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A three-way decision approach with risk strategies in hesitant fuzzy decision information systems

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ABSTRACT

As an efficient decision-making tool, three-way decisions have been extensively studied in diverse generalized fuzzy information systems. Among them, since hesitant fuzzy decision information systems can well depict the hesitancy of individuals, hesitant fuzzy three-way decisions serve as a significant research topic nowadays. However, the conditional probability still remains as a challenging issue in the above field. In order to reasonably evaluate the conditional probability of each object in hesitant fuzzy decision information systems without class labels, an $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-probability dominance-similarity relation is put forward to calculate an (α, β)-probability dominance-similarity class. First, a relative loss function is proposed in hesitant fuzzy decision information systems. Then, a hesitant fuzzy three-way decision approach with three strategies is constructed via considering realistic decision-making features. At last, the rationality and validity of the proposed method are demonstrated by comparative and experimental analyses. The decision-making results with three strategies show that the presented hesitant fuzzy three-way decision approach with neutral strategies is more accurate than other counterparts.

1. Introduction

In real world, diverse complex decision-making environments with uncertain information can be found. In fact, decision-making and information are closely related, and clear information is the basis for making a scientific decision. According to this, Li et al. [11] explored uncertain measurements for a fuzzy relation in an information system (IS). Subsequently, an IS is combined with different environments to derive various systems, such as intuitionistic fuzzy information systems (IFISs) [34] and hesitant fuzzy linguistic term sets [37]. In complex decision-making problems, decision-makers (DMs) may hesitate to make decisions between several evaluation values, or a decision group may not have a unified value. For instance, there are three experts who aim to estimate the safety performance of a product by 0.4, 0.8 and 0.6, respectively. In order to explain the above phenomenon, Torra [27] initiated the definition of hesitant fuzzy sets (HFSs) to fuse the evaluation results and the safety performance of this product can be expressed as {0.4, 0.8, 0.6}. Afterwards, Xia and Xu [39] gave a mathematical form of an HFS. We know that a hesitant fuzzy information system (HFIS) is a combination of an IS under a hesitant environment. Afterwards, many scholars explored this theory and its applications from different perspectives. For instance, Mardani et al. [22] designed a framework of an HFIS to adopt digital health interventions during the COVID-19 outbreak.

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Ma et al. [21] combined a linguistic term set with an HFIS to address green supplier selections. In particular, Liang and Liu [13] put forward a risk decision-making method based on decision-theoretic rough sets in an HFIS. It is pointed out that an HFIS with decision attributes is titled a hesitant fuzzy decision information system (HFDIS). For the sake of exploring an error analysis theory, Liang et al. [16] employed a new aggregation-method-based error analysis for decision-theoretic rough sets in an HFDIS. Wu et al. [37] proposed a hesitant fuzzy linguistic group decision-making method based on the additive consistency. Wu et al. [38] developed a group decision-making method based on hesitant Pythagorean fuzzy information fusion techniques.

The three-way decision (3WD) theory designed by Yao [43] is an effective method to process uncertain problems. We know that "three-way" embodies the principle of three divisions (the accepted region, the boundary region and the rejected region), and "decision-making" is one of ultimate goals for solving realistic problems. Thus, the main idea of 3WD is to reasonably separate a group of objects into three parts, and adopt effective strategies to deal with each part so as to obtain an optimal result. With more than ten years of rapid development, the 3WD theory has become an effective tool for intelligent information processing. Nowadays, there exist many generalized models, such as three-way analysis models [3], three-way learning models [9], three-way clustering models [30], three-way approximations models [49], three-way recognition models [8], three-way concept lattice models [25], 3WD support systems [42], set-theoretic models of 3WD [44], three-way conflict analysis [6], tri-level thinking [45], etc. The above models are all 3WD methods in various contexts with specific mathematical expressions. In sum, the above perspectives can be seen as 3WD in a narrow sense.

The 3WD theory in a broad sense pays more attention to the aspect of structured thinking of granular computing, the methodology of problem solving and the information processing, respectively. Since 2012, the 3WD theory in a broad sense has been widely studied, and the trisecting-acting-outcome (TAO) [46,47] model serves as a typical representation of the 3WD theory in a broad sense. The three basic components of the TAO model are trisecting the whole (T), acting upon the three parts (A) and optimizing the outcome (O), and the specific TAO model is shown in Fig. 1. According to Fig. 1, the TAO model is a combination of numbers, text and thinking, which follows the principle of ternary thinking. Recently, Yao [48] discussed the 3WD theory in a broad sense from the geometry perspective of points, lines and circles.

The description of uncertain problems is the key factor to understand the nature of things and the prerequisite step to solve them. In reality, objects are recognized from multiple levels and aspects. Thus, an HFIS can effectively reflect uncertainties in real-world problems. Moreover, 3WD is a granular computing method to solve uncertain decision-making problems, where the divide-and-conquer idea is a new research perspective, which can aid individuals think, solve and analyze complex uncertain decision-making problems via granular strategies. In fact, decision-making processes in many fields often reflect the idea of 3WD, such as medical diagnosis [42], phased array radar [7], energy policy selections [40], etc. Based on this, the paper proposes a hesitant fuzzy three-way decision (HF-3WD) approach to solve realistic decision-making problems in an HFDIS. Moreover, a DM is a factor that cannot be ignored in a decision-making process, thus we combine the proposed HF-3WD approach with the attitudes of DMs to form the HF-3WD method with three strategies. Finally, we apply the method for solving decision-making problems [16,29] and verifying the rationality and applicability via corresponding comparisons and experiments.

In light of the above statements, the research motivations of this paper can be listed as follows:

- As a decision-making model that conforms to human cognitive processes, the 3WD theory [43] has gradually developed into an effective method to deal with uncertain information. In addition, an HFDIS is one of the expressions of uncertain information in real world. Thus, we intend to propose a 3WD method and solve realistic decision-making problems in an HFDIS.

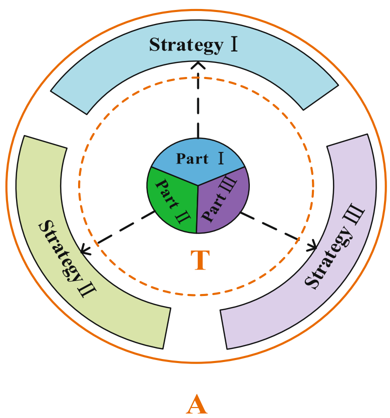

Fig. 1. The TAO model.

- There is no class label in an HFDIS. In the basic framework of 3WD, how to calculate the conditional probability in an HFIS is an important issue. In order to reduce the subjectivity of decision-making [15], Liang et al. [16] defined similarity classes to calculate conditional probabilities. In exceptional cases, the similarity class defined by a distance formula does not conform to the reality. For instance, the evaluation values of the two objects o_1 and o_2 under the two attributes are {0.7, 0.8}, {0.1, 0.2} and {0.5, 0.6}, {0.5, 0.6}. Let the similarity degree $\alpha = 0.1$, we can know that $D(o_1, o_2) \leqslant \alpha$, i.e., $o_2 \in [o_1]_L$. Under the second attribute, o_1 is much worse than o_2, and we do not actually recognize the similar relationship. Wang et al. [29] used the outranked classes to estimate the conditional probability, and the outranking relation measured by the attribute weight ignored the gap between the objects. Thus, the paper defines a new binary relation in an HFDIS to estimate the values of the conditional probability.
- It is noted that the loss function is another core problem of 3WD methods [43]. However, existing loss functions [16,13,29] in an HFDIS have a strong subjective preference. In light of the researches of Liu et al. [19] and Jia and Liu [5], the relative loss functions are obtained by considering the evaluation values of each object for each attribute. The relative loss function can reduce the influence of subjective preferences on decision results. Thus, we plan to introduce a relative loss function in an HFDIS.
- In practice, the personality characteristics of a DM may affect decision outcomes. An optimistic DM tends to focus less on losses than a pessimistic one, whereas a neutral DM is more rational. Thus, in order to better reflect the perception of DMs, we intend to construct an HF-3WD approach with three risk strategies in an HFDIS.

The primary structure of the paper is organized as follows: We review some basic concepts in Section 2. We propose an HF-3WD approach with three risk strategies in Section 3. Moreover, we show the research process of the HF-3WD method in Section 4. In Section 5, we verify the rationality and superiority of the proposed method via two realistic cases. In Section 6, we perform experiments with parameter changes. Section 7 illustrates main contributions and future study directions.

2. Preliminaries

The elementary concepts of 3WD, probability dominance classes, similarity classes, classification error rates and classification precision rates are reviewed in this section.

2.1. The 3WD theory

Rough set theory is a mathematical tool for dealing with uncertain problems, where its main idea is an approximate method by means of upper and lower approximate operators. Due to the lack of fault tolerances in the rough set theory, many scholars have introduced a quantitative model, namely, the probabilistic rough set model [43], which can address the restriction in a rough set where the lower approximation is too strict and the upper approximation is too loose. In order to further explore the basic concepts and semantic interpretations in a probabilistic rough set model, Yao proposed a decision-theoretic rough set model, which gives reasonable solutions for three issues, i.e., the calculation of threshold values, the estimation of conditional probabilities and the semantic interpretation of three domains. The development process from the rough set theory to the decision-theoretic rough set theory is shown in Fig. 2.

In the Bayesian minimum risk decision-making process, a decision-theoretic rough set contains a state set $\mathcal{S} = \{S, \neg S\}$ and an action set $\mathcal{A} = \{A_P, A_B, A_N\}$, where A_P, A_B and A_N are denoted by the acceptance, the non-commitment and the rejection, respectively, which is shown as Table 1.

We know that $\lambda_{\circ\bullet}(\circ, \bullet = P, B, N)$ represents a loss that arises from the corresponding action in a certain state. Under semantic interpretations, the conditions $\lambda_{PP} \leqslant \lambda_{BP} < \lambda_{NP}$ and $\lambda_{NN} \leqslant \lambda_{BN} < \lambda_{PN}$ are satisfied. For the object o, the expected loss values $E(A_\bullet|[o])$ for the three actions A_P, A_B, A_N are given as follows:

$$
\begin{aligned}
E(A_P|[o]) &= \lambda_{PP}\mathcal{P}(S|[o]) + \lambda_{PN}\mathcal{P}(\neg S|[o]),\\
E(A_B|[o]) &= \lambda_{BP}\mathcal{P}(S|[o]) + \lambda_{BN}\mathcal{P}(\neg S|[o]),\\
E(A_N|[o]) &= \lambda_{NP}\mathcal{P}(S|[o]) + \lambda_{NN}\mathcal{P}(\neg S|[o]),
\end{aligned}
$$

where $\mathcal{P}(S|[o])$ is the conditional probability of the object o belonging to S.

It is obvious that $\mathcal{P}(S|[o]) = \frac{|S\cap[o]|}{|o|}$ and $\mathcal{P}(\neg S|[o]) = \frac{|\neg S\cap[o]|}{|[o]|}$. Clearly, $\mathcal{P}(S|[o]) + \mathcal{P}(\neg S|[o]) = 1$, where $[o]$ is an equivalence class.

We know that 3WD gives a reasonable semantic explanation for three domains: the positive domain ($Pos(S)$), the boundary domain ($Bnd(S)$) and the negative domain ($Neg(S)$) of a rough set, namely:

$$
\begin{aligned}
&\text{If } o \in Pos(S), \text{ then } o \Rightarrow A_P,\\
&\text{If } o \in Bnd(S), \text{ then } o \Rightarrow A_B,\\
&\text{If } o \in Neg(S), \text{ then } o \Rightarrow A_N.
\end{aligned}
$$

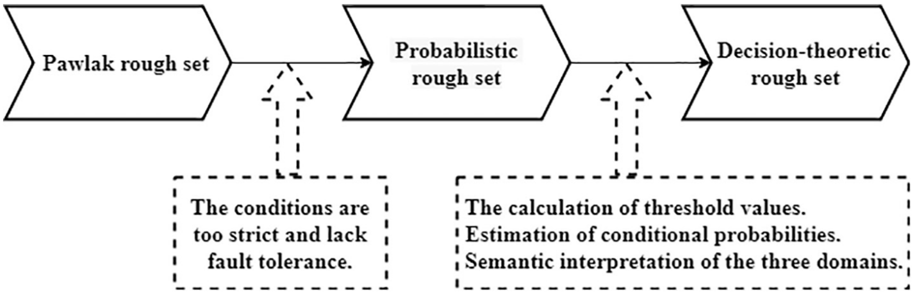

Fig. 2. The development process from rough set theory to decision-theoretic rough set theory.

Table 1
The loss function.

	S	$\neg S$
A_P	λ_{PP}	λ_{PN}
A_B	λ_{BP}	λ_{BN}
A_N	λ_{NP}	λ_{NN}

2.2. HFSs

Definition 2.1 [27]. Let X be a finite set, an HFS on X is a function that when applied to X returns a subset of [0,1]. For the sake of convenience, Xia and Xu [39] gave the mathematical expression of an HFS:

$$S = \{\langle\ o,\ h_S(o)\ \rangle \mid o \in X\},$$

where $h_S(o)$ contains several values in [0,1], representing the possible membership degrees of o to S. $h_S(o)$ is called a hesitant fuzzy element (HFE).

Definition 2.2 [40]. Let h be any an HFE. Then the score function $SC(h)$ of an HFE h is described as follows:

$$SC(h) = \frac{1}{|h|}\sum_{\varepsilon\in h}\varepsilon,$$

where $|h|$ is the number of elements in h. Let h_1, h_2 be any two HFEs, the comparison method meets the condition that $h_1 \succeq h_2$ if and only if $SC(h_1) \geqslant SC(h_2)$.

In most cases, two HFEs h_1 and h_2 have different numbers of elements $|h_1| < |h_2|$. Xu and Xia [40] suggested adding values to h_1 until $|h_1| = |h_2|$. In fact, the risk preferences of DMs determine the added values. In order to solve this problem, Xu and Zhang [41] proposed the following way.

Definition 2.3 [41]. For any HFE h, h^+ and h^- are the maximum value and the minimum value of h, respectively. The added value is $v = \omega h^+ + (1-\omega)h^-$, where ω represents the degree of risk preferences of DMs.

Xia and Xu [39] defined some basic operational laws. However, the dimension of an HFE will increase and the computational complexity will also increase when performing addition and multiplication operations. In order not to increase the dimension of an HFE, Liao et al. [17] proposed the following improved operational laws.

Definition 2.4 [17]. Suppose that h_1, h_2 are two HFEs and ρ is a positive real number, then

- $h_1 \oplus h_2 = \{h_1^{\sigma(i)} + h_2^{\sigma(i)} - h_1^{\sigma(i)}h_2^{\sigma(i)} \mid i = 1,2,\ldots,\max\{|h_1|,|h_2|\}\}$,
- $h_1 \otimes h_2 = \{h_1^{\sigma(i)}h_2^{\sigma(i)} \mid i = 1,2,\ldots,\max\{|h_1|,|h_2|\}\}$,
- $h_1^c = \{1 - h_1^{\sigma(i)} \mid i = 1,2,\ldots,|h_1|\}$,
- $\rho h_2 = \{1 - (1 - h_2^{\sigma(i)})^{\rho} \mid i = 1,2,\ldots,|h_2|\}$,

where h_1 and h_2 ensure the same number of elements by adding elements in advance, $h_1^{\sigma(i)}$ represents the i-th largest/smallest value in h_1.

The premise of Definition 2.4 is that h_1 and h_2 have the same length, and h_1 and h_2 have performed σ permutation. If the lengths of h_1 and h_2 are different, we add some numbers v to the shorter element according to Definition 2.3 so that they keep the same lengths.

Example 2.1. Let $h_1 = \{0.4, 0.1\}, h_2 = \{0.7, 0.5, 0.6\}, \rho = 2$ and a DM's risk preference $\omega = 0$, the lengths of h_1 and h_2 are different. We can expand h_1 to $\{0.4, 0.1, 0.1\}$ by Definition 2.3. Moreover, we perform ascending permutation of the elements in h_1 and h_2, then $h_1 = \{0.1, 0.1, 0.4\}$ and $h_2 = \{0.5, 0.6, 0.7\}$. According to Definition 2.4, we have

$$\begin{aligned} h_1 \oplus h_2 &= \{0.55, 0.64, 0.82\}, \\ h_1 \otimes h_2 &= \{0.05, 0.06, 0.28\}, \\ h_2^c &= \{0.3, 0.4, 0.5\}, \\ \rho h_2 &= \{0.75, 0.84, 0.91\}. \end{aligned}$$

In an HFDIS, the conditions for the equivalence relation between objects are too strict. Hence, Liang et al. [16] redefined a binary relation, namely the similarity relation.

Definition 2.5 [16]. Suppose that $Q = \{\mathcal{U}, \mathcal{I} \bigcup d, \mathcal{V}, \mathcal{F}\}$ is an HFDIS, where $\mathcal{U} = \{o_1, o_2, \ldots, o_n\}, \mathcal{I} = \{I_1, I_2, \ldots, I_m\}, d$ is a decision index, $\mathcal{V} = \bigcup_{i=\{1,2,\ldots,n\}, j=\{1,2,\ldots,m\}} u_{ij}, u_{ij}$ is an HFE and $L \subseteq \mathcal{I}$. For any $o_i, o_j \in \mathcal{U}$, a similarity relation is expressed as:

$$\mathcal{N}_L = \{(o_i, o_j) \in \mathcal{U} \times \mathcal{U} \mid D_L(o_i, o_j) \leqslant \beta\},$$

where $\beta \in [0, 1], D_L(o_i, o_j)$ is the distance between o_i and o_j. The Euclidean function is commonly used to measure the distance between objects:

$$D_L(o_i, o_j) = \sqrt{\frac{1}{|L|} \sum_{k \in L} \{\frac{1}{H} \sum_{m=1}^{H} (u_{ik}^{\sigma(m)} - u_{jk}^{\sigma(m)})^2\}},$$

where $H = \max\{|u_{ik}|, \ |u_{jk}|\}, u_{ik}^{\sigma(m)}$ and $u_{jk}^{\sigma(m)}$ are σ permutations of u_{ik} and u_{jk}, respectively.

In an HFDIS, we have

$$[o_i]_L = \{o_j \in \mathcal{U} \mid D_L(o_i, o_j) \leqslant \beta\},$$

we call $[o_i]_L$ a similarity class of the object o_i with respect to an index set L.

Example 2.2. Suppose that $L = \{\mathcal{I}_1, \mathcal{I}_2\}, \beta = 0.1$, and the evaluation values of o_1 and o_2 under $\mathcal{I}_1$ and $\mathcal{I}_2$ are $u_{11} = \{0.07, 0.13\}, u_{21} = \{0.2, 0.23\}, u_{12} = \{0.15, 0.25\}$ and $u_{22} = \{0.2, 0.3\}$, respectively. By Definition 2.5, we have

$$D_L(o_1, o_2) = \sqrt{\frac{1}{|L|} \sum_{k \in L} \{\frac{1}{H} \sum_{m=1}^{H} (u_{1k}^{\sigma(m)} - u_{2k}^{\sigma(m)})^2\}} = 0.0893.$$

Because $D_L(o_1, o_2) \leqslant \beta, o_1$ and o_2 have a similarity relation under L, namely, $o_2 \in [o_1]_L$.

2.3. Probabilistic dominance classes

Definition 2.6 (*[12,23]*). A quadruple $Q = \{\mathcal{U}, \mathcal{I}, \mathcal{V}, \mathcal{F}\}$ is an IS, where $\mathcal{U} = \{o_1, o_2, \ldots, o_n\}$ is an object set, $\mathcal{I} = \{I_1, I_2, \ldots, I_m\}$ an index set, $\mathcal{V} = \bigcup_{i=\{1,2,\ldots,n\}, j=\{1,2,\ldots,m\}} u_{ij}$ a universe set of the object under the index set, and $\mathcal{F} = \{\mathcal{F}_j \mid \mathcal{U} \rightarrow \mathcal{V}\}$ a mapping set, namely $\mathcal{F}(o_i, I_j) = \mathcal{F}_j(o_i) = u_{ij}$.

Definition 2.7 [35]. Suppose that $Q = \{\mathcal{U}, \mathcal{C}, \mathcal{V}, \mathcal{F}\}$ is an ordered information system (OrIS). For all $\forall o_i, o_j \in \mathcal{U}$, we have

$$P_{I_k}(o_i, o_j) = \begin{cases} 1, & u_{ik} \geqslant u_{jk}, \\ 0, & u_{ik} < u_{jk}, \end{cases}$$

where $P_{I_k}(o_i, o_j)$ is a binary judgment of values of o_i and o_j under I_k.

From Definition 2.2, for any $\mathbf{L} \subseteq \mathcal{I}$, the judgment matrix of Q under $\mathbf{L}$ is given as

$$P_{\mathbf{L}}(o_i, o_j) = \Sigma_{L_k \in \mathbf{L}} \frac{P_{L_k}(o_i, o_j)}{|\mathbf{L}|},$$

where $|\mathbf{L}|$ denotes the cardinal number of $\mathbf{L}$.

Definition 2.8 [35]. In an OrIS Q, we have

$$P_{\mathbf{L}}^{\geqslant \alpha} = \{(o_i, o_j) \in \mathcal{U} \times \mathcal{U} \mid P_{\mathbf{L}}(o_i, o_j) \geqslant \alpha, \alpha \in [0.5, 1]\},$$

where $P_{\mathbf{L}}^{\geqslant \alpha}$ is an α-probability dominance relationship (α-PDR) of Q under $\mathbf{L}$ and α represents a confidence level.

Definition 2.9 [35]. In an OrIS Q, for any α, we have

$$[o_i]_{\mathbf{L}}^{\geqslant \alpha} = \{o_j \in \mathcal{U} \mid (o_i, o_j) \in P_{\mathbf{L}}^{\geqslant \alpha}\},$$

we call $[o_i]_{\mathbf{L}}^{\geqslant \alpha}$ an α-probability dominance class (α-PDC) of the object o_i with respect to an index set $\mathbf{L}$.

2.4. Classification error rates and precision rates

Classification error rates (CER) and precision rates (CPR) act as important indicators in many practical fields. For instance, in the process of medical diagnosis, if the cost of error diagnosis is high, then doctors should minimize decision-making errors as much as possible. A DM must reduce the CER and increase the CPR when making a decision. Therefore, CER and CPR can evaluate the performance of a decision-making method. In other words, if the value of CER is lower and the value of CPR is higher, the performance of the method will be better, vice versa.

Definition 2.10 [14]. The mathematical calculation formula of a CER is presented as follows:

$$Er = \frac{N_{S \to Neg(S)} + N_{\neg S \to Pos(S)}}{|\mathcal{U}|} \times 100\%,$$

where $N_{S \to Neg(S)}$ is the number of objects belonging to S and classified in the $Neg(S)$ domain, $N_{\neg S \to Pos(S)}$ is the number of objects belonging to $\neg S$ and classified in the $Pos(S)$ domain.

Definition 2.11. The mathematical calculation formula of a CPR is presented as follows:

$$Pr = \frac{N_{S \to Pos(S)} + N_{\neg S \to Neg(S)}}{|\mathcal{U}|} \times 100\%,$$

where $N_{S \to Pos(S)}$ is the number of objects belonging to S and classified in the $Pos(S)$ domain, $N_{\neg S \to Neg(S)}$ is the number of objects belonging to $\neg S$ and classified in the $Neg(S)$ domain.

3. A 3WD approach with three risk strategies in an HFDIS

In this section, we propose a 3WD approach with risk attitudes of DMs in an HFDIS. The method can realize the classification of objects. In Section 3.1, we propose a new method of calculating conditional probability. In Section 3.1, we introduce the relative loss function in an HFDIS. In Section 3, we discuss the 3WD method with three attitudes including optimism, pessimism and neutrality.

3.1. Conditional probabilities based on an (α, β)-probability dominance-similarity class

Definition 3.1. A 3WD-based HFDIS is defined as $3WQ = \{\mathcal{U}, \mathcal{I} \bigcup d, \mathcal{V}, \mathcal{F}, \lambda_{PP}, \lambda_{BP}, \lambda_{NP}, \lambda_{PN}, \lambda_{BN}, \lambda_{NN}\}$, where $\mathcal{U} = \{o_1, o_2, \ldots, o_m\}$ is an object set, $\mathcal{I} = \{I_1, I_2, \ldots, I_n\}$ is a conditional attribute set, d is a decision attribute set, $\mathcal{V} = \bigcup_{i=1,2,\ldots,m, j=1,2,\ldots,n} u_{ij}$ is an HFS and $\mathcal{F}$ is a mapping, that is, $\mathcal{F}(o_i, I_j) = \mathcal{F}_j(o_i) = u_{ij}$. Any object $o_i \in \mathcal{U}, (i = 1, 2, \ldots, m)$ can be represented by an $m+7$ dimensional vector, namely $\mathcal{F}(o_i)$=$(\mathcal{V}_{I_1}(o_i), \mathcal{V}_{I_2}(o_i), \ldots, \mathcal{V}_{I_n}(o_i), d(o_i), \lambda_{PP}^i, \lambda_{BP}^i, \lambda_{NP}^i, \lambda_{PN}^i, \lambda_{BN}^i, \lambda_{NN}^i)$.

The calculation of the conditional probability $\mathcal{P}(S|[o])$ and the determination of the loss function in 3WD are two key issues. We have already used the relative loss function in Section 3.1. Then, we discuss the calculation of $\mathcal{P}(S|[o])$.

Yao [43] calculated $\mathcal{P}(S|[o]) = \frac{|S \cap [o]|}{|[o]|}$, where $[o]$ is the equivalence class. In an HFDIS, there is no class label. Hence, it is not feasible to use an equivalence class to calculate $\mathcal{P}(S|[o])$. Based on this reason, Liang et al. [16] used a similarity class to calculate $\mathcal{P}(S|[o]_L)$. A dominance relation is an extension of an equivalence relation in the rough set theory. Liu et al. [18] calculated $\mathcal{P}(S|[o]_{\mathbf{L}})$ via a dominance class. Further, Wang et al. [33] proposed the probability dominance classes to calculate $\mathcal{P}(S|[o_i]_{\mathbf{L}}^{\geqslant \alpha})$. In a dominance rough set model, we find that the dominance relationship is sensitive to noise data. Therefore, we propose an (α, β)-probability dominance-similarity relation.

Definition 3.2. Assume that $Q = \{\mathcal{U}, \mathcal{I} \cup d, \mathcal{V}, \mathcal{F}\}$ is an HFDIS, for any $o_i, o_j \in \mathcal{U}$, a binary judgment value based on a dominance-similarity relation $B_{I_k}^{\leqslant\beta}(o_i, o_j)$ is given as:

$$B_{I_k}^{\leqslant\beta}(o_i, o_j) = \begin{cases} 1, & u_{ik} \succeq u_{jk}, \ and \ \ D(u_{ik}, u_{jk}) \leqslant \beta \\ 0, & other. \end{cases}$$

From Definition 3.2, for any $L \subseteq \mathcal{I}$, a judgment matrix based on the similarity relation $B_L^{\leqslant\beta} = (B_L^{\leqslant\beta}(o_i, o_j))_{n\times n}$ is:

$$B_L^{\leqslant\beta}(o_i, o_j) = \Sigma_{k\in L} \frac{B_{I_k}^{\leqslant\beta}(o_i, o_j)}{|L|}.$$

Definition 3.3. In an HFDIS Q, we have (α, β)-probability dominance-similarity relation:

$$B_L^{(\alpha,\ \beta)} = \{(o_i, o_j) \in \mathcal{U} \times \mathcal{U} \ \mid \ B_L^{\leqslant\beta}(o_i, o_j) \geqslant \alpha, \alpha \in [0.5, 1]\}.$$

Remark 3.1. From Definition 3.3, it is clear that the new binary relation (α, β)-probability dominance-similarity relation satisfies the reflexivity, but does not satisfy the symmetry and transitivity.

Definition 3.4. In an HFDIS Q, an (α, β)-probability dominance-similarity class of o_i with respect to L is defined as:

$$[o_i]_L^{(\alpha,\ \beta)} = \{o_j \in \mathcal{U} \ \mid \ (o_i, o_j) \in B_L^{(\alpha,\ \beta)}\}.$$

From Definition 3.4, a new binary relation can induce an (α, β)-probability dominance-similarity class of $\mathcal{U}$, which forms a covering. That is, $\mathcal{U}/B_L^{(\alpha,\ \beta)} = \{[o_1]_L^{(\alpha,\ \beta)}, [o_2]_L^{(\alpha,\ \beta)}, \ldots, [o_m]_L^{(\alpha,\ \beta)}\}$ is a covering of $\mathcal{U}$, namely, $\bigcup_{i=1}^{m} [o_i]_L^{(\alpha,\ \beta)} = \mathcal{U}$ and $[o_i]_L^{(\alpha,\ \beta)} \bigcap [o_j]_L^{(\alpha,\ \beta)} \neq \varnothing \ (i \neq j)$.

We illustrate the (α, β)-probability dominance-similarity class via the following example.

Example 3.1. A data set is randomly generated by MATLAB, including four conditional attributes (the benefit type), one decision attribute and six objects. An HFDIS $Q = \{\mathcal{U}, \mathcal{I} \cup d, \mathcal{V}, \mathcal{F}\}$ is constructed in Table 2, where $V_d = \{1, 0\}, 1$ means the acceptance and 0 means the rejection.

Let $L = \mathcal{I}, \alpha = 0.5$ and $\beta = 0.1$. According to Definition 3.4, we can obtain:

$$[o_1]_{\mathcal{I}}^{(0.5,\ 0.1)} = \{o_1, o_5, o_6\}, \qquad [o_2]_{\mathcal{I}}^{(0.5,\ 0.1)} = \{o_1, o_2\},$$
$$[o_3]_{\mathcal{I}}^{(0.5,\ 0.1)} = \{o_1, o_2, o_3, o_5, o_6\}, \qquad [o_4]_{\mathcal{I}}^{(0.5,\ 0.1)} = \{o_4\},$$
$$[o_5]_{\mathcal{I}}^{(0.5,\ 0.1)} = \{o_5, o_6\}, \qquad [o_6]_{\mathcal{I}}^{(0.5,\ 0.1)} = \{o_1, o_5, o_6\}.$$

According to the research results of Li et al. [10], the conditional probability under (α, β)-probability dominance-similarity relation is defined as follows.

Definition 3.5. In an HFDIS Q, let $\mathcal{U}/d = \{S, \neg S\}, \forall o_i \in \mathcal{U}$ and $L \subseteq I$, the conditional probability of o_i belonging to S with respect to L is defined as:

$$\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)}) = \frac{|S \cap [o_i]_L^{(\alpha,\ \beta)}|}{|[o_i]_L^{(\alpha,\ \beta)}|}.$$

Remark 3.2. Definition 3.5 provides a method for estimating the conditional probability in an HFDIS. We give $\mathcal{P}(\neg S|[o_i]_L^{(\alpha,\ \beta)})$ as follows:

Table 2
An HFDIS $Q = \{\mathcal{U}, \mathcal{I} \cup d, \mathcal{V}, \mathcal{F}\}$.

$\mathcal{U}$	I_1	I_2	I_3	I_4	d
o_1	{0.3, 0.4}	{0.2, 0.1}	{0.08, 0.12}	{0.15, 0.25}	1
o_2	{0.9, 0.8}	{0.15, 0.25}	{0.1, 0.2}	{0.15, 0.2, 0.25}	0
o_3	{0.25, 0.35}	{0.2, 0.3}	{0.1, 0.13}	{0.2, 0.3}	1
o_4	{0.75, 0.8, 0.85}	{0.48, 0.52}	{0.5, 0.7}	{0.1}	0
o_5	{0.16, 0.24}	{0.13, 0.1}	{0.5, 0.6}	{0.18, 0.22}	1
o_6	{0.3, 0.2}	{0.2, 0.1}	{0.48, 0.52}	{0.2}	0

$$\mathcal{P}(\neg S|[o_i]_L^{(\alpha,\ \beta)}) = \frac{|\neg S \cap [o_i]_L^{(\alpha,\ \beta)}|}{|[o_i]_L^{(\alpha,\ \beta)}|}.$$

Obviously, $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)}) + \mathcal{P}(\neg S|[o_i]_L^{(\alpha,\ \beta)}) = 1$ holds.

We demonstrate the calculation of conditional probabilities by the following example.

Example 3.2. Continued with Example 3.1, we further study the calculation of the conditional probability $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$ of each object $o_i (i = 1, 2, \ldots, 6)$ in Table 2. In the HFDIS, $L = \mathcal{I}, \alpha = 0.5, \beta = 0.1, \mathcal{U}/d = \{S, \neg S\}$. Thus, $S = \{o_1, o_3, o_5\}$. According to Definition 3.5, we can estimate the conditional probability $\mathcal{P}(S|[o_i]_{\mathcal{I}}^{(0.5,\ 0.1)})$.

$$\mathcal{P}(S|[o_1]_{\mathcal{I}}^{(0.5,\ 0.1)}) = \tfrac{2}{3}, \quad \mathcal{P}(S|[o_2]_{\mathcal{I}}^{(0.5,\ 0.1)}) = \tfrac{1}{2},$$
$$\mathcal{P}(S|[o_3]_{\mathcal{I}}^{(0.5,\ 0.1)}) = \tfrac{3}{5}, \quad \mathcal{P}(S|[o_4]_{\mathcal{I}}^{(0.5,\ 0.1)}) = 0,$$
$$\mathcal{P}(S|[o_5]_{\mathcal{I}}^{(0.5,\ 0.1)}) = \tfrac{1}{2}, \quad \mathcal{P}(S|[o_6]_{\mathcal{I}}^{(0.5,\ 0.1)}) = \tfrac{2}{3}.$$

3.2. Relative loss functions in HFDISs

One of the important issues in the 3WD theory is the loss function. Yao et al. [43] gave the loss function values based on personal experiences. Due to the lack of experiences of DMs in realistic situations, a single evaluation value cannot accurately reflect realistic situations. Thus, Liang et al. [13] used an HFE to evaluate the loss function values. Afterwards, Jia and Liu [5] and Liu et al. [19] discussed the relative loss functions based on an information table in different environments. In this section, based on the above research analysis, we propose the relative loss functions in an HFDIS.

It is assumed that there are m objects, n conditional attributes and one decision attribute in an HFDIS. The evaluation value of the i-th object o_i under the j-th conditional attribute I_j is u_{ij}, hence $u_{ij} (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$ constructs an HFE. The weight vector of attributes is $W = \{W_1, W_2, \ldots, W_n\}$. The envelope of an HFS is an intuitionistic fuzzy set (IFS), and the related operations of an HFS are evolved from an IFS [27]. According to the relationship between an HFS and an IFS, the relative loss functions in an HFDIS are shown in Table 3.

In Table 3, $\hat{\lambda}_{PN} = u_{ij}^c, \hat{\lambda}_{BP} = \eta_j u_{ij}, \hat{\lambda}_{BN} = \eta_j u_{ij}^c, \hat{\lambda}_{NP} = u_{ij}, \eta_j \in [0, 1]$ is a risk aversion coefficient and its value represents a DM's ability to bear risks. The conditions $\hat{\lambda}_{PP} \preceq \hat{\lambda}_{BP} \prec \hat{\lambda}_{NP}$ and $\hat{\lambda}_{NN} \preceq \hat{\lambda}_{BN} \prec \hat{\lambda}_{PN}$ are satisfied.

Remark 3.3. The semantic interpretations of the relative loss function are:

(1) When the object $o_i \in S$, we select the action A_P as the reference point, the relative losses of A_P, A_B, A_N to A_P are $\{0\}, \hat{\lambda}_{BP}, \hat{\lambda}_{NP}$ respectively. Similarly, when the object $o_i \in \neg S$, we choose the action A_N as the reference point, the relative losses of A_P, A_B, A_N to A_N are $\hat{\lambda}_{PN}, \hat{\lambda}_{BN}$ and $\{0\}$.

(2) u_{ij} is the evaluation value of the object o_i under the attribute I_j, which also means that the wealth. When the object $o_i \in S$, if the object o_i takes A_N, then you will lose the wealth u_{ij}. If the object o_i takes A_B, then you will lose part of the wealth, which is $\eta_j u_{ij}$. Similarly, u_{ij} is the evaluation value of the object o_i under the attribute I_j, which also means that the wealth under this opposite state is u_{ij}^c. When the object $o_i \in \neg S$, if the object o_i takes A_P, then you will lose the wealth u_{ij}^c. If the object o_i takes A_B, then you will lose part of the wealth, which is $\eta_j u_{ij}^c$.

Each object has a set of loss functions under each condition attribute, then n sets of loss functions are aggregated. The aggregation methods include the hesitant fuzzy weighted average (HFWA) operator [39], the generalized GHFWA operator [39], etc. This paper uses the HFWA operator method to aggregate the loss function. Therefore, the loss functions of the object o_i are shown in Table 4.

In Table 4, $\lambda_{PN}^i = \oplus_{j=1}^n (W_j u_{ij}^c), \lambda_{BP}^i = \oplus_{j=1}^n (W_j \eta_j u_{ij}), \lambda_{BN}^i = \oplus_{j=1}^n (W_j \eta_j u_{ij}^c), \lambda_{NP}^i = \oplus_{j=1}^n (W_j u_{ij})$. The conditions $\lambda_{PP}^i \preceq \lambda_{BP}^i \prec \lambda_{NP}^i$ and $\lambda_{NN}^i \preceq \lambda_{BN}^i \prec \lambda_{PN}^i$ are satisfied.

Table 3
The relative loss functions in an HFDIS.

	S	$\neg S$
A_P	$\{0\}$	$\hat{\lambda}_{PN}$
A_B	$\hat{\lambda}_{BP}$	$\hat{\lambda}_{BN}$
A_N	$\hat{\lambda}_{NP}$	$\{0\}$

Table 4
The relative loss functions for o_i.

	S	$\neg S$
A_P	$\{0\}$	λ^i_{PN}
A_B	λ^i_{BP}	λ^i_{BN}
A_N	λ^i_{NP}	$\{0\}$

3.3. An HF-3WD approach with three strategies in HFDISs

In the decision-making process, apart from objective mathematical formulas, the subjective preferences of DMs are also important factors. The subjective preferences of DMs depend on the degrees to which DMs are willing to take risks. Liu and Liang [18] integrated the loss functions with three strategies in an ordered IS, but the decision-making process is more complicated. In light of this, we aggregate the loss functions with three strategies in an HFDIS.

Definition 3.6. Given a $3WQ = \{\mathcal{U}, \mathcal{I}\bigcup d, \mathcal{V}, \mathcal{F}, \lambda_{PP}, \lambda_{BP}, \lambda_{NP}, \lambda_{PN}, \lambda_{BN}, \lambda_{NN}\}$, in three different situations of an HFDIS, the aggregated relative loss functions of $o_i \in \mathcal{U}$ are defined as:

- The relative loss function with an optimistic attitude: $\lambda^{i,opt}_{\circ\bullet} = \min_{o_k \in [o_i]^{(\alpha,\ \beta)}_L}\{\lambda^k_{\circ\bullet}\}$;
- The relative loss function with a neutral attitude: $\lambda^{i,neu}_{\circ\bullet} = \frac{\bigoplus_{o_k \in [o_i]^{(\alpha,\ \beta)}_L} \lambda^k_{\circ\bullet}}{|[o_i]^{(\alpha,\ \beta)}_L|}$;
- The relative loss function with a pessimistic attitude: $\lambda^{i,pes}_{\circ\bullet} = \max_{o_k \in [o_i]^{(\alpha,\ \beta)}_L}\{\lambda^k_{\circ\bullet}\}$.

Definition 3.6 gives the aggregated relative loss functions for DMs with three different attitudes in an HFDIS, we can obtain the following explanations:

(1) The optimistic DMs are more concerned about profit and less consideration to risk, the loss under the optimistic strategy is equal to the minimum value in the set-valued numbers.

(2) DMs with neutral attitudes have a more rational judgment of risks, the loss under the neutral strategy is equal to the average value in the set-valued numbers.(3) Because the pessimistic DMs consider more risks, the loss under the pessimistic strategy is equal to the maximal value in the set-valued numbers.

Proposition 3.1. *The relative loss functions with different strategies satisfy the following conditions ($\triangleright = opt, neu, pes$):*

$$\lambda^{i,\triangleright}_{PP} \preceq \lambda^{i,\triangleright}_{BP} \prec \lambda^{i,\triangleright}_{NP}, \lambda^{i,\triangleright}_{NN} \preceq \lambda^{i,\triangleright}_{BN} \prec \lambda^{i,\triangleright}_{PN}.$$

Proof. Discussions in Section 3.1 show that $\lambda^i_{PP} \preceq \lambda^i_{BP} \prec \lambda^i_{NP}$ and $\lambda^i_{NN} \preceq \lambda^i_{BN} \prec \lambda^i_{PN}$. By Definition 3.6, we have $\lambda^{i,opt}_{PP}$=$\lambda^{i,opt}_{NN}$=$\{0\}$, $\lambda^{i,opt}_{BP}$=$\min_{o_k \in [o_i]^{(\alpha,\ \beta)}_L}\{\lambda^k_{BP}\}$, $\lambda^{i,opt}_{NP}$=$\min_{o_k \in [o_i]^{(\alpha,\ \beta)}_L}\{\lambda^k_{NP}\}$, $\lambda^{i,opt}_{BN}$=$\min_{o_k \in [o_i]^{(\alpha,\ \beta)}_L}\{\lambda^k_{BN}\}$, $\lambda^{i,opt}_{NN}$=$\min_{o_k \in [o_i]^{(\alpha,\ \beta)}_L}\{\lambda^k_{NN}\}$. So $\lambda^{i,opt}_{PP} \preceq \lambda^{i,opt}_{BP} \prec \lambda^{i,opt}_{NP}, \lambda^{i,opt}_{NN} \preceq \lambda^{i,opt}_{BN} \prec \lambda^{i,opt}_{PN}$ holds. The others can be similarly proven. □

On the basis of the 3WD theory, a novel HF-3WD approach with three strategies is proposed in an HFDIS. According to Section 3.1 and Definition 3.6, the expected loss $E^{\triangleright}(A_\bullet|[o_i]^{(\alpha,\ \beta)}_L)(\triangleright = opt, neu, pes, \bullet = P, B, N)$ can be calculated:

$$E^{\triangleright}(A_P|[o_i]^{(\alpha,\ \beta)}_L) = \lambda^{i,\triangleright}_{PP}\mathcal{P}(S|[o_i]^{(\alpha,\ \beta)}_L) \oplus \lambda^{i,\triangleright}_{PN}\mathcal{P}(\neg S|[o_i]^{(\alpha,\ \beta)}_L), \tag{3-1}$$

$$E^{\triangleright}(A_B|[o_i]^{(\alpha,\ \beta)}_L) = \lambda^{i,\triangleright}_{BP}\mathcal{P}(S|[o_i]^{(\alpha,\ \beta)}_L) \oplus \lambda^{i,\triangleright}_{BN}\mathcal{P}(\neg S|[o_i]^{(\alpha,\ \beta)}_L), \tag{3-2}$$

$$E^{\triangleright}(A_N|[o_i]^{(\alpha,\ \beta)}_L) = \lambda^{i,\triangleright}_{NP}\mathcal{P}(S|[o_i]^{(\alpha,\ \beta)}_L) \oplus \lambda^{i,\triangleright}_{NN}\mathcal{P}(\neg S|[o_i]^{(\alpha,\ \beta)}_L). \tag{3-3}$$

According to the decision rules of the minimum risk in Bayesian decision procedures, we have the following decision rules in an HFDIS.

(HP) If$E^{\triangleright}(A_P|[o_i]^{(\alpha,\ \beta)}_L) \preceq E^{\triangleright}(A_B|[o_i]^{(\alpha,\ \beta)}_L)$, and $E^{\triangleright}(A_P|[o_i]^{(\alpha,\ \beta)}_L) \preceq E^{\triangleright}(A_N|[o_i]^{(\alpha,\ \beta)}_L)$, then $o_i \in Pos(S)$,
(HB) If$E^{\triangleright}(A_B|[o_i]^{(\alpha,\ \beta)}_L) \preceq E^{\triangleright}(A_P|[o_i]^{(\alpha,\ \beta)}_L)$, and $E^{\triangleright}(A_B|[o_i]^{(\alpha,\ \beta)}_L) \preceq E^{\triangleright}(A_N|[o_i]^{(\alpha,\ \beta)}_L)$, then $o_i \in Bnd(S)$,
(HN) If$E^{\triangleright}(A_N|[o_i]^{(\alpha,\ \beta)}_L) \preceq E^{\triangleright}(A_P|[o_i]^{(\alpha,\ \beta)}_L)$, and $E^{\triangleright}(A_N|[o_i]^{(\alpha,\ \beta)}_L) \preceq E^{\triangleright}(A_B|[o_i]^{(\alpha,\ \beta)}_L)$, then $o_i \in Neg(S)$.

The classification of objects can be achieved by decision rules (HP – HN). If $E^{\triangleright}(A_P|[o_i]_L^{(\alpha,\ \beta)}) \sim E^{\triangleright}(A_B|[o_i]_L^{(\alpha,\ \beta)}), E^{\triangleright}(A_P|[o_i]_L^{(\alpha,\ \beta)}) \sim E^{\triangleright}(A_N|[o_i]_L^{(\alpha,\ \beta)})$ or $E^{\triangleright}(A_B|[o_i]_L^{(\alpha,\ \beta)}) \sim E^{\triangleright}(A_N|[o_i]_L^{(\alpha,\ \beta)})$, then DMs should be tie-breaking. Some scholars [29,43] believed the tie-breaking rules according to the preferences of DMs. According to the principle of the minimizing risk, we propose a new method to break the tie.

Definition 3.7. In a $3WQ = \{\mathcal{U}, \mathcal{I} \bigcup d, \mathcal{V}, \mathcal{F}, \lambda_{PP}, \lambda_{BP}, \lambda_{NP}, \lambda_{PN}, \lambda_{BN}, \lambda_{NN}\}$, where d is a decision attribute set. The premise is that the expected losses with the three behaviors are equal, that is, a tie. We use the following rules to classify objects.

- If $d(o_i) = 1$, then $o_i \in Pos(s)$,
- If $d(o_i) = 0$, then $o_i \in Neg(S)$.

Among them, "1" and "0" are not numbers 1 and 0, but these character symbols represent the acceptance or the rejection.

In Definition 3.7, $d(o_i) = 1$ and $d(o_i) = 0$ mean that $o_i \in S$ and $o_i \notin S$, respectively. In realistic decision-making processes, when $o_i \in S$, the risk of taking the action A_P is the minimal. When $o_i \notin S$, the risk of taking the action A_N is the maximal. For instance, in a medical diagnosis, 1 and 0 mean sick and not sick, respectively. The sick subjects should be treated. On the contrary, normal subjects should refuse the treatment.

According to the above discussion, the principles of object classifications are: (1) All objects are classified according to the (HP-HN) decision rules. (2) If there is a tie, it is classified according to Definition 3.7.

4. The research process of the HF-3WD approach in an HFDIS

The decision-making process cannot be separated from the subjective guidance of DMs. The experiences of DMs and the comprehensiveness of information collection all affect the final decision result. An HFDIS is composed of a group of experts with rich experiences in different fields. To a certain extent, it overcomes the blindness of the evaluation process of a single DM. An HFDIS can fully reflect realities. 3WD is a trisecting-acting-outcome model, which can realize the tri-classification of objects and effectively reduce the CER of 2WD. The 3WD model can be used to solve practical hesitant fuzzy decision-making problems.

Therefore, a novel HF-3WD approach with three strategies is used to deal with decision-making problems in an HFDIS. In Section 4.1, we briefly describe the problem in an HFDIS. In Section 4.2, we summarize the decision-making process of the HF-3WD approach with three strategies. In Section 4.3, we list the algorithm and pseudo code of the HF-3WD approach and analyze the algorithm complexity.

4.1. The description of problems in an HFDIS

In an HFDIS $Q = \{\mathcal{U}, \mathcal{I} \cup d, \mathcal{V}, \mathcal{F}\}$, there are m alternatives, n condition attributes, p decision attributes, k DMs, and the weight of each condition attribute I_j is W_j. The weight of attributes represents the degree of importance satisfying $\Sigma_{j=1}^{n} W_j = 1$for $W_j \in [0, 1], j = 1, 2, \ldots, n$. The evaluation value of the i-th alternative under the j-th conditional attribute is u_{ij}, and so u_{ij} is an HFE. The specific HFDIS is shown as Table 5.

In practical problems of an HFDIS, the correct decision of an object is a key issue. 3WD is based on the addition of the boundary domain on 2WD. The three behaviors corresponding to the positive domain ($Pos(S)$), the negative domain ($Neg(S)$) and the boundary domain ($Bnd(S)$) are the acceptance (A_P), the rejection (A_N) and the consideration (A_B), respectively. The considered behavior can reduce the risk of immediate decision-making. The three domains are also in line with practical decision-making situations. Therefore, we propose an HF-3WD method in an HFDIS to solve practical decision-making problems.

Table 5
The hesitant fuzzy decision information system.

$\mathcal{U}$	I_1	I_2	…	I_n	d_1	…	d_p
o_1	u_{11}	u_{12}	…	u_{1n}	$d_1(o_1)$	…	$d_p(o_1)$
o_2	u_{21}	u_{22}	…	u_{2n}	$d_1(o_2)$	…	$d_p(o_2)$
⋮	⋮	⋮	…	⋮	⋮	…	⋮
o_m	u_{m1}	u_{m2}	…	u_{mn}	$d_1(o_m)$	…	$d_p(o_m)$
$\mathcal{W}$	W_1	W_2	…	W_n	⋇	…	⋇

4.2. The process of an HF-3WD method

In this section, we give the decision-making process of an HF-3WD method in an HFDIS.

Evaluations act as a regular work in various fields of modern society and an important basis for making scientific decisions. With the continuous expansion of research fields, individuals are faced with increasingly complex evaluation objects, if only based on a single index IS to evaluate things is often not reasonable, we should consider the problem from a comprehensive perspective, hence the multi-index IS came into being. From the form of Table 5, we know that an HFDIS is a multi-index IS. In an HFDIS, due to the different properties of each conditional attribute, it usually has different dimensions and magnitude orders. That is, the conditional attribute set I includes benefit-type attributes and cost-type attributes. In order to eliminate dimensions, we use a negative indicator to normalize, namely its complement operations. For instance, the expert group believes that the degree of energy saving of a certain project is $\{0.2, 0.4\}$. On the contrary, the expert group believes that the energy waste of the project is $\{0.8, 0.6\}$. Therefore, if the I_i is a cost type, the I_i^c is a benefit type. According to this method, Q is normalized.

After the normalization, we calculate the relative loss function obtained by u_{ij} $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$ by discussions in Section 3.2. We use the HFWA operator to aggregate the relative loss function of each object under each attribute so as to obtain the loss function of each object.

In an HFDIS, the lengths of each evaluation value u_{ij} are different. This is because the evaluation values of two experts are the same, it will be recorded only once. In the HF-3WD method, we first add some numbers to each evaluation value according to Definition 2.3, so that the length of each u_{ij} is equal to k. Then according to the (α, β)-probability dominance-similarity class of Definition 3.4, we calculate the (α, β)-probability dominance-similarity class of each object, and then estimate the conditional probability according to Definition 3.5.

By a psychological experiment (https://www.ydl.com), it is determined that the expert group is optimistic, pessimistic, and neutral. The questions of the psychology experiment include: When you plan to go for a picnic or barbecue, if it rains, will you still follow your plan? If there is an important appointment, do you go out early to prevent traffic jams or other situations? Will you buy insurance before boarding the plane? and so forth. According to the personality of expert groups, the aggregated relative loss function of each object is calculated by Definition 3.6. Finally, the expected losses of each object are calculated and classified according to the decision rules $(HP - HN)$ and Definition 3.7.

4.3. The key steps and algorithm of an HF-3WD method

According to the description in Section 4.2, we list the key steps of an HF-3WD method in an HFDIS to deal with practical problems as follows.

Input: An HFDIS $Q = \{\mathcal{U}, \mathcal{I} \bigcup d, \mathcal{V}, \mathcal{F}\}$, the risk aversion vector $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ and two parameters α, β.

Output: The decision results of each alternative $o_i (i = 1, \ldots, m)$.

Step 1: We conduct a psychological experiment test to determine whether DMs are optimistic, pessimistic, or neutral.

Step 2: For an HFDIS, we use a negative indicator to normalize, namely its complement operation.

Step 3: According to Definition 3.4, we calculate the (α, β)-probability dominance-similarity class of each alternative o_i with respect to $L \subseteq \mathcal{I}$.

Step 4: Based on Definition 3.5, the conditional probability of each alternative o_i can be estimated according to the (α, β)-probability dominance-similarity class.

Step 5: As discussed in Section 3.2, we calculate the relative loss function of each object under each attribute, and then aggregate with the help of the HFWA operator to obtain the relative loss function of each object.

Step 6: From the personality of DMs determined in **Step 1**, we calculate the aggregated relative loss function for each object on the basis of Definition 3.6.

Step 7: Calculate the expected loss value of each alternative o_i by formulas (3–1) to (3–3).

Step 8: With the support of decision rules $(HP) - (HN)$ and Definition 3.7, all objects are classified.

Step 9: Each object o_i is made a decision from the classification results.

Algorithm 1. The algorithm form of an HF-3WD method with three strategies in an HFDIS

Input: An HFDIS $Q = \{\mathcal{U}, \mathcal{I} \bigcup d, \mathcal{V}, \mathcal{F}\}$, the risk aversion vector $\eta = \{\eta_1, \eta_2, \ldots, \eta_n\}$ and two parameters α, β.
Output: The decision results of each alternative o_i $(i = 1, 2, \ldots, m)$.
begin
  First, we conduct a psychological experiment test for DMs.
  for $I_j \in \mathcal{I}$ **do**
    Normalize: u_{ij}, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.
  end
  for $o_i \in \mathcal{U}$ **do**
    compute the (α, β)-probability dominance-similarity class by Definition 3.4.
  end
  for $o_i \in \mathcal{U}$ **do**
    compute: the conditional probability based on Definition 3.5.
  end
  for $u_{ij} \in \mathcal{V}$ **do**
    compute: the relative loss functions according to Section 3.2.
  end
  for $o_i \in \mathcal{U}$ **do**
    compute: the HFWA operator method is used to aggregate the relative loss function.
  end
  for $o_i \in \mathcal{U}$ **do**
    determine: the aggregate loss function according to the personality of DMs.
  end
  for $o_i \in \mathcal{U}$ **do**
    compute: the expected loss value $E^{\triangleright}(A_\bullet|[o_i]_L^{(\alpha,\beta)})(\triangleright = opt, neu, pes, \bullet = P, B, N)$ by formulas (3-1) to (3-3).
  end
  for $o_i \in \mathcal{U}$ **do**
    judgment: with the support of decision rules $(HP) - (HN)$ and Definition 3.7
  end
  for $o_i \in \mathcal{U}$ **do**
    if $E^{\triangleright}(A_P|[o_i]_L^{(\alpha,\beta)}), E^{\triangleright}(A_B|[o_i]_L^{(\alpha,\beta)}), E^{\triangleright}(A_N|[o_i]_L^{(\alpha,\beta)})$, $(\triangleright = opt, neu, pes)$ *are not tied* **then**
      make decisions based on (HP-HN) decision rules.
    end
    if $E^{\triangleright}(A_P|[o_i]_L^{(\alpha,\beta)}) \sim E^{\triangleright}(A_B|[o_i]_L^{(\alpha,\beta)})$ *or* $E^{\triangleright}(A_B|[o_i]_L^{(\alpha,\beta)}) \sim E^{\triangleright}(A_N|[o_i]_L^{(\alpha,\beta)})$ **then**
      if $d(o_i) = 1$ **then**
        $o_i \in$ Pos(S) **else**
          $o_i \in$ Neg(S)
        end
      end
    end
  end
  for $o_i \in \mathcal{U}$ **do**
    decision: The decision result of each object.
    if $o_i \in Pos(S)$ **then**
      $o_i \Rightarrow A_P$.
    end
    if $o_i \in Bnd(S)$ **then**
      $o_i \Rightarrow A_B$.
    end
    if $o_i \in Neg(S)$ **then**
      $o_i \Rightarrow A_N$.
    end
  end
  return: The decision results of all objects.
end

In order to intuitively understand the decision-making process of an HF-3WD with three strategies in an HFDIS, we list the steps in the above algorithm, which is shown in Algorithm 1. With the representation of Algorithm 1, the logical relationship between each step is clear.

Remark 4.1. According to the above algorithm, we analyze the complexity of the HF-3WD method below. In Step 1, the complexity of k DMs is $O(k)$. The complexity caused by Step 2 is $O(n)$. The complexity of calculating Step 3 is $O(nm^2)$. The algorithm complexity of Step 4 is $O(2m)$. In Step 5, the complexity is $O(nm^2)$. In Step 6, the aggregated relative loss function complexity is $O(m^2)$. The complexity of calculating the expected loss in Step 7 is $O(3n)$. The algorithm complexity of Steps 8 and 9 are both $O(m)$. In summary, the complexity of HF-3WD approach with the three strategies in an HFDIS is $O(2nm^2)$.

5. Comparative analysis based on practical cases

In this section, we use the HF-3WD approach with three strategies to solve practical problems. Therefore, the structure of this section is arranged below. In Section 5.1, the rationality of the proposed method is verified by a peer-to-peer online lending platform (PTPOLP) case [16]. In Section 5.2, the superiority of the proposed method is demonstrated by an infectious diseases diagnosis (INDD) [29]. In Section 5.3, a realistic case from UCI data sets shows that the proposed method is better than other existing ones. In Section 5.4, we discuss the differences and advantages of the presented method over previous ones.

5.1. The HF-3WD method to a PTPOLP case

A PTPOLP is an internet finance (ITFIN) service website that includes lending. Online lending refers to the process of borrowing, in which materials, funds, contracts and procedures are all realized via the internet. It is a new financial model developed with the rise of private lending. Nowadays, PTPOLPs have been rapidly expanded in developed countries, such as the United Kingdom and the United States. This new type of financial management model has gradually been accepted by the public in the internet age. On the one hand, a lender realizes appreciations of assets. On the other hand, a borrower can use this convenient and quick way to meet his/her own capital needs. However in July 2011, Haidai website, "the most rigorous online lending platform in China", announced that it would be closed due to lack of funds. The China Banking Regulatory Commission immediately issued the "seven risks" warning, which even put the whole industry into a dire situation for a time. In response to the risk problem, Liang et al. [16] proposed a 3WD method with an error analysis by using two normalization methods with two strategies. In this section, we investigate a new HF-3WD approach with three strategies to the same PTPOLP case.

In the case of a PTPOLP case, the data collected by Liang et al. [16] include twelve platforms, four conditional attributes and one decision attribute. The specific content is shown in Table 8 of [16]. In the decision attribute, "1" represents reliable and "0" represents unreliable. According to the decision attribute, the state set $\mathcal{S} = \{S, \neg S\}$ can be divided, where S represents trustworthy, and $\neg S$ represents untrustworthy.

From Table 8 of [16], we find that the conditional attribute weight vector $\mathcal{W}$ of this HFDIS is unknown. We are not DMs and have no experience to give the value of the weight vector. In order to reduce the influence of uncertain information on decision-making, we use the nonlinear programming model based on the maximum deviation method proposed by Xu and Zhang [41] to calculate the objective weight vector of the PTPOLP in an HFDIS. The maximum deviation method assigns larger weight value to attribute with larger deviations. Since in multi-attribute decision-making problems, attributes with larger deviations play a greater role in decision-making process. In other words, if all objects have the same evaluation value under the same attribute, this attribute has little effect in the process of judging priority. The specific process of the maximum deviation method to obtain the weight vector is shown as follows:

$$\begin{cases} \max D(W') = \sum_{j=1}^{4}\sum_{i=1}^{12}\sum_{k=1}^{12}(W'_j D(u_{ij}, u_{kj})), \\ \text{s.t } W'_j \geqslant 0,\ j = 1,2,3,4, \sum_{j=1}^{4} W'^2_j = 1, \end{cases}$$

where $D(u_{ij}, u_{kj})$ is the Euclidean distance in Definition 2.5. The only solution that satisfies the above nonlinear programming conditions is:

$$W\prime_j = \frac{D_j}{\sqrt{\sum_{j=1}^{n} D_j^2}},$$

where $D_j = \sum_{i=1}^{12}\sum_{k=1}^{12} D(u_{ij}, u_{kj})$. Then in order to satisfy that the weight sum is equal to 1, W'_j $(j = 1,2,3,4)$ is standardized by the following formula:

$$W_j = \frac{W'_j}{\sum_{j=1}^{4} W'_j}.$$

With the support of MATLAB R2014a, the objective weight vector of the PTPOLP in an HFDIS can be calculated as:

$$\mathcal{W} = (0.26, 0.2451, 0.3137, 0.1812).$$

Assume that $\alpha = 0.75, \beta = 0.2$, the risk aversion vector is $\eta = (0.2, 0.29, 0.33, 0.4)$. According to the proposed HF-3WD algorithm, the expected loss values with the three strategies (opt, neu, pes) can be calculated. In order to compare the magnitude relationship between the expected loss values, we calculate the score values of the expected loss with the support of Definition 2.2. We present the $E^{\triangleright}(A_{\bullet}|[o_i]_L^{(\alpha,\ \beta)})(\triangleright = opt, neu, pes, \bullet = P, B, N)$ values of each object in the form of a figure. Assume

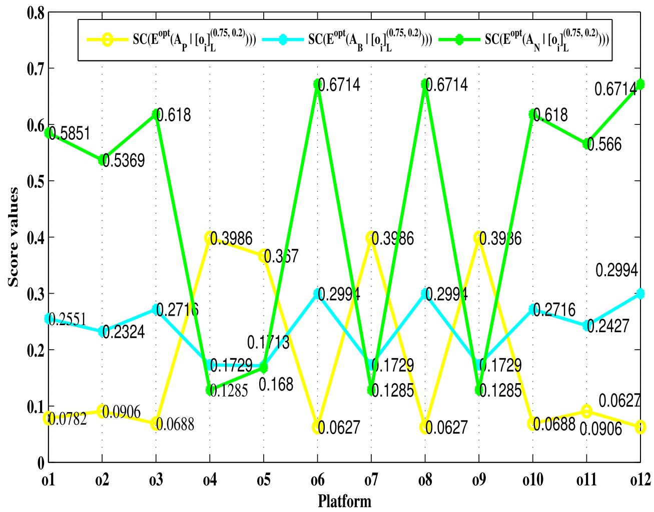

Fig. 3. The results of $SC(E^{opt}(A_\bullet|[o_i]_L^{(0.75,0.2)})), (\bullet = P, B, N)$ with optimistic attitudes for the PTPOLP.

that a DM's attitude is optimistic, the $SC(E^{opt}(A_\bullet|[o_i]_L^{(0.75,0.2)}))$ values are shown in Fig. 3. If a DM's attitude is neutral, the $SC(E^{neu}(A_\bullet|[o_i]_L^{(0.75,0.2)}))$ values are shown in Fig. 4. If a DM's attitude is pessimistic, the $SC(E^{pes}(A_\bullet|[o_i]_L^{(0.75,0.2)}))$ values are shown in Fig. 5.

According to the results of Figs. 3–5, and the decision rules $(HP) - (HN)$, we can see that the classification results of the PTPOLP with three strategies are $Pos(S) = \{o_1, o_2, o_3, o_6, o_8, o_{10}, o_{11}, o_{12}\}, Bnd(S) = \emptyset, Neg(S) = \{o_4, o_5, o_7, o_9\}$.

In the 3WD method with an error analysis proposed by Liang et al. [16], when $\alpha = 0.2$, we calculate the classification results under the four methods as $Pos(S) = \{o_1, o_2, o_3, o_6, o_8, o_{10}, o_{11}, o_{12}\}, Bnd(S) = \emptyset, Neg(S) = \{o_4, o_5, o_7, o_9\}$.

By the analysis of the PTPOLP case, we find that the classification results of the proposed method with the three strategies and the four methods of Liang et al. [16] are the same when $\alpha = 0.2$. Therefore, our proposed HF-3WD method with three strategies is reasonable. Because the PTPOLP case data are special, this case is not enough to prove that the proposed method is superior to the method of Liang et al. [16]. Thus in the following content, we prove the advantages of the proposed HF-3WD method with three strategies by an INDD case, Algerian forest fires case and a series of experiments.

5.2. The HF-3WD model to an INDD case

In this section, the superiority of the HF-3WD method with three strategies is illustrated by an INDD case of Wang et al. [29].

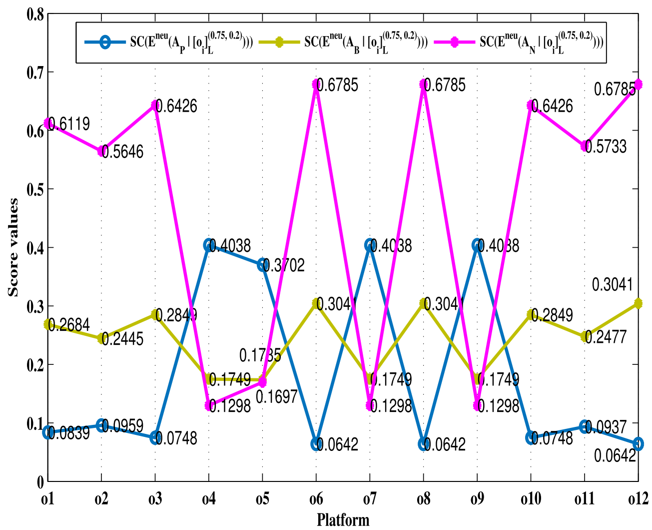

Fig. 4. The results of $SC(E^{neu}(A_\bullet|[o_i]_L^{(0.75,0.2)})), (\bullet = P, B, N)$ with neutral attitudes for the PTPOLP.

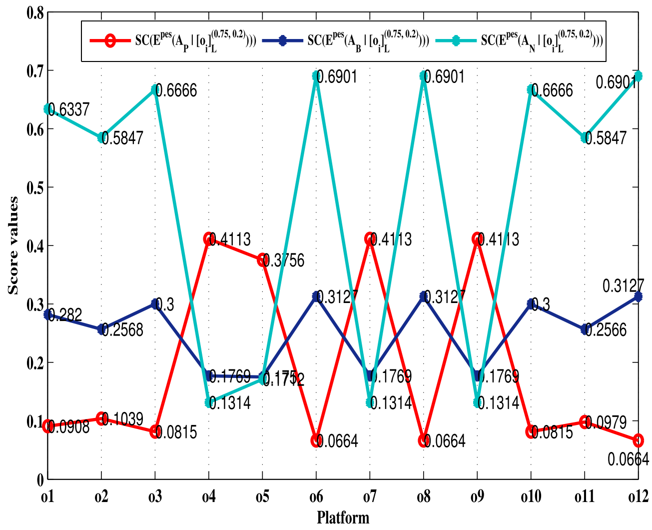

Fig. 5. The results of $SC(E^{pes}(A_\bullet|[o_i]_L^{(0.75,0.2)})), (\bullet = P, B, N)$ with pessimistic attitudes for the PTPOLP.

According to the data provided by the National Bureau of Statistics (https://data.stats.gov.cn), we know that from 0:00 to 12, January 1, 2019 At 24:00 on the 31st, the total number of Chinese citizens with Class A and B legally reported infectious diseases was 3072337 people. The total number of cases over the past five years is shown in Fig. 6. The number of infectious patients accounts for a certain proportion. The research, diagnosis, and treatment of infectious diseases are urgent. Chen et al. [2] proposed the application of the human–computer collaboration model to the medical field to improve the efficiency of diagnosis. It is very important to apply the proposed mathematical model to the diagnosis stage. Next, we apply the proposed HF-3WD method with three strategies and other 3WD methods [16,13,29,50,5] to the data in the medical field.

In [29], there are three DMs, eight patients and five symptoms. The weight vector is $\mathcal{W} = (0.3, 0.2, 0.13, 0.2, 0.17)$. Suppose that the state set $S = \{o_1, o_2, o_4, o_5\}$ is an "infectious disease confirmed" set, the parameters $\alpha = 0.5, \beta = 0.4$ and the risk aversion vector is $\eta = (0.17, 0.2, 0.19, 0.21, 0.15)$. The decision attribute set is a set of key indicators used to detect the classification ability of the method. The classification results of the proposed HF-3WD method with three strategies and other methods [16,13,29,50,5] are shown in Table 6.

Remark 5.1. From the results in Table 6, we can see that our proposed HF-3WD method with three strategies achieves three-way classification, which conforms to practical decision-making processes and reduces the risk of decision errors. The proposed HF-3WD method is also more sensitive with different strategies. The other 3WD methods [16,13,29,50,5] are proposed under the fuzzy information or the HF one. In what follows, we analyze the reasons for the results in Table 6.

The $Bnd(S)$ of Liang et al.'s four methods [16] is always an empty set. This further shows that the method is insufficient in the 3WD theory and the risk attitude research. Liang and Liu's method [13] is highly uncertain. The conditional probability and loss function of the object are subjectively given. Besides, the loss function between each object is not consistent with reality. The result of this decision will change based on a DM's perception. Similarly, the results of Wang et al.'s method [29] are also subjectively affected by the loss function.

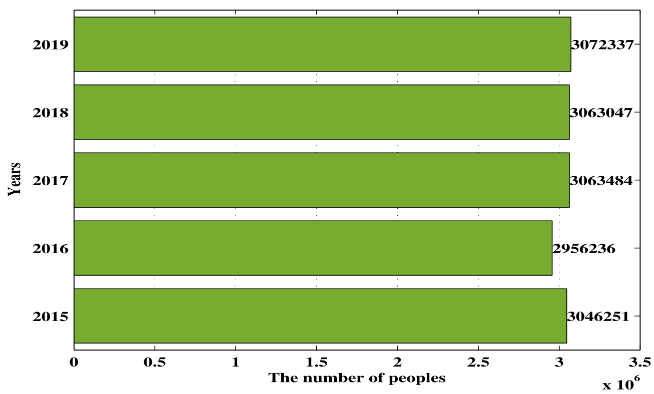

Fig. 6. Total number of cases in the past five years.

Table 6
The classification results of each method in the INDD case.

Method	$Pos(S)$	$Bnd(S)$	$Neg(S)$
Proposed HF-3WD method (optimistic attitude)	o_1, o_2, o_4	o_3, o_5, o_8	o_6, o_7
Proposed HF-3WD method (neutral attitude)	o_1, o_2, o_4, o_5	o_8	o_3, o_6, o_7
Proposed HF-3WD method (pessimistic attitude)	o_1, o_2, o_4, o_5	o_8	o_3, o_6, o_7
Liang et al.'s method [16] (Method 1)	o_2, o_4, o_5	$\varnothing$	o_1, o_3, o_6, o_7, o_8
Liang et al.'s method [16] (Method 2)	o_4, o_5	$\varnothing$	$o_1, o_2, o_3, o_6, o_7, o_8$
Liang et al.'s method [16] (Method 3)	o_2, o_4, o_5	$\varnothing$	o_1, o_3, o_6, o_7, o_8
Liang et al.'s method [16] (Method 4)	o_4, o_5	$\varnothing$	$o_1, o_2, o_3, o_6, o_7, o_8$
Liang and Liu's method [13]	o_1, o_2, o_4	o_3, o_5, o_7, o_8	o_6
Wang et al.'s method [29]	o_1, o_5	o_2, o_4	o_3, o_6, o_7, o_8
Zhan et al.'s method [50] (optimistic attitude)	$\varnothing$	o_1	$o_2, o_3, o_4, o_5, o_6, o_7, o_8$
Zhan et al.'s method [50] (neutral attitude)	o_1	o_6	$o_2, o_3, o_4, o_5, o_7, o_8$
Zhan et al.'s method [50] (pessimistic attitude)	o_1	$\varnothing$	$o_2, o_3, o_4, o_5, o_6, o_7, o_8$
Jia and Liu's method [5]	o_1, o_4	o_2, o_3, o_5, o_8	o_6, o_7

For other methods [50,5], we first use the score function (Definition 2.2) to convert the HF information into fuzzy information. Second, we apply the methods of Zhan et al. [50], Jia and Liu [5] to the INDD case. The loss function of the object in Zhan et al.'s method is subjectively given. The conditional probability in Jia and Liu's method [5] is determined by DMs. Finally, we can know that both methods have uncertain factors, that is, the subjective preference of DMs affects the decision result. When the preferences of the DM change, the decision-making results are also different.

Therefore, our constructed HF-3WD method with three strategies can solve practical decision-making problems.

5.3. The HF-3WD method to an Algerian forest fires case

In order to further demonstrate the merits of the designed HF-3WD method with three strategies, we do another comparative analysis by a data set downloaded from the UCI in this section.

The data set includes 12 attributes (11 condition attributes and 1 decision attribute) and 244 instances that regroup a data of two regions of Algeria, namely the Bejaia region located in the northeast of Algeria and the Sidi Bel-abbes region located in the northwest of Algeria. Each region includes 122 instances. After consulting related literature, we find that the Fine Fuel Moisture Code (FFMC) index, the Duff Moisture Code (DMC) index, the Drought Code (DC) index, the Initial Spread Index (ISI), the Buildup Index (BUI) index and the Fire Weather Index (FWI) can be measured by the temperature noon, the relative humidity, the wind speed and the rain index, respectively. Thus, in order to maintain the consistency of the data, we process the conditional attributes of this data set.

First, these data types of conditional attributes are converted to linguistic variables. According to the literature, when the temperature is less than $-10\ ^{\circ}$C, there is generally no fire, and the corresponding language variable is "low", etc. Then according to Table 1 of Mardani et al. [22], the corresponding hesitant fuzzy evaluation values are given. For example, "low" corresponds to [0.15, 0.3]. Finally, we apply the proposed method and other ones to this dataset. The classification results of each method are shown in Fig. 7.

For the sake of convenience, the thirteen methods mentioned in Section 5.2 are abbreviated as Methods 1 to 13. The proposed method with optimistic attitudes is Method 1, and Method 13 represents the Jia and Liu's method [5].

Remark 5.2. From the classification results of each method in Fig. 7, we once again verify the conclusion of Remark 5.1. It can be seen from the results in Fig. 7 that the proposed method is more sensitive to the psychological characteristics of DMs, and effectively implements the three classifications of objects. Under different risk preferences, the classification results of Liang et al. 's four methods (Methods 4 to 7) [16] are consistent, and the $Bnd(S)$ remains an empty set. Similar results are

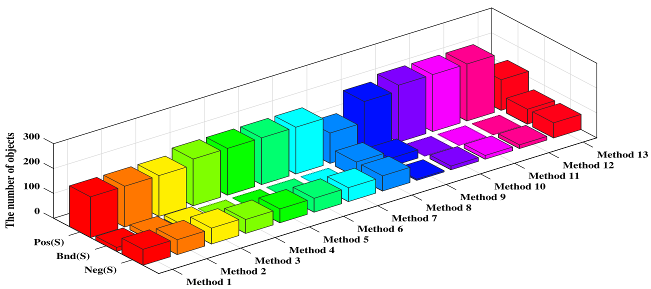

Fig. 7. The classification results of each method in Algerian forest fires case.

Zhan et al.'s method (Methods 10 to 12) [50]. As stated in Remark 5.1, the classification results are not unique because of the deficiencies and subjectivity of other methods [13,29,50,5].

5.4. Discussion

Based on the three specific case studies in Sections 5.1–5.3, we first analyze the differences between the proposed method and other ones [16,13,29,50]. Then, we summarize the advantages of our proposed HF-3WD method with three strategies.

The differences between the given method and others are listed as follows:

(1) Under the basic framework of 3WD in an HF environment, Liang et al. [16] estimated the conditional probability based on the similarity class. In the Liang and Liu's method [13], the conditional probability is subjectively given by DMs. Wang et al. [29] used the ELECTRE outranked class. In a fuzzy environment, Jia and Liu's method [5] is given based on subjective social experience. However, we estimate the conditional probability in an HFDIS by using the (α, β)-probability dominance-similarity class.

(2) The loss functions of each object in the methods of Liang et al. [16], Wang et al. [29] and Zhan et al. [50] are subjectively given. The relative loss function in Jia and Liu's method [5] is studied in a fuzzy environment. However, the loss functions of each object in our proposed HF-3WD method with three strategies are estimated by the relative loss functions based on the HFDIS.

(3) Zhan et al.'s method [50] and Jia and Liu's one [5] are all proposed under fuzzy background, thus the loss function and the expected loss value are both fuzzy numbers. With the support of the error analysis theory, Liang et al. [16] aggregated the loss function and determined the expected loss value of two interval formats. However, our proposed HF-3WD method with three strategies aggregates the loss function through the HFWA operator, and determines the aggregated relative loss function according to the risk attitudes of DMs. The expected loss value is a hesitant fuzzy number (HFN).

By the description of the differences between the above methods, the merits of the designed HF-3WD method with three strategies are shown as follows:

(1) In an HFDIS, the conditions of the equivalence class are too strict, thus Liang et al. [16] proposed a similarity class, which is measured by the standard of distances. In an HFDIS, the similarity relation between two objects under multiple conditional attributes has the problem of average distances. That is, the similarity degree between o_i and o_j under the attribute I_k is higher, and the similarity degree is lower under the attribute I_l. By averaging the distance, it is concluded that there is a similar relationship between o_i and o_j. In real life, if o_i is too much worse than o_j at a certain level, we do not recognize the relationship between o_i and o_j. There are some problems in measuring the relationship between o_i and o_j by means of the similar relations. Furthermore, since the outranking relation of the ELECTRE method is determined by attribute weights, Wang et al.'s method [29] and Zhan et al.'s one [50] ignore the dominance value between objects. The (α, β)-probability dominance-similarity relation can be further in line with practical situations by adding the probability dominance relation. This (α, β)-probability dominance-similarity relation avoids the subjectivity of Jia and Liu's method [5]. In particular, α and β also make the conditions not too strict.

(2) The subjective willingness of the loss function in Liang et al.'s method [16], Wang et al.'s one [29] and Zhan et al.'s one [50] is too strong. The relative loss function proposed by Jia and Liu [5] cannot be directly applied to the HF environment. However, the loss function of our proposed HF-3WD method with three strategies is derived from the evaluation value of each object for each attribute. The risk aversion vector η represents the subjective willingness of DMs. Therefore, our proposed HF-3WD method with three strategies considers subjective preferences and does not completely rely on DMs.

(3) In the psychological test, the individual personality is divided into three types, namely, optimistic, pessimistic and neutral. The method of Liang et al. [16] only considers optimism and pessimism, and the proposed HF-3WD method considers three strategies. Through the study of the above-mentioned practical cases, the method of Liang et al. [16] always maintains the same classification results under different strategies, and our HF-3WD method with three strategies is sensitive to the preferences of DMs.

The above analysis shows that the proposed HD-3WD method with three strategies can reasonably and effectively deal with practical decision-making problems in an HFDIS.

6. Experimental evaluations

The parameters of the HF-3WD model proposed in this paper include the probabilistic dominance degree α, the similarity degree β and the risk aversion vector η. We explore the influence of these parameters on the HF-3WD model with three strategies by experiments. In the discussion part of the previous section, we mention that the data sizes themselves have some influences on the CER. Secondly, we select a data set of the UCI database (http://archive.ics.uci.edu/ml/index.php) for the CER experiment.

6.1. The impact of a risk aversion vector η on classification results

The risk aversion vector η reflects the situation of risk aversions for DMs. The value of each η_j impacts on the classification result. Therefore, this section uses the case in Section 5.2 to illustrate the impact of η on the classification results.

The value range of the risk aversion vector η is [0,1]. Let $\alpha = 0.5, \beta = 0.4$, the classification results of the INDD case with three strategies η from 0 to 1 are shown in Fig. 8.

From the classification results for the INDD with the three strategies in Fig. 8, we can see that there are the following rules:

(1) As η increases, the domains $Pos(S)$ and $Neg(S)$ do not decrease.

(2) As η increases, the domain $Bnd(S)$ does not increase.

This is the same rules discussed in [5,19].

6.2. Experiments on parameters α and β

In this paper, we define a new binary relation, namely, the (α, β)-probability dominance-similarity relation. We use the (α, β)-probability dominance-similarity class to estimate the value of the conditional probability (α, β)-probability dominance-similarity relation. Therefore, this section will first study the influence of values of α and β on $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$. Second, we calculate the fluctuation of the CER by changing the values of α and β. In this section, we also use the INDD case studies.

6.2.1. The impact of α and β on the $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$

In an HFDIS, $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$ is estimated with the aid of the (α, β)-probability dominance-similarity class. From Definition 3.4, we know that $\alpha \in [0.5, 1]$ represents the degree of the probability dominance and $\beta \in [0, 1]$ represents the degree of similarity. In this section, we use the INDD case to explore the influence of values of α and β on the $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$ $(i = 1, 2, \ldots, 8)$. In the INDD case, we set α and β with 0.1 as the step size, and show the $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$ change of each object o_i in Fig. 9.

From the changes in the $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$ of each patient o_i in Fig. 9, we can find that when α becomes larger and β becomes smaller, the $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$ image of each o_i tends to be flat. This is because the larger the α, the smaller the β, the stricter the restriction according to Definition 3.4. Therefore, the trend of $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$ for each o_i is in line with the theory. In practical decision-making problems, it is very important for DMs to give appropriate values of α and β.

6.2.2. The impact of α and β on the CER

In Section 4.2, we discuss the influence of α and β on the $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$. In the basic theory of 3WD, we know that the value of $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$ will affect the classification result of the decision. The CER value is determined by the classification results. Therefore, in this section we calculate the CER values under different values of α and β.

In the case of the INDD, Fig. 10 is the CER results with the three strategies.

From the CER results in Fig. 10, we have the following conclusions:

- In an HFDIS, the CER values under the neutral attitude is always not greater than that under the optimistic and pessimistic attitude. This is consistent with practical decision-making situations. A DM's decision result is more accurate under the neutral attitude, and the CER value is also lower.
- The same as the change rule of $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$, when α becomes larger and β becomes smaller, the CER values with three strategies will decrease. When the restriction conditions become stricter, the decision results are more rigorous, and the CER values become smaller.

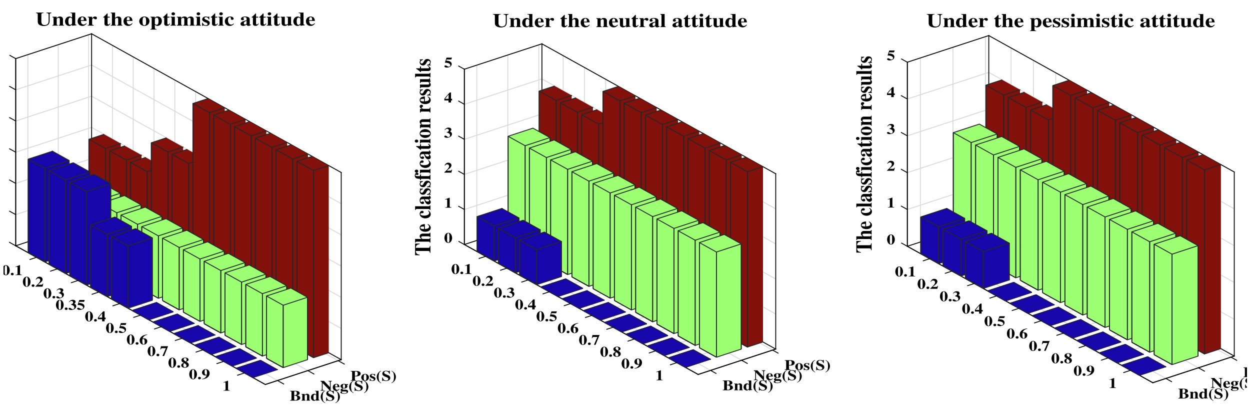

Fig. 8. The classification results for an "infectious disease diagnosis" case with three strategies.

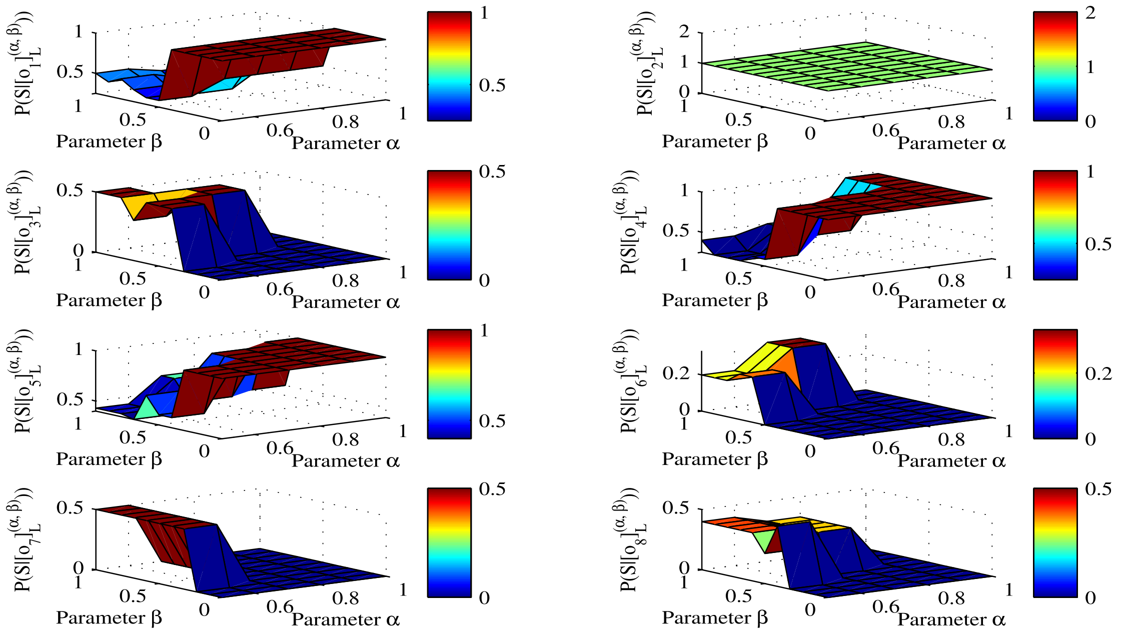

Fig. 9. The changing of $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$ for the "infectious disease diagnosis" case with three strategies.

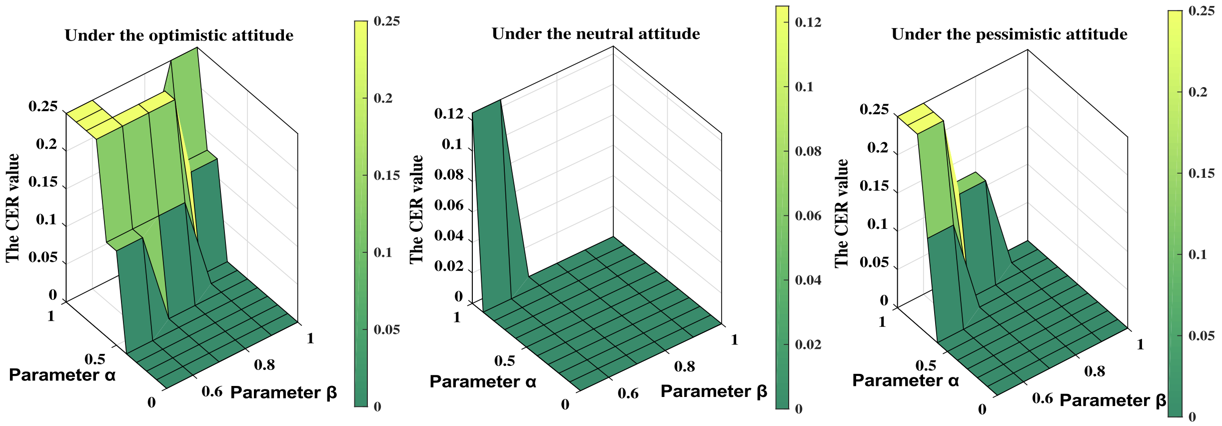

Fig. 10. The fluctuation of the CER for "infectious disease diagnosis" case under three strategies.

By the above experimental analyses, the proposed HF-3WD method with three strategies is consistent with practical situations. The CER value is also small, thus the proposed HF-3WD method with three strategies is reasonable and effective, and can solve practical HF decision-making problems.

6.3. Experimental analysis of the CER and CPR

The above experiments in Sections 5.1 and 5.2 show the rationality of the proposed HF-3WD method with three strategies. In order to verify that the proposed HF-3WD method with three strategies can solve the problem better than the existing 3WD ones [16,13,29,50,5], we create the CER and CPR experiments. This experiment is implemented in MATLAB R2014a on a personal computer.

The proposed HF-3WD method with three strategies aims at solving practical problems. The CER and CPR values are important indicators to measure the performance of a decision-making method. Therefore, we calculate the CER value and CPR value of each method under the INDD case and the Algerian forest fires case. The experimental results are shown in Figs. 11 and 12.

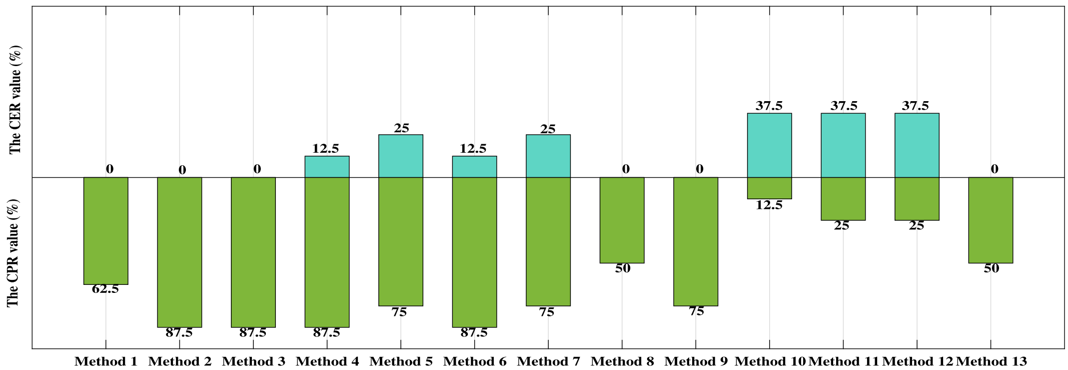

Fig. 11. The CER and CPR results of each method in the INDD case.

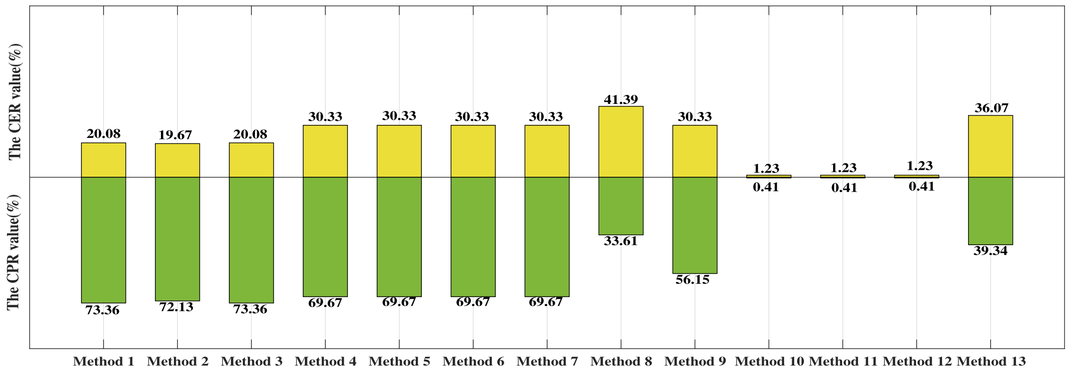

Fig. 12. The CER and CPR results of each method in Algerian forest fires case.

Remark 6.1. (1) By comprehensively considering the CER and CPR results in Fig. 11, we know that the value of CER in the proposed method is 0%, the value of CPR is 62.5% under an optimistic attitude, and 87.5% under neutral and pessimistic attitudes. This reveals that the risk attitude and a DM's cognitive preference will affect decision-making result. The rational decision-making can improve the performance of the decision-making method. Liang ang Liu's method [13], Wang et al.'s one [29] and Jia and Liu's one [5] do not take into account the psychological characteristics of DMs.

(2) There are many factors that affect the CER and CPR, such as data manipulation, the size of data volume, subjective preferences, data collection errors, etc. In the above case, if other influences are not considered, by comparing the CER in Fig. 12, our proposed HF-3WD method with three strategies is not greater than other methods [16,13,29,50]. In the same way, the CPR value of our method in Fig. 12 is not lower than that of other ones [16,13,29,50].

From the above analysis and evaluation, these experimental results illustrate the merit of our proposed method in relation to the existing 3WD solutions [16,13,29,50,5].

7. Conclusions

In this paper, we have proposed an HF-3WD approach under three strategies in an HFDIS. This method further promotes the application ranges of 3WD in an HFDIS. We have applied the HF-3WD method with three strategies to some existing practical cases. By comparing with Liang et al.'s method [16], the rationality and superiority of the HF-3WD method with three strategies have been proven. After a series of experimental analyses, it has been shown that the proposed HF-3WD method with three strategies is stable.

Therefore, the main contributions of this paper are summarized as follows:

(1) Based on the work of Liang et al. [16], we have defined a new binary relationship, namely, an (α,β)-probability dominance-similarity relation in an HFDIS. Then we have estimated the conditional probability $\mathcal{P}(S|[o_i]_L^{(\alpha,\ \beta)})$ via the (α,β)-probability dominance-similarity class.

(2) According to the development of the relative loss function in different environments [5,19], we have introduced the relative loss function in an HFDIS.

(3) In an HFDIS, we have proposed the aggregated relative loss function with three strategies based on the (α,β)-probability dominance-similarity class.

(4) We have established the HF-3WD method with risk strategies and solved practical decision-making problems. Through the comparative and experimental analyses, we have verified the rationality and effectiveness of the presented method.

By the research of this paper, we find that there are some topics worthy of further study as follows:

(1) Under the basic framework of the 3WD theory, we can consider the application of dynamic 3WD models in an HFDIS [24,1].

(2) In an HFDIS, a realistic decision-making problem of incomplete data is worth studying [20].

(3) The utility value of the utility theory [4], the prospect value of the prospect theory [32], the regret value of the regret theory [31] and attribute reductions [28,26] are worthy of further exploring in an HFDIS.

(4) The decision-theoretic five-way approximation of HFSs is a direction that can be further investigated [36].

CRediT authorship contribution statement

Jiajia Wang: Conceptualization, Methodology, Investigation, Writing - original draft. **Xueling Ma:** Methodology, Investigation, Writing - original draft. **Zeshui Xu:** Writing - review & editing. **Jianming Zhan:** Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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