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A decision-theoretic fuzzy rough set in hesitant fuzzy information systems and its application in multi-attribute decision-making

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ABSTRACT

Decision-theoretic rough sets (DTRSs) as a classic model of three-way decisions have been widely applied in the field of risk decision-making. Considering situations where experts hesitate among several evaluation values, hesitant fuzzy sets, as a new generalization of fuzzy sets, can describe uncertain information flexibly in the decision-making process. In this paper, we propose a decision-theoretic fuzzy rough set (DTFRS) model in hesitant fuzzy information systems and discuss its application in multi-attribute decision-making (MADM). More specifically, we first define a novel fuzzy binary relation between two objects by using the hesitant fuzzy distance function. Then, we study the calculations of the fuzzy similarity class and the conditional probability. At the same time, based on the connection between the loss functions and the attribute values, we develop a data-driven calculation method of the relative loss functions. With these discussions, we construct a DTFRS model in hesitant fuzzy information systems and explore the related decision-making mechanism. Furthermore, a three-way decision method based on the proposed DTFRS model is established to handle MADM problems in the context of a hesitant fuzzy environment. The established method not only takes the decision risk into consideration, but also instructs us how to choose the action for each alternative and gives its corresponding semantic explanation. An illustrative example of the stock investment problem is presented to verify the efficacy of our method. Finally, we take a sensitivity analysis and a comparison analysis to show the established method's performance and characteristics.

1. Introduction

Nowadays, due to the increasing complexity and uncertainty of decision-making problems, research on the theories and methodologies of decision-making under uncertainty has caught extensive attention from diverse domains, and various methods for solving decision-making issues in uncertain environment have been proposed [28,36]. This paper details our proposed DTFRS model in hesitant fuzzy information systems and its application to MADM problems. The proposal comes from our work combining three-way decision theory [43–45] with hesitant fuzzy sets (HFSs) [37,38]. In what follows, we briefly review some related studies of DTRSs and HFSs.

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1.1. A short retrospect on the development of DTRSs

In light of Bayesian decision procedure, Yao [43] proposed DTRSs, which bridge the rough set theory and the risk decision theory. DTRSs can generate three kinds of decision rules through the minimum expected risk, namely, acceptance decisions, uncertainty decisions and rejection decisions [35]. With the introduction of DTRSs, they have attracted the attention of many researchers and have been used in many domains, such as feature selection [11], knowledge granulation [10,31,39], conflict analysis [17,13], pattern recognition [32], formal concept analysis [16], and so on. The studies of DTRSs mainly include two aspects: one is the accurate and reasonable expression of the loss functions, and the other is the effective computation and estimation of the conditional probability [22].

Expression of the loss functions using crisp values is difficult for decision-makers because of the complexity and uncertainty found in realistic decision-making situations. To handle this problem, Zhao and Hu [50] considered fuzzy sets and interval-valued fuzzy sets as the evaluation forms of the loss functions, and introduced fuzzy and interval-valued fuzzy DTRS models. Liang et al. [21] proposed a new DTRS model by measuring the loss functions with intuitionistic fuzzy sets. Liang et al. [22] also took into account Pythagorean fuzzy sets as a new evaluation format of the loss functions and explored a Pythagorean fuzzy DTRS model. Sun et al. [34] utilized linguistic terms to represent the loss functions and established a decision-theoretic rough fuzzy set model. Abdel-Basset et al. [1] used neutrosophic sets to denote the loss functions and constructed a corresponding DTRS model. Tang et al. [36] used q-rung orthopair fuzzy sets to depict the loss functions and developed a q-rung orthopair fuzzy DTRS model. Recently, Liang and Liu [19] discussed a new DTRS model by using HFSs as an expression form of the loss functions. However, these studies do not consider the differences of objects and do not give a computational method of the loss functions. Fortunately, under a multi-attribute environment, Jia and Liu [12] proposed a novel three-way decision model by utilizing attribute values to derive the relative loss functions. Liu et al. [26] also used the attribute values to calculate the relative loss functions under an intuitionistic fuzzy multi-attribute environment. Similarly, Liang et al. [20] and Lei et al. [14] studied the relative loss and benefit functions induced from attribute values with interval type-2 fuzzy sets and hesitant fuzzy linguistic term sets, respectively.

The conditional probability in most DTRS models is calculated using the information granule in decision information systems, and one of the preconditions for determining the conditional probability is the decision attribute [22,26]. For example, Liu and Liang [24] used a dominating (or dominated) class of objects to compute the conditional probability in an ordered information system. Liu et al. [25] utilized an L-level similarity class of objects to arrive at the conditional probabilities in an incomplete information system. However, we may confront a universal circumstance in the actual decision environment that the information system does not have the class label or the decision attribute. For instance, in many MADM problems [5], there are no decision attributes and only conditional attributes in information tables. To address this issue, Liang et al. [22] utilized the TOPSIS method to calculate the conditional probability under a Pythagorean fuzzy MADM environment. Recently, Liu et al. [26] effectively used grey relational analysis to estimate the conditional probability in an intuitionistic fuzzy MADM problem.

1.2. A short overview on the development of HFSs

The notion of HFSs was initially introduced by Torra [37], which can effectively describe the uncertainty of complicated problems and the vagueness of people's cognition. Ever since the introduction, HFSs have developed quickly both in theory and application. In theory, Xia and Xu [38] proposed some hesitant fuzzy aggregation operators. Xu and Xia [40] investigated diverse distance and similarity measures of HFSs. Liao et al. [23] discussed the multiplicative consistency of hesitant fuzzy preference relations. Recently, Hu [9] pointed out that there is a lack of mathematical rigor in logic operation definition of HFSs and revised these operations. In application, Ebrahimpour and Eftekhari [3] introduced an innovative method based on HFSs to deal with feature selection on the high dimensional data sets. Sun et al. [33] developed a new approach to pattern recognition issues with HFSs using grey relational analysis. Xu and Zhang [41] explored a new MADM method with the aid of TOPSIS in the hesitant fuzzy context and discussed its application to energy policy selections. Xia and Xu [40] utilized hesitant fuzzy aggregation operators to fuse uncertain information in MADM problems. Feng et al. [4] proposed an MADM method using possibility theory under hesitant fuzzy linguistic environment and studied its application in investments.

In recent years, research on hybrid models by combining HFSs [37] with rough set theory [29,30] has become an increasingly important branch of HFS theory. In 2014, Yang et al. [42] introduced the concept of hesitant fuzzy rough sets based on the constructive approach and the axiomatic approach. In light of the fact that the performance of two different but highly related universes is better than that of a single universe for describing the actual decision-making problems, Zhang et al. [49] proposed a hesitant fuzzy rough set over two universes and studied its applications in decision-making. In 2015, Liang and Liu [19] considered the loss functions of DTRS models with hesitant fuzzy values (HFVs) to construct a hesitant fuzzy DTRS model and studied a risk decision-making method. In the same way, Li and Huang [15] utilized HFVs to characterize the cost and revenue functions in investment decision problems and further developed a hesitant fuzzy three-way investment decision model. In 2020, Zhang et al. [47] studied a multi-granularity DTRS over two universes in the hesitant fuzzy linguistic context and applied it to person-job matching problems. Similarly, Zhang et al. [48] proposed an interval-valued hesitant fuzzy multi-granularity DTRS over two universes by integrating interval-valued hesitant fuzzy sets, multi-granularity rough sets over two universes and DTRSs. Recently, considering decision-makers' bounded rationality and criteria interaction, Lei et al. [14] established a behavioral hesitant fuzzy linguistic multi-granularity DTRS over two universes using prospect theory and Choquet integral, then they established a three-way group decision method to solve green supplier selection problems.

1.3. The motivations of this paper

In this paper, we develop a DTFRS model in the hesitant fuzzy information system by combining DTRSs with HFSs and discuss how it can be applied to MADM problems under hesitant fuzzy environment. The motivations of this paper are summarized as follows:

(1) Uncertainty and complexity can cause an increasingly risk impact on our decision-making processes. Meanwhile, experts may hesitate among several evaluation values for lack of information and the uncertainty that persists in people's cognition. In this situation, we encounter the following two challenges: (a) How to represent the evaluation values of objects reasonably and accurately by considering experts' knowledge and understandings. (b) How to establish a scientific and effective decision model to reduce decision risks and explain the decision results objectively. As a solution to these challenges, we utilize HFVs [40] to represent evaluation values in information systems. Then, we propose a DTFRS model in the hesitant fuzzy information system, which can consider decision risks and direct us regarding the selection of each object's decision action.
(2) The loss functions in traditional DTRS models arise straightly and are constant values for all objects. Based on the results in [12,14,20,26], we develop a data-driven approach to compute the relative loss functions of each object using attribute evaluation values. In addition, in most DTRS models, the conditional probability is calculated by equivalence relations in information tables, which seems to be a stringent condition that may limit the applicability of DTRS models. Therefore, to overcome this limitation, we define a new fuzzy similarity relation by utilizing the hesitant fuzzy distance function, which can depict the relationships among objects flexibly.
(3) The traditional MADM methods are all bulit based on two-way decision theory, which illustrates that the decision result is an either-or matter. Unfortunately, this result can be oversimplified because it can often ignore the necessity of further testing to obtain a final decision result beyond the either-or dichotomy in practical decision-making processes. Therefore, we need to design a new model for MADM. The intuitive approach is to add uncertainty to the final decision results for further investigation. This idea coincides with the thought of three-way decisions [43]. Therefore, we utilize the proposed DTFRS model to establish a three-way decision method to solve MADM problems under hesitant fuzzy environment, dividing all alternatives into three decision regions and obtaining corresponding decision actions objectively.

The remainder of this paper is organized as follows. Section 2 briefly reviews the theory of DTRSs as well as the basic concepts of HFSs. In Section 3, we develop a DTFRS model in hesitant fuzzy information systems based on DTRSs and HFSs. In Section 4, we use the proposed DTFRS model to establish a three-way decision method to MADM under hesitant fuzzy environment. Section 5 presents a stock investment example to show the application of our proposed three-way decision method and verifies its effectiveness by a sensitivity analysis and a comparison analysis. In Section 6, we conclude our work and sketch a plan for future research.

2. Preliminaries

In this section, for the convenience of readers, we give some basic definitions and concepts that are used throughout the paper.

2.1. Decision-theoretic rough sets

Let U be a non-empty finite universe and $R \subseteq U \times U$ be an equivalence relation. Then, U can be parted by R, denoted as $U/R = \{[x]_R | x \in U\}$, and the pair (U, R) is called a rough approximation space [29,43].

Definition 2.1 [43]. Let (U, R) be a rough approximation space. For a subset $X \subseteq U$ and a pair of parameters (α, β) with $0 \leqslant \beta < \alpha \leq 1$, the probabilistic upper and lower approximations of X regarding (U, R), represented as $\overline{R}^{\beta}(X)$ and $\underline{R}^{\alpha}(X)$, respectively, are defined in the following:

$$\overline{R}^{\beta}(X) = \{x \in U | Pro(X|[x]_R) > \beta\}, \tag{1}$$
$$\underline{R}^{\alpha}(X) = \{x \in U | Pro(X|[x]_R) \geqslant \alpha\}, \tag{2}$$

where $Pro(X|[x]_R) = \frac{|[x]_R \cap X|}{|[x]_R|}$ is the conditional probability of $[x]_R \subseteq X$ and $|\bullet|$ denotes the cardinality of a set.

The probabilistic upper and lower approximations of X can deduce three decision regions, i.e., the acceptance region $Acc(X)$, uncertainty region $Unc(X)$ and rejection region $Rej(X)$:

$$Acc(X) = \{x \in U | Pro(X|[x]_R) \geqslant \alpha\}, \quad (3)$$
$$Unc(X) = \{x \in U | \beta < Pro(X|[x]_R) < \alpha\}, \quad (4)$$
$$Rej(X) = \{x \in U | Pro(X|[x]_R) \leqslant \beta\}. \quad (5)$$

To give semantic interpretations of the thresholds and three regions in probabilistic rough sets, Yao [43] proposed decision-theoretic rough sets (DTRSs) with the aid of Bayesian decision procedure. Assume that $St = \{X, \neg X\}$ denotes a state set of objects, which indicates that an object x is in X and not in X; $Ac = \{a_A, a_U, a_R\}$ stands for actions set, where a_A, a_U and a_R represent $x \in Acc(X), x \in Unc(X)$ and $x \in Rej(X)$, respectively. In addition, assume that $\lambda_{AP}, \lambda_{UP}$, and λ_{RP} indicate the loss functions of actions a_A, a_U and a_R, respectively, when $x \in X$; $\lambda_{AN}, \lambda_{UN}$, and λ_{RN} indicate the loss functions of the same actions when $x \notin X$. Generally, the loss functions satisfy $\lambda_{AP} \leqslant \lambda_{UP} < \lambda_{RP}$ and $\lambda_{RN} \leqslant \lambda_{UN} < \lambda_{AN}$ according to a reasonable semantic interpretation. Then, for any object $x \in X$, the expected losses $\mathscr{L}(a_\star|[x]_R)$ $(\star = A, U, R)$ associated with taking the three actions are calculated in the following:

$$\mathscr{L}(a_A|[x]_R) = \lambda_{AP} Pro(X|[x]_R) + \lambda_{AN} Pro(\neg X|[x]_R), \quad (6)$$
$$\mathscr{L}(a_U|[x]_R) = \lambda_{UP} Pro(X|[x]_R) + \lambda_{UN} Pro(\neg X|[x]_R), \quad (7)$$
$$\mathscr{L}(a_R|[x]_R) = \lambda_{RP} Pro(X|[x]_R) + \lambda_{RN} Pro(\neg X|[x]_R). \quad (8)$$

According to the principle of Bayesian minimum risk decision, the following decision rules can be obtained:

(A) If $\mathscr{L}(a_A|[x]_R) \leqslant \mathscr{L}(a_U|[x]_R)$ and $\mathscr{L}(a_A|[x]_R) \leqslant \mathscr{L}(a_R|[x]_R)$, then decide $x \in Acc(X)$;
(U) If $\mathscr{L}(a_U|[x]_R) \leqslant \mathscr{L}(a_A|[x]_R)$ and $\mathscr{L}(a_U|[x]_R) \leqslant \mathscr{L}(a_R|[x]_R)$, then decide $x \in Unc(X)$;
(R) If $\mathscr{L}(a_R|[x]_R) \leqslant \mathscr{L}(a_A|[x]_R)$ and $\mathscr{L}(a_R|[x]_R) \leqslant \mathscr{L}(a_U|[x]_R)$, then decide $x \in Rej(X)$.

Furthermore, if we have the following constraint conditions, i.e., $(\lambda_{AN} - \lambda_{UN})(\lambda_{RP} - \lambda_{UP}) > (\lambda_{UP} - \lambda_{AP})(\lambda_{UN} - \lambda_{RN}), Pro(X|[x]_R) + Pro(\neg X|[x]_R) = 1$ and $\alpha > \beta$, then, the decision rules (A)-(R) can be described with $\alpha = \frac{\lambda_{AN} - \lambda_{UN}}{(\lambda_{AN} - \lambda_{UN}) + (\lambda_{UP} - \lambda_{AP})}$ and $\beta = \frac{\lambda_{UN} - \lambda_{RN}}{(\lambda_{UN} - \lambda_{RN}) + (\lambda_{RP} - \lambda_{UP})}$.

(A′) If $Pro(X|[x]_R) \geqslant \alpha$, then decide $x \in Acc(X)$;
(U′) If $\beta < Pro(X|[x]_R) < \alpha$, then decide $x \in Unc(X)$;
(R′) If $Pro(X|[x]_R) \leqslant \beta$, then decide $x \in Rej(X)$.

2.2. Hesitant fuzzy sets

In this subsection, we briefly review some basic concepts of hesitant fuzzy sets (HFSs) [9,37,38]. HFSs permit the membership degree of an element to a set expressed by a nonempty subset of $[0,1]$. Assume that U and V are two non-empty finite universes and $Map(U,V)$ is the collection of all mappings from U to V, i.e., $Map(U,V) = \{f | f : U \to V\}$.

Definition 2.2 ([9,37]). Assume that U is a non-empty finite universe. Then a mapping $\widetilde{A} : U \to 2^{[0,1]} - \emptyset$ is said to be an HFS of U. The family of all HFSs of U is denoted as $Map\left(U, 2^{[0,1]} - \emptyset\right)$. For simplicity, Xia and Xu [38] presented the following mathematical symbol of HFS

$$\widetilde{A} = \{< x, h_{A^\sim}(x) > | x \in U\}, \quad (9)$$

where $h_{\widetilde{A}}(x)$ is a nonempty subset of $[0,1]$, which stands for the possible membership degrees of $x \in U$ to $\widetilde{A}$. For convenience, $h_{\widetilde{A}}(x)$ is said to be an HFV. If for any $x \in U$, $h_{\widetilde{A}}(x)$ is a non-empty finite subset of $[0,1]$, then $\widetilde{A}$ is called a finite HFS of U.

In this paper, we consider a finite HFS of U. In fact, the values in a finite HFV are generally out of order and we can arrange them by any order according to our needs. Xu and Xia [40] suggested that we can arrange these values in a decreasing order and denote an HFV as $h = \left\{h^{\tau(s)} | s = 1, 2, \ldots, |h|\right\}$, where $h^{\tau(s)}$ is the sth largest value in h.

Definition 2.3 [38]. Let U be a non-empty finite universe. For an HFV $h = \left\{h^{\tau(s)} | s = 1, 2, \ldots, |h|\right\}, Sc(h) = \frac{1}{|h|}\sum_{s=1}^{|h|} h^{\tau(s)}$ is called the score function of h. For two HFVs h_1 and h_2, if $Sc(h_1) < Sc(h_2)$, then $h_1 \prec h_2$; if $Sc(h_1) = Sc(h_2)$, then $h_1 \approx h_2$.

Example 2.1. Assume that $h_1 = \{0.8, 0.6\}$ and $h_2 = \{0.7, 0.6, 0.5\}$ are two HFVs. Based on Definition 2.3, the scores of two HFVs are computed as follows: $Sc(h_1) = \frac{1}{2}(0.8 + 0.6) = 0.7$ and $Sc(h_2) = \frac{1}{3}(0.7 + 0.6 + 0.5) = 0.6$. In this case, $Sc(h_2) < Sc(h_1)$. Therefore, we have $h_2 \prec h_1$.

Let h_1 and h_2 be two HFVs. In most situations, $|h_1| \neq |h_2|$, for convenience, let $l = \max\{|h_1|, |h_2|\}$. To compute the distance between two HFVs accurately, Xu and Xia [40] advised that we need to extend the shorter HFV until the length of both HFVs is the same. A direct method to extend the shorter HFV is to add the same value repeatedly; in principle, any values can be added to it. Actually, how to select the added value mainly depends on decision-makers' risk preferences. To handle this problem, Xu and Zhang [41] proposed the following extension method.

Definition 2.4 [41]. Assume that $h = \left\{h^{\tau(s)} | s = 1, 2, \ldots, |h|\right\}$ is an HFV, and stipulate that $h^{\max}$ and $h^{\min}$ are the maximum and minimum values in h, respectively; then $\widehat{h} = \theta h^{\max} + (1 - \theta) h^{\min}$ is called an extension value, where $\theta (0 \leqslant \theta \leq 1)$ is the parameter given by decision-makers based on their risk preferences.

Therefore, we can add different values to an HFV utilizing parameter θ based on decision-makers' risk preferences. If $\theta = 1$, then $\widehat{h} = h^{\max}$, which means that the risk preference of decision-makers is risk-seeking; if $\theta = 0$, then $\widehat{h} = h^{\min}$, which shows that the risk preference of decision-makers is risk-averse; if $\theta = \frac{1}{2}$, then $\widehat{h} = \frac{1}{2}\left(h^{\max} + h^{\min}\right)$, which illustrates that the risk preference of decision-makers is risk-neutral. Apparently, the selection of the value of the parameter θ reflects decision-makers' risk preferences. In this paper, we assume that decision-makers are risk-averse.

Example 2.2 (*Continuation of Example 2.1*). It is obvious that $|h_1| = 2, |h_2| = 3$ and $|h_2| > |h_1|$. Therefore, according to Definition 2.4 (suppose $\theta = 0$), we can extend h_1 to the following: $\widehat{h}_1 = \{0.8, 0.6, 0.6\}$.

On the basis of the above operational laws and the principle of extension, Xu and Xia [40] defined the hesitant fuzzy distance function of HFVs as follows:

Definition 2.5 [40]. Let h_i and h_j be two HFVs. Then the distance between HFVs h_i and h_j is defined as follows:

$$d(h_i, h_j) = \left(\frac{1}{l}\sum_{s=1}^{l}\left|h_i^{\tau(s)} - h_j^{\tau(s)}\right|^{\sigma}\right)^{1/\sigma}, \tag{10}$$

where $l = \max\{|h_i|, |h_j|\}$. Meanwhile, different values of σ denote different kinds of distance functions. If $\sigma = 1$, it is a Hamming distance; if $\sigma = 2$, it is an Euclidean distance; if $\sigma = \infty$, it is a Chebychev distance.

Example 2.3 (*Continuation of Example 2.1*). It is clear that $\max\{|h_1|, |h_2|\} = 3$. Then, according to Definition 2.5 (suppose $\sigma = 2$), the distance between h_1 and h_2 is calculated as follows: $d(h_1, h_2) = \left[\frac{1}{3}\left(|0.8 - 0.7|^2 + |0.6 - 0.6|^2 + |0.6 - 0.5|^2\right)\right]^{1/2} = 0.0819.$

3. A decision-theoretic fuzzy rough set in hesitant fuzzy information systems

Considering the various advantages of HFSs, we propose a DTFRS model in hesitant fuzzy information systems. First, we present the definition of hesitant fuzzy information systems. Then, according to the connection between the loss functions and the attribute values, we discuss a calculation method to obtain the relative loss functions by HFVs. Finally, in light of the Bayesian decision process, we establish a DTFRS model and discuss its decision rules.

3.1. The fuzzy similarity class and conditional probability

In the realistic decision-making process, we can confront one type of uncertain decision situations, that is, the evaluation result of the information system is not single, but several evaluation values simultaneously. HFSs proposed by Torra [37], exactly provide us a new evaluation format to deal with this problem, especially when we are hesitant about our decisions. Therefore, we define a hesitant fuzzy information system, in which the attribute evaluation values are expressed by HFVs. Under the hesitant fuzzy information system, we first define a new binary relation between two objects based on the hesitant fuzzy distance function. Then, the calculations of the fuzzy similarity class and the conditional probability are developed.

Definition 3.1. A hesitant fuzzy information system is a quadruple $IS = (U, C, V, f)$, where $U = \{x_1, x_2, \ldots, x_n\}$ is a non-empty finite set of objects and $C = \{c_1, c_2, \ldots, c_m\}$ is a non-empty finite set of conditional attributes. $V = \left\{V_{c_k} | k = 1, 2, \ldots, m\right\}$ and V_{c_k} is a domain of the attribute c_k. $f : U \times C \rightarrow V$ is an information function such that $f(x_i, c_k) \in V_{c_k}$ for each $x_i \in U$ and $c_k \in C$, where $f(x_i, c_k)$ is an HFV.

Definition 3.2. Let $IS = (U, C, V, f)$ be a hesitant fuzzy information system. Then, for two objects x_i and x_j in U, the distance-based similarity degree of the objects x_i and x_j on the attribute $c_k \in C$ is defined by

$$Sim_{(c_k)}(x_i, x_j) = 1 - d(f(x_i, c_k), f(x_j, c_k)). \tag{11}$$

According to the hesitant fuzzy distance function in Definition 2.5, the distance-based similarity degree of the objects x_i and x_j on the attribute c_k can be expressed as: $Sim_{(c_k)}(x_i, x_j) = 1 - \left(\frac{1}{l_k}\sum_{s=1}^{l_k}|f^{\tau(s)}(x_i, c_k) - f^{\tau(s)}(x_j, c_k)|^{\sigma}\right)^{1/\sigma}$, where $l_k = \max\{|f(x_i, c_k)|, |f(x_j, c_k)|\}$.

Proposition 3.1. *Let $IS = (U, C, V, f)$ be a hesitant fuzzy information system. Then, the distance-based similarity degree of the objects x_i and x_j on the attribute c_k has the following properties.*

(1) $0 \leqslant Sim_{(c_k)}(x_i, x_j) \leqslant 1$.
(2) $Sim_{(c_k)}(x_i, x_i) = 1$.
(3) $Sim_{(c_k)}(x_i, x_j) = Sim_{(c_k)}(x_j, x_i)$.

Proof. According to Eq. (11), the proof of this proposition is straightforward. □

For clarity, we take an example to illustrate how the distance-based similarity degree of the objects x_i and x_j on the attribute $c_k \in C$ is calculated in a hesitant fuzzy information system.

Example 3.1. A software company desires to hire a system analysis engineer. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ be a set of candidates and $C = \{c_1, c_2, c_3, c_4\}$ be a set of attributes. The evaluation results of each candidate on four attributes are expressed by HFVs. The detailed evaluation results are presented in Table 1.

According to Definition 3.2, we use the Euclidean distance to calculate the distance-based similarity degree of the objects x_i and x_j on the attribute $c_k (k = 1, 2, 3, 4)$. The results are shown as follows:

$$Sim_{(c_1)}(x_i, x_j) = \begin{pmatrix} 1.0000 & 0.6838 & 0.8419 & 0.9184 & 0.7292 & 0.7764 & 0.7450 & 0.6838 \\ 0.6838 & 1.0000 & 0.5472 & 0.7840 & 0.4523 & 0.9000 & 0.4477 & 1.0000 \\ 0.8419 & 0.5472 & 1.0000 & 0.7620 & 0.8709 & 0.6464 & 0.9000 & 0.5472 \\ 0.9184 & 0.7840 & 0.7620 & 1.0000 & 0.6633 & 0.8709 & 0.6633 & 0.7840 \\ 0.7292 & 0.4523 & 0.8709 & 0.6633 & 1.0000 & 0.5491 & 0.9184 & 0.4523 \\ 0.7764 & 0.9000 & 0.6464 & 0.8709 & 0.5491 & 1.0000 & 0.5472 & 0.9000 \\ 0.7450 & 0.4477 & 0.9000 & 0.6633 & 0.9184 & 0.5472 & 1.0000 & 0.4477 \\ 0.6838 & 1.0000 & 0.5472 & 0.7840 & 0.4523 & 0.9000 & 0.4477 & 1.0000 \end{pmatrix},$$

$$Sim_{(c_2)}(x_i, x_j) = \begin{pmatrix} 1.0000 & 0.8586 & 0.6464 & 0.4000 & 0.7354 & 0.5472 & 0.6891 & 0.9000 \\ 0.8586 & 1.0000 & 0.6891 & 0.4523 & 0.8268 & 0.5918 & 0.7620 & 0.8709 \\ 0.6464 & 0.6891 & 1.0000 & 0.7450 & 0.8367 & 0.9000 & 0.9184 & 0.7450 \\ 0.4000 & 0.4523 & 0.7450 & 1.0000 & 0.6127 & 0.8419 & 0.6891 & 0.5000 \\ 0.7354 & 0.8268 & 0.8367 & 0.6127 & 1.0000 & 0.7483 & 0.9184 & 0.8174 \\ 0.5472 & 0.5918 & 0.9000 & 0.8419 & 0.7483 & 1.0000 & 0.8268 & 0.6464 \\ 0.6891 & 0.7620 & 0.9184 & 0.6891 & 0.9184 & 0.8268 & 1.0000 & 0.7840 \\ 0.9000 & 0.8709 & 0.7450 & 0.5000 & 0.8174 & 0.6464 & 0.7840 & 1.0000 \end{pmatrix},$$

Table 1
Evaluation results of Example 3.1.

U/C	c_1	c_2	c_3	c_4
x_1	{0.6, 0.4}	{0.8}	{0.4, 0.3}	{0.6, 0.5, 0.3}
x_2	{0.2}	{0.9, 0.7, 0.6}	{0.5}	{0.4, 0.3}
x_3	{0.7, 0.6}	{0.5, 0.4}	{0.2, 0.1}	{0.6, 0.4}
x_4	{0.5, 0.4, 0.3}	{0.2}	{0.7}	{0.6, 0.5}
x_5	{0.9, 0.7, 0.6}	{0.7, 0.6, 0.4}	{0.3}	{0.7, 0.6, 0.3}
x_6	{0.3}	{0.4, 0.3}	{0.7, 0.5}	{0.9}
x_7	{0.8, 0.7}	{0.6, 0.5, 0.4}	{0.4}	{0.8, 0.6}
x_8	{0.2}	{0.7}	{0.3, 0.2, 0.1}	{0.8, 0.6}

$$Sim_{(c_3)}(x_i,x_j)=\begin{pmatrix}1.0000&0.8419&0.8000&0.6464&0.9293&0.7450&0.9293&0.8586\\0.8419&1.0000&0.6464&0.8000&0.8000&0.8586&0.9000&0.6891\\0.8000&0.6464&1.0000&0.4477&0.8419&0.5472&0.7450&0.9184\\0.6464&0.8000&0.4477&1.0000&0.6000&0.8586&0.7000&0.4934\\0.9293&0.8000&0.8419&0.6000&1.0000&0.6838&0.9000&0.8709\\0.7450&0.8586&0.5472&0.8586&0.6838&1.0000&0.7764&0.6303\\0.9293&0.9000&0.7450&0.7000&0.9000&0.7764&1.0000&0.7840\\0.8586&0.6891&0.9184&0.4934&0.8709&0.6303&0.7840&1.0000\end{pmatrix},$$

$$Sim_{(c_4)}(x_i,x_j)=\begin{pmatrix}1.0000&0.8367&0.9184&0.8845&0.9184&0.5491&0.7840&0.7840\\0.8367&1.0000&0.8419&0.8000&0.7551&0.4477&0.6464&0.6464\\0.9184&0.8419&1.0000&0.9293&0.8586&0.5877&0.8000&0.8000\\0.8845&0.8000&0.9293&1.0000&0.8586&0.6464&0.8419&0.8419\\0.9184&0.7551&0.8586&0.8586&1.0000&0.5959&0.8174&0.8174\\0.5491&0.4477&0.5877&0.6464&0.5959&1.0000&0.7764&0.7764\\0.7840&0.6464&0.8000&0.8419&0.8174&0.7764&1.0000&1.0000\\0.7840&0.6464&0.8000&0.8419&0.8174&0.7764&1.0000&1.0000\end{pmatrix}.$$

Definition 3.3. Assume that $IS=(U,C,V,f)$ is a hesitant fuzzy information system, an attribute subset $B\subseteq C$, for any $x_i,x_j\in U$, the fuzzy similarity relation $\widetilde{R}_B$ educed from the attribute subset B is defined as follows:

$$\widetilde{R}_B(x_i,x_j)=\bigwedge_{b\in B}Sim_{(b)}(x_i,x_j). \tag{12}$$

Obviously, it follows from Proposition 3.1 that $\widetilde{R}_B(x_i,x_i)=\bigwedge_{b\in B}Sim_{(b)}(x_i,x_i)=1$ and $\widetilde{R}_B(x_i,x_j)=\bigwedge_{b\in B}Sim_{(b)}(x_i,x_j)=\bigwedge_{b\in B}Sim_{(b)}(x_j,x_i)=\widetilde{R}_B(x_j,x_i)$. That is to say, $\widetilde{R}_B$ is a reflexive and symmetric fuzzy relation. In what follows, we utilize an example to interpret the definition of the fuzzy similarity relation.

Example 3.2. Let us consider Example 3.1 again. If $B=C=\{c_1,c_2,c_3,c_4\}$, then we have

$$\widetilde{R}_B(x_i,x_j)=\begin{pmatrix}1.0000&0.6838&0.6464&0.4000&0.7292&0.5472&0.6891&0.6838\\0.6838&1.0000&0.5472&0.4523&0.4523&0.5918&0.4477&0.8709\\0.6464&0.5472&1.0000&0.7450&0.8367&0.6464&0.9000&0.5472\\0.4000&0.4523&0.7450&1.0000&0.6127&0.8419&0.6633&0.5000\\0.7292&0.4523&0.8367&0.6127&1.0000&0.5491&0.9184&0.4523\\0.5472&0.5918&0.6464&0.8419&0.5491&1.0000&0.5472&0.6464\\0.6891&0.4477&0.9000&0.6633&0.9184&0.5472&1.0000&0.4477\\0.6838&0.8709&0.5472&0.5000&0.4523&0.6464&0.4477&1.0000\end{pmatrix}.$$

Definition 3.4. Let $IS=(U,C,V,f)$ be a hesitant fuzzy information system and B be an attribute subset of C. Then, for any object $x_i\in U$, the fuzzy similarity class of x_i obtained by the attribute subset B is defined as follows:

$$[x_i]_{R^{\sim}{}_B}=\frac{\widetilde{R}_B(x_i,x_1)}{x_1}+\frac{\widetilde{R}_B(x_i,x_2)}{x_2}+\ldots+\frac{\widetilde{R}_B(x_i,x_n)}{x_n}. \tag{13}$$

From Definition 3.4, we find that $U/\widetilde{R}_B=\left\{[x_i]_{\widetilde{R}_B}|x_i\in U\right\}$ forms a fuzzy covering the universe U.

Proposition 3.2. *Let $IS=(U,C,V,f)$ be a hesitant fuzzy information system. Then, for any $B,D\subseteq C$, the fuzzy similarity class $[x_i]_{\widetilde{R}}$ of x_i has the following properties.*

(1) $[x_i]_{\widetilde{R}_B}=\bigcap_{b\in B}[x_i]_{\widetilde{R}_b}$.

(2) $\bigcup_{x_i\in U}[x_i]_{\widetilde{R}_B}=1_U$.

(3) *If* $B \subseteq D$, *then* $[x_i]_{\widetilde{R}_D} \subseteq [x_i]_{\widetilde{R}_B}$.

Proof.

(1) For any $x_j \in U$, $\left([x_i]_{\widetilde{R}_B}\right)(x_j) = \widetilde{R}_B(x_i, x_j) = \bigwedge_{b \in B} Sim_{(b)}(x_i, x_j) = \bigwedge_{b \in B} \widetilde{R}_b(x_i, x_j) = \bigcap_{b \in B} \left([x_i]_{\widetilde{R}_b}\right)(x_j)$. Thus, $[x_i]_{\widetilde{R}_B} = \bigcap_{b \in B} [x_i]_{\widetilde{R}_b}$.

(2) For any $x_j \in U$, $\left(\bigcup_{x_i \in U} [x_i]_{\widetilde{R}_B}\right)(x_j) = \bigvee_{x_i \in U} [x_i]_{\widetilde{R}_B}(x_j) = \bigvee_{x_i \in U} \widetilde{R}_B(x_i, x_j) = \widetilde{R}_B(x_j, x_j) = 1$. Thus, $\bigcup_{x_i \in U} [x_i]_{\widetilde{R}_B} = 1_U$.

(3) If $B \subseteq D$, then for any $x_i, x_j \in U$, we have $\bigwedge_{d \in D} Sim_{(d)}(x_i, x_j) = \left(\bigwedge_{d \in B} Sim_{(d)}(x_i, x_j)\right) \wedge \left(\bigwedge_{d \in D \setminus B} Sim_{(d)}(x_i, x_j)\right) \leqslant \bigwedge_{d \in B} Sim_{(d)}(x_i, x_j)$. Thus, $[x_i]_{\widetilde{R}_D}(x_j) = \widetilde{R}_D(x_i, x_j) = \bigwedge_{d \in D} Sim_{(d)}(x_i, x_j) \leqslant \bigwedge_{d \in B} Sim_{(d)}(x_i, x_j) = \widetilde{R}_B(x_i, x_j) = [x_i]_{\widetilde{R}_B}(x_j)$, i.e., $[x_i]_{\widetilde{R}_D} \subseteq [x_i]_{\widetilde{R}_B}$. □

Example 3.3 (*Continued from Example 3.2*). According to Definition 3.4, the fuzzy similarity classes of eight candidates with regard to the fuzzy similarity relation $\widetilde{R}_B$ are listed as follows:

$$[x_1]_{R^{\sim}B} = \frac{1}{x_1} + \frac{0.6838}{x_2} + \frac{0.6464}{x_3} + \frac{0.4000}{x_4} + \frac{0.7292}{x_5} + \frac{0.5472}{x_6} + \frac{0.6891}{x_7} + \frac{0.6838}{x_8},$$

$$[x_2]_{R^{\sim}B} = \frac{0.6838}{x_1} + \frac{1}{x_2} + \frac{0.5472}{x_3} + \frac{0.4523}{x_4} + \frac{0.4523}{x_5} + \frac{0.5918}{x_6} + \frac{0.4477}{x_7} + \frac{0.8709}{x_8},$$

$$[x_3]_{R^{\sim}B} = \frac{0.6464}{x_1} + \frac{0.5472}{x_2} + \frac{1}{x_3} + \frac{0.7450}{x_4} + \frac{0.8367}{x_5} + \frac{0.6464}{x_6} + \frac{0.9000}{x_7} + \frac{0.5472}{x_8},$$

$$[x_4]_{R^{\sim}B} = \frac{0.4000}{x_1} + \frac{0.4523}{x_2} + \frac{0.7450}{x_3} + \frac{1}{x_4} + \frac{0.6127}{x_5} + \frac{0.8419}{x_6} + \frac{0.6633}{x_7} + \frac{0.5000}{x_8},$$

$$[x_5]_{R^{\sim}B} = \frac{0.7292}{x_1} + \frac{0.4523}{x_2} + \frac{0.8367}{x_3} + \frac{0.6127}{x_4} + \frac{1}{x_5} + \frac{0.5491}{x_6} + \frac{0.9184}{x_7} + \frac{0.4523}{x_8},$$

$$[x_6]_{R^{\sim}B} = \frac{0.5472}{x_1} + \frac{0.5918}{x_2} + \frac{0.6464}{x_3} + \frac{0.8419}{x_4} + \frac{0.5491}{x_5} + \frac{1}{x_6} + \frac{0.5472}{x_7} + \frac{0.6464}{x_8},$$

$$[x_7]_{R^{\sim}B} = \frac{0.6891}{x_1} + \frac{0.4477}{x_2} + \frac{0.9000}{x_3} + \frac{0.6633}{x_4} + \frac{0.9184}{x_5} + \frac{0.5472}{x_6} + \frac{1}{x_7} + \frac{0.4477}{x_8},$$

$$[x_8]_{R^{\sim}B} = \frac{0.6838}{x_1} + \frac{0.8709}{x_2} + \frac{0.5472}{x_3} + \frac{0.5000}{x_4} + \frac{0.4523}{x_5} + \frac{0.6464}{x_6} + \frac{0.4477}{x_7} + \frac{1}{x_8}.$$

Based on Definition 3.4, we can define the conditional probability of an object x belonging to a fuzzy set X with regard to the fuzzy similarity class $[x]_{\widetilde{R}_B}$ as follows.

Definition 3.5. Let $IS = (U, C, V, f)$ be a hesitant fuzzy information system and $[x]_{\widetilde{R}_B}$ be a fuzzy similarity class of the object x on the attribute subset B. For a fuzzy set X and an object $x \in U$, the conditional probability of the object x belonging to the fuzzy set X with regard to the fuzzy similarity class $[x]_{\widetilde{R}_B}$ is defined as:

$$Pro\left(X|[x]_{R^{\sim}B}\right) = \frac{\sum_{y \in U} X(y) \cdot [x]_{\widetilde{R}_B}(y)}{\sum_{y \in U} [x]_{\widetilde{R}_B}(y)}. \tag{14}$$

According to the definition of the conditional probability $Pro\left(X|[x]_{\widetilde{R}_B}\right)$, the following properties are clear.

Proposition 3.3. *Let* $IS = (U, C, V, f)$ *be a hesitant fuzzy information system and* $Pro\left(X|[x]_{\widetilde{R}_B}\right)$ *be the conditional probability of an object* x *belonging to a fuzzy set* X *with regard to the fuzzy similarity class* $[x]_{\widetilde{R}_B}$*. Then the following statements hold.*

(1) $0 \leqslant Pro\left(X|[x]_{\tilde{R}_B}\right) \leqslant 1$ *and* $Pro\left(\varnothing|[x]_{\tilde{R}_B}\right) = 0$.
(2) *If* $X, Y \in F(U)$ *and* $X \subseteq Y$, *then* $Pro\left(X|[x]_{\tilde{R}_B}\right) \leqslant Pro\left(Y|[x]_{\tilde{R}_B}\right)$.
(3) $Pro\left(X|[x]_{\tilde{R}_B}\right) + Pro\left(X^c|[x]_{\tilde{R}_B}\right) = 1$ *(where* X^c *denotes the complementary set of* X*).*

Proof. It is obvious that items (1) and (2) can be obtained directly by Definition 3.5. For item (3), according to Definition 3.5, it follows that

$$\begin{aligned}Pro\left(X^c|[x]_{\tilde{R}_B}\right) &= \frac{\sum_{y\in U} X^c(y)\cdot[x]_{\tilde{R}_B}(y)}{\sum_{y\in U}[x](y)} = \frac{\sum_{y\in U}(1-X(y))\cdot[x]_{\tilde{R}_B}(y)}{\sum_{y\in U}[x]_{\tilde{R}_B}(y)} \\ &= \frac{\sum_{y\in U}\left([x]_{\tilde{R}_B}(y)-X(y)\cdot[x]_{\tilde{R}_B}(y)\right)}{\sum_{y\in U}[x]_{\tilde{R}_B}(y)} = 1-\frac{\sum_{y\in U}X(y)\cdot[x]_{\tilde{R}_B}(y)}{\sum_{y\in U}[x]_{\tilde{R}_B}(y)} \\ &= 1 - Pro\left(X|[x]_{\tilde{R}_B}\right).\end{aligned}$$

Thus, one gets that $Pro\left(X|[x]_{\tilde{R}_B}\right) + Pro\left(X^c|[x]_{\tilde{R}_B}\right) = 1$. □

In light of the results presented in [7,8] and Proposition 3.3, the conditional probability defined in Definition 3.5 is a decision evaluation function. In the following, we take an example to illustrate the calculation process of it.

Example 3.4 (*Continuation of Example 3.3*). Assume that a fuzzy set X denotes a set of excellent candidates and the membership degrees of eight candidates are shown as follows:

$$X = \frac{0.25}{x_1} + \frac{0.95}{x_2} + \frac{0.7}{x_3} + \frac{0.8}{x_4} + \frac{0.75}{x_5} + \frac{0.45}{x_6} + \frac{0.2}{x_7} + \frac{0.9}{x_8}.$$

Now, we calculate the conditional probability of each candidate x_i belonging to the fuzzy set X with regard to the fuzzy similarity class $[x_i]_{\tilde{R}_B}$. Taking x_1 as an example, by using Definition 3.5, one obtains that

$$\begin{aligned}Pro\left(X|[x_1]_{\tilde{R}_B}\right) &= \frac{\sum_{y\in U}X(y)\cdot[x_1]_{\tilde{R}_B}(y)}{\sum_{y\in U}[x_1]_{\tilde{R}_B}(y)} \\ &= \frac{X(x_1)\times[x_1]_{\tilde{R}_B}(x_1)+X(x_2)\times[x_1]_{\tilde{R}_B}(x_2)+\ldots+X(x_7)\times[x_1]_{\tilde{R}}(x_7)+X(x_8)\times[x_1]_{\tilde{R}_B}(x_8)}{[x_1]_{\tilde{R}_B}(x_1)+[x_1]_{\tilde{R}_B}(x_2)+\ldots+[x_1]_{\tilde{R}_B}(x_7)+[x_1]_{\tilde{R}_B}(x_8)} \\ &= \frac{0.25\times1+0.95\times0.6838+0.7\times0.6464+0.8\times0.4+0.75\times0.7292+0.45\times0.5472+0.2\times0.6891+0.9\times0.6838}{1+0.6838+0.6464+0.4+0.7292+0.5472+0.6891+0.6838} \\ &= 0.5983.\end{aligned}$$

Analogously, we can obtain the conditional probability of the candidate $x_i (i = 2, 3, \ldots, 8)$ belonging to the fuzzy set X with regard to the fuzzy similarity class $[x_i]_{\tilde{R}_B}$. The result is presented as follows: $Pro\left(X|[x_2]_{\tilde{R}_B}\right) = 0.6628$, $Pro\left(X|[x_3]_{\tilde{R}_B}\right) = 0.6080$, $Pro\left(X|[x_4]_{\tilde{R}_B}\right) = 0.6274$, $Pro\left(X|[x_5]_{\tilde{R}_B}\right) = 0.5901$, $Pro\left(X|[x_6]_{\tilde{R}_B}\right) = 0.6291$, $Pro\left(X|[x_7]_{\tilde{R}_B}\right) = 0.5872$ and $Pro\left(X|[x_8]_{\tilde{R}_B}\right) = 0.6606$.

Based on the aforementioned results, we can define two approximation sets of a fuzzy set X with respect to the fuzzy similarity class in a hesitant fuzzy information system.

Definition 3.6. Assume that $IS = (U, C, V, f)$ is a hesitant fuzzy information system and B is an attribute subset of C. For any $X \in F(U)$, set $0 \leqslant \beta < \alpha \leq 1$, the α-lower and β-upper approximations of X with regard to the fuzzy similarity class $[x]_{\tilde{R}_B}$ are defined as follows:

$$\underset{\sim}{R}^{\alpha}(X) = \left\{x \in U|Pro\left(X|[x]_{R^{\sim}B}\right) \geqslant \alpha\right\} = \left\{x \in U|\frac{\sum_{y\in U}X(y)\cdot[x]_{\tilde{R}_B}(y)}{\sum_{y\in U}[x]_{\tilde{R}_B}(y)} \geqslant \alpha\right\}, \quad (15)$$

$$\overline{\tilde{R}}^{\beta}(X) = \left\{x \in U|Pro\left(X|[x]_{R^{\sim}B}\right) > \beta\right\} = \left\{x \in U|\frac{\sum_{y\in U}X(y)\cdot[x]_{\tilde{R}_B}(y)}{\sum_{y\in U}[x]_{\tilde{R}_B}(y)} > \beta\right\}. \quad (16)$$

The pair $\left(\underset{\sim}{R}^{\alpha}(X), \overline{\tilde{R}}^{\beta}(X)\right)$ is called a probabilistic fuzzy rough set of X in a hesitant fuzzy information system if $\underset{\sim}{R}^{\alpha}(X) \neq \overline{\tilde{R}}^{\beta}(X)$. Otherwise, X is said to be a definable set in a hesitant fuzzy information system. Furthermore, the α-acceptance region $Acc(X)$, β-rejection region $Rej(X)$ and (α, β)-uncertainty region $Unc(X)$ of X are defined, respectively, as follows:

Table 2
The relative loss functions of the object x_i on the attribute c_k.

x_i	$C_k^*(P)$	$\neg C_k^*(N)$
a_A	$\lambda_{AP}^k(x_i) = 0$	$\lambda_{AN}^k(x_i) = Sc\left(h_{\max}^k\right) - Sc(h_{ik})$
a_U	$\lambda_{UP}^k(x_i) = \vartheta_k\left(Sc(h_{ik}) - Sc\left(h_{\min}^k\right)\right)$	$\lambda_{UN}^k(x_i) = \vartheta_k\left(Sc\left(h_{\max}^k\right) - Sc(h_{ik})\right)$
a_R	$\lambda_{RP}^k(x_i) = Sc(h_{ik}) - Sc\left(h_{\min}^k\right)$	$\lambda_{RN}^k(x_i) = 0$

$$Acc(X) = \underline{\tilde{R}}^{\alpha}(X) = \left\{x \in U | \frac{\sum_{y \in U} X(y) \cdot [x]_{\tilde{R}_B}(y)}{\sum_{y \in U} [x]_{\tilde{R}_B}(y)} \geqslant \alpha\right\}, \tag{17}$$

$$Rej(X) = U - \overline{\tilde{R}}^{\beta}(X) = \left\{x \in U | \frac{\sum_{y \in U} X(y) \cdot [x]_{\tilde{R}_B}(y)}{\sum_{y \in U} [x]_{\tilde{R}_B}(y)} \leqslant \beta\right\}, \tag{18}$$

$$Unc(X) = \overline{\tilde{R}}^{\beta}(X) - \underline{\tilde{R}}^{\alpha}(X) = \left\{x \in U | \beta < \frac{\sum_{y \in U} X(y) \cdot [x]_{\tilde{R}_B}(y)}{\sum_{y \in U} [x]_{\tilde{R}_B}(y)} < \alpha\right\}. \tag{19}$$

Proposition 3.4. Assume that $IS = (U, C, V, f)$ is a hesitant fuzzy information system, X and Y are two fuzzy sets on U. Then the following statements hold.

(1) For any $0 \leqslant \beta < \alpha \leq 1$, we have $\underline{\tilde{R}}^{\alpha}(X) \subseteq \overline{\tilde{R}}^{\beta}(X)$.
(2) $\underline{\tilde{R}}^{\alpha}(0_U) = 0_U$ and $\overline{\tilde{R}}^{\beta}(1_U) = 1_U$ for any $\alpha \in (0, 1]$ and $\beta \in [0, 1)$.
(3) If $X \subseteq Y$, then $\underline{\tilde{R}}^{\alpha}(X) \subseteq \underline{\tilde{R}}^{\alpha}(Y)$ and $\overline{\tilde{R}}^{\beta}(X) \subseteq \overline{\tilde{R}}^{\beta}(Y)$.
(4) If $0 < \alpha_1 \leqslant \alpha_2 \leq 1$ and $0 \leqslant \beta_1 \leqslant \beta_2 < 1$, then $\underline{\tilde{R}}^{\alpha_2}(X) \subseteq \underline{\tilde{R}}^{\alpha_1}(X)$ and $\overline{\tilde{R}}^{\beta_2}(X) \subseteq \overline{\tilde{R}}^{\beta_1}(X)$.

Proof. It can be easily verified by Definition 3.6. □

Example 3.5. (Continuation of Example 3.4) Assume that $\alpha = 0.7$ and $\beta = 0.6$. Then, based on Definition 3.6, we obtain the α-lower and β-upper approximations of the fuzzy set X as follows:

$$\underline{\tilde{R}}^{0.7}(A) = \left\{x \in U | Pro\left(\tilde{X} | [x]_{R^{\sim}}\right) \geqslant 0.7\right\} = \{x_2, x_8\},$$

$$\overline{\tilde{R}}^{0.6}(A) = \left\{x \in U | Pro\left(\tilde{X} | [x]_{R^{\sim}}\right) > 0.6\right\} = \{x_2, x_3, x_4, x_6, x_8\}.$$

Therefore, three regions of X are given below: $Acc(X) = \{x_2, x_8\}, Unc(X) = \{x_3, x_4, x_6\}$ and $Rej(X) = \{x_1, x_5, x_7\}$.

In the model of probabilistic fuzzy rough set defined in Definition 3.6, the thresholds α and β are two significant notions that are determined by the intuitive arguments rather than the empirical studies. Thus, it raises a question: how to acquire the values of the thresholds based on the empirical studies? In the following parts, we explore an approach to calculate the values of the thresholds by using Bayesian decision theory.

3.2. The calculation of the relative loss functions

In the process of decision-making, decision-makers can select satisfactory objects based on their attribute values. In light of the results in [12,20,26], we discuss a calculation method of the relative loss functions from attribute values expressed by HFVs. In general, there are two types of attributes in hesitant fuzzy information systems, namely, the benefit attributes and the cost attributes. For simplicity, the attributes discussed in this paper are all benefit types.

Assume that the evaluation value of the object x_i with respect to the attribute c_k is an HFV $h_{ik} = \left\{h^{\tau(s)} | s = 1, 2, \ldots, |h_{ik}|\right\}$. The minimal and maximal evaluation values of the attribute c_k are $h_{\min}^k$ and $h_{\max}^k$, which are also expressed by HFVs. In addition, $St = \{C_k^*, \neg C_k^*\}$ denotes two states of the attribute c_k, where $x_i \in C_k^*$ means that the object x_i has the property of c_k and $x_i \in \neg C_k^*$ means that the object x_i does not have the property of c_k. Three actions a_A, a_U and a_R separately denote the decisions of acceptance, uncertainty and rejection. For the attribute c_k, we can calculate the relative loss functions of the object x_i, which are presented in Table 2.

For the relative loss functions presented in Table 2, we give the following explanation. Firstly, we know that the loss of an object in the correct region is the smallest. Therefore, we assign the loss functions $\lambda^k_{AP}(x_i) = 0$ and $\lambda^k_{RN}(x_i) = 0$, which means that the loss of taking acceptance action is 0 in the state of $x_i \in C^*_k$ and the loss of taking rejection action is 0 in the state of $x_i \in \neg C^*_k$. Secondly, $\lambda^k_{RP}(x_i) = Sc(h_{ik}) - Sc\left(h^k_{\min}\right)$ denotes the loss of taking rejection action in the state of $x_i \in C^*_k$, which means that a higher $Sc(h_{ik})$ leads to a greater loss $\lambda^k_{RP}(x_i)$; $\lambda^k_{AN}(x_i) = Sc\left(h^k_{\max}\right) - Sc(h_{ik})$ denotes the loss of taking acceptance action in the state of $x_i \in \neg C^*_k$, which means that a higher $Sc(h_{ik})$ leads to a lower loss $\lambda^k_{AN}(x_i)$. Thirdly, because the loss function in the uncertainty region is the one between in the acceptance region and in the rejection region, we use a parameter $\vartheta_k \in [0, 0.5]$ to calculate the relative loss function in the uncertainty region based on the losses in the acceptance and rejection regions. The parameter ϑ_k was introduced by Li and Zhou [18] to calculate the loss in the uncertainty region, and Jia and Liu [12] also verified its effectiveness and named it as the risk avoidance coefficient. Therefore, the relative loss functions in the uncertainty region are calculated as $\lambda^k_{UP}(x_i) = \vartheta_k\left(Sc(h_{ik}) - Sc\left(h^k_{\min}\right)\right)$ and $\lambda^k_{UN}(x_i) = \vartheta_k\left(Sc\left(h^k_{\max}\right) - Sc(h_{ik})\right)$. Moreover, according to Table 2, one obtains that $\lambda^k_{AP}(x_i) \leqslant \lambda^k_{UP}(x_i) < \lambda^k_{RP}(x_i)$ and $\lambda^k_{RN}(x_i) \leqslant \lambda^k_{UN}(x_i) < \lambda^k_{AN}(x_i)$.

In what follows, we take a numerical example to illustrate the calculation of the relative loss functions proposed in Table 2.

Example 3.6. In regard to a stock investment problem, experts evaluate two stocks x_1 and x_2 based on the attribute c_k (market prospect) by using HFVs. Suppose that the evaluation values of two stocks are $h_1 = \{0.7, 0.6, 0.5\}$ and $h_2 = \{0.9, 0.7, 0.5\}$, respectively. Let $h^k_{\max} = \{1\}, h^k_{\min} = \{0\}$ and $\vartheta_k = 0.35$. Then, according to Table 2, the relative loss functions of x_1 and x_2 can be calculated in Table 3.

Furthermore, we can fuse the relative loss functions of each object on multiple attributes to obtain the aggregated loss function. The aggregated loss function is defined as follows.

Definition 3.7. Let $\lambda^k_{\star\diamondsuit}$ be the relative loss function of the object x_i about taking the action $\star$ in the state $\diamondsuit$ ($\star = A, U, R; \diamondsuit = P, N$) on the attribute c_k. Then, the aggregated loss function of the object x_i about taking the action $\star$ in the state $\diamondsuit$ on the attribute set C is defined as follows:

$$\lambda_{\star\diamondsuit}(x_i) = \frac{1}{|C|}\sum\nolimits_{k=1}^{m} \lambda^k_{\star\diamondsuit}(x_i), \tag{20}$$

where $|C|$ denotes the cardinality of the attribute set C.

According to the relative loss functions of each object derived from attribute values in Table 2, Eq. (20) can be described in the following table (Table 4). X and $\neg X$ stand for two states of all attributes, and can be described by two fuzzy sets, where $X(x_i)$ denotes the satisfaction degree of the object x_i with respect to all attributes and $\neg X(x_i)$ stands for the dissatisfaction degree of the object x_i with respect to all attributes.

Theorem 3.1. *Based on Eq. (20), the aggregated loss functions of the object x_i presented in Table 4 satisfy the following relationship:* $\lambda_{AP}(x_i) \leqslant \lambda_{UP}(x_i) < \lambda_{RP}(x_i)$ *and* $\lambda_{RN}(x_i) \leqslant \lambda_{UN}(x_i) < \lambda_{AN}(x_i)$.

Table 3
The relative loss functions of the stocks x_1 and x_2.

	λ^k_{AP}	λ^k_{UP}	λ^k_{RP}	λ^k_{AN}	λ^k_{UN}	λ^k_{RN}
x_1	0	0.21	0.6	0.4	0.14	0
x_2	0	0.245	0.7	0.3	0.105	0

Table 4
The aggregated loss functions $\lambda_{\star\diamondsuit}(x_i)$ of the object x_i on multiple attributes.

x_i	$X(P)$	$\neg X(N)$
a_A	$\lambda_{AP} = 0$	$\lambda_{AN} = \frac{1}{\vert C\vert}\sum_{k=1}^{m}\left(Sc\left(h^k_{\max}\right) - Sc(h_{ik})\right)$
a_U	$\lambda_{UP} = \frac{1}{\vert C\vert}\sum_{k=1}^{m}\vartheta_k\left(Sc(h_{ik}) - Sc\left(h^k_{\min}\right)\right)$	$\lambda_{UN} = \frac{1}{\vert C\vert}\sum_{k=1}^{m}\vartheta_k\left(Sc\left(h^k_{\max}\right) - Sc(h_{ik})\right)$
a_R	$\lambda_{RP} = \frac{1}{\vert C\vert}\sum_{k=1}^{m}\left(Sc(h_{ik}) - Sc\left(h^k_{\min}\right)\right)$	$\lambda_{RN} = 0$

Proof. Since $\lambda^k_{\star\diamond}(x_i)$ satisfies $\lambda^k_{AP}(x_i) \leqslant \lambda^k_{UP}(x_i) < \lambda^k_{RP}(x_i)$ and $\lambda^k_{RN}(x_i) \leqslant \lambda^k_{UN}(x_i) < \lambda^k_{AN}(x_i)$, one obtains that $\frac{1}{|C|}\lambda^k_{AP}(x_i) \leqslant \frac{1}{|C|}\lambda^k_{UP}(A_i) < \frac{1}{|C|}\lambda^k_{RP}(x_i)$ and $\frac{1}{|C|}\lambda^k_{RN}(x_i) \leqslant \frac{1}{|C|}\lambda^k_{UN}(x_i) < \frac{1}{|C|}\lambda^k_{AN}(x_i)$. Therefore, based on Eq. (20), we have $\lambda_{AP}(x_i) \leqslant \lambda_{UP}(x_i) < \lambda_{RP}(x_i)$ and $\lambda_{RN}(x_i) \leqslant \lambda_{UN}(x_i) < \lambda_{AN}(x_i)$. □

On the basis of Theorem 3.1, we have the following conclusion. With respect to three actions, namely, the acceptance decision, uncertainty decision and rejection decision, the relative loss of adopting an acceptance decision for an object in the correct region is less than that of both adopting an uncertainty decision and a rejection decision in the correct region, and the relative loss of adopting a rejection decision for an object is the largest. Similarly, the relative loss of adopting a rejection decision for an object in the wrong region is less than that of both adopting an uncertainty decision and an acceptance decision in the wrong region, and the relative loss of adopting an acceptance action for an object in the wrong region is the largest.

3.3. The decision rules of DTFRS

In this subsection, we pay attention to construct a DTFRS model in the hesitant fuzzy information system. Then, the corresponding rules are induced and some relative properties are examined.

First, according to Bayesian decision process, the expected losses $\mathscr{L}\left(a_\star|[x_i]_{\widetilde{R}_C}\right)$ $(\star = A, U, R)$ of each object x_i with taking three actions can be calculated as follows:

$$\mathscr{L}\left(a_A|[x_i]_{R^\sim{}_C}\right) = \lambda_{AP}(x_i)Pro\left(X|[x_i]_{R^\sim{}_C}\right) + \lambda_{AN}(x_i)Pro\left(\neg X|[x_i]_{R^\sim{}_C}\right), \tag{21}$$

$$\mathscr{L}\left(a_U|[x_i]_{R^\sim{}_C}\right) = \lambda_{UP}(x_i)Pro\left(X|[x_i]_{R^\sim{}_C}\right) + \lambda_{UN}(x_i)Pro\left(\neg X|[x_i]_{R^\sim{}_C}\right), \tag{22}$$

$$\mathscr{L}\left(a_R|[x_i]_{R^\sim{}_C}\right) = \lambda_{RP}(x_i)Pro\left(X|[x_i]_{R^\sim{}_C}\right) + \lambda_{RN}(x_i)Pro\left(\neg X|[x_i]_{R^\sim{}_C}\right). \tag{23}$$

Subsequently, in light of the idea of minimum expected loss, the decision rules of the object x_i can be deduced as follows:

(A1) If $\mathscr{L}\left(a_A|[x_i]_{\widetilde{R}_C}\right) \leqslant \mathscr{L}\left(a_U|[x_i]_{\widetilde{R}_C}\right)$ and $\mathscr{L}\left(a_A|[x_i]_{\widetilde{R}_C}\right) \leqslant \mathscr{L}\left(a_R|[x_i]_{\widetilde{R}_C}\right)$, then decide $x_i \in Acc(X)$;

(U1) If $\mathscr{L}\left(a_U|[x_i]_{\widetilde{R}_C}\right) \leqslant \mathscr{L}\left(a_A|[x_i]_{\widetilde{R}_C}\right)$ and $\mathscr{L}\left(a_U|[x_i]_{\widetilde{R}_C}\right) \leqslant \mathscr{L}\left(a_R|[x_i]_{\widetilde{R}_C}\right)$, then decide $x_i \in Unc(X)$;

(R1) If $\mathscr{L}\left(a_R|[x_i]_{\widetilde{R}_C}\right) \leqslant \mathscr{L}\left(a_A|[x_i]_{\widetilde{R}_C}\right)$ and $\mathscr{L}\left(a_R|[x_i]_{\widetilde{R}_C}\right) \leqslant \mathscr{L}\left(a_U|[x_i]_{\widetilde{R}_C}\right)$, then decide $x_i \in Rej(X)$.

Afterwards, considering that $Pro\left(x|[x_i]_{\widetilde{R}_C}\right) + Pro\left(\neg X|[x_i]_{\widetilde{R}_C}\right) = 1$, we obtain the simplification of the decision rules (A1) – (R1) as follows:

(A1′) If $Pro\left(X|[x_i]_{\widetilde{R}_C}\right) \geqslant \alpha_i$ and $Pro\left(X|[x_i]_{\widetilde{R}_C}\right) \geqslant \gamma_i$, then decide $x_i \in Acc(X)$;

(U1′) If $Pro\left(X|[x_i]_{\widetilde{R}_C}\right) \leqslant \alpha_i$ and $Pro\left(X|[x_i]_{\widetilde{R}_C}\right) \geqslant \beta_i$, then decide $x_i \in Unc(X)$;

(R1′) If $Pro\left(X|[x_i]_{\widetilde{R}_C}\right) \leqslant \beta_i$ and $Pro\left(X|[x_i]_{\widetilde{R}_C}\right) \leqslant \gamma_i$, then decide $x_i \in Rej(X)$.

The values of α_i, β_i and γ_i are computed by using loss functions as follows:

$$\begin{aligned}
\alpha_i &= \frac{\lambda_{AN}(x_i)-\lambda_{UN}(x_i)}{(\lambda_{AN}(x_i)-\lambda_{UN}(x_i))+(\lambda_{UP}(x_i)-\lambda_{AP}(x_i))} \\
&= \frac{\sum_{k=1}^m (1-\vartheta_k)\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)}{\sum_{k=1}^m (1-\vartheta_k)\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)+\sum_{k=1}^m \vartheta_k\left(Sc(h_{ik})-Sc\left(h^k_{\min}\right)\right)}, \\
\beta_i &= \frac{\lambda_{UN}(x_i)-\lambda_{RN}(x_i)}{(\lambda_{UN}(x_i)-\lambda_{RN}(x_i))+(\lambda_{RP}(x_i)-\lambda_{UP}(x_i))} \\
&= \frac{\sum_{k=1}^m \vartheta_k\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)}{\sum_{k=1}^m \vartheta_k\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)+\sum_{k=1}^m (1-\vartheta_k)\left(Sc(h_{ik})-Sc\left(h^k_{\min}\right)\right)}, \\
\gamma_i &= \frac{\lambda_{AN}(x_i)-\lambda_{RN}(x_i)}{(\lambda_{AN}(x_i)-\lambda_{RN}(x_i))+(\lambda_{RP}(x_i)-\lambda_{AP}(x_i))} = \frac{\sum_{k=1}^m \left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)}{\sum_{k=1}^m \left(Sc\left(h^k_{\max}\right)-Sc\left(h^k_{\min}\right)\right)}.
\end{aligned}$$

Remark 3.1. There are two extreme cases of the thresholds to be explained: (1) if $h_{ik} = h^k_{\min}$ for all k in the hesitant fuzzy information system, then we have $\alpha_i = \beta_i = \gamma_i = 1$, which illustrates that the decision result has only one selection of rejection for the object x_i; (2) if $h_{ik} = h^k_{\max}$ for all k in the hesitant fuzzy information system, then we have $\alpha_i = \beta_i = \gamma_i = 0$, which indicates that the decision result has only one selection of acceptance for the object x_i.

Theorem 3.2. *Assume that α_i and β_i are the functions of ϑ_t, where ϑ_t is the RAC of the attribute $c_t (c_k \in C)$. Then h_{ik} for all k and ϑ_k for all $k \neq t$ are constants in $\alpha_i(\vartheta_t)$ and $\beta_i(\vartheta_t)$, and we have the following three conclusions.*

(1) *α_i is monotonically decreasing with ϑ_t.*
(2) *β_i is monotonously increasing with ϑ_t.*
(3) *γ_i is uncorrelated with ϑ_t.*

Proof.

(1) If $\alpha_i(\vartheta_t)$ is taken as the function of ϑ_t, then the derivative of $\alpha_i(\vartheta_t)$ is computed as follows:

$$\begin{aligned}\alpha_i'(\vartheta_t) &= \frac{y_3(y_1+y_2)-y_1(y_3+y_4)}{(y_1+y_2)^2} = \frac{y_2y_3-y_1y_4}{(y_1+y_2)^2} \\ &= \frac{-\left(Sc\left(h^t_{\max}\right)-Sc(h_{it})\right)\sum_{k=1}^m \vartheta_k\left(Sc(h_{ik})-Sc\left(h^k_{\min}\right)\right)-\left(Sc(h_{it})-Sc\left(h^t_{\min}\right)\right)\sum_{k=1}^m(1-\vartheta_k)\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)}{\left[\sum_{k=1}^m(1-\vartheta_k)\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)+\sum_{k=1}^m\vartheta_k\left(Sc(h_{ik})-Sc\left(h^k_{\min}\right)\right)\right]^2},\end{aligned}$$

where $y_1 = \sum_{k=1}^m (1-\vartheta_k)\left(Sc\left(h^k_{\max}\right) - Sc(h_{ik})\right), y_2 = \sum_{k=1}^m \vartheta_k\left(Sc(h_{ik}) - Sc\left(h^k_{\min}\right)\right), y_3 = -\left(Sc\left(h^t_{\max}\right) - Sc(h_{it})\right)$ and $y_4 = Sc(h_{it}) - Sc\left(h^t_{\min}\right)$. We find that the denominator is greater than 0 and the numerator is less than 0. Therefore, $\alpha_i'(\vartheta_t)$ is less than 0 and α_i is monotonically decreasing with the ϑ_t.

(2) Analogously, if $\beta_i(\vartheta_t)$ is taken as the function of ϑ_t, then the derivative of $\beta_i(\vartheta_t)$ is calculated as follows:

$$\begin{aligned}\beta_i'(\vartheta_t) &= \frac{z_3(z_1+z_2)-z_1(z_3+z_4)}{(z_1+z_2)^2} = \frac{z_2z_3-z_1z_4}{(z_1+z_2)^2} \\ &= \left(Sc\left(h^t_{\max}\right) - Sc(h_{it})\right)\left(\frac{\sum_{k=1}^m(1-\vartheta_k)\left(Sc(h_{ik})-Sc\left(h^k_{\min}\right)\right)+\left(Sc(a_{it})-Sc\left(h^t_{\min}\right)\right)\sum_{k=1}^m\vartheta_k\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)}{\left[\sum_{k=1}^m\vartheta_k\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)+\sum_{k=1}^m(1-\vartheta_k)Sc(h_{ik})-Sc\left(h^k_{\min}\right)\right)\right]^2}\right.,\end{aligned}$$

where $z_1 = \sum_{k=}^m \vartheta_k\left(Sc\left(h^k_{\max}\right) - Sc(h_{ik})\right), z_2 = \sum_{k=1}^m (1-\vartheta_k)\left(Sc(h_{ik}) - Sc\left(h^k_{\min}\right)\right), z_3 = \vartheta_t\left(Sc\left(h^t_{\max}\right) - Sc(h_{it})\right)$ and $z_4 = -\vartheta_t\left(Sc(h_{it}) - Sc\left(h^t_{\min}\right)\right)$. We know that the denominator and the numerator are all greater than 0. Therefore, $\beta_i'(\vartheta_t)$ is greater than 0 and β_i is monotonously increasing with the ϑ_t.

(3) According to the calculation form of γ_i, we observe that it is uncorrelated with ϑ_k.

Theorem 3.3. *The decision thresholds α_i, β_i and γ_i satisfy $0 \leqslant \beta_i \leqslant \gamma_i \leqslant \alpha_i \leq 1$.*

Proof. Based on the calculation form of α_i, β_i and γ_i, one obtains that $0 \leqslant \beta_i, \gamma_i, \alpha_i \leq 1$. Furthermore, according to Theorem 3.2, α_i comes to the minimal value when $\vartheta_k = 0.5$ for all $k(k = 1, 2, \ldots, m)$. Therefore, we have

$$\begin{aligned}\alpha_i^{\min} &= \frac{\sum_{k=1}^m(1-0.5)\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)}{\sum_{k=1}^m(1-0.5)\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)+\sum_{k=1}^m 0.5\left(Sc(h_{ij})-Sc\left(h^j_{\min}\right)\right)} \\ &= \frac{\sum_{k=1}^m\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)}{\sum_{k=1}^m\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)+\sum_{k=1}^m\left(Sc(h_{ik})-Sc\left(h^k_{\min}\right)\right)} \\ &= \frac{\sum_{k=1}^m\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)}{\sum_{k=1}^m\left(Sc\left(h^k_{\max}\right)-Sc\left(h^k_{\min}\right)\right)} = \gamma_i.\end{aligned}$$

It is easy to find that $\alpha_i \geqslant \gamma_i$.

Similarly, according to Theorem 3.2, we find that β_i comes to the maximal value when $\vartheta_k = 0.5$ for all $k(k = 1, 2, \ldots, m)$. Therefore, we have

$$\begin{aligned}\beta_i^{\max} &= \frac{\sum_{k=1}^m 0.5\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)}{\sum_{k=1}^m 0.5\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)+\sum_{k=1}^m(1-0.5)\left(Sc(h_{ik})-Sc\left(h^k_{\min}\right)\right)} \\ &= \frac{\sum_{k=1}^m\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)}{\sum_{k=1}^m\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)+\sum_{k=1}^m\left(Sc(h_{ik})-Sc\left(h^k_{\min}\right)\right)} \\ &= \frac{\sum_{k=1}^m\left(Sc\left(h^k_{\max}\right)-Sc(h_{ik})\right)}{\sum_{k=1}^m\left(Sc\left(h^k_{\max}\right)-Sc\left(h^k_{\min}\right)\right)} = \gamma_i.\end{aligned}$$

It is easy to find that $\beta_i \leqslant \gamma_i$.

Based on the above analysis, we can come to the conclusion that $0 \leqslant \beta_i \leqslant \gamma_i \leqslant \alpha_i \leq 1$.

According to Theorem 3.3, for each object x_i, the decision threshold γ_i is no longer needed in the decision-making process. Therefore, for each object x_i, after computing the conditional probability of the object x_i aiming at the fuzzy set X and based on tie-breaking, the decision rules of the object x_i can be obtained as follows:

(A1″) If $Pro\left(X|[x_i]_{\widetilde{R}_C}\right) \geqslant \alpha_i$, then decide $x_i \in Acc(X)$;

(U1″) If $\beta_i < Pro\left(X|[x_i]_{\widetilde{R}_C}\right) < \alpha_i$, then decide $x_i \in Unc(X)$;

(R1″) If $Pro\left(X|[x_i]_{\widetilde{R}_C}\right) \leqslant \beta_i$, then decide $x_i \in Rej(X)$.

In particular, for a unified risk avoidance coefficient, the thresholds α_i and β_i are calculated as

$$\alpha_i = \frac{(1-\vartheta)\sum_{k=1}^{m}\left(Sc\left(h_{\max}^{k}\right) - Sc(h_{ik})\right)}{(1-\vartheta)\sum_{k=1}^{m}\left(Sc\left(h_{\max}^{k}\right) - Sc(h_{ik})\right) + \vartheta\sum_{k=1}^{m}\left(Sc(h_{ik}) - Sc\left(h_{\min}^{k}\right)\right)}, \tag{24}$$

$$\beta_i = \frac{\vartheta\sum_{k=1}^{m}\left(Sc\left(h_{\max}^{k}\right) - Sc(h_{ik})\right)}{\vartheta\sum_{k=1}^{m}\left(Sc\left(h_{\max}^{k}\right) - Sc(h_{ik})\right) + (1-\vartheta)\sum_{k=1}^{m}\left(Sc(h_{ik}) - Sc\left(h_{\min}^{k}\right)\right)}. \tag{25}$$

Remark 3.2. For the decision rules (A1″) − (R1″), we have the following interpretation. (A1″): Given the fuzzy similarity class $[x_i]_{\widetilde{R}_C}$, when the conditional probability of X occurrence surpasses the threshold α_i, then $x_i \in U$ is classified into the acceptance region $Acc(X)$, meantime, an acceptance decision action is adopted immediately; (U1″): Given the fuzzy similarity class $[x_i]_{\widetilde{R}_C}$, when the conditional probability of X occurs in the interval between two thresholds α_i and β_i, then $x_i \in U$ is classified into the uncertainty region $Unc(X)$, meantime, it illustrates that x_i should be further studied by collecting more information; (R1″): Given the fuzzy similarity class $[x_i]_{\widetilde{R}_C}$, when the conditional probability of X occurrence does not exceed the threshold β_i, then $x_i \in U$ is classified into the rejection region $Rej(X)$, meantime, a rejection decision action is adopted.

4. A three-way decision method to MADM based on the DTFRS model

As an important branch in the field of decision-making research, MADM helps decision-makers select or rank appropriate alternatives according to multiple attributes. With the increasing complexity and uncertainty of decision-making problems, it is necessary to develop an effective and feasible method for avoiding decision risks and acquiring high utilities. This section aims to propose a three-way decision method to MADM under hesitant fuzzy environment by using the constructed DTFRS model. We first give a problem statement of MADM problems under hesitant fuzzy environment. Then, considering that there are no decision attributes in MADM problems, we propose an approach to determine the fuzzy decision class by virtue of the grey relational analysis method. Finally, we outline the detailed steps of the three-way decision method.

4.1. Problem statement

An MADM problem under hesitant fuzzy environment can be interpreted as follows. Let $U = \{x_1, x_2, \ldots, x_n\}$ be a collection of n alternatives, $C = \{c_1, c_2, \ldots, c_m\}$ be a collection of m attributes and $V = \{c_k(x_i)|x_i \in U, c_k \in C; i = 1, 2, \ldots, n; k = 1, 2, \ldots, m\}$ be the evaluation set of the alternative x_i in U aiming at the attribute c_j in the attribute set C. There are two types of attributes, namely, the benefit attributes and the cost attributes. We use J_b and J_c separately to denote the set of benefit attributes and the set of cost attributes. Decision-makers use HFVs to assess each alternative x_i with respect to multiple attributes and obtain a hesitant fuzzy evaluation matrix $\widetilde{H} = (h_{ik})_{n\times m}$. Additionally, the evaluation value of each attribute given by decision-makers is finite, we utilize $h_{\max}^{k}$ and $h_{\min}^{k}$ to denote the maximum and minimum values of the attribute c_k, respectively.

In the following, according to the established DTFRS model, we propose a three-way decision method to address this MAMD problem. First, we determine the fuzzy similarity relation and the fuzzy similarity class based on the hesitant fuzzy evaluation matrix $\widetilde{H} = (h_{ik})_{n\times m}$. Then, in light of the attribute values of each alternative with respect to multiple attributes, we compute the relative loss functions and the aggregated relative loss functions of each alternative. Finally, based on Bayesian decision process, we calculate the expected losses of each alternative with taking three actions and acquire the classification result of each alternative. In fact, the conditional probability is an important ingredient in our DTFRS model. As presented in Eq. (14), the conditional probability is calculated by the fuzzy similarity class $[x]_{\widetilde{R}_C}$ and the fuzzy decision class X. The fuzzy decision class is often determined by decision attributes in information systems. However, there are no decision attributes and only conditional attributes in the decision-making process of MADM problems. In this case, we encounter a

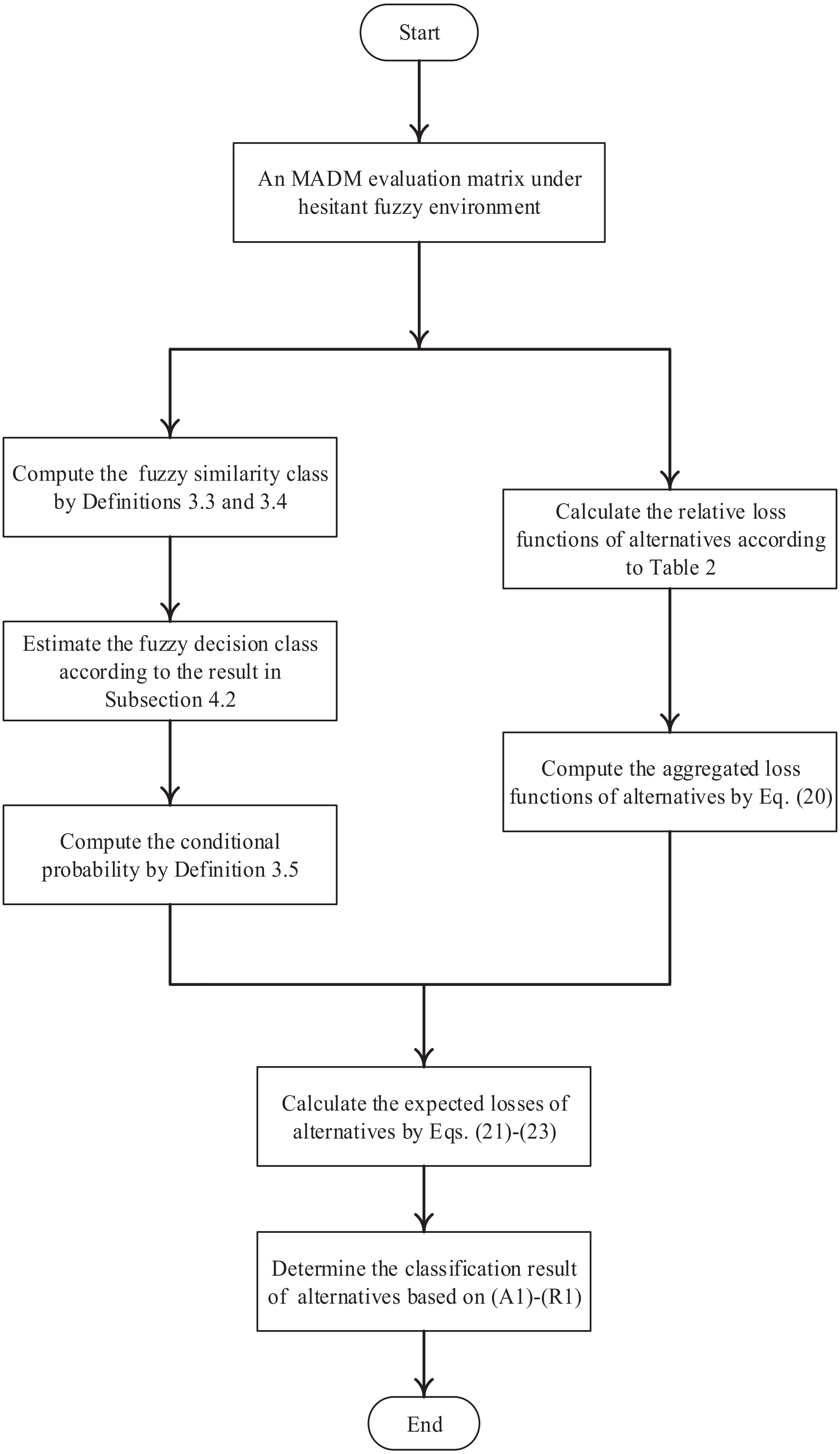


Fig. 1. The whole decision-making process of the proposed 3WD method.

challenge that how to determine the fuzzy decision class in an MADM problem under hesitant fuzzy environment? In the following, we utilize grey relational analysis [6] to estimate a fuzzy decision class of alternatives.

4.2. The estimation of the fuzzy decision class

In the existing studies, the decision class or state set is a crucial part of DTRS models, which is often determined by the decision attribute in an information table. However, there are no decision attributes in most MADM problems. Therefore, in order to use the proposed DTFRS model to solve MADM problems in the hesitant fuzzy context, we need to effectively and reasonably compute and estimate the decision class in the MADM matrix. In 2019, Jia and Liu [12] put forward a data-driven approach to determine the relative loss functions, in which each attribute is divided into two states. They describe two states $\{C_k^*, \neg C_k^*\}$ of the attribute c_k as follows: $x_i \in C_k^*$ indicates that the alternative x_i is satisfactory on the attribute c_k and $x_i \in \neg C_k^*$ shows that the alternative x_i is unsatisfactory on the attribute c_k. Inspired by this thought, we can use the positive ideal solution of each attribute to denote the decision state of each attribute and utilize grey relational analysis [6] to obtain a fuzzy decision class of all alternatives, where the membership degree of each alternative to the fuzzy set denotes the satisfaction degree of the alternative to all attributes. The specific estimation approach of the fuzzy decision class is presented as follows:

(1) Determine the positive ideal solution of each attribute $c_k(k = 1, 2, \ldots, m)$
The positive ideal solution should be the optimal value of the attribute values assessed by decision-makers. Based on the extension method of HFVs introduced in Definition 2.4, the positive ideal solution h_k^+ of each attribute c_k is obtained as follows:

$$h_k^+ = \left\{ \max_{1 \leqslant i \leqslant n} \left\{ h_{ik}^{\tau(s)} \right\} \text{ if } k \in J_b, \min_{1 \leqslant i \leqslant n} \left\{ h_{ik}^{\tau(s)} \right\} \text{ if } k \in J_c | s = 1, 2, \ldots, \max_{1 \leqslant i \leqslant n} \{ l_{h_{ik}} \} \right\}. \tag{26}$$

(2) Compute the grey relational coefficient of each alternative $x_i(i = 1, 2, \ldots, n)$ from the positive ideal solution aiming at the attribute c_k. The grey relational coefficient of each alternative x_i is calculated as follows:

$$\xi_{ik} = \frac{\min_{1 \leqslant i \leqslant n} \min_{1 \leqslant k \leqslant m} d(h_{ik}, h_k^+) + \phi \max_{1 \leqslant i \leqslant n} \max_{1 \leqslant k \leqslant m} d(h_{ik}, h_k^+)}{d(h_{ik}, h_k^+) + \phi \max_{1 \leqslant i \leqslant n} \max_{1 \leqslant k \leqslant m} d(h_{ik}, h_k^+)}, \tag{27}$$

where ϕ represents the distinguishing coefficient and $\phi \in (0, 1)$. In general, $\phi = 0.5$ is adopted. In addition, the grey relational coefficient indicates that how close the evaluation value of h_{ik} is to h_k^+ and the bigger the grey relational coefficient is, the closer h_{ik} and h_k^+ is.

(3) Compute the grey relation degree of each alternative x_i from the positive ideal solution about all attributes by utilizing the following equation:

$$\delta(x_i) = \frac{1}{|C|} \sum_{k=1}^{m} \xi_{ik}. \tag{28}$$

(4) Based on the grey relation degree of each alternative x_i from the positive ideal solution, we define a fuzzy set of "good alternatives" as follows:

$$X = \frac{\delta(x_1)}{x_1} + \frac{\delta(x_2)}{x_2} + \ldots + \frac{\delta(x_n)}{x_n}. \tag{29}$$

Remark 4.1. In general, most MADM problems do not have the decision attribute in evaluation matrices, but the final decision target can be regarded as a decision attribute. For instance, for the problem of project investment presented in [12], a good project or a suitable project can be regarded as a decision attribute. Meanwhile, we know that the concepts of good projects or suitable projects are all fuzzy sets according to the knowledge of fuzzy mathematics, so the final decision evaluation values of alternatives obtained by diverse MADM techniques and methodologies can be regarded as the values of alternatives with respect to the decision attribute. Based on the aforementioned calculation process, the bigger the grey relation degree of the alternative is, the better the alternative is in the decision-making processes. This illustrates that the grey relation degree of each alternative reflects the closeness degree or the satisfaction degree of each alternative to all conditional attributes. Therefore, we can use the grey relation degree of each alternative to denote the value of each alternative to the decision attribute.

4.3. The key steps of the three-way decision method

In this section, based on the proposed DTFRS model, the detailed steps of the three-way decision method for solving MADM problems under hesitant fuzzy environment are presented in the following.

Input An MADM evaluation matrix under hesitant fuzzy environment $\widetilde{H} = (h_{ik})_{n \times m}$.

Table 5
Hesitant fuzzy evaluation matrix $\widetilde{H}$ of stock investment.

U/C	c_1	c_2	c_3	c_4
x_1	$\{0.8\}$	$\{0.7, 0.6\}$	$\{0.4, 0.3\}$	$\{0.8\}$
x_2	$\{0.6, 0.4, 0.3\}$	$\{0.4, 0.3\}$	$\{0.2, 0.1\}$	$\{0.4\}$
x_3	$\{0.8, 0.7, 0.6, 0.5\}$	$\{0.4\}$	$\{0.6, 0.5, 0.4\}$	$\{0.7, 0.6, 0.4\}$
x_4	$\{0.6\}$	$\{0.5, 0.4\}$	$\{0.6, 0.4, 0.3, 0.1\}$	$\{0.5, 0.4, 0.2\}$
x_5	$\{0.3\}$	$\{0.4, 0.3, 0.1\}$	$\{0.2\}$	$\{0.6, 0.5, 0.4\}$
x_6	$\{0.7\}$	$\{0.3, 0.2\}$	$\{0.50.40.3\}$	$\{0.8, 0.7, 0.6, 0.4\}$
x_7	$\{0.9, 0.8\}$	$\{0.7, 0.6, 0.5, 0.4\}$	$\{0.5\}$	$\{0.8, 0.6\}$
x_8	$\{0.8, 0.7\}$	$\{0.7\}$	$\{0.9, 0.7, 0.6, 0.4\}$	$\{0.6, 0.4, 0.3\}$

Output The classification result of alternatives.
Step 1 Calculate the fuzzy similarity class $[x_i]_{\widetilde{R}_C}$ of each alternative x_i based on Definitions 3.3 and 3.4.
Step 2 Determine the fuzzy decision class X according to the result in SubSection 4.2.
Step 3 Compute the conditional probability $Pro\left(X|[x_i]_{\widetilde{R}_C}\right)$ of each alternative x_i belonging to the fuzzy decision class X with respect to the fuzzy similarity class $[x_i]_{\widetilde{R}_C}$ based on Definition 3.5.
Step 4 Figure out the relative loss functions $\lambda^k_{\star\diamond}(x_i)$ of each alternative x_i according to Table 2.
Step 5 Compute the aggregated loss functions $\lambda_{\star\diamond}(x_i)$ of each alternative x_i under different attributes by Eq. (20).
Step 6 Calculate the expected losses $\mathscr{L}(a_\star|x_i)$ of each alternative x_i based on Eqs. (21)–(23).
Step 7 Determine the classification result of each alternative x_i according to the decision rules (A1) − (R1).
Furthermore, the whole decision procedure of the proposed three-way decision method is depicted in Fig. 1.

5. An illustrative example

In this section, we take the proposed three-way decision method to solve a stock investment problem under hesitant fuzzy environment to illustrate its effectiveness. Moreover, a sensitivity analysis and a comparison analysis are conducted to illustrate the characteristics and advantages of our method.

5.1. A case description of stock investment

Stock investment refers to the investment action and investment process in which investors buy stocks to obtain dividends and capital gains. In recent years, with the rapid development of China's securities market, stock investment has become an important channel for enterprises' direct investment and personal investment and financing.

Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ be a collection of eight stocks, which are "medicine stock (x_1)", "home appliances stock (x_2)", "communication stock (x_3)", "automobile stock (x_4)", "high-technology stock (x_5)", "education stock (x_6)", "energy stocks (x_7)" and "environmental protection stock (x_8)", respectively. After careful analysis, there are four factors to evaluate these stocks, including "asset operating capacity (c_1)", "debt paying ability (c_2)", "enterprise growth (c_3)" and "equity expansion capacity (c_4)". Then, experts with extensive experience in stock investment evaluation are invited from a consulting firm to evaluate these eight stocks. Experts assess eight stocks based on four attributes by means of HFVs and the evaluation results are shown in Table 5, which is a hesitant fuzzy evaluation matrix $\widetilde{H} = (h_{ik})_{8\times4}$ and h_{ik} is an HFV standing for the possible evaluation values given by experts for the stock x_i on the attribute c_k ($i = 1,2,3,4,5,6,7,8$ and $k = 1,2,3,4$). The risk avoidance coefficient vector of four attributes given by experts is $\vartheta = (0.45, 0.4, 0.35, 0.25)^T$. For a stock, there are two possible states: a good market prospect (X) and a bad market prospect ($\neg X$). Let $Ac = \{a_A, a_U, a_R\}$ be a set of actions, where a_A, a_U and a_R denote investment, further analysis, and rejection of investment, respectively. In the following, we employ the established three-way decision method in Section 4 to solve this stock investment selection problem.

5.2. Decision analysis based on the proposed three-way decision method

With the aid of the proposed three-way decision method, we analyze the whole evaluation process of the above stock investment problem. The whole evaluation process to deal with the problem is clearly displayed in accordance with each step as follows.

[Step 1] Based on Definitions 3.3 and 3.4, we calculate the fuzzy similarity class $[x_i]_{\widetilde{R}_C}$ of each stock x_i from $\widetilde{H} = (h_{ik})_{8\times4}$. At the beginning, according to Definition 3.3, the fuzzy similarity relation $\widetilde{R}_C$ is calculated as follows:

$$\widetilde{R}_C = \begin{pmatrix} 1.0000 & 0.6127 & 0.7450 & 0.8000 & 0.5000 & 0.6000 & 0.8882 & 0.9293 \\ 0.6127 & 1.0000 & 0.7450 & 0.7918 & 0.8174 & 0.7056 & 0.5918 & 0.6464 \\ 0.7450 & 0.7450 & 1.0000 & 0.8775 & 0.6326 & 0.8419 & 0.8064 & 0.7000 \\ 0.8000 & 0.7918 & 0.8775 & 1.0000 & 0.7000 & 0.8000 & 0.7450 & 0.7450 \\ 0.5000 & 0.8174 & 0.6326 & 0.7000 & 1.0000 & 0.6000 & 0.4477 & 0.5472 \\ 0.6000 & 0.7056 & 0.8419 & 0.8000 & 0.6000 & 1.0000 & 0.6646 & 0.5472 \\ 0.8882 & 0.5918 & 0.8064 & 0.7450 & 0.4477 & 0.6646 & 1.0000 & 0.8129 \\ 0.9293 & 0.6464 & 0.7000 & 0.7450 & 0.5472 & 0.5472 & 0.8129 & 1.0000 \end{pmatrix}.$$

Then, on the basis of Definition 3.4, the fuzzy similarity classes of eight stocks are displayed as follows:

$$[x_1]_{R^{\sim}_C} = \frac{1}{x_1} + \frac{0.6127}{x_2} + \frac{0.7450}{x_3} + \frac{0.8000}{x_4} + \frac{0.5000}{x_5} + \frac{0.6000}{x_6} + \frac{0.8882}{x_7} + \frac{0.9293}{x_8},$$

$$[x_2]_{R^{\sim}_C} = \frac{0.6127}{x_1} + \frac{1}{x_2} + \frac{0.7450}{x_3} + \frac{0.7918}{x_4} + \frac{0.8174}{x_5} + \frac{0.7056}{x_6} + \frac{0.5918}{x_7} + \frac{0.6464}{x_8},$$

Table 6
The relative loss functions of eight stocks on four attributes.

U	C	λ^k_{AP}	λ^k_{UP}	λ^k_{RP}	λ^k_{AN}	λ^k_{UN}	λ^k_{RN}	U	C	λ^k_{AP}	λ^k_{UP}	λ^k_{RP}	λ^k_{AN}	λ^k_{UN}	λ^k_{RN}
x_1	c_1	0	0.36	0.8	0.2	0.09	0	x_5	c_1	0	0.135	0.3	0.7	0.315	0
	c_2	0	0.26	0.65	0.35	0.14	0		c_2	0	0.1067	0.2667	0.7333	0.2933	0
	c_3	0	0.1225	0.35	0.65	0.2275	0		c_3	0	0.07	0.2	0.8	0.28	0
	c_4	0	0.2	0.8	0.2	0.05	0		c_4	0	0.125	0.5	0.5	0.125	0
x_2	c_1	0	0.195	0.4333	0.5667	0.255	0	x_6	c_1	0	0.315	0.7	0.3	0.135	0
	c_2	0	0.14	0.35	0.65	0.26	0		c_2	0	0.1	0.25	0.75	0.3	0
	c_3	0	0.0525	0.15	0.85	0.2975	0		c_3	0	0.14	0.4	0.6	0.21	0
	c_4	0	0.1	0.4	0.6	0.15	0		c_4	0	0.1563	0.625	0.375	0.0938	0
x_3	c_1	0	0.2925	0.65	0.35	0.1575	0	x_7	c_1	0	0.3825	0.85	0.15	0.0675	0
	c_2	0	0.16	0.4	0.6	0.24	0		c_2	0	0.22	0.55	0.45	0.18	0
	c_3	0	0.175	0.5	0.5	0.175	0		c_3	0	0.175	0.5	0.5	0.175	0
	c_4	0	0.1417	0.5667	0.4333	0.1083	0		c_4	0	0.175	0.7	0.3	0.075	0
x_4	c_1	0	0.27	0.6	0.4	0.18	0	x_8	c_1	0	0.3375	0.75	0.25	0.1125	0
	c_2	0	0.18	0.45	0.55	0.22	0		c_2	0	0.28	0.7	0.3	0.12	0
	c_3	0	0.1225	0.35	0.65	0.2275	0		c_3	0	0.2275	0.65	0.35	0.1225	0
	c_4	0	0.0917	0.3667	0.6333	0.1583	0		c_4	0	0.1083	0.4333	0.5667	0.1417	0

Table 7
The aggregated loss functions of eight stocks.

U	λ_{AP}	λ_{UP}	λ_{RP}	λ_{AN}	λ_{UN}	λ_{RN}
x_1	0	0.2356	0.65	0.35	0.1269	0
x_2	0	0.1219	0.3333	0.6667	0.2406	0
x_3	0	0.1923	0.5292	0.4708	0.1702	0
x_4	0	0.166	0.4417	0.5583	0.1965	0
x_5	0	0.1092	0.3167	0.6833	0.2533	0
x_6	0	0.1778	0.4937	0.5063	0.1847	0
x_7	0	0.2381	0.65	0.35	0.1244	0
x_8	0	0.2383	0.6333	0.3667	0.1242	0

Table 8
The expected losses of eight stocks with regard to three actions.

Stock	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$\mathscr{L}(a_A\|[x_i]_R)$	0.1435	0.3062	0.2067	0.2464	0.3215	0.2274	0.1443	0.1511
$\mathscr{L}(a_U\|[x_i]_R)$	0.1910	0.1764	0.1826	0.1795	0.1770	0.1809	0.1912	0.1913
$\mathscr{L}(a_R\|[x_i]_R)$	0.3834	0.1802	0.2968	0.2468	0.1677	0.2720	0.3820	0.3723

$$[x_3]_{\widetilde{R}_C} = \frac{0.7450}{x_1} + \frac{0.7450}{x_2} + \frac{1}{x_3} + \frac{0.8775}{x_4} + \frac{0.6326}{x_5} + \frac{0.8419}{x_6} + \frac{0.8064}{x_7} + \frac{0.7000}{x_8},$$

$$[x_4]_{\widetilde{R}_C} = \frac{0.8000}{x_1} + \frac{0.7918}{x_2} + \frac{0.8775}{x_3} + \frac{1}{x_4} + \frac{0.7000}{x_5} + \frac{0.8000}{x_6} + \frac{0.7450}{x_7} + \frac{0.7450}{x_8},$$

$$[x_5]_{\widetilde{R}_C} = \frac{0.5000}{x_1} + \frac{0.8174}{x_2} + \frac{0.6326}{x_3} + \frac{0.7000}{x_4} + \frac{1}{x_5} + \frac{0.6000}{x_6} + \frac{0.4477}{x_7} + \frac{0.5472}{x_8},$$

$$[x_6]_{\widetilde{R}_C} = \frac{0.6000}{x_1} + \frac{0.7056}{x_2} + \frac{0.8419}{x_3} + \frac{0.8000}{x_4} + \frac{0.6000}{x_5} + \frac{1}{x_6} + \frac{0.6646}{x_7} + \frac{0.5472}{x_8},$$

$$[x_7]_{\widetilde{R}_C} = \frac{0.8882}{x_1} + \frac{0.5918}{x_2} + \frac{0.8064}{x_3} + \frac{0.7450}{x_4} + \frac{0.4477}{x_5} + \frac{0.6646}{x_6} + \frac{1}{x_7} + \frac{0.8129}{x_8},$$

$$[x_8]_{\widetilde{R}_C} = \frac{0.9293}{x_1} + \frac{0.6464}{x_2} + \frac{0.7000}{x_3} + \frac{0.7450}{x_4} + \frac{0.5472}{x_5} + \frac{0.5472}{x_6} + \frac{0.8129}{x_7} + \frac{1}{x_8}.$$

[Step 2] Based on the description of the stock investment, we can regard the market prospect as a decision attribute and the fuzzy set of "good market prospect stocks" as a fuzzy decision class. Then, by use of Eqs. (26)–(29), the grey relation degrees of eight stocks from the positive ideal solution are shown as follows: $\delta(x_1) = 0.7615$, $\delta(x_2) = 0.3913$, $\delta(x_3) = 0.5311$, $\delta(x_4) = 0.4692$, $\delta(x_5) = 0.3817$, $\delta(x_6) = 0.5193$, $\delta(x_7) = 0.6922$ and $\delta(x_8) = 0.7469$. Thus, based on the result in SubSection 4.2, we can estimate the fuzzy decision class in the hesitant fuzzy evaluation matrix $\widetilde{H} = (h_{ik})_{8\times4}$ as:

$$X = \frac{0.7615}{x_1} + \frac{0.3913}{x_2} + \frac{0.5311}{x_3} + \frac{0.4692}{x_4} + \frac{0.3817}{x_5} + \frac{0.5193}{x_6} + \frac{0.6922}{x_7} + \frac{0.7469}{x_8}.$$

[Step 3] According to Definition 3.5, we compute the conditional probability of each stock belonging to the fuzzy decision class X. On the basis of Eq. (14), the conditional probabilities for eight stocks are calculated as follows: $Pro\left(X|[x_1]_{\widetilde{R}_C}\right) = 0.5899$, $Pro\left(X|[x_2]_{\widetilde{R}_C}\right) = 0.5378$, $Pro\left(X|[x_3]_{\widetilde{R}_C}\right) = 0.5610$, $Pro\left(X|[x_4]_{\widetilde{R}_C}\right) = 0.5587$, $Pro\left(X|[x_5]_{\widetilde{R}_C}\right) = 0.5294$, $Pro\left(X|[x_6]_{\widetilde{R}_C}\right) = 0.5509$, $Pro\left(X|[x_7]_{\widetilde{R}_C}\right) = 0.5878$ and $Pro\left(X|[x_8]_{\widetilde{R}_C}\right) = 0.5878$.

[Step 4] On the basis of Table 2, we calculate the relative loss functions $\lambda^k_{\star\diamond}(x_i)$ of eight stocks educed from attribute values h_{ik} $(i = 1, 2, \ldots, 8; k = 1, 2, 3, 4)$. The results are listed in Table 6.

[Step 5] According to Eq. (20), we calculate the aggregated loss function $\lambda_{\star\diamond}(x_i)$ of each stock x_i by aggregating the relative loss functions of each stock x_i under multiple attributes, where $\star \in \{A, U, R\}$, $\diamond \in \{P, N\}$ and $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. The aggregated loss functions of eight stocks are presented in Table 7.

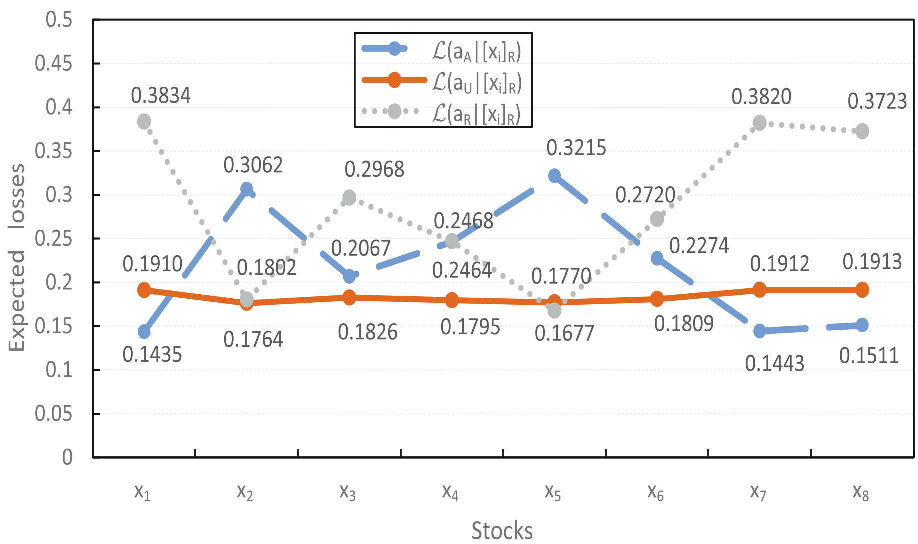


Fig. 2. The expected losses of eight stocks with taking three actions.

Table 9
The classification results of eight stocks with the different values of parameter ϑ_k.

The value of ϑ_k	$Acc(X)$	$Unc(X)$	$Rej(X)$
$\vartheta_k = 0$	$\varnothing$	$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$	$\varnothing$
$\vartheta_k = 0.05$	$\varnothing$	$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$	$\varnothing$
$\vartheta_k = 0.1$	$\varnothing$	$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$	$\varnothing$
$\vartheta_k = 0.15$	$\varnothing$	$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$	$\varnothing$
$\vartheta_k = 0.2$	$\varnothing$	$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$	$\varnothing$
$\vartheta_k = 0.25$	$\varnothing$	$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$	$\varnothing$
$\vartheta_k = 0.3$	x_1, x_7, x_8	x_2, x_3, x_4, x_5, x_6	$\varnothing$
$\vartheta_k = 0.35$	x_1, x_7, x_8	x_2, x_3, x_4, x_6	x_5
$\vartheta_k = 0.4$	x_1, x_7, x_8	x_3, x_4, x_6	x_2, x_4
$\vartheta_k = 0.45$	x_1, x_3, x_7, x_8	x_4, x_6	x_2, x_5
$\vartheta_k = 0.5$	$x_1, x_3, x_4, x_6, x_7, x_8$	$\varnothing$	x_2, x_5

[Step 6] Applying Eqs. (21)–(23), we calculate the expected losses $\mathscr{L}(a_\star|[x_i]_R)$ ($\star = A, U, R; i = 1, 2, \ldots, 8$) of each stock x_i with regard to three actions. The results are depicted in Table 8.

Meanwhile, the expected losses $\mathscr{L}(a_\star|[x_i]_R)$ of eight stocks with taking three actions are depicted in Fig. 2.

[Step 7] On the basis of decision rules (A1) – (R1), the classification results of eight stocks in the stock investment problem are shown as follows: $Acc(X) = \{x_1, x_7, x_8\}, Unc(X) = \{x_2, x_3, x_4, x_6\}$ and $Rej(X) = \{x_5\}$.

From the classification results of eight stocks, we can obtain that the stocks x_1, x_7 and x_8 should be considered as acceptable stocks, i.e., they can be selected for investment immediately; the stock x_5 is recognized as an infeasible stock, i.e, it should not be invested; and the stocks x_2, x_3, x_4 and x_6 should be further analyzed.

5.3. Sensitivity analysis

During the decision analysis of the proposed three-way decision method, it involves a key parameter $\vartheta_k (k = 1, 2, 3, 4)$. In this part, we conduct a sensitivity analysis to analyse the influence of the parameter ϑ_k on the decision results of the stocks in details. On the basis of the evaluation result in Table 5, we use different values of ϑ_k in Step 5 of the proposed three-way decision method to deal with the stock investment problem presented in SubSection 5.1. To facilitate analysis, we set the parameter ϑ_k to the same value on four attributes. When the parameter ϑ_k changes from 0 to 0.5 with the step of 0.05, we can obtain the decision results of eight stocks, which are depicted in Table 9. Meanwhile, we compute the expected losses $\mathscr{L}(a_A|[x_i]_R), \mathscr{L}(a_U|[x_i]_R)$ and $\mathscr{L}(a_R|[x_i]_R)$ of each stock with different values of parameter ϑ_k. The calculation results of the expected losses for eight stocks are shown in Fig. 3. With regard to Fig. 3, , x-coordinate denotes the value of the risk avoidance coefficient and y-coordinate stands for the values of the expected losses of each stock.

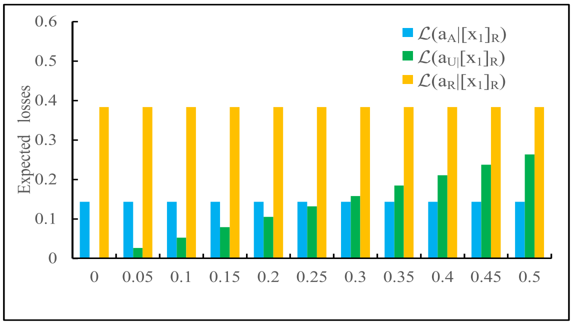
According to the results presented in Table 9 and Fig. 3, we analyze the influence of the parameter ϑ_k on the stock investment selection in detail.

(1) The classification results of eight stocks are different with the parameter ϑ_k increasing in our method, which illustrates that the decision results are sensitive to the values of parameter ϑ_k. When $\vartheta_k = 0, 0.05, \ldots, 0.25$, the classification result of eight stocks is $Unc(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, which illustrates that all stocks need to be further analyzed; when $\vartheta_k = 0.3$, the classification result of eight stocks is $Acc(X) = \{x_1, x_7, x_8\}$ and $Unc(X) = \{x_2, x_3, x_4, x_5, x_6\}$, which means that the stocks x_1, x_7 and x_8 should be invested immediately and others need to be further studied; when $\vartheta_k = 0.35, 0.4 and 0.45$, the eight stocks are classified into three decision regions, which means that the classification result is a three-way decision; and when $\vartheta_k = 0.5$, the classification result of eight stocks is $Acc(X) = \{x_1, x_3, x_4, x_6, x_7, x_8\}$ and $Rej(X) = \{x_2, x_5\}$, which means that the stocks x_1, x_3, x_4, x_6, x_7 and x_8 should be invested directly and the stocks x_5 and x_4 should not be invested.

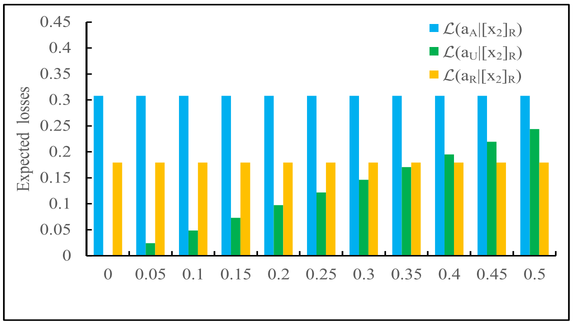
(2) The number of stocks in the acceptance region $Acc(X)$ increases with the parameter ϑ_k increasing, the number of stocks in the uncertainty region $Unc(X)$ decreases with the parameter ϑ_k increasing, and the number of stocks in the rejection region $Rej(X)$ increases with the parameter ϑ_k increasing. This also reconfirms the validity of the parameter ϑ_k as the risk avoidance coefficient in the decision-making processes.

(3) All the values of expected losses $\mathscr{L}(a_A|[x_i]_R)$ and $\mathscr{L}(a_R|[x_i]_R)$ for every stock are the same with the increase of parameter ϑ_k, and the value of expected losses $\mathscr{L}(a_U|[x_i]_R)$ $(i = 1, 2, \ldots, 8)$ for every stock decreases with the increase of parameter ϑ_k. This implies that the parameter ϑ_k as the risk avoidance coefficient only affects the decision action of further investigation in the stock investment problem.

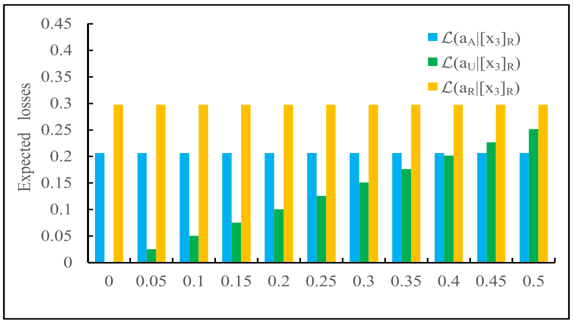
In summary, we can conclude that the parameter ϑ_k has a certain influence on the classification results. Decision-makers can enlarge or reduce the size of the uncertainty region in the actual decision-making environment according to their preferences. In general, our proposed three-way decision method can support the needs of decision-makers with different risk appetites and provide us with more diverse decision results.



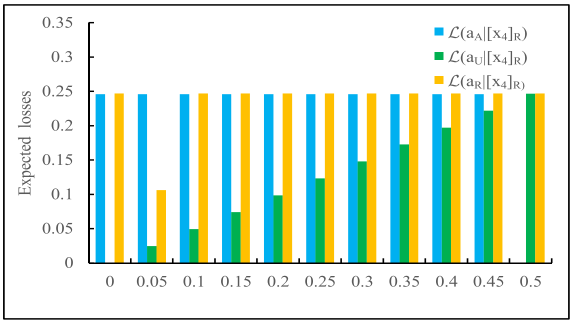
(1) The expected losses of x_1



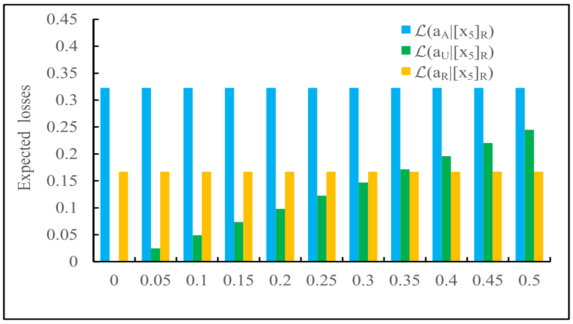
(2) The expected losses of x_2



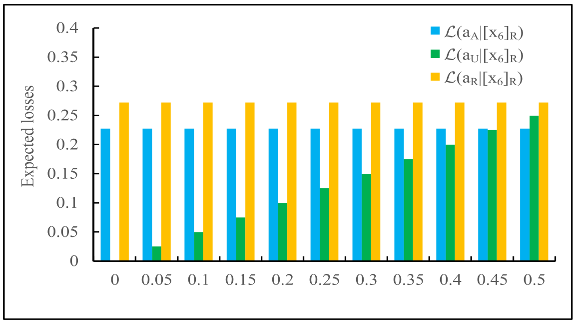
(3) The expected losses of x_3



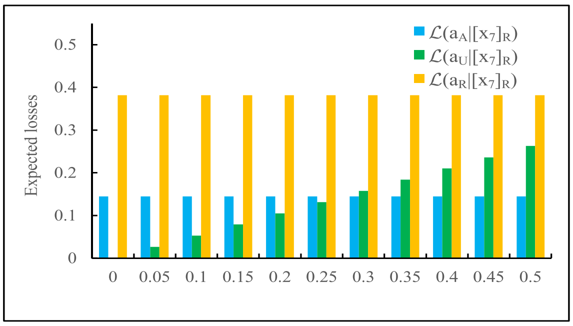
(4) The expected losses of x_4



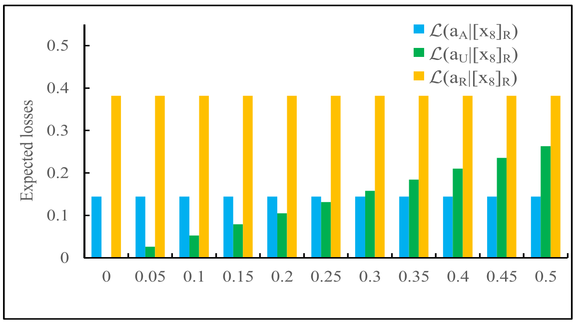
(5) The expected losses of x_5



(6) The expected losses of x_6



(7) The expected losses of x_7



(8) The expected losses of x_8

Fig. 3. The expected losses of eight stocks with the different values of ϑ_k.

5.4. Comparison analysis

In the previous sections, we propose a three-way decision method to address MADM problems under hesitant fuzzy environment and illustrate the procedure of it by an example of stock investment. To validate the effectiveness and superiority of the proposed three-way decision method, it is necessary to assess the credibility of the result obtained by the proposed method. The most common way to assess the credibility of the result is to compare it with other existing MADM methods. Thus, in the subsequent part, we compare our proposed three-way decision method with the existing ones.

The proposed three-way decision method is developed by virtue of the model of DTRSs under hesitant fuzzy environment. The loss functions in our proposed method are obtained from the evaluation values of attributes, and all alternatives are divided into three decision regions. Actually, the model of DTRSs is a novel developing strategy to handle realistic decision-making problems by the idea of three-way decisions, which has been applied in MADM problems by many scholars. For example, Jia and Liu [12] first used the attribute values to express the relative loss functions and proposed a three-way decision model under a fuzzy environment. Liu et al. [26] further investigated the relative loss functions derived from intuitionistic fuzzy attribute values and discussed a three-way decision model in the intuitionistic fuzzy context. Liang et al. [20] studied the relative loss and benefit functions obtained by interval type-2 fuzzy attribute values and put forward a three-way decision model under an interval type-2 fuzzy environment. Recently, Lei et al. [14] discussed the relative loss and benefit functions educed from hesitant fuzzy linguistic attribute values and constructed a three-way group decision model under a hesitant fuzzy linguistic environment. But the above four models cannot be directly utilized to deal with MADM problems under a hesitant fuzzy environment.

In what follows, we compare the decision result obtained by the proposed method with four classical MADM methods. The TOPSIS method as a useful MADM method is widely applied to MADM problems, so we compare the proposed method with the hesitant fuzzy TOPSIS (HF-TOPSIS) method [41]. Meanwhile, we also compare the proposed method with the hesitant fuzzy weighted average (HFWA) operator method [38] and the hesitant fuzzy power weighted average (HFPWA) operator method [46]. Moreover, the fuzzy decision class in the proposed method is estimated by grey relational analysis, so we compare the proposed method with the hesitant fuzzy grey relational analysis (HF-GRA) method [6]. It should be noted that the existing four MADM methods only aim to obtain the ranking of alternatives and to determine an optimal alternative. Thus, to compare with them, we can acquire the ranking of all alternatives based on the following three principles: (1) each stock's associated loss/cost is obtained by its final classification result, for example, if the stock x_i is divided into $Acc(X)$, it means that the stock x_i should be selected an acceptance decision and its associated loss/cost is $\mathscr{L}(a_A|[x_i]_R)$, similarly, if the stock x_i is divided into $Unc(X)$ or $Rej(X)$, then the associated loss/cost of it is $\mathscr{L}(a_U|[x_i]_R)$ or $\mathscr{L}(a_R|[x_i]_R)$; (2) in regard to three classifications, the ranking of the corresponding stocks in their own classification can be acquired by the associated costs of stocks in increasing order, respectively; (3) based on the preference order of (A1) $\succ$ (U1) $\succ$ (R1), the ranking of all stocks can be implemented. In what follows, according to the above three principles, we analyze the ranking of eight stocks.

At the beginning, based on the results in Table 8 and Fig. 2, the classification rule and associated cost of each stock are displayed in the following table (see Table 10).

Table 10
The classification rule and associated cost of each stock.

Stock	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
Classification rule	(A1)	(U1)	(U1)	(U1)	(R1)	(U1)	(A1)	(A1)
Cost(x_i)	0.1435	0.1764	0.1826	0.1795	0.1677	0.1809	0.1443	0.1511

Table 11
The decision results obtained by different methods.

Methods		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
HF-TOPSIS	Rank	1	5	7	6	8	4	2	3
HFAW	Rank	1	4	5	6	8	7	3	2
HFPWA	Rank	1	4	6	5	8	7	3	2
HF-GRA	Rank	1	4	7	6	8	5	3	2
Proposed method	Rank	1	4	7	5	8	6	2	3
	Classification results		$Acc(X) = \{x_1, x_7, x_8\}$		$Unc(X) = \{x_2, x_3, x_4, x_6\}$		$Rej(X) = \{x_5\}$		

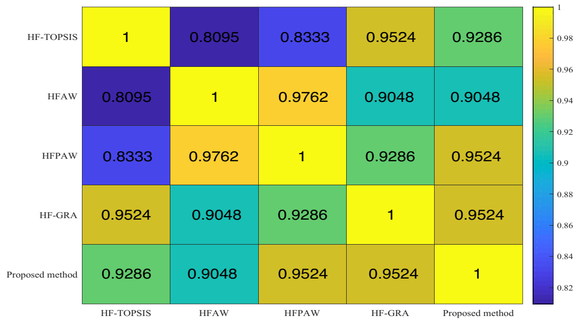


Fig. 4. The Sccs of the ranking results obtained by different methods.

Then, considering the associated cost of each stock, we rank the stocks in their own classification region. The result is listed as follows: for classification rule (A1): $x_1 \succ x_7 \succ x_8$; for classification rule (U1): $x_2 \succ x_4 \succ x_6 \succ x_3$; for classification rule (R1): x_5. Then, based on the preference order of $(A1) \succ (U1) \succ (R1)$, the ranking of eight stocks is obtained as follows: $x_1 \succ x_7 \succ x_8 \succ x_2 \succ x_4 \succ x_6 \succ x_3 \succ x_5$.

Finally, according to the evaluation result in Table 5, four existing MADM methods are separately used to deal with the stock investment problem presented in SubSection 5.1. The decision results of the proposed method and four existing ones are displayed in Table 11.

Furthermore, to obtain the relationship among the ranking results obtained by different methods, Spearman's correlation coefficient (Scc) [27] as an important indicator is utilized to explain the connection between two different methods and show the validity of the proposed three-way decision method. According to Table 11, the Sccs of the ranking results obtained by different methods are presented in Fig. 4.

From Table 11, we know that the stocks x_1 and x_5 are remained in the same position of the ranking results derived from five methods. In addition, the optimal stock obtained by these five methods is also the same, namely, the stock x_1, which illustrates that the proposed method is effective and feasible for the stock investment problem described in Subsection 5.1. In general, an Scc of two ranking results greater than 0.8 illustrates a strong similarity between the results obtained by different methods. From the result in Fig. 4, all Sccs are greater than 0.8. Thus, we can obtain that there is a good similarity between the ranking results obtained by the proposed method and four existing methods, which further demonstrates that the ranking result derived from the proposed method is valid and reliable.

Compared with the existing four methods, the proposed method considers the risk appetites of decision-makers and implements effective investment for all stocks by dividing all stocks into three pairwise disjoint classifications, which can effectively reduce decision risks and improve efficiency. In addition, the proposed method can depict uncertain and hesitant information that existed in MADM problems and provide a scientific and effective guideline for decision-makers. In specific, we sum up the merits of the proposed three-way decision method as follows:

- The relative loss functions in our proposed method are calculated by attribute values expressed by HFVs, which can be adjusted according to decision-makers' risk preferences and the actual decision-making problems. In this view, the proposed three-way decision method is more flexible.
- The proposed three-way decision method is established by taking advantage of DTRSs and HFSs simultaneously, which can not only consider decision risks in the decision-making processes but also instruct us how to select an action for each alternative and give its corresponding semantic interpretation.

6. Conclusions

In this article, we propose a DTFRS model in the hesitant fuzzy information system and establish a novel three-way decision method to deal with MADM problems. The main contributions of this article are summarized as follows. (1) A new fuzzy similarity relation between two objects is defined by utilizing the hesitant fuzzy distance function. (2) Based on the attribute values expressed by HFVs, we develop a method to calculate the relative loss functions and establish a DTFRS model in the hesitant fuzzy information system. The corresponding classification rules are induced and some relative properties are discussed. (3) A three-way decision method is proposed to address MADM problems under hesitant fuzzy environment, and the method is verified with an illustrative example of the stock investment problem, a sensitivity analysis, and a comparison analysis, respectively.

In the future research, we will focus on the following two issues. First, as a new generalization of HFSs, hesitant fuzzy linguistic term sets can effectively characterize qualitative evaluation information by combining HFSs [14,47] with fuzzy linguistic methods. Consequently, we can express evaluation values in information systems using hesitant fuzzy linguistic term sets and investigate a DTFRS model under hesitant fuzzy linguistic environment. Second, in the context of group decision-making, proportional interval type-2 hesitant fuzzy sets [2] provide a new dimension for constructing group information representation by integrating the proportional information of each generalized linguistic term. Therefore, it is advisable to develop a DTFRS model and a three-way group decision method in the framework of multi-attribute group decision-making with proportional interval type-2 hesitant fuzzy sets.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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