Information Sciences 541 (2020) 362–390

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

A novel decision-making approach based on three-way decisions in fuzzy information systems

Jin Ye [a], Jianming Zhan [a,*], Zeshui Xu [b]

[a] School of Mathematics and Statistics, Hubei Minzu University, Enshi, Hubei 445000, China
[b] Business School, Sichuan University, Chengdu, Sichuan 610064, China

ARTICLE INFO

Article history:
Received 13 April 2020
Received in revised form 22 June 2020
Accepted 25 June 2020
Available online 11 July 2020

Keywords:
Fuzzy α-neighborhood operator
Probabilistic rough fuzzy set
Three-way decision
Multi-criteria decision-making
Fuzzy information system

ABSTRACT

The decision-making theory with three-way decisions (3WD) improves the either-or decision results of the traditional multi-criteria decision-making (MCDM) methods and provides a more reasonable decision basis. This paper presents a novel 3WD method to solve MCDM problems in fuzzy information systems. Considering the usefulness of fuzzy neighborhood operators when dealing with fuzzy numerical data, we define a reflexive fuzzy α-neighborhood operator for overcoming the weaknesses of the existing fuzzy β-neighborhood operators in some applications. A probabilistic rough fuzzy set model and MCDM-based 3WD model are established by means of the constructed fuzzy ε-neighborhood classes. We give a determination method of the conditional probability according to data tables of MCDM problems. Particularly, the steps and algorithm of 3WD method are proposed. Then the feasibility and validity of the method are illustrated by the validity test and the comparative analysis on an example of project investments. Finally, we demonstrate the stability of the method through the experimental analysis on different data sets.

1. Introduction

In recent years, the idea of 3WD plays a unique role in the study of decision-making theory. The decision-making methods with 3WD can not only provide a complete ranking, but also classify all alternatives according to three decision rules. What is more, the 3WD theory takes into account the losses or costs of alternatives caused by taking different actions, which is line with decision-making situations in real life. Based on this, the reasonable semantics for making a decision can be clearly interpreted. In addition, since 3WD has become an important granular computing methodology and granular computing has been adopted as a useful and effective tool to handle decision-making problems under ambiguity and uncertainty, 3WD provides insights into deeper understanding for solving MCDM problems from the point of granular computing. In view of the above advantages of 3WD, the decision-making based on 3WD has attracted many scholars to carry out the research [8,11,14,17,18,24,27,30].

In 2009, Yao [40] first introduced the model of 3WD to explain the rules of decision-theoretic rough sets. The three decision rules correspond to positive rules for acceptance, boundary rules for delayed decisions, and negative rules for rejection, respectively. However, the original 3WD model has some shortcomings in some ways, such as the estimation of loss func-

* Corresponding author.
E-mail addresses: yejin9501ah@126.com (J. Ye), zhanjianming@hotmail.com (J. Zhan), xuzeshui@263.net (Z. Xu).

https://doi.org/10.1016/j.ins.2020.06.050
0020-0255/

tions, the conditional probability calculation method in fuzzy environments, the process of numerical data, etc. To solve the above weaknesses, many extended 3WD models [4,7,16,22,31,33,46] have been developed from two perspectives of connotation and extension.

The research on the connotation of 3WD focuses on the loss function and the conditional probability. In most cases, the loss functions are evaluated subjectively by decision-makers based on their own knowledge and experience. Since it is difficult to quantify the losses in practice, the types of loss functions have been generalized to all kinds of fuzzy environments, such as, interval environments [8,23], hesitant fuzzy environments [17,46], intuitionistic fuzzy environments [24], Pythagorean fuzzy environments [18], and so on. Nevertheless, no matter how the type of loss functions changes, the loss functions are still subjective in nature. In order to reduce the subjectivity of loss functions, Jia and Liu [11] presented an objective computation method to obtain the relative loss functions [21] which are the variant of loss functions. This method provides a new idea and insight for the combination of 3WD and MCDM. In this paper, we employ the same method as [11] to calculate the relative loss functions.

For the conditional probability, the existing researches have also given a lot of discussions. In 3WD theory, the conditional probability consists of a state and a description of an object. Generally, a state can be represented by either of the following two forms: classical sets and fuzzy sets [45]. However, whether the set of states is classic [40,41] or fuzzy [31], few studies indicate that the set of states can be computed. Based on this, we propose an objective approach to calculate the set of fuzzy states by means of weighted aggregations. Additionally, the description of an object can be viewed as a class of the object induced by a binary relation, such as an equivalence class [33,40], a similarity class [16,23], a neighborhood class [1,15], and so forth. Hence, the formula of the conditional probability in [40] can be extended based on different relation classes. Besides, Hu et al. [8] introduced the concept of the relative conditional probability. Lang et al. [14] proposed to replace the conditional probability with the distance function. Liang et al. [18] presented a method to obtain the conditional probability by using ideal TOPSIS solutions. Although there are many defined formulae of the conditional probability, the objective conditional probability based on decision-making problems is still lacking. To further enrich the study of the conditional probability, we propose a novel determination method of the conditional probability based on MCDM-driven data tables.

The extension of 3WD models mainly concentrates on the development of 3WD theory. On the one hand, from the perspective of vertical development, Hu [7] came up with the notion of 3WD spaces for making the 3WD theory more general, including the study of different types and properties of 3WD models. Yao [42] analyzed a wide sense of 3WD and put forward a trisecting-acting-outcome (TAO) model. Yao [43] systematically studed the connection of 3WD granular computing, rough set and formal concept analysis. On the other hand, from the perspective of horizontal development, the extension of 3WD models reflects the characteristics of knowledge transfer. That is, 3WD models based on different relations are proposed, such as dominance relations [22], fuzzy relations [30], neighborhood relations [15], etc. Likewise, 3WD models based on different information systems are also presented, such as hybrid information systems [16,33], incomplete information systems [23,39], neighborhood systems [1], and so forth. It is not difficult to find from the current study that the existing fuzzy β-neighborhood operators [2,25,38], as fuzzy binary relations, can handle fuzzy numerical data effectively. However, the study of 3WD models based on fuzzy β-neighborhood operators is still a blank. One of the main reasons for this phenomenon is that the existing fuzzy β-neighborhood operators are defective in applications, which is described in Sections 2.1 and 3 in detail. Therefore, this paper establishes a novel 3WD model by virtue of the reflexive fuzzy α-neighborhood operator defined in Section 3.1.

It should be noted that the purpose of this paper is to propose a decision-making method for solving project investment decision-making problems rather than just for classifications. As we know, the traditional MCDM methods are two-way decisions, namely, acceptance and rejection, such as the EDAS method [5], the TOPSIS method [10], the aggregation operator method [6,37], the PROMETHEE method [19], etc. Decision-makers can only make decisions based on their own needs from the ranking result, regardless of whether the projects are good or bad. Thus, this outcome may increase the risk of decision-making. On the contrary, decision results based on 3WD can reduce the risk of decision-making by considering three decision rules. For example, we can see from [11] that the ranking result $A_7 > A_1 > A_2 > A_4 > A_5 > A_6 > A_8 > A_3$ is obtained by the TOPSIS method [10] and the ranking result (based on α) $A_7 > A_1 > A_2 > A_4 > A_5 > A_6 > A_3 > A_8$ is calculated according to the decision-making method with 3WD proposed by [11]. Assume that the conditional probability of each investment project is 0.4, by means of the method proposed in [11], we have three regions as follows: $POS(C) = \{A_1, A_7\}, BND(C) = \{A_2, A_4, A_5\}$ and $NEG(C) = \{A_3, A_6, A_8\}$. Suppose that decision-makers plan to invest in three projects, the projects A_7, A_1 and A_2 are chosen according to the TOPSIS method [10], while the projects A_7 and A_1 are definitely selected and the project A_2 needs further consideration by the method proposed in [11]. If decision-makers do not invest in the project A_2, it can reduce the risk of the decision. Otherwise, the risk of the decision increases. Moreover, based on the above example, it can be seen that the number of investment projects to be accepted is determined by subjective judgments of decision-makers in traditional MCDM methods, but the numbers of investment projects in three regions are determined objectively in the decision-making methods with 3WD. Based on these reasons, we propose a novel 3WD method to MCDM in this paper.

From the above descriptions, we outline four motivations of this paper as follows:

(1) Compared to the traditional MCDM methods with two-way decisions [5,10,19,26,37], a delayed decision is added and the loss or risk of each alternative for taking certain action is taken into account, which is more in line with realistic decision situations. At the same time, 3WD can provide a more reasonable decision basis and aid decision-makers to adjust their requirements according to three decision rules. In light of the above strengths of 3WD, this paper plans to propose an MCDM method via the idea of 3WD.
(2) Since neighborhood relations can deal with numerical data effectively [15], many models based on neighborhood relations have been developed, such as the neighborhood-based rough set models [9,20,36], the neighborhood-based decision-theoretic rough set models [15], the neighborhood-based 3WD models [1], and so forth. As a generalization of neighborhood relations under fuzzy environments, fuzzy β-neighborhood operators [25,38] are introduced by means of fuzzy logical conjunctions, which are effective for dealing with fuzzy numerical data. At present, the fuzzy rough set models based on fuzzy β-neighborhood operators have been widely studied [12,25,38,47]. However, the research of 3WD model based on fuzzy β-neighborhood operators is lacking. In order to solve the above problem, we intend to establish a novel 3WD based on fuzzy β-neighborhood operators in fuzzy information systems.
(3) We can see from [25,38] that the existing fuzzy β-neighborhood operators are not reflexive when $\beta \in (0,1)$. Moreover, for the fuzzy neighborhood operators in [2], their applications are restricted by a strict condition which is at least one value of 1 for the membership of each object under all sub-coverings. To overcome the above shortcomings, we prefer to define a reflexive fuzzy β-neighborhood operator (hereinafter called the fuzzy α-neighborhood operator).
(4) In the decision-making methods with 3WD, few studies have been published on the computation method of the set of states. In general, two states in 3WD model are given subjectively. To reduce the subjectivity in the decision-making process, this paper intends to present an objective method to calculate the set of states. Besides, in [11], there is a lack of a reasonable semantic explanation for the ranking results based on thresholds when the conditional probabilities are inconsistent. To solve this problem, we try to propose a determination method of the conditional probability based on MCDM problems to develop a new 3WD method.

The rest of this paper is organized as follows: Section 2 reviews several studies related to this paper and gives some new insights. In Section 3, we first define a reflexive fuzzy α-neighborhood operator and construct fuzzy ε-neighborhood classes, then we present a novel MCDM-based 3WD model. Section 4 proposes a 3WD method to handle decision-making problems and illustrates the feasibility of the proposed method by a practical example. Meanwhile, the validity test is carried out to verify the validity of the proposed method. Sections 5 and 6 confirm that the proposed method is effective and stable according to the comparative and experimental analyses. Section 7 concludes this paper and indicates future research works.

2. Preliminary

In this section, we review several existing studies, such as the related work of fuzzy neighborhood operators [2,25,38], the notion of fuzzy information systems [4,32] and the computation approaches of loss functions under MCDM environments [11]. Besides, we give some new notes for the above concepts.

2.1. Fuzzy neighborhood operators based on fuzzy β-covering

As special fuzzy binary relations, fuzzy neighborhood operators [2,25,38] are derived from classical neighborhood operators [44] under fuzzy information.

Definition 2.1. [25] Let U be a nonempty finite set of objects and $C = \{C_1, C_2, \ldots, C_m\}$ be a family of m fuzzy sets on U. If C is called a fuzzy β-covering for $\beta \in (0,1]$, i.e., $\left(\bigcup_{j=1}^{m} C_j\right)(u) \geqslant \beta$ for any $u \in U$, then we call (U, C) a fuzzy β-covering approximation space (FβCAS).

In an FβCAS, Ma [25] proposed a fuzzy β-neighborhood operator as follows:

Definition 2.2. [25,38] Let (U, C) be an FβCAS and $\beta \in (0,1]$. For any $u, v \in U$, a fuzzy β-neighborhood operator is defined by:

$$N_{Ma}^{\beta}(u)(v) = \inf\{C_j(v) | C_j \in C, \quad C_j(u) \geqslant \beta\} = \bigwedge_{C_j \in N_C^{\beta}(u)} C_j(v), \tag{2-1}$$

where $N_C^{\beta}(u) = \{C_j \in C | C_j(u) \geqslant \beta\}$ is the β-neighborhood system of u.

It is worth noting that the fuzzy β-neighborhood operator N_{Ma}^{β} does not satisfy the reflexivity when $\beta \in (0,1)$, i.e., $N_{Ma}^{\beta}(u)(u) \neq 1$ for any $u \in U$. Based on the research of Ma [25], the notion of fuzzy β-minimal description [38] is introduced as follows:

Definition 2.3. [38] Let (U, C) be an FβCAS and $\beta \in (0,1]$. For any $u \in U$, the fuzzy β-minimal description of u is defined as follows:

$$\mathrm{Md}_{\mathrm{C}}^{\beta}(u)=\left\{\mathrm{C_j}\in \mathrm{N}_{\mathrm{C}}^{\beta}(u):\forall \mathrm{C_s}\in \mathrm{N}_{\mathrm{C}}^{\beta}(u)\wedge \mathrm{C_s}\subseteq \mathrm{C_j}\Rightarrow \mathrm{C_j}=\mathrm{C_s}\right\}. \tag{2-2}$$

Definition 2.3 makes a lot of sense since the semantic interpretation of fuzzy β-minimal description is reasonable. That is, we only need to use the essential characteristics with respect to this object when we describe an object, rather than all the characteristics.

Furthermore, Yang and Hu [38] proposed three new fuzzy β-neighborhood operators, which can form a fuzzy lattice with the operator presented in Definition 2.2. However, it is a pity that these three operators are not reflexive either. In addition, D'eer et al. [2] also constructed four types of fuzzy neighborhood operators by means of fuzzy logical connectives in a fuzzy covering approximation space[1] (FCAS). Although these four operators satisfy the reflexivity, their applications are restricted by a strict condition which is at least one value of 1 for the membership of each object under all sub-coverings.

2.2. Fuzzy information systems

Information systems, also called data tables, as the carrier of knowledge are widely used. For the classical information systems [28], the relation between an object and its attribute values is certain. However, data are usually vague and imprecise in real life. Therefore, we introduce fuzzy information systems to represent uncertain information.

Definition 2.4. [4,32] A fuzzy information system (FIS) can be denoted by a pair $S=(\mathrm{U},\mathrm{C})$, where $\mathrm{U}=\{u_1,u_2,\ldots,u_n\}$ is a nonempty finite set of objects and $\mathrm{C}=\{\mathrm{C}_1,\mathrm{C}_2,\ldots,\mathrm{C}_m\}\subseteq\mathcal{F}(\mathrm{U})$ ($\mathcal{F}(\mathrm{U})$ denotes the family of all fuzzy sets on U) is a nonempty finite set of attributes for describing objects. The more detailed form is a quaternion form $S=(\mathrm{U},\mathrm{C},\mathrm{V},\mathrm{I})$, in which $\mathrm{V}=\bigvee_{\mathrm{C}_j\in\mathrm{C}}\mathrm{V}_{\mathrm{C}_j}=\{u_{ij}|i\in\Phi=\{1,2,\ldots,n\},j\in\Delta=\{1,2,\ldots,m\}\}\subseteq[0,1]$ where $\mathrm{V}_{\mathrm{C}_j}$ is the value domain of the attribute C_j and $\mathrm{I}=\left\{\mathrm{I}_j:\mathrm{U}\to\mathrm{V}_{\mathrm{C}_j}\right\}$ is a mapping set, such that $\mathrm{I}_j(u_i)=u_{ij}\in[0,1]$ for $u_i\in\mathrm{U}$.

In general, FISs satisfy the following two conditions: (1) $\mathrm{C}_j\neq\emptyset$ for each $\mathrm{C}_j\in\mathrm{C}$, namely, $\forall\mathrm{C}_j\in\mathrm{C},\exists u_i\in\mathrm{U}$, s.t. $u_{ij}\neq0$. Otherwise, the attribute C_j is worthless. (2) Each object has a non-zero value under at least one attribute, that is, $\forall u_i\in\mathrm{U},\exists\mathrm{C}_j\in\mathrm{C}$, s.t. $u_{ij}\neq0$. Otherwise, the object u_i is unmeaning. Next, Table 1 shows an FIS with the consistent types of attributes, where $\mathrm{C_j}=\frac{u_{1j}}{u_1}+\frac{u_{2j}}{u_2}+\ldots+\frac{u_{nj}}{u_n}$ is a fuzzy set on U. Furthermore, we have the following remark:

Remark 2.5. According to Definitions 2.1 and 2.4, if the types of attributes are consistent in FISs, the set C of attributes describing objects is a family of m fuzzy sets and satisfies that $\left(\bigcup_{j=1}^{m}\mathrm{C_j}\right)(u_i)\geqslant\beta\,(\beta\in(0,1])$ for any $u_i\in\mathrm{U}$. Thus, C constitutes a fuzzy β-covering. That is, an FIS with the consistent types of attributes can be regarded as an FβCAS and vice versa.

2.3. The 3WD model under multiple-criteria environment

Given the advantages of the theory of 3WD, Jia and Liu [11] proposed a 3WD model to handle MCDM problems. What is more, an approach is presented to compute loss functions and thresholds under the multiple-criteria environment. Before reviewing the computation approach, we first describe the following MCDM problems.

Let $\mathrm{U}=\{u_1,u_2,\ldots,u_n\}$ be a set of n alternatives and $\mathrm{C}=\{\mathrm{C}_1,\mathrm{C}_2,\ldots,\mathrm{C}_m\}$ be a set of m criteria. In general, the dimensions and types of criteria may be different. Decision-makers evaluate each alternative in line with different criteria and obtain an evaluation set $\mathrm{V}=\{\mathrm{C}_j(u_i)|i\in\Phi;j\in\Delta\}$, in which $\mathrm{C}_j(u_i)$ denotes the evaluation value of the alternative u_i under the criterion C_j. Thus, for each C_j, a domain of evaluation values of alternatives can be obtained, which is labeled as $\mathrm{D}_j=\{\mathrm{C}_j(u_i)|i\in\Phi\}$. In D_j, $\mathrm{C}_j(u_i)\in\left[u_{\min}^j,u_{\max}^j\right]$ for each $u_i\in\mathrm{U}$, where $u_{\min}^j$ and $u_{\max}^j$ denote the minimum value and the maximum value under the criterion C_j. Besides, since each criterion plays a different role in decision-making, a weight vector $\mathrm{W}=\{w_1,w_2,\ldots,w_m\}$ is used for representing the set of importance of criteria, such that $0\leqslant w_j\leq1$ and $\sum_{j=1}^{m}w_j=1$.

As we know, data tables can be utilized to express the information of MCDM problems. If the types of criteria are consistent, the data tables can be regarded as FISs (see Table 1) when decision-makers use fuzzy numbers to evaluate all alternatives under criteria. That is, $\mathrm{C}_j(u_i)=u_{ij}\in[0,1]$ for $i\in\Phi$ and $j\in\Delta$. Otherwise, we need to convert the types of criteria so that the types of criteria are consistent.

By the research results of [11] and the above discussions, the relative loss functions and three thresholds of each alternative are introduced as follows:

Let $\Omega=\{X,\neg X\}$ be a set of states and $\mathfrak{A}=\{a_P,a_B,a_N\}$ be a set of actions, where the three actions a_P,a_B and a_N represent $u_i\in\mathrm{Pos}(X)$ (acceptance), $u_i\in\mathrm{Bnd}(X)$ (non-commitment) and $u_i\in\mathrm{Neg}(X)$ (rejection), respectively. The following Table 2 shows the original and relative loss functions regarding the cost (or risk) of three actions in two states, where $\widetilde{\lambda}_{BP}=\lambda_{BP}-\lambda_{PP}$, $\widetilde{\lambda}_{NP}=\lambda_{NP}-\lambda_{PP}$, $\widetilde{\lambda}_{PN}=\lambda_{PN}-\lambda_{NN}$ and $\widetilde{\lambda}_{BN}=\lambda_{BN}-\lambda_{NN}$.

[1] Fuzzy covering approximation space: When $\beta=1$, fuzzy β-covering approximation spaces are fuzzy covering approximation spaces.

Table 1
The fuzzy information system $S = (\mathrm{U}, \mathrm{C})$.

U/C	C_1	C_2	$\cdots$	C_m
u_1	u_{11}	u_{12}	$\cdots$	u_{1m}
u_2	u_{21}	u_{22}	$\cdots$	u_{2m}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_n	u_{n1}	u_{n2}	$\cdots$	u_{nm}

Table 2
Two types of loss functions.

	Original loss functions		Relative loss functions	
	X (P)	$\neg X$ (N)	X (P)	$\neg X$ (N)
a_P	λ_{PP}	λ_{PN}	0	$\widetilde{\lambda}_{PN}$
a_B	λ_{BP}	λ_{BN}	$\widetilde{\lambda}_{BP}$	$\widetilde{\lambda}_{BN}$
a_N	λ_{NP}	λ_{NN}	$\widetilde{\lambda}_{NP}$	0

In Table 2, the original loss functions are interpreted as follows: $\lambda_{PP}, \lambda_{BP}$ and λ_{NP} represent the costs or losses caused by taking actions of a_P, a_B and a_N, respectively, when $u_i \in X$. In like manner, $\lambda_{PN}, \lambda_{BN}$ and λ_{NN} represent the costs or losses caused by taking actions of a_P, a_B and a_N, respectively, when $u_i \in \neg X$. In addition, the relative loss functions can be explained as follows: when $u_i \in X$, by taking the loss of the action a_P as the standard, the relative losses of a_P, a_B and a_N are 0, $\widetilde{\lambda}_{BP}$ and $\widetilde{\lambda}_{NP}$, respectively; when $u_i \in \neg X$, by taking the loss of action a_N as the standard, the relative losses of a_P, a_B and a_N are $\widetilde{\lambda}_{PN}, \widetilde{\lambda}_{BN}$ and 0, respectively.

Based on an MCDM problem in an FIS, Jia and Liu [11] proposed an approach to compute the relative loss functions of each alternative, as shown in Table 3, where the parameter σ_j $(\in [0,1])$ is the risk avoidance coefficient under the criterion C_j. More details can be obtained from [11].

According to Table 3 and the theory of 3WD, three thresholds θ_i, ξ_i and γ_i with respect to each alternative are calculated as follows:

$$\theta_i = \frac{\sum_{j=1}^{m}(1-\sigma_j)w_j(1-u_{ij})}{\sum_{j=1}^{m}(1-\sigma_j)w_j(1-u_{ij}) + \sum_{j=1}^{m}\sigma_j w_j u_{ij}}, \tag{2-3}$$

$$\xi_i = \frac{\sum_{j=1}^{m}\sigma_j w_j(1-u_{ij})}{\sum_{j=1}^{m}\sigma_j w_j(1-u_{ij}) + \sum_{j=1}^{m}(1-\sigma_j)w_j u_{ij}}, \tag{2-4}$$

$$\gamma_i = \sum_{j=1}^{m} w_j(1-u_{ij}). \tag{2-5}$$

3. Fuzzy -neighborhood operator-based 3WD model

In this section, we first discuss the importance of reflexivity of binary relations in 3WD theory. Then, a novel 3WD model based on reflexive fuzzy α-neighborhood operators is proposed. Section 3.1 constructs fuzzy α-neighborhood classes by means of the proposed reflexive fuzzy α-neighborhood operator. Based on this, a probabilistic rough fuzzy set model is presented in Section 3.2. To solve some weaknesses of probabilistic rough fuzzy set model, Section 3.3 establishes a 3WD model based on MCDM.

As described in Section 2.1, the fuzzy neighborhood operators in [2,25,38] are deficient. We give an example to illustrate the importance of reflexivity of binary relations below.

Example 3.1. Let (U, C) be an FβCAS, as shown in Table 4, where $\mathrm{U} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $\mathrm{C} = \{C_1, C_2, C_3, C_4, C_5\}$.

According to Definition 2.1 and Table 4, we can know that $\beta \in (0, 0.6]$ and $C_2 \supseteq C_4$. Let $\beta = 0.5$, we can obtain the results in Table 5.

By ε-cut sets $(\varepsilon \in [0,1])$ [13] and the operator presented in Definition 2.2, we introduce the following fuzzy ε-neighborhood class $[u_i]^{\varepsilon}_{\mathrm{N}^{\beta}_{Ma}}$ of u_i:

$$[u_i]^{\varepsilon}_{\mathrm{N}^{\beta}_{Ma}} = \left\{u_t | \mathrm{N}^{\beta}_{Ma}(u_i)(u_t) \geqslant \varepsilon\right\}. \tag{3-1}$$

When $\varepsilon = \beta \in (0, 0.6]$, the fuzzy ε-neighborhood class of u_i is equivalent to the β-neighborhood of u_i proposed by Ma [25]. We have that $u_i \in [u_i]^{\varepsilon}_{\mathrm{N}^{\beta}_{Ma}} \neq \emptyset$. However, when $\varepsilon \in (0.6, 1]$, $[u_i]^{\varepsilon}_{\mathrm{N}^{\beta}_{Ma}} = \emptyset$ may hold for some $u_i \in \mathrm{U}$. Let $\varepsilon = 0.7$, we obtain that $[u_1]^{0.7}_{\mathrm{N}^{0.5}_{Ma}} = [u_2]^{0.7}_{\mathrm{N}^{0.5}_{Ma}} = [u_3]^{0.7}_{\mathrm{N}^{0.5}_{Ma}} = [u_4]^{0.7}_{\mathrm{N}^{0.5}_{Ma}} = [u_6]^{0.7}_{\mathrm{N}^{0.5}_{Ma}} = \emptyset$ by Table 5. Moreover, $[u_5]^{0.7}_{\mathrm{N}^{0.5}_{Ma}} = \{u_3\}$. Hence, $\bigcup_{i=1}^{6} [u_i]^{0.7}_{\mathrm{N}^{0.5}_{Ma}} = \{u_3\} \neq \mathrm{U}$.

Table 3
The relative loss functions of u_i.

u_i	X (P)	$\neg X$ (N)
a_P	0	$\sum_{j=1}^{m} w_j(1-u_{ij})$
a_B	$\sum_{j=1}^{m} \sigma_j w_j u_{ij}$	$\sum_{j=1}^{m} \sigma_j w_j(1-u_{ij})$
a_N	$\sum_{j=1}^{m} w_j u_{ij}$	0

Table 4
An FβCAS of the example.

U/C	C_1	C_2	C_3	C_4	C_5
u_1	0.6	0.7	0.5	0.6	0.8
u_2	0.4	0.3	0.6	0.2	0.7
u_3	0.7	0.8	0.3	0.7	0.5
u_4	0.2	0.5	0.7	0.4	0.3
u_5	0.3	0.6	0.1	0.6	0.4
u_6	0.5	0.4	0.6	0.3	0.6

Table 5
The membership of u_t on fuzzy 0.5-neighborhood of u_i $(i, t \in \{1, 2, \ldots, 6\})$.

$N_{Ma}^{0.5}(u_i)(u_t)$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	0.5	0.2	0.3	0.2	0.1	0.3
u_2	0.5	0.6	0.3	0.3	0.1	0.6
u_3	0.6	0.2	0.5	0.2	0.3	0.3
u_4	0.5	0.3	0.3	0.5	0.1	0.4
u_5	0.6	0.2	0.7	0.4	0.6	0.3
u_6	0.5	0.4	0.3	0.2	0.1	0.5

From Example 3.1, it can be seen that the construction of fuzzy ε-neighborhood classes based on the fuzzy β-neighborhood operators with β-reflexivity[2] is unreasonable. If we only set $\varepsilon = \beta \in (0, 0.6]$, the fuzzy ε-neighborhood classes ignore the case where the belief level ε belongs to (0.6,1]. In applications, especially decision-making applications, this can lead to one-sided results that can affect to make the right decisions. Furthermore, we consider $\varepsilon \in (0.6, 1]$. If we regard fuzzy ε-neighborhood class of u_i as information granule of u_i, the union of the information granules of all objects may not cover the universe U, which is unreasonable in 3WD theory. The main reason for the above weaknesses is that the fuzzy β-neighborhood operator defined by Ma [25] is β-reflexive rather than reflexive. Therefore, the reflexivity of fuzzy neighborhood operators is very important for the study of 3WD.

3.1. A reflexive fuzzy α-neighborhood operator

To overcome the shortcomings of fuzzy neighborhood operators in [2,25,38], a reflexive fuzzy α-neighborhood operator is constructed in FISs that the types of criteria are consistent. Then, the concept of fuzzy ε-neighborhood classes is presented by using the idea of ε-cut sets.

Definition 3.2. Let $S = (U, C)$ be an FIS and $\mathcal{I}$ be an $\mathcal{R}$-implicator.[3] By introducing a critical value α, where $\alpha \in \left(0, \min_i\left(\max_j u_{ij}\right)\right)$, we define a fuzzy α-neighborhood operator $\mathcal{N}_\alpha$ as follows:

$$\mathcal{N}_\alpha(u_i)(u_t) = \bigwedge_{C_j \in Md_C^\alpha(u_i)} \mathcal{I}(C_j(u_i), \quad C_j(u_t)) = \bigwedge_{C_j \in Md_C^\alpha(u_i)} \mathcal{I}(u_{ij}, u_{tj}), \tag{3-2}$$

where $u_i, u_t \in U$ and $Md_C^\alpha(u_i)$ is the fuzzy α-minimal description of u_i.

Obviously, fuzzy α-neighborhood $\mathcal{N}_\alpha(u_i)$ of u_i is a fuzzy set on U, i.e., $\mathcal{N}_\alpha(u_i) = \frac{\mathcal{N}_\alpha(u_i)(u_1)}{u_1} + \frac{\mathcal{N}_\alpha(u_i)(u_2)}{u_2} + \cdots + \frac{\mathcal{N}_\alpha(u_i)(u_n)}{u_n}$. In the following, the properties of fuzzy α-neighborhood operators are demonstrated by the following proposition.

Proposition 3.3. Let $S = (U, C)$ be an FIS, $\mathcal{I}$ be an $\mathcal{R}$-implicator and $\mathcal{T}$ be a left-continuous t-norm. We have the following properties for the fuzzy α-neighborhood operator $\mathcal{N}_\alpha$:

[2] β-reflexivity: Let U be a universe and N be a fuzzy neighborhood operator. If $N(u)(u) \geqslant \beta$ $(\beta \in (0, 1])$ for each $u \in U$, then N is β-reflexive.
[3] R-implicator: for any $x, y \in [0, 1], \mathcal{I}_\mathcal{T}(x, y) = \sup\{z \in [0, 1] | \mathcal{T}(x, z) \leqslant y\}$, where $\mathcal{T}$ is a t-norm.

(1) The fuzzy α-neighborhood operator $\mathcal{N}_\alpha$ is reflexive, i.e., $\mathcal{N}_\alpha(u_i)(u_i) = 1$ for each $u_i \in \mathrm{U}$.

(2) The fuzzy α-neighborhood operator $\mathcal{N}_\alpha$ satisfies the $\mathcal{T}$-transitivity, i.e., for $u_i, u_s, u_t \in \mathrm{U}, \mathcal{T}(\mathcal{N}_\alpha(u_i)(u_s), \mathcal{N}_\alpha(u_s)(u_t)) \leqslant \mathcal{N}_\alpha(u_i)(u_t)$.

(3) $\mathrm{N}^\alpha_{Ma}(u_i) \subseteq \mathcal{N}_\alpha(u_i)$ for each $u_i \in \mathrm{U}$.

(4) $\mathrm{N}^1_{\mathrm{C}}(u_i) \subseteq \mathcal{N}_{\alpha=1}(u_i)$ for each $u_i \in \mathrm{U}$, where $\mathrm{N}^1_{\mathrm{C}}$ is the first type of fuzzy neighborhood operator proposed by D'eer et al. [2].

Proof. (1) Since $\mathcal{I}$ is an $\mathcal{R}$-implicator, $\mathcal{I}(u)(u) = 1$ for each $u \in [0,1]$. Thus, for any $u_i \in \mathrm{U}, \mathcal{N}_\alpha(u_i)(u_i) = \bigwedge_{\mathrm{C}_j \in \mathrm{Md}^\alpha_{\mathrm{C}}(u_i)} \mathcal{I}(u_{ij}, u_{ij}) = 1$.

(2) For $u_i, u_s, u_t \in \mathrm{U}, \mathcal{T}(\mathcal{N}_\alpha(u_i)(u_s), \mathcal{N}_\alpha(u_s)(u_t)) \leqslant \bigwedge_{\mathrm{C}_j \in \mathrm{Md}^\alpha_{\mathrm{C}}(u_i)} \mathcal{T}(\mathcal{I}(u_{ij}, u_{sj}), \mathcal{I}(u_{sj}, u_{tj})) \leqslant \bigwedge_{\mathrm{C}_j \in \mathrm{Md}^\alpha_{\mathrm{C}}(u_i)} \mathcal{I}(u_{ij}, u_{tj}) = \mathcal{N}_\alpha(u_i)(u_t)$.

(3) For any $u_i, u_t \in \mathrm{U}$, since $\mathrm{Md}^\alpha_{\mathrm{C}}(u_i) \subseteq \mathrm{N}^\alpha_{\mathrm{C}}(u_i)$, we have $\mathrm{N}^\alpha_{Ma}(u_i)(u_t) = \bigwedge_{\mathrm{C}_j \in \mathrm{N}^\alpha_{\mathrm{C}}(u_i)} (u_{tj}) \leqslant \bigwedge_{\mathrm{C}_j \in \mathrm{Md}^\alpha_{\mathrm{C}}(u_i)} (u_{tj}) = \bigwedge_{\mathrm{C}_j \in \mathrm{Md}^\alpha_{\mathrm{C}}(u_i)} \mathcal{I}(1, u_{tj}) \leqslant \bigwedge_{\mathrm{C}_j \in \mathrm{Md}^\alpha_{\mathrm{C}}(u_i)} \mathcal{I}(u_{ij}, u_{tj}) = \mathcal{N}_\alpha(u_i)(u_t)$. Thus, $\mathrm{N}^\alpha_{Ma}(u_i) \subseteq \mathcal{N}_\alpha(u_i)$ for each $u_i \in \mathrm{U}$.

(4) For any $u_i, u_t \in \mathrm{U}$, when $\alpha = \min_i\left(\max_j u_{ij}\right) = 1, \mathcal{N}_{\alpha=1}(u_i)(u_t) = \bigwedge_{\mathrm{C}_j \in \mathrm{Md}^\alpha_{\mathrm{C}}(u_i)} \mathcal{I}(u_{ij}, u_{tj}) \geqslant \bigwedge_{\mathrm{C}_j \in \mathrm{C}} \mathcal{I}(u_{ij}, u_{tj}) = \mathrm{N}^1_{\mathrm{C}}(u_i)(u_t)$. That is, $\mathrm{N}^1_{\mathrm{C}}(u_i) \subseteq \mathcal{N}_{\alpha=1}(u_i)$. □

In the following, we construct fuzzy ε-neighborhood classes to transform the fuzzy α-neighborhoods into the classical sets.

Definition 3.4. Let $S = (\mathrm{U}, \mathrm{C})$ be an FIS, $\mathcal{N}_\alpha$ be the fuzzy α-neighborhood operator in Definition 3.2 and $\varepsilon \in [0,1]$. We define fuzzy ε-neighborhood class $[u_i]^\varepsilon_{\mathcal{N}_\alpha}$ of u_i as follows:

$$[u_i]^\varepsilon_{\mathcal{N}_\alpha} = \{u_t \in \mathrm{U} | \mathcal{N}_\alpha(u_i)(u_t) \geqslant \varepsilon\}. \tag{3-3}$$

According to Proposition 3.3, we have the following remark:

Remark 3.5. Let $S = (\mathrm{U}, \mathrm{C})$ be an FIS and $\varepsilon \in [0,1]$. For the fuzzy ε-neighborhood classes, the following conclusions hold:

(1) $u_i \in [u_i]^\varepsilon_{\mathcal{N}_\alpha}$ for any $u_i \in \mathrm{U}$.

(2) $\bigcup_{u_i \in \mathrm{U}} [u_i]^\varepsilon_{\mathcal{N}_\alpha} = \mathrm{U}$.

Obviously, the union of the fuzzy ε-neighborhood classes of all objects can cover the universe U rather than partition.

Example 3.6. Let (U, C) be an FIS, as shown Table 4 in Example 3.1, and $\alpha = 0.5$. By using Definition 3.2, we first obtain Tables 6 and 7 as follows:

Let $\varepsilon = 0.7$, then the following results can be calculated based on Definition 3.4:

$$[u_1]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_3, u_6\}, \quad [u_2]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_2, u_3, u_6\}, \quad [u_3]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_3\},$$
$$[u_4]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_2, u_4, u_6\}, \quad [u_5]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_3, u_4, u_5, u_6\}, \quad [u_6]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_2, u_3, u_4, u_6\}.$$

3.2. Probabilistic rough fuzzy set models based on fuzzy ε-neighborhood classes

By virtue of the operator $\mathcal{N}_\alpha$, we present the following probabilistic rough fuzzy set model.

Definition 3.7. Let (U, C) be an FIS, N_α be a fuzzy α-neighborhood operator and $\varepsilon \in [0,1]$. For $u_i, u_t \in \mathrm{U}$ and $X \in \mathcal{F}(\mathrm{U}), P\left(X | [u_i]^\varepsilon_{\mathcal{N}_\alpha}\right) = \frac{\sum_{u_t \in [u_i]^\varepsilon_{\mathcal{N}_\alpha}} X(u_t)}{|[u_i]^\varepsilon_{\mathcal{N}_\alpha}|}$ is a probability measure that u_i in $[u_i]^\varepsilon_{\mathcal{N}_\alpha}$ belongs to the fuzzy set X. Thus, if $0 \leqslant \xi_i < \theta_i \leq 1$ for any $i \in \Phi$, the lower and upper approximations of X are defined with respect to θ_i and ξ_i as follows:

$$\underline{Apr}_{\{(\xi_i,\theta_i)|i\in\Phi\}}(X) = \left\{u_i \in \mathrm{U} | P\left(X | [u_i]^\varepsilon_{\mathcal{N}_\alpha}\right) \geqslant \theta_i\right\}, \tag{3-4}$$
$$\overline{Apr}_{\{(\xi_i,\theta_i)|i\in\Phi\}}(X) = \left\{u_i \in \mathrm{U} | P\left(X | [u_i]^\varepsilon_{\mathcal{N}_\alpha}\right) > \xi_i\right\}. \tag{3-5}$$

If $\underline{Apr}_{\{(\xi_i,\theta_i)|i\in\Phi\}}(X) \neq \overline{Apr}_{\{(\xi_i,\theta_i)|i\in\Phi\}}(X)$, X is a probabilistic rough fuzzy set. Otherwise, X is a probabilistic definable fuzzy set. With the help of Definition 3.7, the universe U can be divided into three regions below:

$$\mathrm{Pos}(X) = \underline{Apr}_{\{(\xi_i,\theta_i)|i\in\Phi\}}(X) = \left\{u_i \in \mathrm{U} | P\left(X | [u_i]^\varepsilon_{\mathcal{N}_\alpha}\right) \geqslant \theta_i\right\}, \tag{3-6}$$
$$\mathrm{Bnd}(X) = \overline{Apr}_{\{(\xi_i,\theta_i)|i\in\Phi\}}(X) - \underline{Apr}_{\{(\xi_i,\theta_i)|i\in\Phi\}}(X) = \left\{u_i \in \mathrm{U} | \xi_i < P\left(X | [u_i]^\varepsilon_{\mathcal{N}_\alpha}\right) < \theta_i\right\}, \tag{3-7}$$
$$\mathrm{Neg}(X) = \mathrm{U} - \overline{Apr}_{\{(\xi_i,\theta_i)|i\in\Phi\}}(X) = \left\{u_i \in \mathrm{U} | P\left(X | [u_i]^\varepsilon_{\mathcal{N}_\alpha}\right) \leqslant \xi_i\right\}. \tag{3-8}$$

Table 6
The 0.5-neighborhood system and fuzzy 0.5-minimal description of u_i.

	u_1	u_2	u_3	u_4	u_5	u_6
$N_C^{0.5}(u_i)$	$\{C_1, C_2, C_3, C_4, C_5\}$	$\{C_3, C_5\}$	$\{C_1, C_2, C_4, C_5\}$	$\{C_2, C_3\}$	$\{C_2, C_4\}$	$\{C_1, C_3, C_5\}$
$Md_C^{0.5}(u_i)$	$\{C_1, C_3, C_4, C_5\}$	$\{C_3, C_5\}$	$\{C_1, C_4, C_5\}$	$\{C_2, C_3\}$	$\{C_4\}$	$\{C_1, C_3, C_5\}$

Table 7
The membership of u_t on fuzzy 0.5-neighborhood $N_{0.5}(u_i)$ of u_i.

$N_{0.5}(u_i)(u_t)$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	1	0.6	0.7	0.5	0.6	0.7
u_2	0.9	1	0.7	0.6	0.5	0.9
u_3	0.9	0.5	1	0.5	0.6	0.6
u_4	0.8	0.8	0.6	1	0.4	0.9
u_5	1	0.6	1	0.8	1	0.7
u_6	0.9	0.9	0.7	0.7	0.5	1

It should be noted that one or two of $\mathrm{Pos}(X), \mathrm{Bnd}(X)$ and $\mathrm{Neg}(X)$ may be the empty set. So, we have the following remark:

Remark 3.8. Let $\aleph = \{\mathrm{Pos}(X), \mathrm{Bnd}(X), \mathrm{Neg}(X)\}$ be a family of three regions of X on U, we have: $\mathrm{Pos}(X) \cap \mathrm{Bnd}(X) = \emptyset, \mathrm{Pos}(X) \cap \mathrm{Neg}(X) = \emptyset, \mathrm{Bnd}(X) \cap \mathrm{Neg}(X) = \emptyset$ and $\mathrm{Pos}(X) \cup \mathrm{Bnd}(X) \cup \mathrm{Neg}(X) = \mathrm{U}$. Thus, $\aleph$ is called a weak tri-partition [43] of U.

Next, we discuss several degenerations of the proposed probabilistic rough fuzzy set model.

Proposition 3.9. Let $\theta = \theta_i$ and $\xi = \xi_i$ for all $i \in \Phi$. $\mathcal{N}_\alpha$ is the defined fuzzy α-neighborhood operator. For $X \in \mathcal{F}(\mathrm{U})$, the following conclusions hold:

(1) If $\mathcal{N}_\alpha$ is replaced with an equivalence relation, then the proposed probabilistic rough fuzzy set is degenerated to the probabilistic rough fuzzy set proposed by Sun et al. [31]. Furthermore, if $\xi = 1 - \theta$, then we can obtain the variable precision probabilistic rough fuzzy set [31].

(2) If X is a crisp set of U and $\mathcal{N}_\alpha$ is replaced with an equivalence relation, then the proposed probabilistic rough fuzzy set can be degenerated to the classical probabilistic rough set [35].

(3) Based on (2), the proposed probabilistic rough fuzzy set is degenerated to the variable precision rough set [50] if $\xi = 1 - \theta$.

In this paper, we only give the definition of the lower and upper approximations for the proposed probabilistic rough fuzzy set model without discussing their properties because the properties are similar to the properties of models in [31,35,50].

Example 3.10. (Continue with Example 3.6) Let $X = \frac{0.5}{u_1} + \frac{0.7}{u_2} + \frac{0.6}{u_3} + \frac{0.6}{u_4} + \frac{0.8}{u_5} + \frac{0.7}{u_6}$ be a fuzzy set of U. Table 8 gives the two threshold values for each object as follows:

According to Definition 3.7, we have:

$$P\left(X|[u_1]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.6, \quad P\left(X|[u_2]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.625, \quad P\left(X|[u_3]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.55,$$
$$P\left(X|[u_4]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.625, \quad P\left(X|[u_5]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.64, \quad P\left(X|[u_6]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.62.$$

Thus,

$$\underline{Apr}_{\{(\xi_i,\theta_i)|i\in\Phi\}}(X) = \{u_2, u_4, u_5\}, \qquad \overline{Apr}_{\{(\xi_i,\theta_i)|i\in\Phi\}}(X) = \{u_1, u_2, u_4, u_5\}.$$

We can obtain the following three regions of U by using the formulae (3-6)–(3-8):

$$\mathrm{Pos}(X) = \{u_2, u_4, u_5\}, \quad \mathrm{Bnd}(X) = \{u_1\}, \quad \mathrm{Neg}(X) = \{u_3, u_6\}.$$

3.3. MCDM-based 3WD models

From the research above, some concepts lack systematic calculation methods and reasonable semantic interpretations, like the given thresholds θ_i and ξ_i. Moreover, the set of states in 3WD still lacks reasonable calculation ways up to now. To deal with the above problems, we establish a novel MCDM-based 3WD model in light of the 3WD theory proposed by Yao [40,41]. Next, we define an MCDM-based hybrid data table through the combination of MCDM and 3WD.

Table 8
The two parameter values of u_i.

	u_1	u_2	u_3	u_4	u_5	u_6
θ_i	0.65	0.62	0.64	0.6	0.62	0.7
ξ_i	0.55	0.52	0.58	0.5	0.55	0.65

Definition 3.11. A three-way-based fuzzy data Table (3WFDT) is defined as 3WFDT=(FDT, RLFT), where FDT=(FIS,W) is a fuzzy data table about MCDM problems, including an FIS with the same types of criteria and a weight vector W, and RLFT = $\left(\mathrm{U}, \widetilde{\lambda}^i_{\diamond\diamond}\right)$ is a relative loss function table. For any $u_i \in \mathrm{U}$, the information of u_i in 3WFDT can be denoted by m + 6 elements: $\mathrm{I}(u_i) = \left(u_{i1}, u_{i2}, \ldots, u_{im}; \widetilde{\lambda}^i_{PP}, \widetilde{\lambda}^i_{BP}, \widetilde{\lambda}^i_{NP}, \widetilde{\lambda}^i_{PN}, \widetilde{\lambda}^i_{BN}, \widetilde{\lambda}^i_{NN}\right)$, as shown in Table 9.

As stated in Section 2.3, the relative loss functions of each object in Table 9 can be explained. In addition, the six relative loss functions of u_i satisfy the following conditions:

$$0 = \widetilde{\lambda}^i_{PP} \leqslant \widetilde{\lambda}^i_{BP} < \widetilde{\lambda}^i_{NP}, \qquad 0 = \widetilde{\lambda}^i_{NN} \leqslant \widetilde{\lambda}^i_{BN} < \widetilde{\lambda}^i_{PN}. \tag{3 − 9}$$

In what follows, we propose an objective method to obtain "fuzzy concept X".

Definition 3.12. Let $X = \frac{X(u_1)}{u_1} + \frac{X(u_2)}{u_2} + \cdots + \frac{X(u_n)}{u_n}$ be a fuzzy set on U. Based on FDT for an MCDM problem, the membership $X(u_i)$ of u_i with respect to X is calculated by:

$$X(u_i) = \sum_{j=1}^{m} w_j u_{ij}. \tag{3 − 10}$$

Similarly, we can also define "fuzzy concept" $\neg X = \sum_{i=1}^{n} \frac{\neg X(u_i)}{u_i}$, where $\neg X(u_i) = 1 - X(u_i)$.

Let $\Omega = \{X, \neg X\}$ be a set of states, where $X, \neg X \in \mathcal{F}(\mathrm{U})$. For each $u_i \in \mathrm{U}$, the formula $X(u_i) + \neg X(u_i) = 1$ shows that the fuzzy sets X and $\neg X$ form a fuzzy partition of U. The set of actions is given by $\mathfrak{A} = \{a_P, a_B, a_N\}$, where a_P, a_B and a_N denote the three actions in classifying an object u_i, namely, deciding $u_i \in \mathrm{Pos}(X)$, deciding $u_i \in \mathrm{Bnd}(X)$ and deciding $u_i \in \mathrm{Neg}(X)$, respectively. Based on a 3WDFDT, we can calculate the following expected loss $\mathrm{R}\left(a_\bullet|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right)$ of u_i taking the action of $a_\bullet$ $(\bullet = P, B, N)$ by using Bayesian decision process:

$$\mathrm{R}\left(a_P|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) = \widetilde{\lambda}^i_{PN} P\left(\neg X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right), \tag{3 − 11}$$

$$\mathrm{R}\left(a_B|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) = \widetilde{\lambda}^i_{BP} P\left(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) + \widetilde{\lambda}^i_{BN} P\left(\neg X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right), \tag{3 − 12}$$

$$\mathrm{R}\left(a_N|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) = \widetilde{\lambda}^i_{NP} P\left(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right). \tag{3 − 13}$$

According to the Bayesian decision theory with the minimum loss, we whereafter deduce three decision rules (P)-(N) as follows:

(P) If $\mathrm{R}\left(a_P|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) \leqslant \mathrm{R}\left(a_B|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right)$ and $\mathrm{R}\left(a_P|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) \leqslant \mathrm{R}\left(a_N|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right)$, decide $u_i \in \mathrm{Pos}(X)$;
(B) If $\mathrm{R}\left(a_B|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) \leqslant \mathrm{R}\left(a_P|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right)$ and $\mathrm{R}\left(a_B|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) \leqslant \mathrm{R}\left(a_N|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right)$, decide $u_i \in \mathrm{Bnd}(X)$;
(N) If $\mathrm{R}\left(a_N|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) \leqslant \mathrm{R}\left(a_P|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right)$ and $\mathrm{R}\left(a_N|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) \leqslant \mathrm{R}\left(a_B|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right)$, decide $u_i \in \mathrm{Neg}(X)$.

Afterwards, by virtue of the formula $P\left(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) + P\left(\neg X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) = 1$ for any $u_i \in \mathrm{U}$, the above rules (P)-(N) can be simplified as the following rules (P1)-(N1):

(P1) If $P\left(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) \geqslant \theta_i$ and $P\left(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) \geqslant \gamma_i$, decide $b_i \in \mathrm{POS}(D_1)$;
(B1) If $P\left(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) \leqslant \theta_i$ and $P\left(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) \geqslant \xi_i$, decide $b_i \in \mathrm{BND}(D_1)$;
(N1) If $P\left(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) \leqslant \gamma_i$ and $P\left(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right) \leqslant \xi_i$, decide $b_i \in \mathrm{NEG}(D_1)$.

The thresholds θ_i, γ_i and ξ_i for $i \in \Phi$ can be computed as follows:

$$\theta_i = \frac{\widetilde{\lambda}^i_{PN} - \widetilde{\lambda}^i_{BN}}{\left(\widetilde{\lambda}^i_{PN} - \widetilde{\lambda}^i_{BN}\right) + \widetilde{\lambda}^i_{BP}}, \tag{3 − 14}$$

$$\gamma_i = \frac{\widetilde{\lambda}^i_{PN}}{\widetilde{\lambda}^i_{PN} + \widetilde{\lambda}^i_{NP}}, \tag{3 − 15}$$

$$\xi_i = \frac{\widetilde{\lambda}^i_{BN}}{\left(\widetilde{\lambda}^i_{NP} - \widetilde{\lambda}^i_{BP}\right) + \widetilde{\lambda}^i_{BN}}. \tag{3 − 16}$$

Furthermore, we consider the following condition:

$$\frac{\widetilde{\lambda}^i_{BP}}{\widetilde{\lambda}^i_{PN} - \widetilde{\lambda}^i_{BN}} < \frac{\widetilde{\lambda}^i_{NP} - \widetilde{\lambda}^i_{BP}}{\widetilde{\lambda}^i_{BN}}, i \in \Phi. \tag{3 − 17}$$

Table 9
A three-way-based fuzzy data table.

	FDT				RLFT					
	C_1	C_2	$\cdots$	C_m	$\widetilde{\lambda}^i_{PP}$	$\widetilde{\lambda}^i_{BP}$	$\widetilde{\lambda}^i_{NP}$	$\widetilde{\lambda}^i_{PN}$	$\widetilde{\lambda}^i_{BN}$	$\widetilde{\lambda}^i_{NN}$
u_1	u_{11}	u_{12}	$\cdots$	u_{1m}	0	$\widetilde{\lambda}^1_{BP}$	$\widetilde{\lambda}^1_{NP}$	$\widetilde{\lambda}^1_{PN}$	$\widetilde{\lambda}^1_{BN}$	0
u_2	u_{21}	u_{22}	$\cdots$	u_{2m}	0	$\widetilde{\lambda}^2_{BP}$	$\widetilde{\lambda}^2_{NP}$	$\widetilde{\lambda}^2_{PN}$	$\widetilde{\lambda}^2_{BN}$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_n	u_{n1}	u_{n2}	$\cdots$	u_{nm}	0	$\widetilde{\lambda}^n_{BP}$	$\widetilde{\lambda}^n_{NP}$	$\widetilde{\lambda}^n_{PN}$	$\widetilde{\lambda}^n_{BN}$	0
W	w_1	w_2	$\cdots$	w_m	*	*	*	*	*	*

It implies $0 \leqslant \xi_i < \gamma_i < \theta_i \leq 1$. After tie-breaking rules, we can obtain the following simplified three decision rules (P2)-(N2):

(P2) If $P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}) \geqslant \theta_i$, decide $u_i \in \mathrm{Pos}(X)$;

(B2) If $\xi_i < P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}) < \theta_i$, decide $u_i \in \mathrm{Bnd}(X)$;

(N2) If $P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}) \leqslant \xi_i$, decide $u_i \in \mathrm{Neg}(X)$.

Thus, the established 3WD model can degrade into the probabilistic rough fuzzy set model proposed in Section 3.2. Besides, we also consider another condition:

$$\frac{\widetilde{\lambda}^i_{BP}}{\widetilde{\lambda}^i_{PN} - \widetilde{\lambda}^i_{BN}} \geqslant \frac{\widetilde{\lambda}^i_{NP} - \widetilde{\lambda}^i_{BP}}{\widetilde{\lambda}^i_{BN}}, i \in \Phi. \qquad (3-18)$$

It implies $0 \leqslant \theta_i \leqslant \gamma_i \leqslant \xi_i \leq 1$. In this case, the rules (P1)-(N1) can be re-expressed as the following rules (P3)-(N3):

(P3) If $P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}) \geqslant \gamma_i$, decide $u_i \in \mathrm{Pos}(X)$;

(N3) If $P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}) < \gamma_i$, decide $u_i \in \mathrm{Neg}(X)$.

From the above descriptions, we can see that the rules (P2)-(N2) correspond to 3WD, while the rules (P3) and (N3) correspond to two-way decisions. In the following, the algorithm of the MCDM-based 3WD model is given by Algorithm 1.

Input: $3WFDT = \{FDT, RLFT\}$, parameters α and ε
Output: Three regions Pos, Bnd and Neg

begin
  for $u_i \in$ U **do**
    compute: $X(u_i) = \sum_{j=1}^{m} w_j u_{ij}$
  end
  for $u_i \in$ U **do**
    Given a parameter value α
    for $t = 1, 2, \cdots, n$ **do**
      compute: $\mathcal{N}_\alpha(u_i)(u_t) = \bigwedge_{C_j \in \mathrm{Md}^{\alpha}_{C}(u_i)} \mathcal{I}(u_{ij}, u_{tj})$
    end
    Given a parameter value ε
    compute: $[u_i]^{\varepsilon}_{\mathcal{N}_\alpha} = \{u_t \mid \mathcal{N}_\alpha(u_i)(u_t) \geq \varepsilon\}$
  end
  for $u_i \in$ U **do**
    compute: $\theta_i = \frac{\widetilde{\lambda}^i_{PN} - \widetilde{\lambda}^i_{BN}}{(\widetilde{\lambda}^i_{PN} - \widetilde{\lambda}^i_{BN}) + \widetilde{\lambda}^i_{BP}}$, $\gamma_i = \frac{\widetilde{\lambda}^i_{PN}}{\widetilde{\lambda}^i_{PN} + \widetilde{\lambda}^i_{NP}}$, $\xi_i = \frac{\widetilde{\lambda}^i_{BN}}{(\widetilde{\lambda}^i_{NP} - \widetilde{\lambda}^i_{BP}) + \widetilde{\lambda}^i_{BN}}$
  end
  for $u_i \in$ U **do**
    if $P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}) \geq \theta_i$ **then**
      decide: $u_i \in \mathrm{Pos}(X)$
      else if $P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}) \leq \xi_i$ **then**
        decide: $u_i \in \mathrm{Neg}(X)$
        else
          decide: $u_i \in \mathrm{Bnd}(X)$
        end
      end
    end
  end
  return: $\mathrm{Pos}(X)$, $\mathrm{Bnd}(X)$ and $\mathrm{Neg}(X)$
end

Remark 3.13. In Algorithm 1, we compute the fuzzy set X, with the time complexity of $O(n)$. We calculate the fuzzy ε-neighborhood classes, with the time complexity of $O(m^2n + n^2m)$. The time complexity of the computation of three thresholds is $O(n)$. We calculate three regions with the time complexity of $O(n)$. Therefore, the time complexity of Algorithm 1 is $O(m^2n + n^2m)$.

4. A novel three-way multi-criteria decision-making (3WMCDM) method

In recent years, 3WD models to MCDM [8,24,27] have become one of the hot topics in the study of decision-making theory. In this section, we develop a new MCDM method in the context of investment project decision-making according to the MCDM-based 3WD model established in Section 3.3 and design the corresponding algorithm for the method. Based on this, an example with actual rankings is given to demonstrate the feasibility of the proposed method. In addition, we perform a validity test of the proposed method by using the three test criteria presented in [34,46]. The results show that the proposed 3WMCDM method is feasible and effective.

4.1. Problem statement

In real life, investment project decision-making, as a kind of common MCDM problem, is widely studied by decision-makers. How to make more scientific decisions is generally concerned. Hence, lots of methodologies [11,17] are put forward to make an optimal decision from a number of alternatives. In what follows, we mainly make some illustrations to the investment project decision-making problems. This is the foundation for us to come up with a new methodology.

Similar to the description of MCDM in Section 2.3, we briefly introduce an investment project decision-making problem as follows: $\mathrm{U} = \{u_1, u_2, \ldots, u_n\}$ is a set of n investment projects and $\mathrm{C} = \{\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_m\}$ is a set of m criteria. There are two possible states for an investment project: a good investment project or a bad one. Moreover, the types of criteria may be different (cost criterion and benefit criterion). To eliminate the dimension of criteria, the evaluation values of investment projects can be represented by fuzzy values. That is, $\mathrm{C}_j(u_i) \in [0,1]$ for $i \in \Phi$ and $j \in \Delta$. Therefore, we can obtain an FIS $S = (\mathrm{U}, \mathrm{C})$. However, the weight of each criterion is often unknown in practical decision-making problems. When the weights are unknown, the objective methods can be employed to obtain weights of criteria from the data set itself. In general, decision-makers can accept, reject or not-commit an investment project based on its quality. As a matter of fact, no matter what state an investment project is in, there is a loss or risk associated with taking a certain action. Therefore, an investment project can generate six loss functions based on two states and three actions.

Note 1: The background to the investment project decision-making is to give readers a clearer understanding of the method to be proposed. In fact, the following method can be applied to other decision-making problems with FISs in the context of multiple criteria.

4.2. A description of the 3WMCDM method

Following the above statement, we develop a novel 3WMCDM method by using the established MCDM-based 3WD model to handle investment project decision-making problems and describe the decision-making process of 3WMCDM in detail as follows:

Firstly, assume that an FIS with different types of criteria is obtained, which is called the original FIS, according to the decision-making problem described above. If the weights of criteria are unknown, the following entropy weight method [29,49] is used.

Definition 4.1. [49] Let $S = (\mathrm{U}, \mathrm{C})$ be an original FIS. The following formula is given to calculate the entropy value (e_j) of the jth criterion:

$$e_j = -\frac{1}{\ln n}\sum_{i=1}^{n} p_{ij} \ln p_{ij}, \tag{4 − 1}$$

where $e_j > 0$ and $p_{ij} = \frac{\mathrm{C}_j(u_i)}{\sum_{i=1}^{n} \mathrm{C}_j(u_i)}$ for $i \in \Phi$ and $j \in \Delta$. For the case where $\mathrm{C}_j(u_i) = 0, p_{ij} \ln p_{ij} = 0$. Then, the entropy weight (w_j) for the jth criterion is computed as follows:

$$w_j = \frac{1 - e_j}{\sum_{j=1}^{m}(1 - e_j)}, \quad j \in \Delta. \tag{4 − 2}$$

Obviously, $w_j \in [0,1]$ and $\sum_{j=1}^{m} w_j = 1$. So, we obtain a weight vector $\mathrm{W} = \{w_1, w_2, \ldots, w_m\}$.

Secondly, due to different types of criteria, we unify the types of criteria by the following way:

If C_j is a benefit criterion, then we have

$$u_{ij} = \mathrm{C}_j(u_i), i \in \Phi \tag{4 − 3}$$

If C_j is a cost criterion, then we have

$$u_{ij} = 1 - \mathrm{C}_j(u_i), i \in \Phi \tag{4 − 4}$$

As can be seen that we convert the cost criteria into the benefit criteria. Thus an FIS with the same types of criteria can be obtained, which is called the transformed FIS.

Thirdly, we calculate the six relative loss functions generated by each investment project taking three actions in different states by using the formulae in Table 3. The calculated relative loss functions satisfy the formula (3-9). Therefore, a 3WFDT can be obtained by Definition 3.11.

Fourthly, according to the obtained 3WFDT, a fuzzy set X (denotes the set of good investment projects) is computed by Definition 3.12. Meanwhile, three thresholds (θ_i, γ_i and ξ_i) of each investment project are also calculated by using the formulae (3-14)–(3-16).

In fact, we can calculate θ_i, γ_i and ξ_i directly by using the formulae (2-3)–(2-5). Besides, the fuzzy set X is explained as follows: $X(u_i)$ denotes the degree of membership that the investment project u_i is a good investment project. It is certain that u_i is a good investment project when $X(u_i) = 1$. Thus, the union of membership of each investment project to a good investment project constitutes a fuzzy concept on U.

Next, we compute fuzzy ε-neighborhood class $[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}$ of each investment project by means of the formulae (3-2) and (3-3), which α is a critical value and ε is a belief level.

In practical decision-making problems, α can be interpreted as follows: in order to ensure a fair campaign for each investment project, decision-makers set a critical value α according to the actual situation so that each investment project meets the α-level requirements under at least one criterion. Moreover, a belief level ε can be interpreted as the degree of reliability between two investment projects. In general, the value of ε is given by the experience of decision-makers.

Afterwards, the conditional probability $P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha})$ is calculated. Furthermore, we obtain three regions of U by virtue of the three decision rules derived from the established 3WD model.

Finally, we rank all investment projects by using the principle of $\mathrm{Pos}(X) \succ \mathrm{Bnd}(X) \succ \mathrm{Neg}(X)$ and the associated costs with minimum risk. In this paper, the associated cost of each investment project is calculated by the following formula:

$$\mathrm{cost}(u_i) = \begin{cases} \mathrm{R}\left(a_P|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right), & \text{if} \quad u_i \in \mathrm{Pos}(X), \\ \mathrm{R}\left(a_B|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right), & \text{if} \quad u_i \in \mathrm{Bnd}(X), \\ \mathrm{R}\left(a_N|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right), & \text{if} \quad u_i \in \mathrm{Neg}(X). \end{cases} \qquad (4-5)$$

Let two investment projects u_i and u_t be in the same region. If $\mathrm{cost}(u_i) < \mathrm{cost}(u_t)$, then u_i is strictly better than u_t, i.e., $u_i \succ u_t$. Suppose that $\exists u_i \in \mathrm{Pos}(X), u_i \succ u_t$ if $\mathrm{cost}(u_i) < \mathrm{cost}(u_t)$ for $\forall u_t \in \mathrm{Pos}(X)$ $(t \neq i)$. Then, u_i is the optimal investment project.

4.3. Steps and algorithm

To simplify the 3WMCDM method presented in Section 4.2, the key steps are listed as follows:

Input: An FIS $S = (\mathrm{U}, \mathrm{C})$, two parameters α and ε.

Output: The ranking results of the 3WMCDM method.

Step 1: Check the weight vector W. If W is unknown, we compute the W by the formulae (4-1) and (4-2). Meanwhile, if the types of criteria are same in the FIS, then we go to Step 3. Otherwise, we go to Step 2.

Step 2: Obtain the transformed FIS by the formulae (4-3) and (4-4).

Step 3: Compute the fuzzy set X, the relative loss functions and thresholds (θ_i, γ_i and ξ_i) by Definition 3.12, the formulae in Table 3 and the formulae ((2-3)–(2-5).

Step 4: Given two parameter values α and ε, compute the fuzzy ε-neighborhood classes $[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}$ by the formulae (3-2) and (3-3).

Step 5: Calculate the conditional probability $P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha})$.

Step 6: Obtain three regions by generating decision rules.

Step 7: Rank alternatives by virtue of $\mathrm{Pos}(X) \succ \mathrm{Bnd}(X) \succ \mathrm{Neg}(X)$ and the associated costs.

In the following, if the weight vector of criteria is unknown, Algorithm 2 is presented to implement the proposed 3WMCDM method.

Input: An FIS $S = \{\mathrm{U}, \mathrm{C}\}$, two parameters α and ε
Output: Ranking of alariatives

begin
  for $\mathrm{C}_j \in \mathrm{C}$ **do**
    compute: all entropy values e_j, $j = 1, 2, \cdots, m$ // by the formulae (4-1)
    compute: all entropy weights w_j, $j = 1, 2, \cdots, m$ // by the formulae (4-2)
  end
  for $u_i \in \mathrm{U}$ **do**
    if C_j is a benefit criterion, **then** $u_{ij} = \mathrm{C}_j(u_i)$, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$
    if C_j is a cost criterion, **then** $u_{ij} = 1 - \mathrm{C}_j(u_i)$, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$
  end
  for $u_i \in \mathrm{U}$ **do**
    compute: all memberships $X(u_i)$, $i = 1, 2, \cdots, n$ // by the formula (3-10)
    compute: all relative loss functions $\widetilde{\lambda}^i_{BP}$, $\widetilde{\lambda}^i_{NP}$, $\widetilde{\lambda}^i_{PN}$ and $\widetilde{\lambda}^i_{BN}$, $i = 1, 2, \cdots, n$ // by the formula in Table 3
    compute: all thresholds θ_i, γ_i, ξ_i, $i = 1, 2, \cdots, n$ // by the formulae (2-3)-(2-5)
  end
  Given thresholds α and ε
  for $u_i \in \mathrm{U}$ **do**
    compute: all fuzzy ε-neighborhood classes $[u_i]^{\varepsilon}_{\mathcal{N}_\alpha} = \{u_t \mid \mathcal{N}_\alpha(u_i)(u_t) \geq \varepsilon\}$, $i = 1, 2, \cdots, n$ // by the formulae (3-2) and (3-3)
  end
  for $u_i \in \mathrm{U}$ **do**
    compute: all conditional probabilities $P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}) = \frac{\sum_{u_t \in [u_i]^{\varepsilon}_{\mathcal{N}_\alpha}} X(u_t)}{|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}|}, i = 1, 2, \cdots, n$
  end
  for $u_i \in \mathrm{U}$ **do**
    compute: three regions
    $\mathrm{Pos}(X)$: all investment projects that satisfy the following rule:
    if $P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}) \geq \theta_i$, **decide** $u_i \in \mathrm{Pos}(X)$
    $\mathrm{Bnd}(X)$: all investment projects that satisfy the following rule:
    if $\xi_i < P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}) < \theta_i$, **decide** $u_i \in \mathrm{Bnd}(X)$
    $\mathrm{Neg}(X)$: all investment projects that satisfy the following rule:
    if $P(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}) \leq \xi_i$, **decide** $u_i \in \mathrm{Neg}(X)$
  end
  for $u_i \in \mathrm{U}$ **do**
    compute: all associated costs $\mathrm{cost}(u_i)$, $i = 1, 2, \cdots, n$
    if $u_i \in \mathrm{Pos}(X)$, **then** $\mathrm{cost}(u_i) = \mathrm{R}(a_P|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha})$ // by the formula (3-11)
    if $u_i \in \mathrm{Bnd}(X)$, **then** $\mathrm{cost}(u_i) = \mathrm{R}(a_B|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha})$ // by the formula (3-12)
    if $u_i \in \mathrm{Neg}(X)$, **then** $\mathrm{cost}(u_i) = \mathrm{R}(a_N|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha})$ // by the formula (3-13)
  end
  for $u_i, u_t \in \mathrm{U}$ **do**
    if u_i, u_t are in the same region and $\mathrm{cost}(u_i) < \mathrm{cost}(u_t)$, **then** $u_{i \succ u_t}$
    else by means of the principle of $\mathrm{Pos}(X) \succ \mathrm{Bnd}(X) \succ \mathrm{Neg}(X)$
  end
  return: the ranking results
end

Remark 4.2. Algorithm 2 computes the ranking of all alternatives by virtue of the proposed 3WMCDM method. In the algorithm, the time complexity of Step 1 is $\mathrm{O}(nm)$. Step 2 causes the time complexity of $\mathrm{O}(m)$. Step 3 causes the time complexity of $\mathrm{O}(n)$. Step 4 computes the fuzzy α-neighborhood operator, with the time complexity of $\mathrm{O}(m^2n + n^2m)$. The time complexity of Steps 5–7 is $\mathrm{O}(n)$. Thus, the time complexity of Algorithm 2 is $\mathrm{O}(m^2n + n^2m)$.

When the weight vector W is known, we ignore the weight calculation part in Algorithm 2 and the corresponding input of the known weight vector. At this point, the time complexity of Step 1 is $\mathrm{O}(1)$. The time complexity of other steps and the overall algorithm remains the same.

Note 2: Whether the weight vector W is known or unknown, the steps of the proposed method are reasonable and necessary for making scientific decisions. The rationality and necessity of each step are explained below.

The weight vector and the types of criteria are two important factors that need to be considered in the process of decision-making. Thus, Step 1 in the proposed method considers two cases of weights of criteria and different types of criteria. Furthermore, to facilitate the processing of data, we unify the different types of criteria by Step 2. Then, according to the transformed data, we use the idea of weighted aggregation [6] to aggregate the fuzzy evaluation values of each alternative on different criteria into the comprehensive fuzzy membership of each alternative. Based on the comprehensive fuzzy membership of each alternative, all alternatives can be ranked.

However, the ranking result obtained in this way can only be viewed as two-way decisions, namely, it is either/or. Such a decision does not correspond to the actual decision with the delayed decision. Usually, the delayed decision is presented in many decision-making problems. For example, in an investment project decision-making problem, the traditional MCDM methods can be used to make a decision. Nevertheless, the traditional MCDM methods only determine the optimal investment project or more projects worth investing in from top to bottom according to the ranking by the subjective judgments of decision-makers. They ignore the need for further evaluation to determine the final investment projects. This makes the decision results derived from the traditional MCDM methods too subjective and harsh since the final decision-making result is a subjectively two-way decisions result, namely, deterministic invest in or not invest in a project. Thus, we need to find a novel way to MCDM. The direct way is to add indeterminacy to the decision results. This thought coincides with the idea of 3WD. In 3WD, the delayed decision as a non-deterministic decision is added based on two-way decisions. In view of the advantages of 3WD, we do not use the comprehensive fuzzy membership of each alternative to rank alternatives and make decisions. We use the 3WD theory to make more reasonable and scientific decision-making. Hence, Steps 3–7 are designed to obtain a more reasonable ranking of alternatives and a 3WD result, which are the key steps of the proposed 3WMCDM method.

Using the aggregation information of each alternative, Step 3 constructs a fuzzy state X on U and computes the relative loss functions and thresholds. Following the idea of 3WD, Step 4 uses the defined fuzzy neighborhood operator to construct the information granularity of each alternative. Based on Steps 3 and 4, the conditional probability of each alternative is calculated in Step 5. By comparing the conditional probability of each alternative with the corresponding three thresholds, Step 6 classifies all alternatives into three regions ($\mathrm{Pos}(X), \mathrm{Bnd}(X)$ and $\mathrm{Neg}(X)$). Each region corresponds to a decision rule. Finally, by considering the associated cost of each alternative in its corresponding region and the principle of $\mathrm{Pos}(X) \succ \mathrm{Bnd}(X) \succ \mathrm{Neg}(X)$, we rank all alternatives in Step 7. According to the semantic interpretation of 3WD, the semantic interpretation of Steps 3–7 is reasonable. Besides, compared with the ranking results obtained by the membership degree of X, the ranking results obtained by Steps 3–7 not only reduce the decision risk, but also is more in line with the practical situation. Therefore, each step of the proposed 3WMCDM method is reasonable and necessary.

4.4. Numerical example

For simplicity, this section adopts the data set in [11] to verify the feasibility of the proposed 3WMCDM method. An example of investment projects is enumerated as follows: Let $\mathrm{U} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be a set of eight investment projects and $\mathrm{C} = \{\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4, \mathrm{C}_5\}$ be a set of five criteria, i.e., expected benefits, environmental influence, market saturation, social benefits and energy conservation, respectively. C_2 and C_3 are cost criteria, the others are benefit criteria. The evaluation value of each investment project under each criterion is shown in Table 10, which is an original FIS of the example given by [11].

We firstly consider the case where the weight vector of criteria is unknown. According to Steps 1–3, we can obtain a 3WFDT in Table 11, where an FDT is obtained by the formulae (4-1)–(4-4) and a RLFT is obtained by using the research results of [11]. Here, the risk avoidance coefficient vector $\sigma = \{0.35, 0.4, 0.4, 0.4, 0.5\}$ of criteria in [11] is adopted.

At the same time, based on the FDT in Table 11, a set of “good investment project” X (the fuzzy set) and all thresholds (θ_i, γ_i and ξ_i) can be computed by means of Step 3. By the formula (3-10), we obtain the fuzzy set X as follows:

$$X = \frac{0.7452}{u_1} + \frac{0.6029}{u_2} + \frac{0.4032}{u_3} + \frac{0.7059}{u_4} + \frac{0.5305}{u_5} + \frac{0.3442}{u_6} + \frac{0.7783}{u_7} + \frac{0.3093}{u_8}.$$

Besides, the values of thresholds calculated by the formulae (2-3)–(2-5) are shown in Table 12. It is obvious that $\xi_i < \gamma_i < \theta_i$ for $i = 1, 2, \ldots, 8$.

Secondly, Step 4 calculates the fuzzy ε-neighborhood classes by means of the formulae (3-2) and (3-3). Let $\mathcal{I} = \mathcal{I}_{\mathcal{L}}, \alpha = 0.5$ and $\varepsilon = 0.7$, we have:

$$[u_1]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_2, u_4, u_5, u_7\}, \qquad [u_2]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_2, u_5, u_7\},$$
$$[u_3]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_2, u_3, u_4, u_5, u_7\}, \qquad [u_4]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_2, u_4, u_7\},$$
$$[u_5]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_2, u_4, u_5, u_7\}, \qquad [u_6]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\},$$
$$[u_7]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_7\}, \qquad [u_8]^{0.7}_{\mathcal{N}_{0.5}} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}.$$

Then, we make use of Step 5 to compute the conditional probabilities $P\left(X|[u_i]^{\varepsilon}_{\mathcal{N}_\alpha}\right)$ as follows:

Table 10
The original FIS of the example in [11].

U/C	C_1	C_2	C_3	C_4	C_5
u_1	0.8	0.4	0.3	0.8	0.9
u_2	0.9	0.5	0.5	0.7	0.6
u_3	0.3	0.4	0.6	0.4	0.3
u_4	0.5	0.2	0.2	0.7	0.6
u_5	0.7	0.6	0.6	0.5	0.8
u_6	0.4	0.8	0.7	0.7	0.3
u_7	0.9	0.5	0.1	0.8	0.7
u_8	0.6	0.8	0.8	0.3	0.4

Table 11
The 3WFDT derived from the original FIS.

	FDT					RLFT					
	C_1	C_2	C_3	C_4	C_5	$\widetilde{\lambda}^i_{PP}$	$\widetilde{\lambda}^i_{BP}$	$\widetilde{\lambda}^i_{NP}$	$\widetilde{\lambda}^i_{PN}$	$\widetilde{\lambda}^i_{BN}$	$\widetilde{\lambda}^i_{NN}$
u_1	0.8	0.6	0.7	0.8	0.9	0	0.3084	0.7452	0.2548	0.1022	0
u_2	0.9	0.5	0.5	0.7	0.6	0	0.2452	0.6029	0.3971	0.1654	0
u_3	0.3	0.6	0.4	0.4	0.3	0	0.1645	0.4032	0.5968	0.2461	0
u_4	0.5	0.8	0.8	0.7	0.6	0	0.2895	0.7059	0.2941	0.1211	0
u_5	0.7	0.4	0.4	0.5	0.8	0	0.2215	0.5305	0.4695	0.1891	0
u_6	0.4	0.2	0.3	0.7	0.3	0	0.1401	0.3442	0.6558	0.2705	0
u_7	0.9	0.5	0.9	0.8	0.7	0	0.3172	0.7783	0.2217	0.0934	0
u_8	0.6	0.2	0.2	0.3	0.4	0	0.1264	0.3093	0.6907	0.2842	0
W	0.1525	0.1835	0.3633	0.1183	0.1823	*	*	*	*	*	*

Table 12
The threshold values of each investment project.

U	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
θ_i	0.331	0.4858	0.6807	0.374	0.5587	0.7333	0.2888	0.7628
γ_i	0.2548	0.3971	0.5968	0.2941	0.4695	0.6558	0.2217	0.6907
ξ_i	0.1896	0.3163	0.5076	0.2253	0.3797	0.5699	0.1684	0.6084

$$P\left(X|[u_1]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.6726, P\left(X|[u_2]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.6642, P\left(X|[u_3]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.6277, P\left(X|[u_4]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.7081,$$
$$P\left(X|[u_5]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.6726, P\left(X|[u_6]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.5872, P\left(X|[u_7]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.7617, P\left(X|[u_8]^{0.7}_{\mathcal{N}_{0.5}}\right) = 0.5525.$$

Subsequently, in light of the 3WD model established in Section 3.3, the decision rules (P2)-(N2) are obtained since $\xi_i < \gamma_i < \theta_i$. The following Fig. 1 is depicted to intuitively compare the conditional probability of each investment project with the corresponding thresholds:

In Fig. 1, the rules (P2)-(N2) can be understood as follows: for an investment project $u_i \in U$, if the value point of the conditional probability is higher than the value point of θ_i, then u_i is accepted, i.e., we decide $u_i \in \text{Pos}(X)$; Similarly, if the value point of conditional probability is lower than the value point of ξ_i, then u_i is rejected, i.e., we decide $u_i \in \text{Neg}(X)$; Or else, u_i needs further investigation, i.e., we decide $u_i \in \text{Bnd}(X)$. Therefore, three regions of U obtained by Step 6 are listed as follows:

$$\text{Pos}(X) = \{u_1, u_2, u_4, u_5, u_7\}, \qquad \text{Bnd}(X) = \{u_3, u_6\}, \qquad \text{Neg}(X) = \{u_8\}.$$

Finally, all investment projects can be ranked by Step 7. To obtain the ranking result of investment projects in each region, we calculate the associated cost of each investment project in its corresponding region by the formula (4-5). The results are listed in Table 13. Based on Table 13 and the principle of $\text{Pos}(X) \succ \text{Bnd}(X) \succ \text{Neg}(X)$, Table 14 shows the ranking results of investment projects in different regions.

By Table 14, all alternatives are ranked and the optimal investment project is u_7. Moreover, decision-makers can obtain an important message, which indicates that the company can choose no more than five investment projects to dispose of idle money based on the classification results of three regions. That is, the projects u_7, u_1, u_4, u_2 and u_5 can be invested.

In addition, if the weight vector W is known, we use the weight vector $W = \{0.3, 0.1, 0.3, 0.2, 0.1\}$ of [11] to obtain the ranking of alternatives. First of all, we can know from [11] that the actual rankings of the example are listed in the following Table 15.

Similarly, the ranking of investment projects obtained by the proposed 3WMCDM method is shown as follows: $u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_8 \succ u_6$. At the same time, the following three regions can be obtained: $\text{Pos}(X) = \{u_1, u_2, u_4, u_5, u_7\}, \text{Bnd}(X) = \{u_3, u_6, u_8\}$ and $\text{Neg}(X) = \emptyset$.

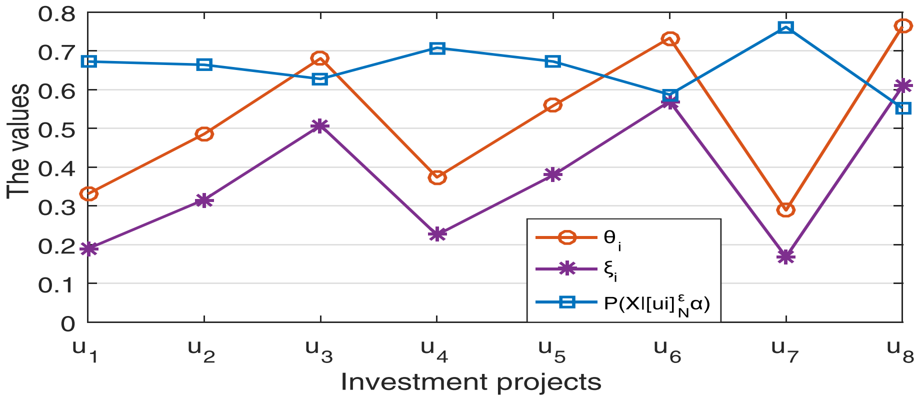

Fig. 1. The comparison between the conditional probabilities and the thresholds.

Table 13
The decision rule and associated cost of each investment project.

U	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
Decision results	P2	P2	B2	P2	P2	B2	P2	N2
cost (u_i)	0.0834	0.1333	0.1949	0.0858	0.1537	0.1939	0.0528	0.1709

Table 14
The ranking results of investment projects in different regions.

Regions	The ranking results
Pos (X)	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5$
Bnd (X)	$u_6 \succ u_3$
Neg (X)	u_8
U	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$

Table 15
The actual rankings of the example.

Strategies	The ranking results
Based on θ_i	$u_7 \succ u_1 \succ u_2 \succ u_4 \succ u_5 \succ u_6 \succ u_3 \succ u_8$
Based on γ_i	$u_7 \succ u_1 \succ u_2 \approx u_4 \succ u_5 \succ u_6 \succ u_3 \succ u_8$
Based on ξ_i	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$

From the above ranking results, we can see that the optimal result is consistent with the optimal result of the actual rankings of the example, and the proposed 3WMCDM method is able to classify investment projects. In conclusion, the above example shows that the proposed 3WMCDM method is feasible for solving the MCDM problems.

4.5. Validity test of the proposed 3WMCDM method

Wang et al. [34] used a validity test which includes three test standards to prove the effectiveness of decision-making methods. We recall the three test standards as follows:

Test standard 1: In a known ranking, by using a worse alternative to replace a non-optimal alternative, the optimal alternative derived from an effective MCDM method should not be changed with the weight vector unchanged.

Test standard 2: Suppose that the number and weights of criteria are unchanged, a set of alternatives of an original MCDM problem are decomposed into several smaller sets of alternatives to form several smaller MCDM problems. The same decision-making method is utilized to obtain the rankings of the several smaller MCDM problems. For a valid MCDM method, the ranking of alternatives among the several smaller MCDM problems should satisfy the transitivity.

Test standard 3: Based on the test standard 2, the ranking results of alternatives of the above several smaller MCDM problems are merged. If a complete ranking is obtained and the obtained ranking is consistent with the ranking of the original MCDM problem, then the effectiveness of the decision-making method is illustrated.

Likewise, we adopt the above three test standards for verifying the validity of the proposed 3WMCDM method in the following subsection.

4.5.1. Validity test of the proposed 3WMCDM method using the test standard 1

As shown in Section 4.4, whether the weight vector of criteria is known or unknown, the optimal alternative is u_7 and other alternatives are non-optimal. In light of the test standard 1, we use a worse alternative u_1' instead of the alternative u_1 in Table 10. The evaluation values of the alternative u_1' under criteria are 0.7, 0.3, 0.2, 0.7 and 0.8, successively. Let $K = \{u_1', u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be a new set of alternatives. In what follows, we verify the validity of the proposed 3WMCDM method according to two cases of the weight vector.

On the one hand, when the weight vector $W = \{0.3, 0.1, 0.3, 0.2, 0.1\}$ is known, the ranking of alternatives in K by the proposed 3WMCDM method is $u_7 \succ u_1' \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_8 \succ u_6$. We can see that the optimal alternatives in K and U are the same, namely, u_7. In addition, the relative ranking of the other unchanged alternatives has not changed either. Thus, the proposed 3WMCDM method based on the known weight vector passes the test standard 1.

On the other hand, the weight vector $W = \{0.1525, 0.1835, 0.3633, 0.1183, 0.1823\}$ is calculated when the weight vector is unknown. We obtain the ranking of alternatives in K as follows: $u_7 \succ u_1' \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$. Similar to the above description, the proposed 3WMCDM method based on the objective weight vector passes the test standard 1. Therefore, the proposed 3WMCDM method passes the validity test of the test standard 1.

4.5.2. Validity test of the proposed 3WMCDM method under the test standards 2 and 3

According to the above test standard 2, we decompose the set of alternatives in the original MCDM problem in Section 4.4 into three smaller sets of alternatives, which are denoted by $K_1 = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}, K_2 = \{u_1, u_2, u_4, u_5, u_6, u_7, u_8\}$ and $K_3 = \{u_1, u_2, u_3, u_4, u_5, u_7, u_8\}$. The ranking results of Table 16 can be obtained by using the proposed 3WMCDM method.

On the basis of Table 16, we can see that the ranking of investment projects derived from the above three sets satisfies the transitivity property. For example, when the weight vector $W = \{0.3, 0.1, 0.3, 0.2, 0.1\}$ is known, we have $u_3 \succ u_8$ in $K_3, u_8 \succ u_6$ in K_2 and $u_3 \succ u_6$ in K_1. Hence, the proposed 3WMCDM method passes the test standard 2.

Besides, for different weight vectors in Table 16, we integrate the ranking results derived from the above three sets into a complete ranking. When $W = \{0.3, 0.1, 0.3, 0.2, 0.1\}$, the complete ranking $u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_8 \succ u_6$ is obtained which is identical to the ranking of the un-decomposed original MCDM problem. For the objective weight vector $W = \{0.1525, 0.1835, 0.3633, 0.1183, 0.1823\}$, we obtain the complete ranking $u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$, which is also consistent with the ranking of the un-decomposed original MCDM problem. Therefore, the proposed 3WMCDM method passes the test standard 3.

To sum up, the validity test of the proposed 3WMCDM method is passed. In the next section, we will further discuss the effectiveness of the method by comparison analysis.

5. Comparison analysis

In this section, we analyze the effectiveness and superiority of the proposed 3WMCDM method on the basis of the example in Section 4.4. Section 5.1 compares the proposed 3WMCDM method with seven decision-making methods from the aspects of known weights and unknown weights, and analyzes the comparative results of rankings of different methods one by one from three perspectives of decision-making theory. Section 5.2 systematically discusses the differences between the proposed 3WMCDM method and the other methods. Furthermore, the strengths and weaknesses of the proposed 3WMCDM method are summarized.

5.1. Comparison results with seven different decision-making methods

In what follows, we choose seven decision-making methods to compare with the proposed 3WMCDM method. The seven decision-making methods are chosen from three perspectives of decision-making theory, including four traditional MCDM methods, two methods based on fuzzy β-neighborhood operators and one method based on thresholds of 3WD.

Although the traditional MCDM methods cannot divide alternatives into three categories, they can get a ranking of alternatives. This is consistent with the end use of our method. On the side, the traditional MCDM methods can be regarded as two-way decisions which is a special case of 3WD. Thus, we choose the following four common MCDM methods: the TOPSIS method [10], the WAA operator method [6], the EDAS method [5] and the VIKOR method [26]. Moreover, the fuzzy α-neighborhood operator plays an important role in our method. Thus, we select the following two decision-making methods with fuzzy β-neighborhood operators: the Jiang et al.'s method [12] and the Zhang et al.'s method [47]. Besides, since the proposed 3WMCDM method is based on the 3WD theory, the 3WD-based decision-making methods are more worthy of consideration when we choose the comparison methods. Meanwhile, since the proposed 3WMCDM method is built on an FIS, decision-making methods in other fuzzy environments [17,30] are not applicable. Hence, we only select one method from the perspective of 3WD to compare with the proposed 3WMCDM method, which is the Jia et al.'s method [11]. In the following, the comparison results of different methods are respectively shown for different cases of weights.

(1) Comparison results of different methods when the weight vector of criteria is known

Table 16
The rankings of investment projects derived from three smaller sets of alternatives.

Weights	Sets	The ranking of investment projects
$W = \{0.3, 0.1, 0.3, 0.2, 0.1\}$	K_1	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_6$
	K_2	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_8 \succ u_6$
	K_3	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_8$
$W = \{0.1525, 0.1835, 0.3633, 0.1183, 0.1823\}$	K_1	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3$
	K_2	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_8$
	K_3	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_8$

(a) The comparison between the ranking of the 3WMCDM method and the actual rankings

As shown in Section 4.4, we can obtain the ranking of the proposed 3WMCDM method and the actual rankings if $W = \{0.3, 0.1, 0.3, 0.2, 0.1\}$. It can be easily found that the difference between the ranking of the proposed 3WMCDM method and the actual rankings. To verify the validity of the proposed 3WMCDM method, we adopt the Spearman's rank correlation coefficient (SRCC) to analyze the correlation between the ranking of the 3WMCDM method and the actual rankings. We introduce the formula for calculating SRCC as follows:

$$SRCC = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2 - 1)}. \quad (5-1)$$

In the formula (5-1), n is the number of alternatives and $d_i = x_i - y_i$, in which x_i denotes the ranking position of u_i in the ranking of the proposed 3WMCDM method and y_i denotes the ranking position of u_i in an actual ranking.

Generally speaking, an SRCC greater than 0.8 indicates that the degree of correlation between two rankings is significant [11,30]. Next, we calculate the SRCCs between the ranking of the 3WMCDM method and the actual rankings, as shown in Table 17.

Note 3: The actual rankings are the rankings derived from the Jia et al.'s method [11].

Remark 5.1. Based on Table 17, the SRCCs between the ranking of 3WMCDM method and the actual rankings are greater than 0.9, which indicates that the degree of correlation between the ranking of 3WMCDM method and the actual rankings is very significant. Therefore the proposed 3WMCDM method is very valid.

(b) The comparison between the rankings derived from the other decision-making methods and the actual and 3WMCDM method's rankings

According to the example in Section 4.4, the rankings of investment projects derived from the rest of six decision-making methods are shown in Table 18.

Note 4: The letters at the bottom of Table 18 are the parameters of different methods.

On the basis of Table 18, the rankings of the above six methods are different from the actual and 3WMCDM method's ranking. Analogously, we adopt the SRCC to analyze the correlation between the rankings of the above six methods and the actual and 3WMCDM method's ranking. All of the SRCCs are shown in Table 19.

Remark 5.2. Analogously, we can see from Tables 18 and 19 that the four traditional MCDM methods and the Jiang et al.'s method are effective for ranking alternatives. Meanwhile, the proposed method is as effective as the four traditional MCDM methods and the Jiang et al.' method. Moreover, the proposed method is more effective than the Zhang et al.'s method.

(2) Comparison results of different methods when the weight vector of criteria is unknown

The weight vector $W = \{0.1525, 0.1835, 0.3633, 0.1183, 0.1823\}$ is computed by the entropy weight method if the W is unknown. Similarly, we can obtain the results of Tables 18′ and Table 19′.

Remark 5.3. According to Tables 18′ and Table 19′, the optimal result of the proposed 3WMCDM method is consistent with the optimal results derived from the above four traditional MCDM methods and the Jia et al.'s method. Furthermore, except that the SRCC between the proposed 3WMCDM method and the Zhang's method is less than 0.8, the SRCCs between the proposed 3WMCDM method and the other decision-making methods are greater than 0.8.

Remark 5.3 is an intuitive summary of the above comparison results. To illustrate the validity of the proposed method when the weights of criteria are unknown, we analyze the above comparison results in detail from the following three perspectives of decision-making theory.

5.1.1. Comparative analysis with four traditional MCDM methods

To visually show the difference in the ranking results between the proposed 3WMCDM method and any of the four traditional MCDM methods, we plot Fig. 2 based on Tables 18′.

Fig. 2 shows four pairwise comparison sub-graphes which intuitively reflect the changes in the ranking position of each investment project obtained by two different methods. By Tables 18′ and Fig. 2, the optimal and worst investment project of the proposed 3WMCDM method are consistent with those of the four traditional MCDM methods, i.e., u_7 is the optimal

Table 17
The SRCCs between the ranking of the 3WMCDM method and the actual rankings.

Decision-making methods	3WMCDM method	Jia et al.'s method (based on θ_i)	Jia et al.'s method (based on γ_i)	Jia et al.'s method (based on ξ_i)
3WMCDM method	1	0.9048	0.9286	0.9167
Jia et al.'s method [11] (based on θ_i)	-	1	0.9762	0.9881
Jia et al.'s method [11] (based on γ_i)	-	-	1	0.9881
Jia et al.'s method [11] (based on ξ_i)	-	-	-	1

Table 18
The ranking results of different methods when W is known.

Different methods	Ranking results	Optimal investment project
TOPSIS method [10]	$u_7 \succ u_1 \succ u_2 \succ u_4 \succ u_5 \succ u_6 \succ u_8 \succ u_3$	u_7
WAA operator method [6]	$u_7 \succ u_1 \succ u_2 \approx u_4 \succ u_5 \succ u_6 \succ u_3 \succ u_8$	u_7
EDAS method [5]	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$	u_7
VIKOR method[b] [26]	$u_7 \succ u_1 \succ u_2 \succ u_4 \succ u_5 \succ u_6 \succ u_3 \succ u_8$	u_7
Jiang et al.'s method[c] [12]	$u_1 \succ u_7 \succ u_4 \succ u_2 \approx u_5 \succ u_3 \succ u_8 \succ u_6$	u_1
Zhang et al.'s method[d] [47]	$u_5 \succ u_3 \succ u_2 \approx u_4 \approx u_7 \succ u_6 \succ u_8 \succ u_1$	u_5

[b] $\upsilon = 0.5$; [c] $\beta = 0.5$, $k = 0$; [d] $\beta = 0.5$, $q = 0.5$.

Table 19
The SRCCs between the rankings derived from the above six methods and the actual and 3WMCDM method's rankings when the W is known.

Decision-making methods	3WMCDM method	Jia et al.'s method [11] (based on θ_i)	Jia et al.'s method [11] (based on γ_i)	Jia et al.'s method [11] (based on ξ_i)
TOPSIS method [10]	0.8810	0.9762	0.9643	0.9524
WAA operator method [6]	0.9167	0.9881	1	0.9881
EDAS method [5]	0.9286	0.9762	0.9881	1
VIKOR method [26]	0.9048	1	0.9881	0.9762
Jiang et al.'s method [12]	0.9643	0.8690	0.8810	0.8929
Zhang et al.'s method [47]	0.0833	0.0119	0.0238	0.0119

Table 18'
The ranking results of different methods when W is unknown.

Different methods	Ranking results	Optimal project
3WMCDM method[a]	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$	u_7
TOPSIS method [10]	$u_7 \succ u_4 \succ u_1 \succ u_2 \succ u_5 \succ u_3 \succ u_6 \succ u_8$	u_7
WAA operator method [6]	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_6 \succ u_8$	u_7
EDAS method [5]	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_6 \succ u_8$	u_7
VIKOR method[b] [26]	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_6 \succ u_8$	u_7
Jiang et al.'s method[c] [12]	$u_1 \succ u_7 \succ u_4 \succ u_2 \approx u_5 \succ u_3 \succ u_8 \succ u_6$	u_1
Zhang et al.'s method[d] [47]	$u_5 \succ u_3 \succ u_2 \approx u_4 \approx u_7 \succ u_1 \succ u_6 \succ u_8$	u_5
Jia et al.'s method (θ_i,γ_i,ξ_i) [11]	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_6 \succ u_8$	u_7

[a] $\alpha = 0.5$, $\varepsilon = 0.7$; [b] $\upsilon = 0.5$; [c] $\beta = 0.5$, $k = 0$; [d] $\beta = 0.5$, $q = 0.5$.

investment project and u_8 is the worst investment project. Furthermore, we can see from Fig. 2 (a) that there are some changes in the ranking position of u_1 and u_4, as well as u_3 and u_6, and the ranking positions of other investment projects are unchanged. Also, for Fig. 2 (b), (c) and (d), only the ranking positions of u_3 and u_6 are swapped. Although the ranking result of the proposed 3WMCDM method is somewhat different from those of the four traditional MCDM methods, the difference is not obvious in terms of the length of the ranking position change. This can be proven through the SRCCs in Table 19'. Obviously, all of the SRCCs between the proposed 3WMCDM method and the four traditional MCDM methods are greater than 0.9.

Table 19'
The SRCC between any two of eight methods when W is unknown.

Decision-making methods	3WMCDM method	TOPSIS method	WAA operator method	EDAS method	VIKOR method	Jiang et al.'s method	Zhang et al.'s method	Jia et al.'s method
3WMCDM method	1	0.9524	0.9762	0.9762	0.9762	0.8929	0.25	0.9762
TOPSIS method [10]	–	1	0.9762	0.9762	0.9762	0.8929	0.4405	0.9762
WAA operator method [6]	–	–	1	1	1	0.9405	0.369	1
EDAS method [5]	–	–	–	1	1	0.9405	0.369	1
VIKOR method [26]	–	–	–	–	1	0.9405	0.369	1
Jiang et al.'s method [12]	–	–	–	–	–	1	0.3571	0.9405
Zhang et al.'s method [47]	–	–	–	–	–	–	1	0.3690
Jia et al.'s method [11]	–	–	–	–	–	–	–	1

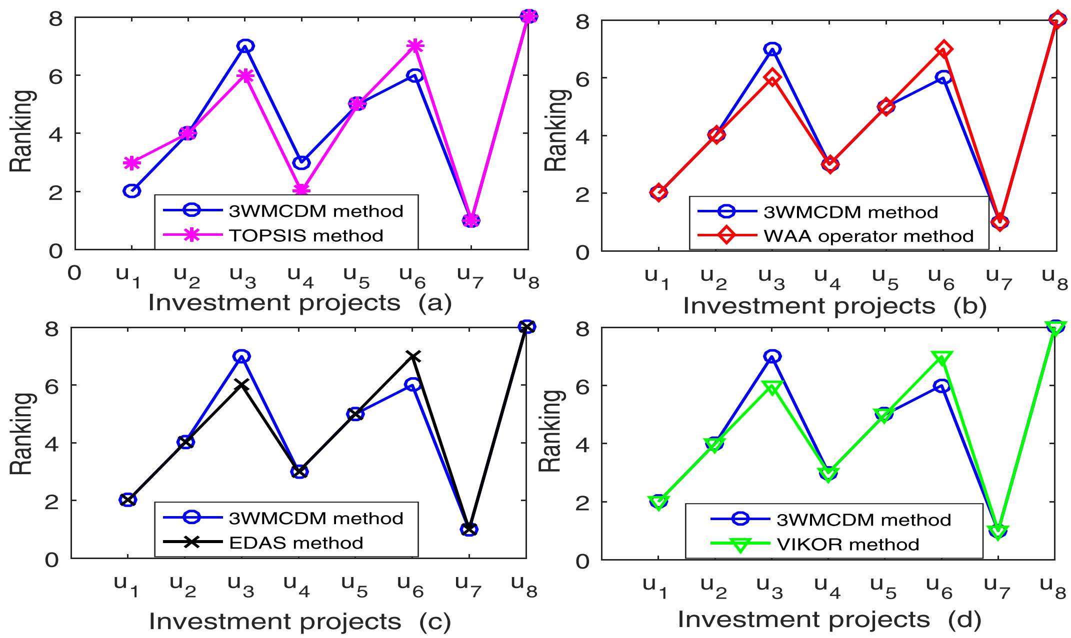

Fig. 2. The pairwise comparisons of ranking results derived from the proposed 3WMCDM method and the four traditional MCDM methods.

Remark 5.4. In addition to the consistent optimal investment projects, the degree of correlation of ranking results between the proposed 3WMCDM method and any of the four traditional MCDM methods is very significant. So, the proposed 3WMCDM method is as effective as the above four traditional MCDM methods to deal with the example of investment projects.

5.1.2. Comparative analysis with fuzzy β-neighborhood operator-based two methods

In like manner, Fig. 3 is drawn to exhibit the difference in ranking results derived from the proposed 3WMCDM method and the two methods based on fuzzy β-neighborhood operators.

According to Table 18$'$ and Fig. 3, the Jiang et al.'s method and the Zhang et al.'s method can not get a complete ranking, namely, some investment projects are indistinguishable by these two methods. Thus, the ability of these two methods to distinguish investment projects is worse than the proposed 3WMCDM method. Moreover, the optimal investment projects obtained from the proposed 3WMCDM method, the Jiang et al.'s method and the Zhang et al.'s method are different. Meanwhile, the optimal investment projects of these two methods are also inconsistent with the optimal results of the above four traditional MCDM methods. Therefore, in terms of optimal results, the proposed 3WMCDM method is more effective than the Jiang et al.'s method and the Zhang et al.'s method. In addition, Table 19$'$ can confirms that the correlation between the rankings derived from the proposed 3WMCDM method and the Zhang et al.'s method is not significant. At the same time, the SRCCs between the Zhang et al.'s method and the other decision-making methods are also lower than 0.8.

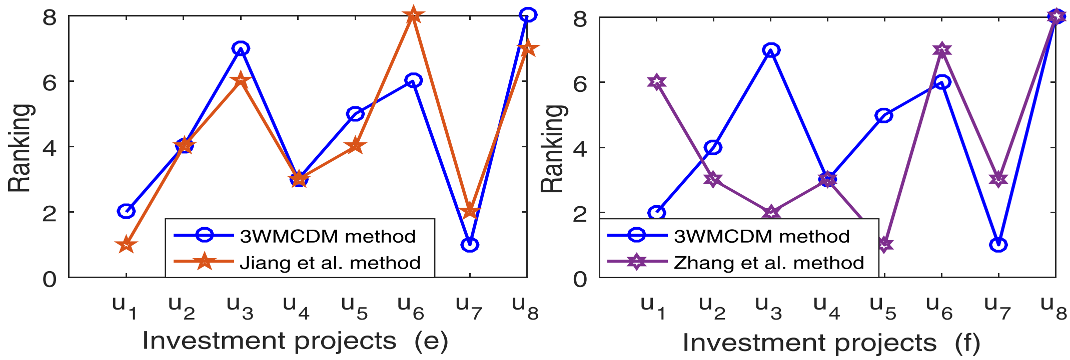

Fig. 3. The pairwise comparisons of ranking results derived from the proposed 3WMCDM method and the fuzzy β-neighborhood operator-based two methods.

Remark 5.5. From the above discussions and descriptions, the proposed 3WMCDM method is more effective than the two methods based on fuzzy β-neighborhood operators [12,47].

5.1.3. Comparative analysis with one 3WD-based decision-making method

Similarly, compared with the Jia et al.'s method, we can obtain the following Fig. 4.

From Fig. 4, the difference in the ranking results between the proposed 3WMCDM method and the Jia et al.'s method can be seen intuitively, which is similar to the description in Section 5.1.1. In the following, we mainly analyze the weaknesses of the Jia et al.'s method.

As described in [11], all investment projects can be ranked according to the orders of the values of all θ_i (or γ_i or ξ_i) under the assumption that all conditional probabilities are the same. A lower value of θ_i (or γ_i or ξ_i) implies a higher possibility of accepting the investment project u_i, that is, the higher the ranking of u_i. From the above description, we can find two weaknesses of the Jia et al.'s method as follows: on the one hand, the above assumption that all investment projects have the same conditional probability is too strict, which is not in line with the actual decision-making problems. In general, different investment projects can induce different conditional probabilities because of different evaluation data. Furthermore, the use of conditional probability and three decision rules are not really embodied in the process of getting the ranking according to all thresholds θ_i (or γ_i or ξ_i). On the other hand, when conditional probabilities of investment projects are different, the Jia et al.'s method is not applicable for decision-making problems due to the lack of reasonable semantic explanation. For example, let $\theta_1 = 0.8, \theta_2 = 0.6$ and $\theta_3 = 0.5$. When $P(X|[u_1]^{\varepsilon}_{\mathcal{N}_\alpha}) = P(X|[u_2]^{\varepsilon}_{\mathcal{N}_\alpha}) = P(X|[u_3]^{\varepsilon}_{\mathcal{N}_\alpha}) = 0.9$, we have $u_3 \succ u_2 \succ u_1$ by the Jia et al.'s method. However, if $P(X|[u_2]^{\varepsilon}_{\mathcal{N}_\alpha}) = 0.65$ and $P(X|[u_3]^{\varepsilon}_{\mathcal{N}_\alpha}) = 0.55$, we only obtain $\{u_1, u_2, u_3\} \subseteq \mathrm{Pos}(X)$ by the theory of 3WD in [11] rather than ranking. To overcome the above shortcomings, we combine the established MCDM-based 3WD model with the research results of [11] to propose the 3WMCDM method. By introducing the notion of associated costs, we discuss the ranking of investment projects with different conditional probabilities.

Remark 5.6. By the above analysis, the proposed method is as valid as the Jia et al.'s method. In terms of the performance of method, the proposed method overcomes the weaknesses of the Jia et al.'s method and gives a reasonable semantic explanation of decision results.

5.2. Discussion

Likewise, we can obtain conclusions similar to Remarks 5.4,5.5,5.6 for the known weight vector. Next, we give another example to show the superiority of the proposed 3WMCDM method.

Example 5.7. Let $U_1 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a set of six manager candidates. The following six benefit criteria are considered, i.e., health statuses (C_1), education backgrounds (C_2), communicative abilities (C_3), work styles (C_4), writing abilities (C_5), business experiences (C_6). Table 20 shows the evaluation value of each manager candidate under criteria.

The weight vector $W = \{0.1545, 0.1545, 0.19, 0.19, 0.1555, 0.1555\}$ is calculated. Let $\mathcal{I} = \mathcal{I}_{\mathcal{L}}, \alpha = 0.8$ and $\varepsilon = 0.9$, the ranking results of different methods are presented in Table 21.

By Table 21, the selected seven methods can not obtain the complete rankings. The candidate x_1 is indistinguishable from the candidate x_4. Meanwhile, with the exception of the Zhang et al.'s method, the other six methods cannot determine an optimal candidate. Nonetheless, the proposed 3WMCDM method not only obtains a complete ranking, but also selects an optimal candidate x_1. Therefore, the proposed 3WMCDM method not only excels in dealing with the above example more efficiently than the others, but also acts as an essential and unique MCDM method when solving the above example.

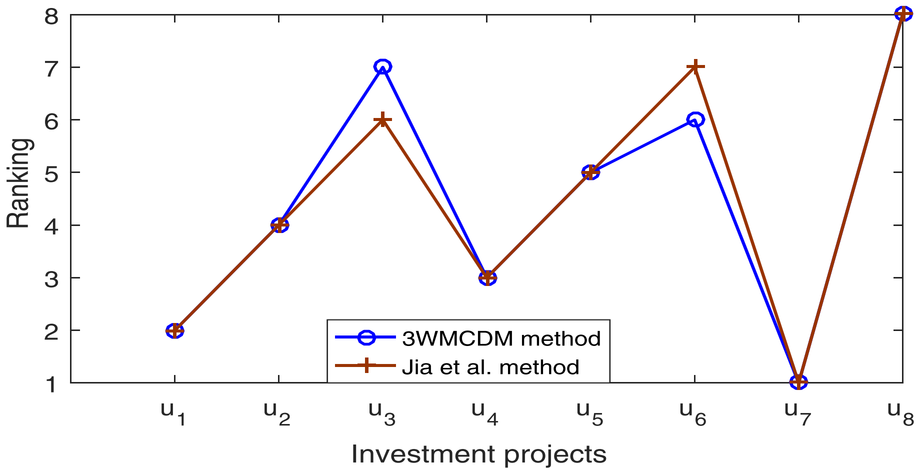

Fig. 4. The pairwise comparison of ranking results between the proposed 3WMCDM method and the Jia et al.'s method.

Table 20
The evaluation value of each manager candidate under criteria.

U_1/C	C_1	C_2	C_3	C_4	C_5	C_6
x_1	0.68	0.74	0.64	0.83	0.65	0.75
x_2	0.48	0.83	0.62	0.78	0.48	0.72
x_3	0.83	0.48	0.78	0.58	0.75	0.48
x_4	0.74	0.68	0.83	0.64	0.75	0.65
x_5	0.76	0.66	0.48	0.62	0.72	0.83
x_6	0.66	0.76	0.58	0.48	0.83	0.75

Table 21
The ranking results of different methods.

Different methods	Ranking results	The optimal candidate
3WMCDM method	$x_1 \succ x_4 \succ x_5 \succ x_6 \succ x_2 \succ x_3$	x_1
TOPSIS method [10]	$x_1 \approx x_4 \succ x_5 \succ x_2 \succ x_6 \succ x_3$	x_1, x_4
WAA operator method [6]	$x_1 \approx x_4 \succ x_5 \succ x_6 \succ x_2 \succ x_3$	x_1, x_4
EDAS method [5]	$x_1 \approx x_4 \succ x_5 \succ x_6 \succ x_2 \succ x_3$	x_1, x_4
VIKOR method [26]	$x_1 \approx x_4 \succ x_2 \succ x_3 \succ x_5 \succ x_6$	x_1, x_4
Jiang et al.'s method [12]	$x_1 \approx x_4 \succ x_5 \succ x_6 \succ x_2 \succ x_3$	x_1, x_4
Zhang et al.'s method [47]	$x_5 \succ x_6 \succ x_1 \approx x_4 \succ x_2 \succ x_3$	x_5
Jia et al.'s method (γ_i) [11]	$x_1 \approx x_4 \succ x_5 \succ x_6 \succ x_2 \succ x_3$	x_1, x_4

In what follows, we discuss the main differences between the proposed 3WMCDM method and the existing decision-making methods as follows:

(1) In MCDM, the weights of criteria is an important factor that affects decision-making results. The proposed 3WMCDM method considers two cases of weights. However, a lot of decision-making methods only take into account the known weight vector, for instance, the above selected seven decision-making methods [5,6,10–12,26,47]. Besides, some existing decision-making methods do not consider the weights of criteria in the decision-making process, such as the method in [17] and the method in [30].

(2) The proposed 3WMCDM method considers the relative loss functions of alternatives and the risk of decisions. When we make decisions, certain losses or risks are always derived. For example, in an investment project decision-making problem, if an investor treats a bad project as a good one to invest in it, the investor will surely bear the risk of losing money. Nevertheless, the traditional MCDM methods and the existing decision-making methods with fuzzy neighborhood operators, such as the above four MCDM methods [5,6,10,26] and two methods based on fuzzy β-neighborhood operators [12,47], do not consider the risk or loss of decisions.

(3) The proposed 3WMCDM method can divide all alternatives into positive region, negative region and boundary region which correspond to accept decision, reject decision and not commitment decision. The traditional MCDM methods and the existing decision-making methods with fuzzy neighborhood operators, however, can only classify alternatives into positive region and negative region, which corresponding to accept and reject decisions.

(4) The proposed 3WMCDM method can objectively determine the number of alternatives in different regions by virtue of the theory of 3WD. Nevertheless, in some existing MCDM methods, such as the above four MCDM methods [5,6,10,26] and two methods based on fuzzy β-neighborhood operators [12,47], the number of acceptable or rejected alternatives is determined according to the subjective needs of decision-makers. In addition, for the Jia et al.'s method [11], three regions of alternatives are obtained based on the subjectively conditional probabilities. For the proposed 3WMCDM method, the conditional probabilities of alternatives are calculated by the conditional probability formula, rather than the subjectively conditional probabilities.

(5) In the proposed 3WMCDM method, we consider the reflexive fuzzy α-neighborhood operators. Despite the fuzzy β-neighborhood operators are considered in some decision-making methods, such as the Jiang et al.'s method [12] and the Zhang et al.'s method [47], the fuzzy β-neighborhood operators are not reflexive.

(6) Whether the conditional probabilities are the same or different, all alternative can be ranked by the proposed 3WMCDM method. Nevertheless, due to the lack of reasonable semantic explanations, the Jia et al.'s method [11] can not be used to rank all alternative in the case of different conditional probabilities of alternatives.

Remark 5.8. In conclusion, the proposed 3WMCDM method has six advantages as follows:

(1) Compared with the four traditional MCDM methods [5,6,10,26] and the two methods based on fuzzy β-neighborhood operators [12,47], the proposed 3WMCDM method can divide all alternatives into three regions which correspond to three decision actions. The delayed decision is added, which can reduce the risk of decisions.

(2) The fuzzy α-neighborhood operator used in the proposed 3WMCDM method satisfies the reflexivity, which overcomes the defect that the fuzzy β-neighborhood operator adopted in the Jiang et al.'s method and the Zhang et al.'s method does not satisfy the reflexivity.

(3) The proposed 3WMCDM method can rank all alternatives in the case of different conditional probabilities, which overcomes the shortcomings of the Jia et al.'s method.

(4) Compared with the selected seven methods, the proposed 3WMCDM method takes into account the risk of making a decision for each alternative. The obtained ranking result is more reasonable and scientific. At the same time, some decision-making methods can not handle some MCDM problems well, but the proposed method can solve them well (see Example 5.7).

(5) The proposed 3WMCDM method is flexible, the conditional probability can be adjusted by changing the values of α and ε to generate different regions. Thus, decision-makers can use the proposed 3WMCDM method to obtain flexible decisions.

(6) The proposed 3WMCDM method provides a new decision-making thinking by combining the rough fuzzy set theory [3], the 3WD theory and the decision-making theory, which further enriches the research results of the 3WD and decision-making theories.

In addition, the proposed method has the following disadvantages: on the one hand, the algorithm of the proposed 3WMCDM method has a relatively high time complexity. On the other hand, the proposed 3WMCDM method can not be applied to data with the type of symbols.

6. Experimental analysis

In this section, experiments based on sensitivity analysis are performed on different decision-making problems. We analyze the influence of the change of parameter values on the decision results from three aspects. The results of these experiments demonstrate the stability of the proposed 3WMCDM method for the parameters α and ε. All experiments are carried out on a personal computer with 3.00 GHz CPU, 8 GB RAM and 64-bit Windows 10. The experimental programs are written by Matlab R2016a.

By Algorithm 2, the conditional probability is a key point in generating 3WD, while the determination of the conditional probability depends on two parameters α and ε. To evaluate the stability of the proposed 3WMCDM method, it is necessary to analyze the influence of the change of two parameter values on the decision results.

In the following, we select two data sets from the existing literature as follows: one is the data set in [11] (Table 15 in [11]), namely, the data set of example in Section 4.4, the other is the experimental data set presented in [48] (Table 17 in [48]). In our experiments, we set $\alpha \in (0, 0.6]$ with step size 0.1 and $\varepsilon \in [0, 1]$ with step size 0.1 for the data set in [11]. and we set $\alpha \in (0, 1]$ with step size 0.2 and $\varepsilon \in [0, 1]$ with step size 0.2 for the experimental data set in [48]. Here, we focus on the experimental analysis of the data set in [11].

6.1. Sensitivity analysis of α when ε is fixed

For a fixed $\varepsilon = 0.7$, we analyze the influence of the change of α on the decision results based on two data sets. We first analyze the influence of the change of α on the conditional probability. As α changes, if the values of conditional probabilities keep the same, then the proposed 3WMCDM method is very stable. Otherwise, we calculate the ranking of alternatives when α changes. If the rankings of alternatives for different values of α are the same, then the method is very stable. Or, if the optimal alternatives for different values of α are consistent and $SRCC \geqslant \lambda_v$, then the method is stable. Here, v denotes the level of significance and λ_v denotes the critical value on the significance level v. If the inequation $SRCC \geqslant \lambda_v$ holds, then the correlation between the ranking based on parameters and the actual ranking is significant.

(a) Sensitivity analysis on the data set in [11]

Table 22 lists the calculation results of $P\left(X|[u_i]^{0.7}_{\mathcal{N}_\alpha}\right)$ for any $u_i \in U$ when α changes.

From Table 22, we can see that the conditional probability of each alternative does not change when the value of α changes. This indicates that the proposed 3WMCDM method is very stable for different values of α.

(b) Sensitivity analysis on the experimental data set in [48]

Similarly, as α changes, we can calculate the values of conditional probabilities. However, the calculation results of conditional probability are different when α changes. Thus, as α changes, the ranking of alternatives is calculated, as shown in Table 23.

Note 5: In Table 23, the SRCC denotes the SRCC between the ranking based on the parameter α and the actual ranking.

Let $n = 50$ and $v = 0.05$, then we can obtain that $\lambda_v = 0.363$. By Table 23, h_{45} is the optimal alternative for different values of α, which is consistent with the optimal result of the actual ranking in [48]. Besides, for different values of $\alpha, SRCC > \lambda_v$ holds, which shows the correlation between the ranking based on parameters and the actual ranking is significant. Hence, the proposed 3WMCDM method is valid and stable for the experimental data set in [48].

Remark 6.1. From the above different data sets, we can know that for a fixed $\varepsilon = 0.7$, the proposed 3WMCDM method is stable for different values of α.

6.2. Sensitivity analysis of ε when α is fixed

Similarly, for a fixed $\alpha = 0.5$, we analyze the influence of the change of ε on the decision results based on two data sets.

(a) Sensitivity analysis on the data set in [11]

For simplicity, the ranking result of each region and the optimal result are listed in Table 24 when ε changes.

Note 6: When $\varepsilon = 0$, 0.1 and 0.2, the ranking of alternatives in each region and the optimal alternative are consistent with the results of $\varepsilon = 0.3$.

By Table 24, the optimal results for different values of ε are consistent. Next, Fig. 5 depicts the change of three regions when ε changes.

Fig. 5 visually shows that the three regions of 3WD vary with the values of ε. We can see that the variation of the three regions is almost stable. To analyze the stability of the ranking results based on different values of ε, we calculate the SRCC between the 3WMCDM method with a certain value of ε and any of the six other decision-making methods besides the Zhang et al.'s method selected in Section 5. The results are listed in Table 25.

Note 7: For example, when $\varepsilon = 0.5$ and $\varepsilon = 0.1$, the SRCCs between the proposed 3WMCDM method with a certain value of ε and the above six methods are the same.

According to Table 25, in fact, we can obtain the following Fig. 6.

From Table 25 and Fig. 6, we can see that the SRCCs between the 3WMCDM method with $\varepsilon \in [0, 1]$ and the above six methods are all greater than 0.8. Thus, the proposed 3WMCDM method are stable for different values of ε from the aspects of optimal results and SRCCs.

(b) Sensitivity analysis on the experimental data set in [48]

Similar to Section 6.1 (b), we have the results of Table 26 when ε changes.

By Table 26, the optimal results obtained from different values of ε are still the optimal result of actual ranking. At the same time, $SRCC > \lambda_v$ holds for different values of ε. Thus, with the change of ε, the proposed 3WMCDM method is stable for the experimental data set in [48].

Remark 6.2. From the above different data sets, we can know that for a fixed $\alpha = 0.5$, the proposed 3WMCDM method is stable for the different values of ε.

6.3. Sensitivity analysis when α and ε change simultaneously

(a) Sensitivity analysis on the data set in [11].

In this section, Fig. 7 outlines the experiment's results of the conditional probabilities of eight alternatives when $\alpha \in (0, 0.6]$ and $\varepsilon \in [0, 1]$.

According to Fig. 7, we can determine the specific region that the alternative falls into by comparing the conditional probability of each alternative and thresholds. That is, all alternatives can be divided into three regions. Based on the three regions obtained from different values of α and ε, all investment projects can be ranked and the optimal alternative can be selected. On the basis of the results of the experiment, the selected optimal alternatives are the same, i.e., u_7, when $\alpha \in (0, 0.6]$ and $\varepsilon \in [0, 1]$. Thus, the proposed 3WMCDM method is stable on the optimal alternative when α and ε change simultaneously.

(b) Sensitivity analysis on the experimental data set in [48]

Table 22
The calculated results of the conditional probability $P\left(X|[u_i]^{0.7}_{\mathcal{N}_\alpha}\right)$.

$P\left(X|[u_i]^{0.7}_{\mathcal{N}_\alpha}\right)$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
$\alpha = 0.1$	0.6726	0.6642	0.6277	0.7081	0.6726	0.5872	0.7617	0.5525
$\alpha = 0.2$	0.6726	0.6642	0.6277	0.7081	0.6726	0.5872	0.7617	0.5525
$\alpha = 0.3$	0.6726	0.6642	0.6277	0.7081	0.6726	0.5872	0.7617	0.5525
$\alpha = 0.4$	0.6726	0.6642	0.6277	0.7081	0.6726	0.5872	0.7617	0.5525
$\alpha = 0.5$	0.6726	0.6642	0.6277	0.7081	0.6726	0.5872	0.7617	0.5525
$\alpha = 0.6$	0.6726	0.6642	0.6277	0.7081	0.6726	0.5872	0.7617	0.5525

Table 23
The rankings of alternatives based on different α when $\varepsilon = 0.7$.

α	Ranking of alternative	Optimal result	SRCC
$\alpha = 0.2$	$h_{45} \succ h_{38} \succ h_{14} \succ h_{46} \succ h_1 \succ h_2 \succ h_{48} \succ h_{23} \succ h_{16} \succ h_{11} \succ h_{35} \succ h_{13} \succ h_{25} \succ h_6 \succ h_7 \succ h_{31} \succ h_{24} \succ h_{19} \succ h_{37} \succ h_{15} \succ h_{28} \succ h_{47} \succ h_{49} \succ h_{41} \succ h_5 \succ h_{12} \succ h_{17} \succ h_{21} \succ h_8 \succ h_{10} \succ h_9 \succ h_{39} \succ h_{29} \succ h_{26} \succ h_{36} \succ h_{34} \succ h_{50} \succ h_{22} \succ h_{27} \succ h_{18} \succ h_{42} \succ h_3 \succ h_{32} \succ h_{33} \succ h_{20} \succ h_{44} \succ h_{43} \succ h_{40} \succ h_{30} \succ h_4$	h_{45}	0.8493
$\alpha = 0.4$	$h_{45} \succ h_{38} \succ h_{14} \succ h_{46} \succ h_1 \succ h_2 \succ h_{48} \succ h_{23} \succ h_{16} \succ h_{11} \succ h_{35} \succ h_{13} \succ h_{25} \succ h_6 \succ h_7 \succ h_{31} \succ h_{24} \succ h_{19} \succ h_{37} \succ h_{15} \succ h_{28} \succ h_{47} \succ h_{49} \succ h_{41} \succ h_5 \succ h_{12} \succ h_{17} \succ h_{21} \succ h_8 \succ h_{10} \succ h_9 \succ h_{39} \succ h_{29} \succ h_{26} \succ h_{36} \succ h_{34} \succ h_{50} \succ h_{22} \succ h_{27} \succ h_{18} \succ h_{42} \succ h_3 \succ h_{32} \succ h_{33} \succ h_{20} \succ h_{44} \succ h_{43} \succ h_{40} \succ h_{30} \succ h_4$	h_{45}	0.8493
$\alpha = 0.6$	$h_{45} \succ h_{38} \succ h_{14} \succ h_{46} \succ h_1 \succ h_2 \succ h_{48} \succ h_{23} \succ h_{16} \succ h_{11} \succ h_{35} \succ h_{13} \succ h_{25} \succ h_6 \succ h_7 \succ h_{31} \succ h_{24} \succ h_{19} \succ h_{37} \succ h_{15} \succ h_{28} \succ h_{47} \succ h_{49} \succ h_5 \succ h_{12} \succ h_{17} \succ h_{21} \succ h_8 \succ h_{41} \succ h_{10} \succ h_9 \succ h_{39} \succ h_{29} \succ h_{26} \succ h_{36} \succ h_{34} \succ h_{50} \succ h_{27} \succ h_{18} \succ h_3 \succ h_{32} \succ h_{33} \succ h_{20} \succ h_{43} \succ h_{44} \succ h_{42} \succ h_{40} \succ h_{22} \succ h_{30} \succ h_4$	h_{45}	0.8380
$\alpha = 0.8$	$h_{45} \succ h_{38} \succ h_{14} \succ h_{46} \succ h_1 \succ h_{48} \succ h_{23} \succ h_{13} \succ h_2 \succ h_{37} \succ h_{16} \succ h_{15} \succ h_{28} \succ h_{12} \succ h_{25} \succ h_{24} \succ h_{35} \succ h_{11} \succ h_{19} \succ h_6 \succ h_9 \succ h_{29} \succ h_4 \succ h_{41} \succ h_7 \succ h_5 \succ h_{49} \succ h_{17} \succ h_{34} \succ h_{21} \succ h_{31} \succ h_{47} \succ h_8 \succ h_{26} \succ h_{39} \succ h_3 \succ h_{32} \succ h_{33} \succ h_{27} \succ h_{20} \succ h_{18} \succ h_{50} \succ h_{43} \succ h_{44} \succ h_{10} \succ h_{42} \succ h_{40} \succ h_{36} \succ h_{22} \succ h_{30}$	h_{45}	0.8365
$\alpha = 1$	$h_{45} \succ h_{38} \succ h_{23} \succ h_{46} \succ h_{14} \succ h_{34} \succ h_1 \succ h_2 \succ h_{35} \succ h_{16} \succ h_{48} \succ h_6 \succ h_{15} \succ h_{11} \succ h_{24} \succ h_{49} \succ h_{19} \succ h_{13} \succ h_{40} \succ h_7 \succ h_{28} \succ h_{21} \succ h_5 \succ h_{31} \succ h_{25} \succ h_{17} \succ h_{41} \succ h_{30} \succ h_4 \succ h_9 \succ h_3 \succ h_{32} \succ h_{33} \succ h_{20} \succ h_{27} \succ h_{50} \succ h_{18} \succ h_{44} \succ h_{43} \succ h_{10} \succ h_{42} \succ h_{12} \succ h_{26} \succ h_{47} \succ h_{29} \succ h_{36} \succ h_{22} \succ h_{39} \succ h_8 \succ h_7$	h_{45}	0.7966

Likewise, h_{45} is the optimal alternative no matter how the values of α and ε change for the data set in [48]. Therefore, for the optimal alternative, the proposed 3WMCDM method is stable.

Remark 6.3. As far as the optimal alternative is concerned, the stability of the proposed 3WMCDM method is demonstrated when α and ε change simultaneously.

7. Conclusion

In this paper, the established MCDM-based 3WD models generalize probabilistic rough fuzzy set models based on fuzzy α-neighborhood operators by using the idea of 3WD under the context of MCDM. Based on this point, a novel 3WMCDM method has been proposed in FISs to deal with investment decision-making problems. According to a practical example, the feasibility and effectiveness of the proposed 3WMCDM method have been demonstrated by the validity test and the comparative analysis. Besides, we have analyzed the stability of the proposed 3WMCDM method through the experimental analysis on different data sets. In the following, we outline the main contributions of this paper as follows:

(1) We have defined the reflexive fuzzy α-neighborhood operator in FISs to overcome the weaknesses of the existing fuzzy neighborhood operators [2,25,38].

(2) Based on the fuzzy ε-neighborhood classes constructed by the fuzzy α-neighborhood operators, a probabilistic rough fuzzy set model and MCDM-based 3WD model have been presented to enrich the research contents of neighborhood-based rough set [9,15,20] and 3WD theory.

(3) We have proposed a new determination method of conditional probability by considering an objective set of states according to MCDM problems with fuzzy evaluation values.

Table 24
The ranking result of each region when ε changes.

ε	Regions	Ranking results	The optimal result
$\varepsilon = 0.3$	Pos(X)	$u_7 \succ u_1 \succ u_4 \succ u_2$	u_7
	Bnd(X)	$u_3 \succ u_5$	
	Neg(X)	$u_8 \succ u_6$	
	U	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_3 \succ u_5 \succ u_8 \succ u_6$	
$\varepsilon = 0.4$	Pos(X)	$u_7 \succ u_1 \succ u_4 \succ u_2$	u_7
	Bnd(X)	$u_3 \succ u_5$	
	Neg(X)	$u_8 \succ u_6$	
	U	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_3 \succ u_5 \succ u_8 \succ u_6$	
$\varepsilon = 0.5$	Pos(X)	$u_7 \succ u_1 \succ u_4 \succ u_2$	u_7
	Bnd(X)	$u_3 \succ u_5$	
	Neg(X)	$u_8 \succ u_6$	
	U	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_3 \succ u_5 \succ u_8 \succ u_6$	
$\varepsilon = 0.6$	Pos(X)	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5$	u_7
	Bnd(X)	u_3	
	Neg(X)	$u_8 \succ u_6$	
	U	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_8 \succ u_6$	
$\varepsilon = 0.7$	Pos(X)	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5$	u_7
	Bnd(X)	$u_6 \succ u_3$	
	Neg(X)	u_8	
	U	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$	
$\varepsilon = 0.8$	Pos(X)	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5$	u_7
	Bnd(X)	$u_6 \succ u_3$	
	Neg(X)	u_8	
	U	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$	
$\varepsilon = 0.9$	Pos(X)	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5$	u_7
	Bnd(X)	$u_6 \succ u_3$	
	Neg(X)	u_8	
	U	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$	
$\varepsilon = 1$	Pos(X)	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5$	u_7
	Bnd(X)	$u_6 \succ u_3$	
	Neg(X)	u_8	
	U	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$	

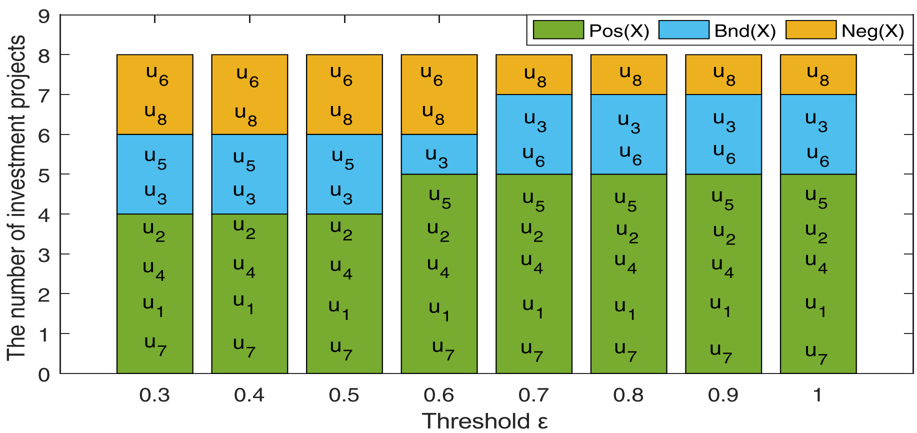

Fig. 5. The three regions obtained by the different values of ε.

(4) A novel 3WMCDM method has been proposed by virtue of decision-making theory with 3WD to overcome the shortcomings of [11].

In the future, we will focus on novel 3WMCDM methods under other information systems, such as ordered information systems [22], incomplete information systems [23,39], etc. Furthermore, the combination of 3WD and traditional MCDM methods is worthy of further investigations.

Table 25
The SRCCs between the 3WMCDM method with a certain value of ε and the below six methods.

3WMCDM method for a certain ε	TOPSIS method [10]	WAA operator method [6]	EDAS method [5]	VIKOR method [26]	Jiang et al.'s method [12]	Jia et al.'s method [11]
$\varepsilon = 0.5$ (0, 0.1, 0.2, 0.3, 0.4)	0.9286	0.9524	0.9524	0.9524	0.9167	0.9524
$\varepsilon = 0.6$	0.9524	0.9762	0.9762	0.9762	0.9643	0.9762
$\varepsilon = 0.7$ (0.8, 0.9, 1)	0.9524	0.9762	0.9762	0.9762	0.8929	0.9762

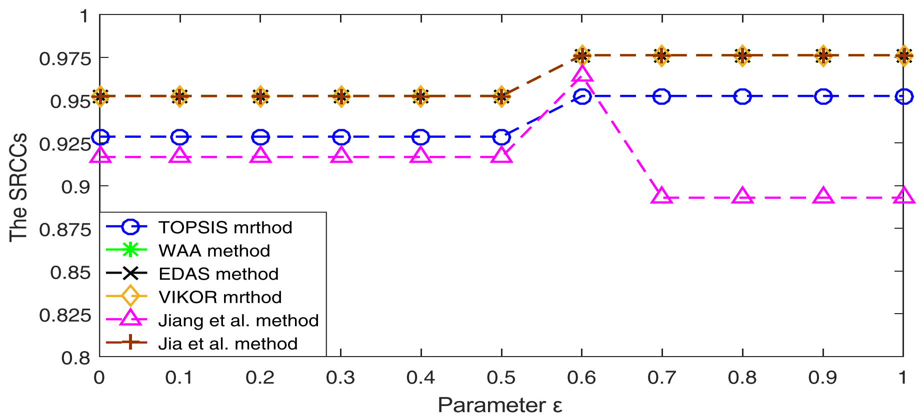

Fig. 6. The SRCCs between the 3WMCDM method with a certain ε and the above six methods.

Table 26
The rankings of alternatives based on different values of ε when $\alpha = 0.5$.

ε	Ranking of alternative	Optimal result	SRCC
$\varepsilon = 0$	$h_{45} \succ h_{38} \succ h_{14} \succ h_{46} \succ h_1 \succ h_2 \succ h_{48} \succ h_{23} \succ h_{16} \succ h_{11} \succ h_{15} \succ h_{34} \succ h_{35} \succ h_6 \succ h_{13} \succ h_7 \succ h_{25} \succ h_3 \succ h_{32} \succ h_{33} \succ h_{20} \succ h_{27} \succ h_{50} \succ h_{18} \succ h_{43} \succ h_{44} \succ h_{10} \succ h_{42} \succ h_{26} \succ h_{12} \succ h_{47} \succ h_5 \succ h_{36} \succ h_{39} \succ h_{29} \succ h_{22} \succ h_8 \succ h_{40} \succ h_{49} \succ h_{28} \succ h_{30} \succ h_{19} \succ h_{21} \succ h_{24} \succ h_9 \succ h_{37} \succ h_{17} \succ h_{41} \succ h_{31} \succ h_4$	h_{45}	0.4509
$\varepsilon = 0.2$	$h_{45} \succ h_{38} \succ h_{14} \succ h_{46} \succ h_1 \succ h_2 \succ h_{23} \succ h_{48} \succ h_{16} \succ h_{15} \succ h_{34} \succ h_{11} \succ h_{35} \succ h_6 \succ h_{13} \succ h_7 \succ h_{25} \succ h_4 \succ h_3 \succ h_{32} \succ h_{33} \succ h_{20} \succ h_{27} \succ h_{50} \succ h_{18} \succ h_{43} \succ h_{44} \succ h_{10} \succ h_{42} \succ h_{26} \succ h_{12} \succ h_{47} \succ h_{36} \succ h_{39} \succ h_5 \succ h_{29} \succ h_{22} \succ h_8 \succ h_{49} \succ h_{40} \succ h_{28} \succ h_{30} \succ h_{21} \succ h_{19} \succ h_{24} \succ h_{37} \succ h_{17} \succ h_9 \succ h_{31} \succ h_{41}$	h_{45}	0.4889
$\varepsilon = 0.4$	$h_{45} \succ h_{38} \succ h_{14} \succ h_1 \succ h_{46} \succ h_{23} \succ h_2 \succ h_{48} \succ h_{15} \succ h_{34} \succ h_{16} \succ h_{11} \succ h_4 \succ h_{35} \succ h_{13} \succ h_6 \succ h_{30} \succ h_7 \succ h_{17} \succ h_{24} \succ h_{31} \succ h_{49} \succ h_{25} \succ h_{28} \succ h_{40} \succ h_{19} \succ h_{41} \succ h_{37} \succ h_9 \succ h_{12} \succ h_{21} \succ h_8 \succ h_{32} \succ h_3 \succ h_{33} \succ h_{20} \succ h_{27} \succ h_{50} \succ h_{18} \succ h_{44} \succ h_{43} \succ h_{10} \succ h_{42} \succ h_{26} \succ h_{47} \succ h_5 \succ h_{36} \succ h_{39} \succ h_{22} \succ h_{29}$	h_{45}	0.8871
$\varepsilon = 0.6$	$h_{45} \succ h_{38} \succ h_{14} \succ h_{46} \succ h_1 \succ h_2 \succ h_{23} \succ h_{48} \succ h_{15} \succ h_{34} \succ h_{35} \succ h_{13} \succ h_{37} \succ h_{16} \succ h_7 \succ h_{30} \succ h_4 \succ h_{25} \succ h_{11} \succ h_{31} \succ h_{24} \succ h_{47} \succ h_{49} \succ h_6 \succ h_{12} \succ h_{19} \succ h_{17} \succ h_{28} \succ h_{41} \succ h_{21} \succ h_8 \succ h_5 \succ h_{29} \succ h_{10} \succ h_{26} \succ h_9 \succ h_{40} \succ h_{36} \succ h_{18} \succ h_{39} \succ h_3 \succ h_{32} \succ h_{33} \succ h_{20} \succ h_{27} \succ h_{50} \succ h_{44} \succ h_{43} \succ h_{42} \succ h_{22}$	h_{45}	0.9520
$\varepsilon = 0.8$	$h_{45} \succ h_{38} \succ h_{48} \succ h_{23} \succ h_{14} \succ h_{46} \succ h_{11} \succ h_1 \succ h_2 \succ h_6 \succ h_{13} \succ h_7 \succ h_{25} \succ h_{41} \succ h_{31} \succ h_{17} \succ h_{24} \succ h_{15} \succ h_{28} \succ h_{19} \succ h_{29} \succ h_{47} \succ h_{49} \succ h_{21} \succ h_5 \succ h_{12} \succ h_{10} \succ h_{16} \succ h_{26} \succ h_{39} \succ h_{36} \succ h_{34} \succ h_{27} \succ h_{35} \succ h_{50} \succ h_{37} \succ h_{32} \succ h_{20} \succ h_{44} \succ h_{43} \succ h_{42} \succ h_{18} \succ h_{22} \succ h_{33} \succ h_{40} \succ h_8 \succ h_3 \succ h_{30} \succ h_9 \succ h_4$	h_{45}	0.7365
$\varepsilon = 1$	$h_{45} \succ h_{38} \succ h_{14} \succ h_{46} \succ h_1 \succ h_2 \succ h_{48} \succ h_{23} \succ h_{16} \succ h_{11} \succ h_{15} \succ h_{34} \succ h_{35} \succ h_6 \succ h_{13} \succ h_7 \succ h_{25} \succ h_3 \succ h_{32} \succ h_{33} \succ h_{20} \succ h_{27} \succ h_{50} \succ h_{18} \succ h_{43} \succ h_{44} \succ h_{10} \succ h_{42} \succ h_{26} \succ h_{12} \succ h_{47} \succ h_5 \succ h_{36} \succ h_{39} \succ h_{29} \succ h_{22} \succ h_8 \succ h_{40} \succ h_{49} \succ h_{28} \succ h_{30} \succ h_{19} \succ h_{21} \succ h_{24} \succ h_9 \succ h_{37} \succ h_{17} \succ h_{41} \succ h_{31} \succ h_4$	h_{45}	0.6295

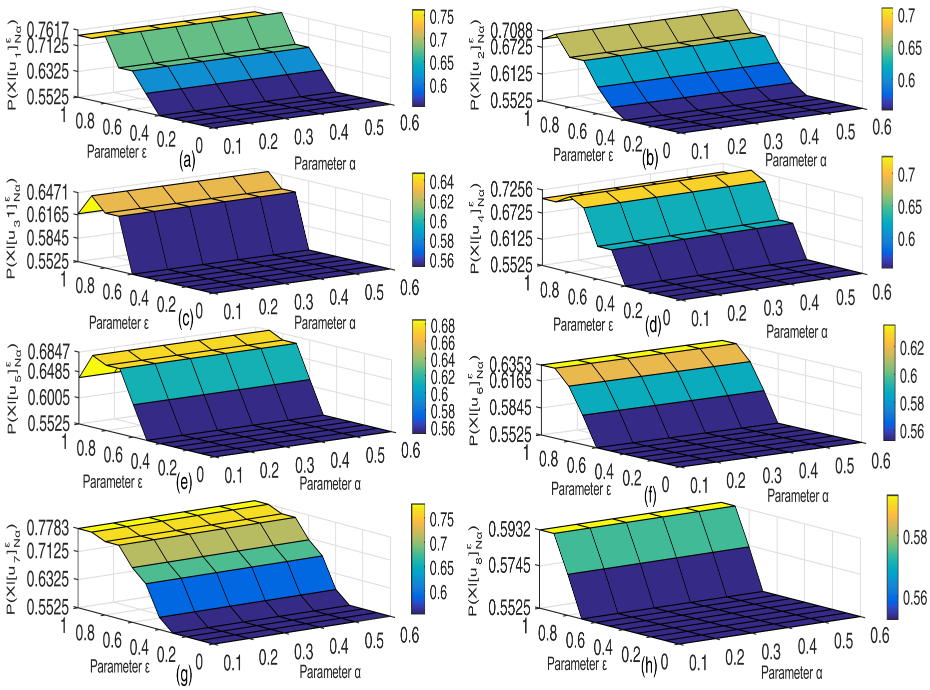

Fig. 7. The results of the conditional probabilities of investment projects when α and ε change.

CRediT authorship contribution statement

Jin Ye: Conceptualization, Methodology, Investigation, Writing - original draft. **Jianming Zhan:** Conceptualization, Methodology, Investigation, Writing - original draft. **Zeshui Xu:** Conceptualization, Methodology, Investigation, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors are extremely grateful to the editors and three reviewers for their valuable comments and helpful suggestions which helped to improve the presentation of this paper. The work was partially supported by grants from the NNSFC (61866011, 11961025).

References

[1] Y.M. Chen, Z.Q. Zeng, Q.X. Zhu, C.H. Tang, Three-way decision reduction in neighborhood systems, Appl. Soft Comput. 38 (2016) 942–954.
[2] L. D'eer, C. Cornelis, L. Godo, Fuzzy neighborhood operators based on fuzzy coverings, Fuzzy Sets Syst., 312 (2017) 17-35.
[3] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst. 17 (1990) 191–209.
[4] T. Feng, H.T. Fan, J.S. Mi, Uncertainty and reduction of variable precision multigranulation fuzzy rough sets based on three-way decisions, Int. J. Approx. Reason. 85 (2017) 36–58.
[5] M.K. Ghorabaee, E.K. Zavadskas, L. Olfat, Z. Turskis, Multi-criteria inventory classification using a new method of evaluation based on distance from average solution (EDAS), Informatica 26 (3) (2015) 435–451.
[6] J.C. Harsanyi, Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility, J. Politicl Econ. 63 (1955) 309–321.
[7] B.Q. Hu, Three-way decisions space and three-way decisions, Inf. Sci. 281 (2014) 21–52.
[8] J.H. Hu, Y. Yang, X.H. Chen, A novel TODIM method-based three-way decision model for medical treatment selection, Int. J. Fuzzy Syst. 20 (4) (2018) 1240–1255.
[9] Q.H. Hu, D.R. Yu, J.F. Liu, C.X. Wu, Neighborhood rough set based heterogeneous feature subset selection, Inf. Sci. 178 (2008) 3577–3594.
[10] C.L. Hwang, K. Yoon, Multiple attribute decision making methods and applications, Springer, Berlin Heidelberg, 1981.
[11] F. Jia, P.D. Liu, A novel three-way decision model under multiple-criteria environment, Inf. Sci. 471 (2019) 29–51.
[12] H.B. Jiang, J.M. Zhan, D.G. Chen, Covering-based variable precision (I; T)-fuzzy rough sets with applications to multiattribute decision-making, IEEE Trans. Fuzzy Syst. 27 (8) (2019) 1558–1572.
[13] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice-Hall, New Jersey, 1995.

[14] G.M. Lang, D.Q. Miao, M.J. Cai, Three-way decision approaches to conflict analysis using decision-theoretic rough set theory, Inf. Sci. 406 (2017) 185–207.
[15] W.W. Li, Z.Q. Huang, X.Y. Jia, X.Y. Cai, Neighborhood based decision-theoretic rough set models, Int. J. Approx. Reason. 69 (2016) 1–17.
[16] Z.W. Li, P.F. Zhang, N.X. Xie, G.Q. Zhang, C.F. Wen, A novel three-way decision method in a hybrid information system with images and its application in medical diagnosis, Eng. Appl. Artif. Intell. 92 (2020) 103651.
[17] D.C. Liang, D. Liu, A novel risk decision making based on decision-theoretic rough sets under hesitant fuzzy information, IEEE Trans. Fuzzy Syst. 23 (2) (2015) 237–247.
[18] D.C. Liang, Z.S. Xu, D. Liu, Y. Wu, Method for three-way decisions using ideal TOPSIS solutions at pythagorean fuzzy information, Inf. Sci. 435 (2018) 282–295.
[19] H.C. Liao, Z.S. Xu, Multi-criteria decision making with intuitionistic fuzzy PROMETHEE, Int. J. Fuzzy Syst. 27 (2014) 1703–1717.
[20] G.P. Lin, Y.H. Qian, J.J. Li, NMGRS: Neighborhood-based multi-granulation rough sets, Int. J. Approx. Reason. 53 (7) (2012) 1080–1093.
[21] D. Liu, T.R. Li, D. Ruan, Probabilistic model criteria with decision-theoretic rough sets, Inf. Sci. 181 (17) (2011) 3709–3722.
[22] D. Liu, D.C. Liang, Three-way decisions in ordered decision system, Knowl.-Based Syst. 137 (2017) 182–195.
[23] D. Liu, D.C. Liang, C.C. Wang, A novel three-way decision model based on incomplete information system, Knowl.-Based Syst. 91 (2016) 32–45.
[24] P.D. Liu, Y.M. Wang, F. Jia, H. Fujita, A multiple attribute decision making three-way model for intuitionistic fuzzy numbers, Int. J. Approx. Reason. 119 (2020) 177–203.
[25] L.W. Ma, Two fuzzy covering rough set models and their generalizations over fuzzy lattices, Fuzzy Sets Syst. 294 (2016) 1–17.
[26] S. Opricovic, G.H. Tzeng, Compromise solution by MCDM methods: A comparative analysis of VIKOR and TOPSIS, Eur. J. Oper. Res. 156 (2) (2004) 445–455.
[27] J.F. Pang, X.Q. Guan, J.Y. Liang, B.L. Wang, P. Song, Multi-attribute group decision-making method based on multi-granulation weights and three-way decisions, Int. J. Approx. Reason. 117 (2020) 122–147.
[28] Z. Pawlak, Information systems, theoretical foundations, Inform. Syst. 6 (3) (1981) 205–218.
[29] C.E. Shannon, The mathematical theory of communication, I, II, Bell Syst. Techn. J., 27(3,4) (1948) 379–423.
[30] B.Z. Sun, W.M. Ma, B.J. Li, X.N. Li, Three-way decisions approach to multiple attribute group decision making with linguistic information-based decision-theoretic rough fuzzy set, Int. J. Approx. Reason. 93 (2018) 424–442.
[31] B.Z. Sun, W.M. Ma, H.Y. Zhao, Decision-theoretic rough fuzzy set model and application, Inf. Sci. 283 (2014) 180–196.
[32] C.Z. Wang, D.G. Chen, Q.H. Hu, Fuzzy information systems and their homomorphisms, Fuzzy Sets Syst. 249 (2014) 128–138.
[33] P. Wang, P.F. Zhang, Z.W. Li, A three-way decision method based on Gaussian kernel in a hybrid information system with images: An application in medical diagnosis, Appl. Soft Comput. 77 (2019) 734–749.
[34] X.T. Wang, E. Triantaphyllou, Ranking irregularities when evaluating alternatives by using some ELECTRE methods, Omega 36 (2008) 45–63.
[35] S.K.M. Wong, W. Ziarko, Comparison of the probabilistic approximate classification and the fuzzy set model, Fuzzy Sets Syst. 21 (1987) 357–362.
[36] W.Z. Wu, W.X. Zhang, Neighborhood operator systems and approximations, Inf. Sci. 144 (2002) 201–217.
[37] Z.S. Xu, Q.L. Da, An overview of operators for aggregating information, Int. J. Fuzzy Syst. 18 (2003) 953–969.
[38] B. Yang, B.Q. Hu, Fuzzy neighborhood operators and thier corresponding derived fuzzy coverings, Fuzzy Sets Syst. 370 (2019) 1–33.
[39] X.P. Yang, T.J. Li, A.H. Tan, Three-way decisions in fuzzy incomplete information systems, Int. J. Mach. Learn. Cybern. 11 (3) (2020) 667–674.
[40] Y.Y. Yao, Three-way decision: An interpretation of rules in rough set theory, Rough Sets Knowl. Technol. 5589 (2009) 642–649.
[41] Y.Y. Yao, Three-way decisions with probabilistic rough sets, Inf. Sci. 180 (3) (2010) 341–353.
[42] Y.Y. Yao, Three-way decision and granular computing, Int. J. Approx. Reason. 103 (2018) 107–123.
[43] Y.Y. Yao, Three-way granular computing, rough set and formal concept analysis, Int. J. Approx. Reason. 116 (2020) 106–125.
[44] Y.Y. Yao, B. Yao, Covering based rough set approximations, Inf. Sci. 200 (2012) 91–107.
[45] L.A. Zadeh, Fuzzy sets, Inform, Control 8 (1965) 338–353.
[46] C. Zhang, D.Y. Li, J.Y. Liang, Multi-granularity three-way decisions with adjustable hesitant fuzzy linguistic multigranulation decision-theoretic rough sets over two universes, Inf. Sci. 507 (2020) 665–683.
[47] K. Zhang, J.M. Zhan, W.Z. Wu, J.C.R. Alcantud, Fuzzy β-covering based $(\mathcal{I},\mathcal{T})$-fuzzy rough set models and applications to multi-attribute decision-making, Comput. Ind. Eng. 128 (2019) 605–621.
[48] K. Zhang, J.M. Zhan, Y.Y. Yao, TOPSIS method based on a fuzzy covering approximation space: An application to biological nano-materials selection, Inf. Sci. 502 (2019) 297–329.
[49] M. Zhang, Z.F. Zhou, Evaluation on the independent innovation capacity of high-tech enterprises based on rough set and entropy weight-TOPSIS, Math. Pract. Theory (in Chin.) 38 (24) (2008) 52–58.
[50] W. Ziarko, Variable precision rough set model, J. Comp. Syst. Sci. 46 (1993) 39–59.