FRS model, five real data sets downloaded from UCI Machine Learning Repository are applied for comparative analysis; the detailed descriptions of them are shown in Table 6.

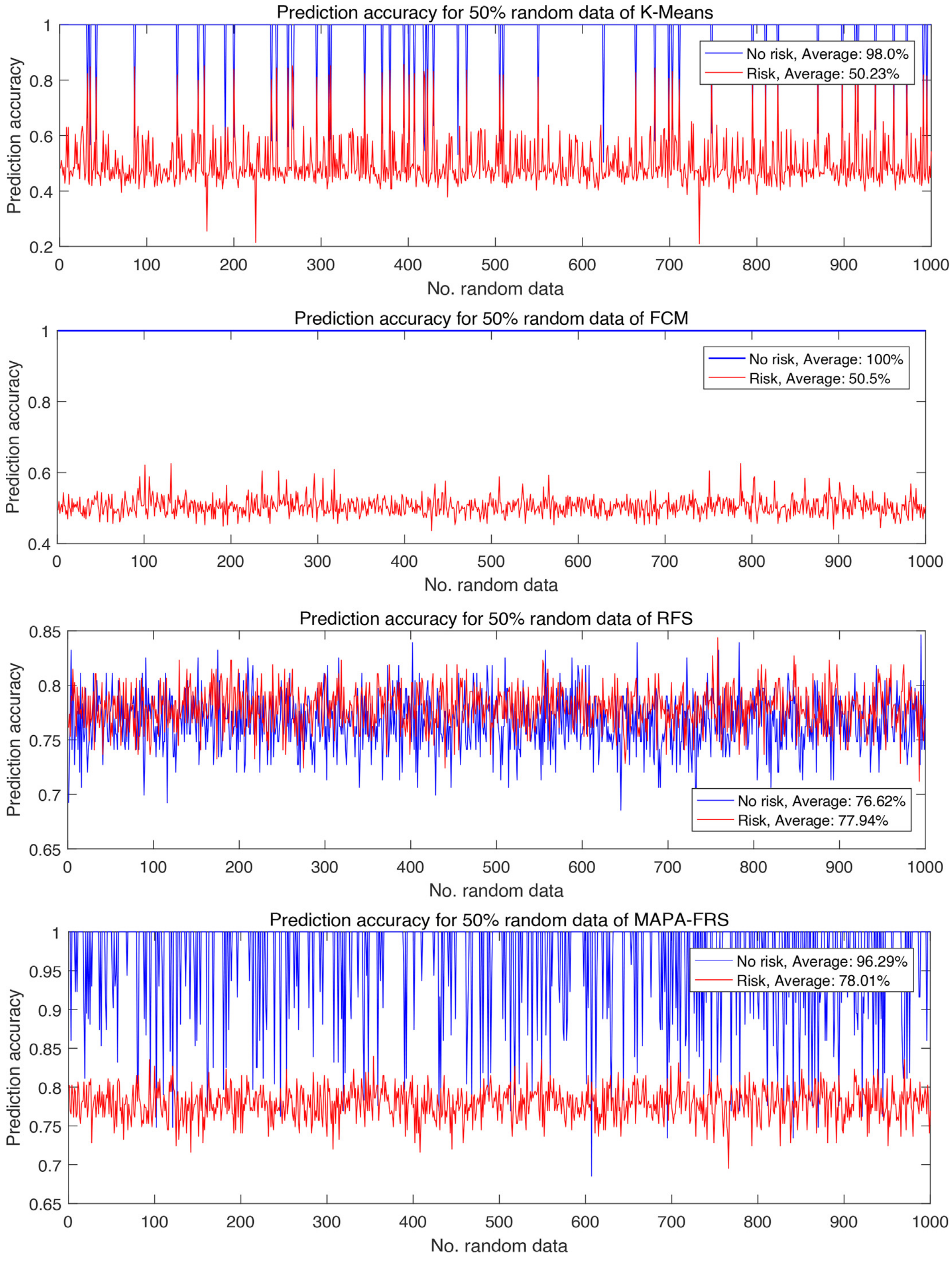

Fig. 5. Prediction Accuracy with compared models under 50% random training data.

Statement: In the following experiments, 30% of each data set was randomly selected as a training set and the remaining 70% as a test set, and randomly selected for 1000 times. The experimental methods and procedures follow subSection 4.2. Noise experiments with randomly deleted attributes are added to verify the robustness. "Raw" expresses the case of original attribute set. 10%, 20% and 30% indicate the proportion of randomly deleted attributes.

Table 7 displays the prediction accuracy under different cases of noises of different data sets for four prediction models. Fig. 8 shows the prediction accuracy of BreastCancer under different t in the case of Raw data in Table 7, which obtained by

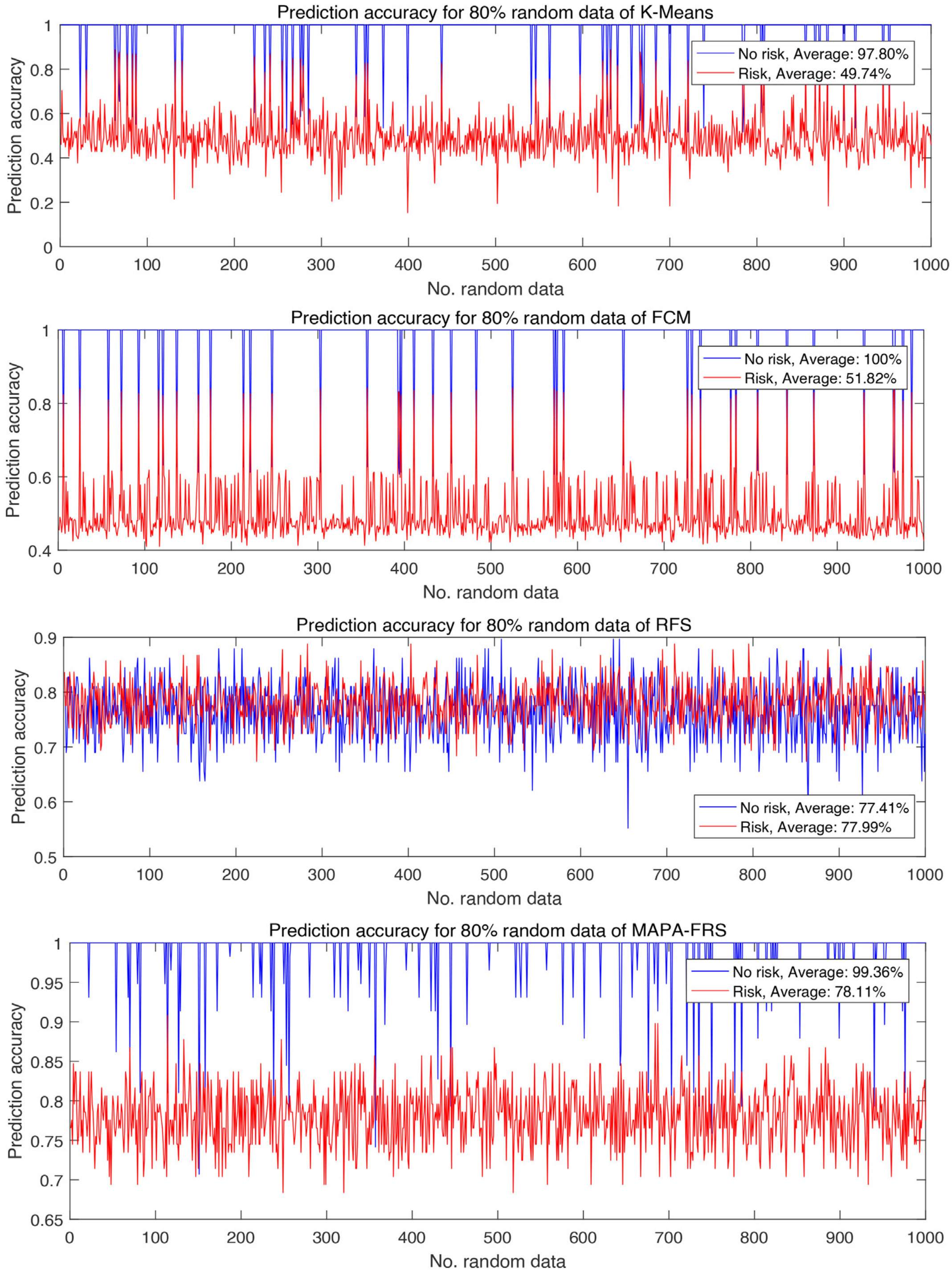

Fig. 6. Prediction Accuracy with compared models under 80% random training data.

the given parameter tuning method. Figs. 9–13 exhibit the ROC Curves with different cases of noises of the data sets. Table 8 shows the AUC of prediction models with different data sets under different cases of noises.

From Table 7, one can find that, in the five data sets, the average prediction accuracy of the proposed MAPA-FRS model of BreastCancer data set is the highest. Our MAPA-FRS model precedes RFS model for four times, and three times for K-meams and FCM models. Furthermore, the average prediction accuracy of divorce data set and wisconsin data set under MAPA-FRS model is more than 88.7%. Meanwhile, one can see that the smaller the attribute set is, the higher the proportion of attribute

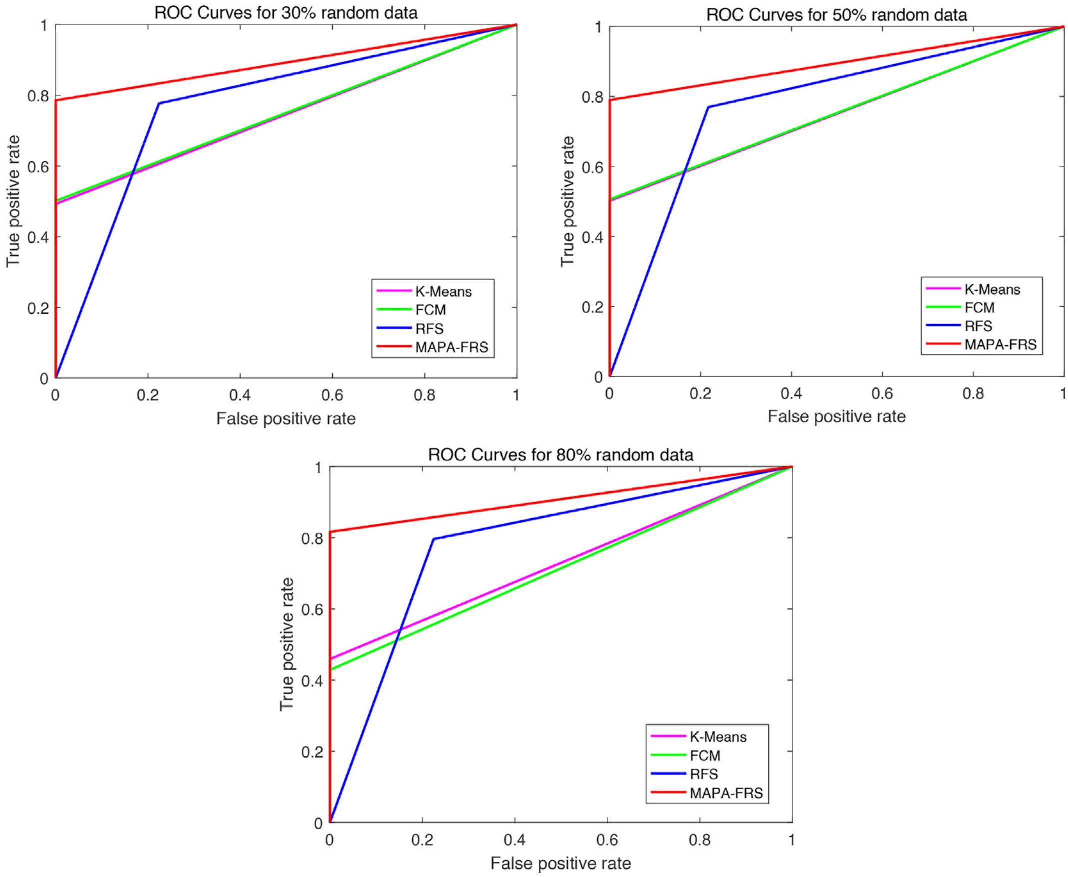

Fig. 7. ROC Curves with different percentages of random training data of prediction models.

Table 5
AUC with different percentages of random training data of prediction models

	K-Means	FCM	RFS	MAPA-FRS
30%	0.7463	0.7507	0.8210	**0.8930**
50%	0.7510	0.7531	0.8206	**0.8951**
80%	0.7296	0.7143	0.8270	**0.9082**

Table 6
The descriptions of Data sets

Data sets	Objects	Attributes	Classes
spectfheart	267	44	2
divorce	170	54	2
wisconsin	683	9	2
crx	690	15	2
BreastCancer	116	10	2

set deletion is, and the more obvious the influence is in the prediction result of data sets, since the higher the probability that key attributes deleted. In addition, our MAPA-FRS model holds the highest average prediction accuracy under all data sets and noise cases. It implies that the robustness of the proposed MAPA-FRS model is superior to other models.

Fig. 8 shows that in a certain range of hyper-parameters t, the average prediction accuracy tends to be stable, and the optimal solution is further adjusted based on this range.

Figs. 9–13 and Table 8 show that the ROC curves and AUC results of MAPA-FRS model on spectfheart and BreastCancer data sets are superior to other compared models, while the AUC of MAPA-FRS are mostly higher than 80% in the other three

Table 7
Prediction accuracy under different scenarios of different data sets

Data sets	Cases	K-Means	FCM	RFS (γ, δ)		MAPA-FRS (t, γ, θ)	
spectfheart	Raw	0.6322	0.5319	(0.05,0.8)	**0.7819**	(-0.225,0.05,0.75)	0.6568
	10%	0.6277	0.5160		**0.7872**	(-0.2,0.05,0.75)	0.6446
	20%	0.6317	0.5215		**0.7713**	(-0.17,0.05,0.75)	0.6438
	30%	0.6277	0.5319		**0.7872**	(-0.145,0.05,0.75)	0.6480
divorce	Raw	**0.9750**	**0.9750**	(0.15,0.75)	0.6583	(0,0.2,0.8)	0.9250
	10%	**0.9750**	**0.9750**		0.6583		0.9083
	20%	**0.9750**	**0.9750**		0.6750		0.9083
	30%	**0.9750**	**0.9750**		0.6667		0.9010
wisconsin	Raw	**0.9603**	0.9564	(0.05,0.7)	0.8852	(0.005,0.05,0.55)	0.9269
	10%	**0.9603**	0.9563		0.8631		0.8998
	20%	**0.9634**	0.9607		0.8613		0.9112
	30%	**0.9571**	0.9535		0.7285		0.8873
crx	Raw	0.7305	**0.8057**	(0.05.0.7)	0.7497	(-0.02,0.05,0.85)	0.7524
	10%	0.7198	**0.8064**		0.7543	(-0.08,0.05,0.85)	0.7798
	20%	0.7280	**0.7950**	(0.25,0.65)	0.7056	(-0.28, 0.25,0.85)	0.7527
	30%	0.5912	**0.6696**	(0.05,0.7)	0.6086		0.6399
BreastCancer	Raw	0.5321	0.5244	(0.25,0.7)	0.6463	(0.134,0.15,0.8)	**0.6951**
	10%	0.5244	0.5244		0.6098		**0.7073**
	20%	0.5230	0.5122		0.5854	(0.1,0.15,0.8)	**0.6829**
	30%	0.5366	0.5366		0.51578		**0.6341**
	Average	0.7573	0.7501	-	0.7150	-	**0.7753**

Note: The prediction accuracy is the average hit ratio of the two prediction trends. The values of the parameters are obtained by traversal.

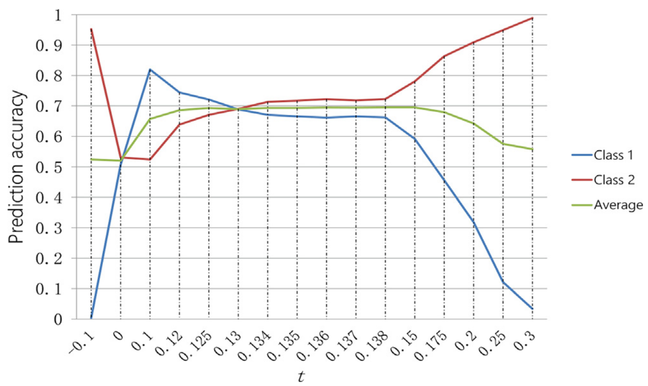

Fig. 8. The prediction accuracy of BreastCancer under different t.

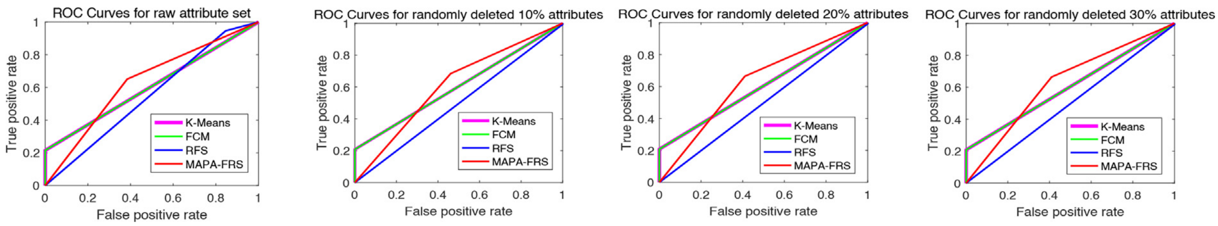

Fig. 9. ROC Curves under different scenarios of spectfheart.

data sets. Especially, the AUC result of spectfheart data set of MAPA-FRS model precedes other compared models about 14.0%. Meanwhile, AUC results of MAPA-FRS model under all data sets and noise cases are over 73.41%, it's even more than 88% for divorce and wisconsin data sets. Particularly, our MAPA-FRS model transcends the three compared models in terms of average AUC under all data sets and noise cases.

From these results, we summarize and analyze the following aspects:

- In spectfheart data set, the average prediction accuracy of RFS model is higher than MAPA-FRS model, but AUC results is opposite. That is because the hitting rate of predicting for one of the categories is too low, which means almost all of them are predicted to be in the same category. Although the average prediction accuracy of RFS model is higher, the prediction

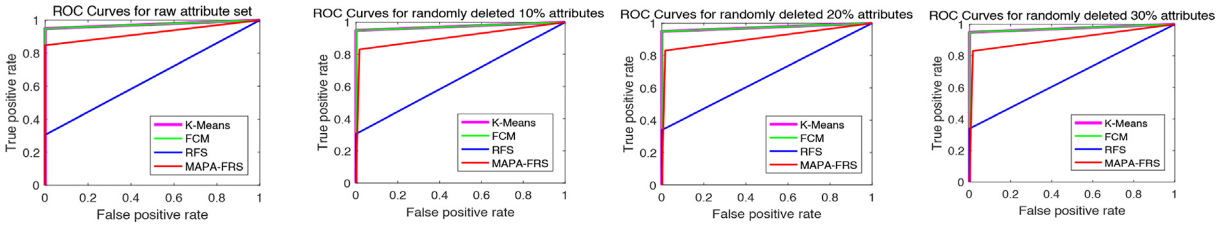

Fig. 10. ROC Curves under different scenarios of divorce.

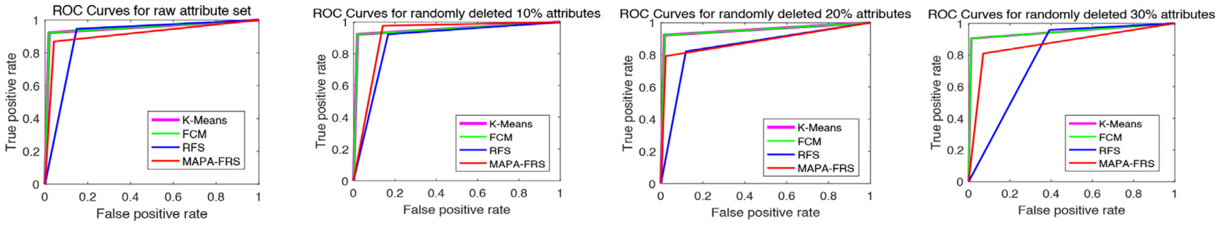

Fig. 11. ROC Curves under different scenarios of wisconsin.

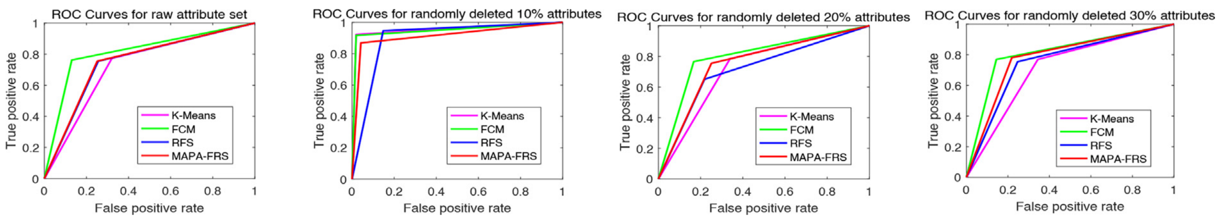

Fig. 12. ROC Curves under different scenarios of crx.

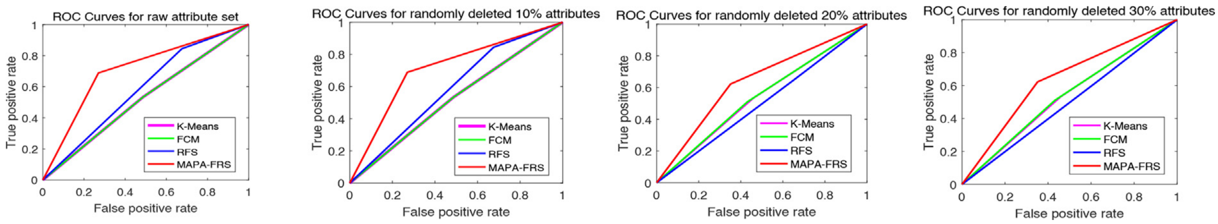

Fig. 13. ROC Curves under different scenarios of BreastCancer.

ability is not ideal, which is caused by the data imbalance and the lack of robustness of the model. The average prediction accuracy and average AUC under all data sets and noise cases of the proposed MAPA-FRS model are transcends other models, which indicates that the robustness and forecasting performance of the proposed MAPA-FRS model outstrip the other compared models.

- Considering from the starting point of this paper, the MAPA-FRS model is a trend prediction model. Under the circumstance of unknown labels of test samples, the MAPA-FRS prediction model predicts the labels of each test sample on the basis of training data set. The prediction accuracy is the label hit ratio of trend prediction of test samples, while the prediction accuracies of K-means and FCM models are the hit ratio of test samples clustering into two categories. Frankly speaking, the trend prediction model of this paper is a prediction model for two categories. Although MAPA-FRS model is essentially a clustering model, it is different from K-means and FCM clustering models. The results of K-means and FCM clustering models also verify the effectiveness of our proposed prediction model. Because MAPA-FRS model predicts the trend of each test sample and can attach a label for it. That is to say, these labels have semantic information.
- The AUC results of MAPA-FRS model are over 73%, it is feasible in terms of predictive analysis.

Table 8
AUC of prediction models with different data sets

Data sets	Cases	K-Means	FCM	RFS	MAPA-FRS
spectfheart	Raw	0.6007	0.6007	0.5806	**0.7494**
	10%	0.6040	0.6040	0.5034	**0.7341**
	20%	0.6074	0.6074	0.5943	**0.7440**
	30%	0.6040	0.6040	0.5034	**0.7417**
divorce	Raw	**0.9746**	**0.9746**	0.6525	0.9237
	10%	**0.9746**	**0.9746**	0.6525	0.9096
	20%	**0.9746**	**0.9746**	0.6695	0.9096
	30%	**0.9746**	**0.9746**	0.6610	0.8983
wisconsin	Raw	**0.9531**	0.9502	0.9070	0.9187
	10%	**0.9531**	0.9503	0.8901	0.9207
	20%	**0.9545**	0.9516	0.8706	0.8888
	30%	**0.9471**	0.9413	0.7990	0.8816
crx	Raw	0.7907	**0.8432**	0.8045	0.8058
	10%	0.7829	**0.8421**	0.8071	0.8237
	20%	0.7877	**0.8338**	0.7790	0.8058
	30%	0.7303	**0.7610**	0.7318	0.7463
BreastCancer	Raw	0.7013	0.6988	0.7336	**0.7736**
	10%	0.6975	0.6975	0.6813	**0.7803**
	20%	0.6975	0.6923	0.6736	**0.7675**
	30%	0.7027	0.6991	0.6888	**0.7431**
	Average	0.8006	0.8088	0.7092	**0.8233**

From these results and discussions, we can conclude that the new prediction model is feasible and effective. The reasons can be elaborated from the following aspects. Firstly, to explore and measure correlation between attributes, an attribute-oriented fuzzy similarity relation is constructed. Next, due to the correlation between attributes can determine the partition of objects, then an attribute-oriented FRS model is established, which paves the way for further constructing prediction model. In addition, cosine deviation computed by the least deviation to optimistic and pessimistic directions is employed for trend prediction. What's more, parameters are brought into make the model more adaptable. Those aspects can ensure our prediction model obtaining better forecasting performance.

5. Conclusions

The FRS is an important tool for tackling the uncertainty information and can be employed in forecasting. In this paper, we introduce an attribute-oriented FRS model to deal with FISs, the upper and lower approximation sets of FRS model are regarded as optimistic decision and pessimistic decision, and the deviation between the alternative and the two decisions is measured by cosine, and then making prediction with the minimum deviation. Furthermore, to evaluate the prediction model proposed in this paper, we conduct experiments with real data and conducted comparative experimental analysis with K-means clustering, FCM clustering and a prediction model based on RFS. The experiment results show that the new prediction model is feasible and effective.

Some problems related to the proposed prediction model need to be considered and discussed further. The fuzzy similarity relation measures the relationship between attributes, if the data dimension increases, how to ensure that no interfering attributes affect the prediction results, or how to effectively remove the interfering attributes. In the future, we will investigate models that can guarantee to forecast performance over real, high-dimensional data sets to ensure their applicability to a wider range of data applications.

CRediT authorship contribution statement

Yun Kang: Conceptualization, Methodology, Writing - original draft. **Bin Yu:** Conceptualization, Methodology, Supervision, Writing - review & editing. **Mingjie Cai:** Funding acquisition, Formal analysis.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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