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Fuzzy rough set theory for the interval-valued fuzzy information systems ☆

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Abstract

The concept of the rough set was originally proposed by Pawlak as a formal tool for modelling and processing incomplete information in information systems, then in 1990, Dubois and Prade first introduced the rough fuzzy sets and fuzzy rough sets as a fuzzy extension of the rough sets. The aim of this paper is to present a new extension of the rough set theory by means of integrating the classical Pawlak rough set theory with the interval-valued fuzzy set theory, i.e., the interval-valued fuzzy rough set model is presented based on the interval-valued fuzzy information systems which is defined in this paper by a binary interval-valued fuzzy relations $R \in F^{(i)}(U \times U)$ on the universe U. Several properties of the rough set model are given, and the relationships of this model and the others rough set models are also examined. Furthermore, we also discuss the knowledge reduction of the classical Pawlak information systems and the interval-valued fuzzy information systems respectively. Finally, the knowledge reduction theorems of the interval-valued fuzzy information systems are built.

Keywords: Fuzzy sets; Rough sets; Interval-valued fuzzy sets; Interval-valued fuzzy information systems; Knowledge reduction

1. Introduction

The theory of rough sets was firstly proposed by Pawlak [17–19]. It is an extension of the set theory for the study of intelligent systems characterized by insufficient and incomplete information. The successful application of rough set theory in a variety of problems has been amply demonstrated its usefulness. This theory evoked into a far-reaching methodology centering on analysis of incomplete information [9,23,24,31]. It soon evoked a natural question concerning possible connections between rough sets and fuzzy sets. Generally

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speaking, both theories address the problem of information granulation: the theory of fuzzy sets is centrad upon fuzzy information granulation, whereas rough set theory is focused on crisp information granulation. Originally, the basic notion in rough set theory was indiscernibility (i.e., indiscernibility between objects in information systems induced by different values of attributes characterizing these objects), yet in recent extensions [7,14,20,22] the focus moves to the notion of similarity, which is in fact a fuzzy concept. Therefore, it is apparent that these two theories have become much more closely related to each other.

A key notion in Pawlak's rough set model is equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. By replacing the equivalence relation with an arbitrary binary relation, different kinds of generalizations in Pawlak's rough set models were obtained. Dubois and Prade [3,4] were among the first who investigated the problem of a fuzzyfication of a rough set, the concept of rough fuzzy set and fuzzy rough set were proposed by replacing crisp binary relations with fuzzy relations in the universe [1,3,4,12,13,25,32]. Moreover, Dubois and Prade also pointed out that the rough fuzzy set is a special case of the fuzzy rough set in the universe in their literatures. The fuzzy rough set theory, proposed by the others authors, has been made up the deficiencies of the traditional rough set theory in several aspects. However, there are both of the symbolic values, real values and possibly interval values of the attributes in the real life database [6], therefore, the traditional fuzzy rough set theory could not deal with those data effectively. It is then necessary to extend the traditional fuzzy rough set theory in a general sense. In this paper, we propose the interval-valued fuzzy rough set theory by combining the interval-value fuzzy set theory with the traditional rough set theory. Therefore, the traditional fuzzy rough set theory is extended and its weaknesses are overcome.

There are at least two approaches for the development of the fuzzy rough sets theory, the constructive and the axiomatic approaches [29,30,33]. Moris and Yakout [11] provide the axiomatization for the fuzzy rough set model. Wu et al. [28] present a general framework for the study of fuzzy rough sets in which both constructive and axiomatic approaches are used. In the constructive approach, the relations in the universe is the primitive notion, the lower and upper approximation operators are constructed by means of this notion, therefore, one can obtain a pair of lower and upper generalized approximation operators in this approach. Dubois and Prade were among the first researchers to propose the concept of the fuzzy rough sets from the constructive approach. In the axiomatic approach, various classes of fuzzy rough approximation operators are characterized by different sets of axioms, these axioms guarantee the existence of certain types of fuzzy relations producing the same operators. This paper is devoted to the discussion of the interval-valued fuzzy rough sets model by the constructive approach.

Fuzzy rough sets have been used to solve practical problems such as data mining [9,16], approximate reasoning [21], medical time series, case generation [15], mining stock price [27], and descriptive dimensionality reduction [6].

As a generalization of the Zadeh fuzzy set, the notion of interval- valued fuzzy sets was suggested for the first time by Gorzalczany [5] and Turksen [26], and it was applied to the fields of approximate inference, signal transmission and controller, etc. In this paper we combine the classical Pawlak rough sets theory with the interval-valued fuzzy sets theory, define the interval-valued fuzzy information system, and discuss the rough set theory of the interval-valued fuzzy information system. So, the interval-valued fuzzy rough sets model is obtained in the interval-valued fuzzy information system by the constructive approach. Several properties of this model are given, and the relationships of this model and the other rough set models are also examined.

Knowledge reduction [2] is performed in information systems by means of the notion of a reduction based on a specialization of the general notion of independence due to Maczewski [10]. The knowledge reduction of consistent information systems based on the rough sets theory have been many practices conclusion. In recent years, more attention has been paid to knowledge reduction in inconsistent systems in rough sets research [8]. Many types of knowledge reduction have been proposed in the area of rough sets [30,31]. In this paper, we are concerned with approaches to knowledge reduction based on the interval-valued fuzzy rough sets model. We first define the interval-valued fuzzy reduction on the classical Pawlak information systems, then discuss the knowledge reduction of the interval-valued fuzzy information systems. Finally, the knowledge reduction theorems of the interval-valued fuzzy information systems are built.

The structure of the rest of this paper is as follows: Section 2 briefly introduces necessary notions of rough sets, fuzzy sets and interval-valued fuzzy sets. In Section 3, we define the interval-valued fuzzy binary relations of the universe, and then define the interval-valued fuzzy rough sets model while several properties of the model are also examined. In Section 4, the relationships of the model defined in Section 3 and the others rough set models are studied. In Section 5, an interval-valued fuzzy reduction in the classical Pawlak information systems is defined and the knowledge reduction of the interval-valued fuzzy information systems is investigated. We conclude the paper with a summary and an outlook for further research in Section 6.

2. Preliminaries

2.1. Fuzzy sets

Let U be a finite and non-empty set called universe. A fuzzy set A is a mapping from U into the unit interval $[0,1]$:

$$\mu_A : U \to [0,1],$$

where each $x \in U$ is the membership degree of x in A. Practically, we may consider U as a set of objects of concern and a crisp subset of U represents a "non-vague" concept imposed on objects in U. Then a fuzzy set A of U is thought of as a mathematical representation of a "vague" concept described linguistically.

The set of all the fuzzy sets defined on U is denoted by $F(U)$.

Given a number $\alpha \in [0,1]$, the α-cut or α-level set, of A is defined as follows:

$$[A]^\alpha = \{x : \mu_A(x) \geqslant \alpha\},$$

which is a subset of U. A strong α-cut set is defined by

$$\sigma_\alpha(A) = \{x : \mu_A(x) > \alpha\}.$$

Theorem 2.1. *Let $A \in F(U)$, then the following equations hold*:

$$A = \cup_{r\in[0,1]}([A]^r \cap r^*),$$
$$A = \cup_{r\in[0,1]}(\sigma_r(A) \cap r^*),$$

where r^* denotes the fuzzy set whose membership function is the constant, $\cup$ means maximum and $\cap$ means minimum.

Proof. Since r^* denotes the fuzzy set in which the membership function is the constant, that is, $r^*(x) = r$ for any $x \in U$,

$$\begin{aligned}(\cup_{r\in[0,1]}([A]^r \cap r^*))(x) &= \vee_{r\in[0,1]}(r \wedge [A]^r(x)) \\ &= \max(\vee_{r\leqslant\mu_A(x)}(r \wedge [A]^r(x)), \vee_{\mu_A(x)<r}(r \wedge [A]^r(x))) \\ &= \max(\vee_{r\leqslant\mu_A(x)}(r \wedge 1), \vee_{\mu_A(x)<r}(r \wedge 0)) \\ &= \max(\vee_{r\leqslant\mu_A(x)}r, \vee_{\mu_A(x)<r}0) \\ &= \max(\mu_A(x), 0) \\ &= \mu_A(x).\end{aligned}$$

This proves that $A = \cup_{r\in[0,1]}([A]^r \cap r^*)$, and the latter one has an analogous proof. □

2.2. Rough sets

Formally, the theory begins with the notion of an approximation space, which is a pair (U, R), where U is a non-empty set(the universe of discourse) and R an equivalence relation on U, i.e., R is reflexive, symmetric, and transitive, The relation R decomposes the set U into a disjoint classes in such a way that two elements x, y are in the same class iff $(x, y) \in R$. Let U/R denote the quotient set of U by the relation R, and

$$U/R = \{X_1, X_2, \ldots, X_m\},$$

where X_i is an equivalence class of $R, i = 1, 2, \ldots, m$.

If two elements x, y in U belong to the same equivalence class $X_i \in U/R$, we say that x and y are indiscernible. The equivalence class of R and the empty set $\emptyset$ are the elements in the approximation space (U, R).

Note that in the context of rough set based data analysis, the equivalence relation in an approximation space is often interpreted via the notion of information system. An information system with U as the universe is a triple $(U, \mathbb{A}, \{V_a\}_{a\in\mathbb{A}})$, where U is a set of objects, $\mathbb{A}$ is a set of attributes, and V_a is the set of attribute values for attribute a understood as a mapping $a : U \to V_a$. It is easily seen that each subset $\mathbb{A}$ of the attribute set $\mathbb{A}$ induces an equivalence relation $ind(\mathbb{A})$ called as indiscernibility relation as follows:

$$ind(\mathbb{A}) = \{(x, y) \in U \times U : a(x) = a(y), a \in \mathbb{A}\}$$

and $ind(\mathbb{A}) = \bigcap_{a\in\mathbb{A}} ind(a)$, where $ind(a)$ means $ind(\{a\})$. If $(x, y) \in \mathbb{A}$, we then say that the objects x and y are indiscernible with respect to attributes from $\mathbb{A}$. In other words, we cannot distinguish x from y and vice versa, in terms of attributes in $\mathbb{A}$.

Let us return to an arbitrary approximation space (U, R). Given an arbitrary set $X \in 2^U$, in general, it may not be possible to describe X precisely in (U, R). One may characterize X by a pair of lower and upper approximations defined as follows [14–16]:

$$\underline{R}X = \cup\{Y \in U/R : Y \subseteq X\} = \{x \in U : [x]_R \subseteq X\},$$
$$\overline{R}X = \cup\{Y \in U/R : Y \cap X \neq \emptyset\} = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

The lower approximation $\underline{R}X$ is the union of all the elementary sets which are subsets of X, and the upper approximation $\overline{R}X$ is the union of all the elementary sets which have a non-empty intersection with X. The interval $[\underline{R}X, \overline{R}X]$ is the representation of an ordinary set X in the approximation space (U, R) or simply, the rough set of X.

The lower(upper) approximation $\underline{R}X(\overline{R}X)$ is interpreted as the collection of those elements of U that definitely(possibly) belong to X. Further, we also define

- A set $X \in 2^U$ is said to be definable(or exact) in (U, R) iff $\underline{R}X = \overline{R}X$.
- For any $X, Y \in 2^U$, X is said to be roughly included in Y, denoted by $X\tilde{\subset}X$, iff $\underline{R}X \subseteq \underline{R}Y$ and $\overline{R}X \subseteq \overline{R}Y$.
- X and Y are said to be roughly equal, denoted by $X \approx_R Y$, in (U, R) iff $\underline{R}X = \underline{R}Y$ and $\overline{R}X = \overline{R}Y$.

2.3. Interval-valued and interval-valued fuzzy set

Throughout this paper, let I be a closed unit interval, i.e., $I = [0, 1]$. Let $[I] = \{[a, b] : a \leqslant b, a, b \in I\}$. For any $a \in I$, define $\bar{a} = [a, a]$, obviously, $a \in [I]$.

Definition 2.1. If $\forall a_i \in I, i \in J$, we define

$$\bigvee_{i\in J} a_i = \sup\{a_i : i \in J\}, \quad \bigwedge_{i\in J} a_i = \inf\{a_i : i \in J\}.$$
$$\bigvee_{i\in J} [a_i, b_i] = [\vee_{i\in J} a_i, \vee_{i\in J} b_i], \quad \bigwedge_{i\in J} [a_i, b_i] = [\wedge_{i\in J} a_i, \wedge_{i\in J} b_i],$$

where $\bigvee$ means maximum and $\bigwedge$ means minimum.

In particular for $[a_i, b_i] \in [I], i = 1, 2$, we define

$$[a_1, b_1] = [a_2, b_2] \quad \text{iff } a_1 = a_2,\ b_1 = b_2;$$
$$[a_1, b_1] \leqslant [a_2, b_2] \quad \text{iff } a_1 \leqslant a_2,\ b_1 \leqslant b_2;$$
$$[a_1, b_1] < [a_2, b_2] \quad \text{iff } [a_1, b_1] \leqslant [a_2, b_2] \text{ but } [a_1, b_1] \neq [a_2, b_2].$$

Definition 2.2. Let X be an ordinary non-empty set. Then the mapping $A : X \to [I]$ is called an interval-valued fuzzy set in X. All interval-valued fuzzy set on X are denoted as $F^{(i)}(X)$.

Definition 2.3. If $A \in F^{(i)}(X)$, let $A(x) = [A^-(x), A^+(x)]$, where $x \in X$, then two fuzzy sets $A^- : X \to I$, and $A^+ : X \to I$ are called the lower fuzzy set and the upper fuzzy set about A, respectively.

Definition 2.4. Let $A \in F^{(i)}(X), [\lambda_1, \lambda_2] \in [I]$, we call $A_{[\lambda_1,\lambda_2]} = \{x \in X : A^-(x) \geqslant \lambda_1, A^+(x) \geqslant \lambda_2\}$, and $A_{(\lambda_1,\lambda_2)} = \{x \in X : A^-(x) > \lambda_1, A^+(x) > \lambda_2\}$, the $[\lambda_1, \lambda_2]$-level set of A and (λ_1, λ_2)-level set of A, respectively, where (λ_1, λ_2) in $A_{(\lambda_1,\lambda_2)}$ is not an interval, it is only a sign, and we may admit $\lambda_1 = \lambda_2$, clearly, $x \in A_{[\lambda_1,\lambda_2]}$ iff $A_{[\lambda_1,\lambda_2]} \geqslant [\lambda_1, \lambda_2]$.

Definition 2.5. Let $A \in F^{(i)}(X), [\lambda_1, \lambda_2] \in [I]$. We define

$$([\lambda_1, \lambda_2]A)(x) = [\lambda_1, \lambda_2] \wedge [A^-(x), A^+(x)].$$

Proposition 2.1 (The decomposition theorem of interval-valued fuzzy sets). *Let $A \in F^{(i)}(X)$. Then*

$$A = \bigcup_{[\lambda_1,\lambda_2]\in[I]} [\lambda_1, \lambda_2]A_{[\lambda_1,\lambda_2]} = \bigcup_{[\lambda_1,\lambda_2]\in[I]} [\lambda_1, \lambda_2]A_{(\lambda_1,\lambda_2)}.$$

Proof. We have only to prove the equation $A(x) = (\bigcup_{[\lambda_1,\lambda_2]\in[I]} [\lambda_1, \lambda_2]A_{[\lambda_1,\lambda_2]})(x)$ holds for any $x \in U$.

$$\begin{aligned}
\left(\bigcup_{[\lambda_1,\lambda_2]\in[I]} [\lambda_1, \lambda_2]A_{[\lambda_1,\lambda_2]}\right)(x) &= \vee_{[\lambda_1,\lambda_2]\in[I]}([\lambda_1, \lambda_2]A_{[\lambda_1,\lambda_2]}(x)) \\
&= \vee_{[\lambda_1,\lambda_2]\in[I]}[\lambda_1 \wedge A_{[\lambda_1,\lambda_2]}(x), \lambda_2 \wedge A_{[\lambda_1,\lambda_2]}(x)] \\
&= \vee_{[\lambda_1,\lambda_2]\in[I]}[\lambda_1 \wedge (A^-_{\lambda_1} \cap A^+_{\lambda_2})(x), \lambda_2 \wedge (A^-_{\lambda_1} \cap A^+_{\lambda_2})(x)] \\
&= \vee_{[\lambda_1,\lambda_2]\in[I]}[\lambda_1 \wedge A^-_{\lambda_1}(x) \wedge A^+_{\lambda_2}(x), \lambda_2 \wedge A^-_{\lambda_1}(x) \wedge A^+_{\lambda_2}(x)] \\
&= [\vee_{[\lambda_1,\lambda_2]\in[I]}(\lambda_1 \wedge A^-_{\lambda_1}(x) \wedge A^+_{\lambda_2}(x)), \vee_{[\lambda_1,\lambda_2]\in[I]}(\lambda_2 \wedge A^-_{\lambda_1}(x) \wedge A^+_{\lambda_2}(x))].
\end{aligned}$$

We could prove that $\vee_{[\lambda_1,\lambda_2]\in[I]}(\lambda_1 \wedge A^-_{\lambda_1}(x) \wedge A^+_{\lambda_2}(x)) = A^-(x)$. In fact,

$$\begin{aligned}
\vee_{[\lambda_1,\lambda_2]\in[I]}(\lambda_1 \wedge A^-_{\lambda_1}(x) \wedge A^+_{\lambda_2}(x)) &= \vee_{\lambda_1\in[0,1]}\vee_{\lambda_2\in[\lambda_1,1]}(\lambda_1 \wedge A^-_{\lambda_1}(x) \wedge A^+_{\lambda_2}(x)) \\
&= \vee_{\lambda_1\in[0,1]}(\lambda_1 \wedge A^-_{\lambda_1}(x) \wedge \vee_{\lambda_2\in[\lambda_1,1]}A^+_{\lambda_2}(x)) \\
&= \vee_{\lambda_1\in[0,1]}(\lambda_1 \wedge A^-_{\lambda_1}(x) \wedge A^+_{\lambda_1}(x)) \\
&= \vee_{\lambda_1\in[0,1]}(\lambda_1 \wedge A^-_{\lambda_1}(x)) \\
&= A^-(x).
\end{aligned}$$

Similarly, $\vee_{[\lambda_1,\lambda_2]\in[I]}(\lambda_2 \wedge A^-_{\lambda_1}(x) \wedge A^+_{\lambda_2}(x)) = A^+(x)$ and the (λ_1, λ_2)-level set of A can be proved in a similar way. □

3. The interval-valued fuzzy rough set models

3.1. The rough approximation of a crisp set on the interval-valued fuzzy information system

Let U be a non-empty finite universe. A binary interval-valued fuzzy subset R of $U \times U$ is called an interval-valued fuzzy relation in U.

Definition 3.1. Let U be a non-empty finite universe. For the interval-valued fuzzy relation $R(R \in F^{(i)}(U \times U))$ of the universe U,

(1) R is reflexive, if $R(x, y) = \bar{1}$, for any $x, y \in U$,
(2) R is symmetric, if $R(x, y) = R(y, x)$, for any $x, y \in U$,
(3) R is transitive, if $R(x, z) \geqslant R(x, y) \wedge R(y, z)$, for any $x, y, z \in U$.

If the fuzzy relation R is reflexive, symmetric and transitive, then R is an interval-valued fuzzy equivalence relation.

In this section, the interval-valued fuzzy information system means that R is the interval-valued fuzzy relation in the classical approximation space.

Definition 3.2. (U,R) is an interval-valued fuzzy information system, if R is a reflexive interval-valued fuzzy relation of the universe U. If R is an interval-valued fuzzy equivalence relation in the universe U, then (U,R) is an interval-valued fuzzy equivalence relation information system.

Definition 3.3. Let (U,R) be an interval-valued fuzzy information system. For any $x \in U$, call $[x]^{(i)} : U \to [I], y \to R(x,y)$ the interval-valued fuzzy neighborhood of x.

Let U be a non-empty finite universe. For any crisp set $X(X \subseteq U)$ of U, define

$$\underline{R}(X)(y) = \min_{x \notin X} (\bar{1} - R(x,y)), \quad \overline{R}(X)(y) = \max_{x \in X} R(x,y) \quad \text{for any } y \in U.$$

$\underline{R}(X)$ and $\overline{R}(X)$ are called the interval-valued fuzzy lower approximation and the interval-valued fuzzy upper approximation of X about the interval-valued fuzzy information system (U,R), respectively, and $\underline{R} : P(U) \to F^{(i)}(U), \overline{R} : P(U) \to F^{(i)}(U)$ is called the interval-valued fuzzy lower approximation operator and upper approximation operator, respectively. Obviously

$$\underline{R}(X)(y) = \left[\min_{x \notin X}(1 - R^+(x,y)),\ \min_{x \notin X}(1 - R^-(x,y))\right],$$

$$\overline{R}(X)(y) = \left[\max_{x \in X} R^-(x,y),\ \max_{x \in X} R^+(x,y)\right].$$

Theorem 3.1. *Let (U,R) be an interval-valued fuzzy information system. For any subset $X, Y \in P(U)$, the following properties hold for the lower and upper approximations of an interval-valued fuzzy operator*:

(1) $\underline{R}(U) = U,\ \overline{R}(\emptyset) = \emptyset$,
(2) $\underline{R}(\sim X) =\sim \overline{R}(X),\ \overline{R}(\sim X) =\sim \underline{R}(X)$,
(3) $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y),\ \overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$,
(4) $\underline{R}(X \cup Y) \supseteq \underline{R}(X) \cup \underline{R}(Y),\ \overline{R}(X \cap Y) \subseteq \overline{R}(X) \cup \overline{R}(Y)$,
(5) $\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$.

Proof. For any interval-valued fuzzy set A, we define $\max_{x \in \emptyset} A(x) = \bar{0}, \min_{x \in \emptyset} A(x) = \bar{1}$.

Then (1) is easy to prove.

(2) Since $\underline{R}(\sim X)(y) = [\min_{x \notin \sim X}(1 - R^+(x,y)), \min_{x \notin \sim X}(1 - R^-(x,y))]$, there are

$$\sim \underline{R}(\sim X)(y) = \bar{1} - \left[\min_{x \notin \sim X}(1 - R^+(x,y)), \min_{x \notin \sim X}(1 - R^-(x,y))\right] = \left[\max_{x \in X} R^-(x,y), \max_{x \in X} R^+(x,y)\right]$$
$$= \max_{x \in X} R(x,y) = \overline{R}(X)(y).$$

So, the equation $\underline{R}(\sim X) =\sim \overline{R}(X)$ is proved, and the equation $\overline{R}(\sim X) =\sim \underline{R}(X)$ can be proved in a similar way.

(3) For any $y \in X$, $\quad \underline{R}(X \cap Y)(y) = \min_{x \in X \cap Y} (\bar{1} - R(x,y)) = \left(\min_{x \notin X}(\bar{1} - R(x,y)) \wedge \left(\min_{x \in Y}(\bar{1} - R(x,y))\right)\right)$
$$= (\underline{R}(X) \cap \underline{R}(Y))(y).$$

Therefore, the equation $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$ is proved, and the equation $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$ can be proved in a similar way.

(4) is easy to prove according to (3).

(5) For any classical $X(X \subseteq U)$, there is $X(y) = 1 \Longleftrightarrow y \in X$. Therefore, if $y \notin X$, we have $\underline{R}(X)(y) = \bar{0}$ for the reflexive of R and the definition of the lower approximation, and $\overline{R}(X)(y) = \bar{1}$ for the reflexive of R and the definition of the upper approximation.

Then, $\underline{R}(X)(y) \leqslant X(y) \leqslant \overline{R}(X)(y)$, and (5) is proved. □

Theorem 3.2. *Let (U,R) be an interval-valued fuzzy information system. For any subset $X \subseteq U$, $\underline{R}(X) = \overline{R}(X)$ iff $R(x,y) = \bar{0}$ (for any $x \in X, y \notin X$).*

Proof. Let $X(y)$ be the characteristic function of X, we have $X(y) \leqslant \max_{x \in X} R(x,y)$ according to the reflexive of R. Therefore, $X(y) = \max_{x \in X} R(x,y)$ iff for any $y \notin X, x \in X$, and $R(x,y) = \bar{0}$. Since $\min_{x \notin X}(\bar{1} - R(x,y)) = X(y)$ for any $y \in X, x \notin X$ and $R(x,y) = \bar{0}$, we prove that $\underline{R}(X) = \overline{R}(X)$. □

Let (U,R) be an interval-valued fuzzy information system. If $\underline{R}(X) = \underline{R}(Y)$, then X is called the interval-valued fuzzy lower rough equal to Y, denote as $X \eqsim Y$.

If $\overline{R}(X) = \overline{R}(Y)$, then X is called the interval-valued fuzzy upper rough equal to Y, denoted as $X \simeq Y$.

If X is both the lower and upper rough equal to Y, then X is called the rough equal to Y, denote as $X \approx Y$.

Proposition 3.1. *Let $\eqsim, \simeq, \approx$ be the lower rough equal, the upper rough equal and the rough equal, which was defined in the above definition of two interval-valued fuzzy sets in the universe U. Then they are all equivalence relations on $F^{(i)}(U)$.*

Proof. For any $X,Y,Z(X,Y,Z \subseteq U)$, there is $\underline{R}(X) = \underline{R}(X)$. Then the reflexive is proved. If $\underline{R}(X) = \underline{R}(Y)$, then there must be $\underline{R}(Y) = \underline{R}(X)$, so the symmetric is proved. And if both of $\underline{R}(X) = \underline{R}(Y)$ and $\underline{R}(Y) = \underline{R}(Z)$ hold, then there must be $\underline{R}(X) = \underline{R}(Z)$, and the transitive is proved. Therefore, $\eqsim$ is the equivalence relation on $F^{(i)}(U)$.

Similarly, $\simeq$ and $\approx$ can be proved in a similar way. □

Theorem 3.3. *Let (U,R) be an interval-valued fuzzy information system. For any subset $X \subseteq U$, the interval-valued fuzzy lower approximation, upper approximation and the interval-valued fuzzy neighborhood of X satisfied the following relations:*

$$\underline{R}(X) = \cap_{x \notin X}(\sim [x]^{(i)}), \quad \overline{R}(X) = \cup_{x \in X}[x]^{(i)}.$$

Proof. Since $[x]^{(i)}(y) = R(x,y)$, for any $y \in U$, for $\sim [x]^{(i)}(y) = 1 - R(x,y)$, then $\underline{R}(X) = \cap_{x \notin X}(\sim [x]^{(i)}) = \min_{x \notin X}(1 - R(x,y))$. Then we can easily prove it by the definition. □

Theorem 3.4. *Let (U,R) be an interval-valued fuzzy information system, denoted as*

$$R_{[\alpha_1,\alpha_2]} = \{(x,y) : R(x,y) \geqslant [\alpha_1,\alpha_2]\} = \{(x,y) : R^-(x,y) \geqslant \alpha_1, R^+(x,y) \geqslant \alpha_2\},$$
$$R_{(\alpha_1,\alpha_2)} = \{(x,y) : R(x,y) > [\alpha_1,\alpha_2]\} = \{(x,y) : R^-(x,y) > \alpha_1, R^+(x,y) > \alpha_2\}$$

for any $[\alpha_1,\alpha_2] \in [I]$. Then $R_{[\alpha_1,\alpha_2]}$ and $R_{(\alpha_1,\alpha_2)}$ are the classical binary relations of U, and they satisfy the following properties:

(1) *If R is reflexive, then $R_{[\alpha_1,\alpha_2]}$ and $R_{(\alpha_1,\alpha_2)}$ are reflexive, too.*
(2) *If R is symmetric, then $R_{[\alpha_1,\alpha_2]}$ and $R_{(\alpha_1,\alpha_2)}$ are symmetric, too.*
(3) *If R is transitive, then $R_{[\alpha_1,\alpha_2]}$ and $R_{(\alpha_1,\alpha_2)}$ are transitive, too.*

In particular, if R is the interval-valued fuzzy equivalence relation, then $R_{[\alpha_1,\alpha_2]}$ and $R_{(\alpha_1,\alpha_2)}$ are the binary equivalence relations, too.

Proof. (1) and (2) are easily obtained, we have only prove that (3) holds. Since R is transitive, there is $R(x.z) \geqslant R(x,y) \wedge R(y.z)$ for any $x,y,z \in U$. Therefore, if $(x,y) \in R_{[\alpha_1,\alpha_2]}, (y,z) \in R_{[\alpha_1,\alpha_2]}$, $R(x,y) \geqslant [\alpha_1,\alpha_2]$ and $R(y,z) \geqslant [\alpha_1,\alpha_2]$, then $R(x,z) \geqslant [\alpha_1,\alpha_2]$, i.e., $(x,z) \in R_{[\alpha_1,\alpha_2]}$. Hence, $R_{[\alpha_1,\alpha_2]}$ is transitive.

Meanwhile, $R_{(\alpha_1,\alpha_2)}$ can be proved in a similar way. $\square$

Let (U,R) be an interval-valued fuzzy information system. Then we know that $R_{[\alpha_1,\alpha_2]}$ and $R_{(\alpha_1,\alpha_2)}([\alpha_1,\alpha_2]\in[I])$ are the equivalence relations of the universe U. Therefore, they construct a partition of the universe U by $R_{[\alpha_1,\alpha_2]}$ and $R_{(\alpha_1,\alpha_2)}$.

$$U/R_{[\alpha_1,\alpha_2]}=\{[x]^{(i)}_{[\alpha_1,\alpha_2]}:x\in U\},\quad U/R_{(\alpha_1,\alpha_2)}=\{[x]^{(i)}_{(\alpha_1,\alpha_2)}:x\in U\},$$

where

$$[x]^{(i)}_{[\alpha_1,\alpha_2]}=\{y\in U:R(x,y)\geqslant[\alpha_1,\alpha_2]\}=\{y\in U:R^-(x,y)\geqslant\alpha_1,R^+(x,y)\geqslant\alpha_2\},$$
$$[x]^{(i)}_{(\alpha_1,\alpha_2)}=\{y\in U:R(x,y)>[\alpha_1,\alpha_2]\}=\{y\in U:R^-(x,y)>\alpha_1,R^+(x,y)>\alpha_2\}.$$

Definition 3.4. Let (U,R) be an interval-valued fuzzy equivalence relation information system. For any crisp set $X(X\subseteq U)$ and $[\alpha_1,\alpha_2]\in[I]$, define the lower approximation (strong lower approximation) and the upper approximation (strong upper approximation) at the level $[\alpha_1,\alpha_2]$ of X as follows, respectively:

$$\underline{R}_{[\alpha_1,\alpha_2]}(X)=\{x\in U:[x]^{(i)}_{[\alpha_1,\alpha_2]}\subseteq X\},\quad \underline{R}_{(\alpha_1,\alpha_2)}(X)=\{x\in U:[x]^{(i)}_{(\alpha_1,\alpha_2)}\subseteq X\},$$
$$\overline{R}_{[\alpha_1,\alpha_2]}(X)=\{x\in U:[x]^{(i)}_{[\alpha_1,\alpha_2]}\cap X\neq\emptyset\},\quad \overline{R}_{(\alpha_1,\alpha_2)}(X)=\{x\in U:[x]^{(i)}_{(\alpha_1,\alpha_2)}\cap X\neq\emptyset\}.$$

If $\underline{R}_{[\alpha_1,\alpha_2]}(X)=\overline{R}_{[\alpha_1,\alpha_2]}(X)$, then X is said to be definable in the interval-valued fuzzy equivalence relation information system (U,R) at the level of $[\alpha_1,\alpha_2]$, otherwise, it is called roughness at the level of $[\alpha_1,\alpha_2]$.

If $\underline{R}_{(\alpha_1,\alpha_2)}(X)=\overline{R}_{(\alpha_1,\alpha_2)}(X)$, then X is said to be strongly definable in the interval-valued fuzzy equivalence relation information system (U,R) at the level of (α_1,α_2), otherwise, it is called strong roughness at the level of (α_1,α_2).

Since $\underline{R}_{[\alpha_1,\alpha_2]}(X),\overline{R}_{[\alpha_1,\alpha_2]}(X)$ and $\underline{R}_{(\alpha_1,\alpha_2)}(X),\overline{R}_{(\alpha_1,\alpha_2)}(X)$ are constructed based on the relation of $R_{[\alpha_1,\alpha_2]}$ and $R_{(\alpha_1,\alpha_2)}$ of the universe U. Therefore, the corresponding conclusions about the classical Pawlak rough set are also holding for the above definition.

Theorem 3.5. *Let (U,R) be an interval-valued fuzzy equivalence relation information system. If $[\alpha_1,\alpha_2],[\beta_1,\beta_2]\in[I]$, and $[\alpha_1,\alpha_2]<[\beta_1,\beta_2]$, Then,*

(1) $\underline{R}_{[\alpha_1,\alpha_2]}(X)\subseteq\underline{R}_{[\beta_1,\beta_2]}(X)\subseteq X\subseteq\overline{R}_{[\beta_1,\beta_2]}(X)\subseteq\overline{R}_{[\alpha_1,\alpha_2]}(X),\ X\subseteq U;$
(2) $\underline{R}_{(\alpha_1,\alpha_2)}(X)\subseteq\underline{R}_{(\beta_1,\beta_2)}(X)\subseteq X\subseteq\overline{R}_{(\beta_1,\beta_2)}(X)\subseteq\overline{R}_{(\alpha_1,\alpha_2)}(X),\ X\subseteq U.$

Proof. Since $[\alpha_1,\alpha_2]<[\beta_1,\beta_2]$, $\underline{R}_{[\beta_1,\beta_2]}\subseteq\underline{R}_{[\alpha_1,\alpha_2]},\underline{R}_{(\beta_1,\beta_2)}\subseteq\underline{R}_{(\alpha_1,\alpha_2)}$, then $[x]^{(i)}_{[\beta_1,\beta_2]}\subseteq[x]^{(i)}_{[\alpha_1,\alpha_2]}$, $[x]^{(i)}_{(\beta_1,\beta_2)}\subseteq[x]^{(i)}_{(\alpha_1,\alpha_2)}$. That is $[x]^{(i)}_{[\alpha_1,\alpha_2]}\subseteq X$ and $[x]^{(i)}_{[\beta_1,\beta_2]}\subseteq[x]^{(i)}_{[\alpha_1,\alpha_2]}$ hold for $\forall x\in\underline{R}_{[\alpha_1,\alpha_2]}(X)$. This implies that $[x]^{(i)}_{[\beta_1,\beta_2]}\subseteq X$. Therefore, $x\in\underline{R}_{[\beta_1,\beta_2]}(X)$, i.e., $\underline{R}_{[\alpha_1,\alpha_2]}(X)\subseteq\underline{R}_{[\beta_1,\beta_2]}(X)$.

It is easy to prove the other relations in a similar way. $\square$

According to the Theorem 3.5, $\{\underline{R}_{[\alpha_1,\alpha_2]^c}:[\alpha_1,\alpha_2]\in[I]\}$, $\{\overline{R}_{[\alpha_1,\alpha_2]}:[\alpha_1,\alpha_2]\in[I]\}$, $\{\underline{R}_{(\alpha_1,\alpha_2)^c}:[\alpha_1,\alpha_2]\in[I]\}$, $\{\overline{R}_{(\alpha_1,\alpha_2)}:[\alpha_1,\alpha_2]\in[I]\}$ construct a family of nested sets, respectively. Hence, they determine the following interval-valued fuzzy set of the universe U, respectively.

(1) $\underline{R}'(X)(y)=\sup\{[\alpha_1,\alpha_2]:y\in\underline{R}_{[\alpha_1,\alpha_2]^c}(X)\},y\in U;$
(2) $\overline{R}'(X)(y)=\sup\{[\alpha_1,\alpha_2]:y\in\overline{R}_{[\alpha_1,\alpha_2]}(X)\},y\in U;$
(3) $\underline{R}''(X)(y)=\sup\{[\alpha_1,\alpha_2]:y\in\underline{R}_{(\alpha_1,\alpha_2)^c}(X)\},y\in U;$
(4) $\overline{R}''(X)(y)=\sup\{[\alpha_1,\alpha_2]:y\in\overline{R}_{(\alpha_1,\alpha_2)}(X)\},y\in U.$

We call the interval-valued fuzzy set $\underline{R}'(X)$ and $\overline{R}'(X)$, which defined by (1) and (2) are the interval-valued fuzzy lower and upper approximations of X in the universe U, respectively, whereas call $\underline{R}''(X)$ and $\overline{R}''(X)$, which defined by (3) and (4) are the interval-valued fuzzy strong lower approximation and strong upper approximation of X on the universe U, respectively.

Proposition 3.2. *Let (U, R) be an interval-valued fuzzy equivalence relation information system. For any crisp set $X (X \subseteq U)$, the interval-valued fuzzy lower approximation $\underline{R}'(X)$, the interval-valued fuzzy upper approximation $\overline{R}'(X)$, interval-valued fuzzy strong lower approximation $\underline{R}''(X)$ and interval-valued fuzzy strong upper approximation $\overline{R}''(X)$ of X satisfy the following relation*:

$$\underline{R}''(X) \subseteq \underline{R}'(X) \subseteq X \subseteq \overline{R}''(X) \subseteq \overline{R}'(X).$$

Theorem 3.6. *Let (U, R) be an interval-valued fuzzy equivalence relation information system. For any $X \subseteq U$, then*

$$\overline{R}'(X) = \overline{R}(X).$$

Proof. For any $y \in U$, denoted as $[r_1, r_2] = \overline{R}(X)(y) = \max_{x \in X} R(x, y), [\beta_1, \beta_2] = \overline{R}'(X)(y) = \sup\{[\alpha_1, \alpha_2] : [y]^{(i)}_{[\alpha_1, \alpha_2]} \cap X \neq \emptyset\}, \forall [r_1, r_2], [\beta_1, \beta_2], [\alpha_1, \alpha_2] \in [I]$. Since $[r_1, r_2] = \max_{x \in X} R(x, y)$, there exists $x \in X$ such that $R(x, y) = [r_1, r_2]$, i.e., $x \in [y]^{(i)}_{[r_1, r_2]}$.Thus $[y]^{(i)}_{[r_1, r_2]} \cap X \neq \emptyset$, i.e., $[\beta_1, \beta_2] \geqslant [r_1, r_2]$.If $[\beta_1, \beta_2] > [r_1, r_2]$, then there exists $[\beta'_1, \beta'_2] \in [I]$ such that $[\beta_1, \beta_2] > [\beta'_1, \beta'_2] > [r_1, r_2]$.According to $[\beta_1, \beta_2] = \sup\{[\alpha_1, \alpha_2] : [y]^{(i)}_{[\alpha_1, \alpha_2]} \cap X \neq \emptyset\}$, we know that there exists $x \in X$ such that $R(x, y) \geqslant [\beta'_1, \beta'_2]$.Therefore, $[r_1, r_2] = \max_{x \in X} R(x, y) \geqslant [\beta'_1, \beta'_2] > [r_1, r_2]$, and $[r_1, r_2] \geqslant [\beta'_1, \beta'_2] > [r_1, r_2]$.This is a contradiction.Then $[\beta_1, \beta_2] = [r_1, r_2]$, i.e., $\overline{R}'(X) = \overline{R}(X)$. □

Theorem 3.7. *Let (U, R) be an interval-valued fuzzy equivalence relation information system. For any $X \subseteq U$, then*

$$\underline{R}''(X) = \underline{R}(X).$$

Proof. For any $y \in U$, denoted as $[r_1, r_2] = \underline{R}(X)(y) = \min_{x \notin X}(\bar{1} - R(x, y)), [\beta_1, \beta_2] = \underline{R}''(X)(y) = \sup\{[\alpha_1, \alpha_2] : [y]^{(i)}_{(\alpha_1, \alpha_2)^c} \cap (\sim X) = \emptyset\}, \forall [r_1, r_2], [\beta_1, \beta_2], [\alpha_1, \alpha_2], \bar{1} \in [I]$. Since $[r_1, r_2] = \min_{x \notin X}(\bar{1} - R(x, y))$, $\bar{1} - R(x, y) \geqslant [r_1, r_2]$ for any $x \notin X$, i.e., $R(x, y) \leqslant \bar{1} - [r_1, r_2]$. Hence, $[y]^{(i)}_{(r_1, r_2)^c} \cap (\sim X) = \emptyset$, i.e., $[\beta_1, \beta_2] \geqslant [r_1, r_2]$. If there exists $[\beta'_1, \beta'_2] \in [I]$ such that $[\beta_1, \beta_2] > [\beta'_1, \beta'_2] > [r_1, r_2]$, then $[y]^{(i)}_{(\beta'_1, \beta'_2)^c} \cap (\sim X) = \emptyset$ according to $[\beta_1, \beta_2] = \underline{R}''(x, y) = \sup\{[\alpha_1, \alpha_2] : [y]^{(i)}_{(\alpha_1, \alpha_2)^c} \cap (\sim X) = \emptyset\}$. That is, $R(x, y) \leqslant [\beta'_1, \beta'_2]^c = \bar{1} - [\beta'_1, \beta'_2]$ for any $x \notin X$, i.e., $\bar{1} - R(x, y) \geqslant [\beta'_1, \beta'_2]$for any $x \notin X$. Thus $[r_1, r_2] = \min_{x \notin X}(\bar{1} - R(x, y)) \geqslant \min\{[\beta'_1, \beta'_2] : [\beta'_1, \beta'_2] > [r_1, r_2]\}$, and $[r_1, r_2] \geqslant [\beta'_1, \beta'_2] > [r_1, r_2]$. This is a contradiction. Therefore, $[\beta_1, \beta_2] = [r_1, r_2]$, i.e., $\underline{R}''(X) = \underline{R}(X)$. □

As for the above discussion, the following conclusions are clear.

Theorem 3.8. *Let (U, R) be an interval-valued fuzzy equivalence relation information system. Then*

(1) $\overline{R}'(X) = \sim \underline{R}''(\sim X)$,
(2) $\overline{R}''(X) = \sim \underline{R}'(\sim X)$.

Proof. From (2) of Theorem 3.1, we have $\overline{R}(X) = \sim \underline{R}(\sim X)$, $\overline{R}(X) = \overline{R}'(X)$, and$\underline{R}''(X) = \underline{R}(X)$ according to the Theorems 3.6 and 3.7, too. Therefore, we have $\overline{R}(X) = \sim \underline{R}(\sim X)$, and $\sim \underline{R}''(\sim X) = \sim \underline{R}(\sim X)$. Then $\overline{R}'(X) = \sim \underline{R}''(\sim X)$.

The equation $\overline{R}''(X) = \sim \underline{R}'(\sim X)$ can be proved in a similar way. □

3.2. The rough approximation of an interval-valued fuzzy set based on the interval-valued fuzzy information system

Definition 3.5. Let (U, R) be an interval-valued fuzzy information system and A the interval-valued fuzzy set of the universe U. Define the interval-valued fuzzy lower approximation $\underline{R}(A)$ and the interval-valued fuzzy upper approximation $\overline{R}(A)$ of A in the interval-valued fuzzy information system (U, R) as follows, respectively. For any $x \in U$

$$\underline{R}(A)(x) = \min\{A(y) \vee (\bar{1} - R(x,y)) : y \in U\},$$
$$\overline{R}(A)(x) = \max\{A(y) \wedge R(x,y) : y \in U\}.$$

$\underline{R} : F^{(i)}(U) \to F^{(i)}(U)$ and $\overline{R} : F^{(i)}(U) \to F^{(i)}(U)$ are called the interval-valued fuzzy lower approximation operator and the interval-valued fuzzy upper approximation operator, respectively.

Clearly, the above definition implies equivalences of the following form:

$$\begin{aligned}
\underline{R}(A)(x) &= \bigwedge_{y\in U}(A(y) \vee (\bar{1} - R(x,y))) \\
&= \left[\bigwedge_{y\in U}(A^-(y) \vee (1 - R^+(x,y))),\ \bigwedge_{y\in U}(A^+(y) \vee (1 - R^-(x,y)))\right] \quad \forall x \in U, \\
\overline{R}(A)(x) &= \bigvee_{y\in U}(A(y) \wedge R(x,y)) \\
&= \left[\bigvee_{y\in U}(A^-(y) \wedge R^-(x,y)),\ \bigvee_{y\in U}(A^+(y) \wedge R^+(x,y))\right] \quad \forall x \in U.
\end{aligned}$$

Theorem 3.9. *Let (U,R) be an interval-valued fuzzy information system and A the interval-valued fuzzy set of the universe U. $A_{[\alpha_1,\alpha_2]}$ and $A_{(\alpha_1,\alpha_2)}([\alpha_1,\alpha_2] \in [I])$ are the $[\alpha_1,\alpha_2]$-level set and strong $[\alpha_1,\alpha_2]$-level set of A, respectively. Then*

$$\begin{aligned}
\overline{R}(A) &= \bigvee_{[\alpha_1,\alpha_2]\in[I]}([\alpha_1,\alpha_2] \wedge \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})) \\
&= \bigvee_{[\alpha_1,\alpha_2]\in[I]}([\alpha_1,\alpha_2] \wedge \overline{R}_{[\alpha_1,\alpha_2]}(A_{(\alpha_1,\alpha_2)})) \\
&= \bigvee_{[\alpha_1,\alpha_2]\in[I]}([\alpha_1,\alpha_2] \wedge \overline{R}_{(\alpha_1,\alpha_2)}(A_{[\alpha_1,\alpha_2]})) \\
&= \bigvee_{[\alpha_1,\alpha_2]\in[I]}([\alpha_1,\alpha_2] \wedge \overline{R}_{(\alpha_1,\alpha_2)}(A_{(\alpha_1,\alpha_2)})).
\end{aligned}$$

Furthermore,

(1) $[\overline{R}(A)]_{(\alpha_1,\alpha_2)} \subseteq \overline{R}_{(\alpha_1,\alpha_2)}(A_{(\alpha_1,\alpha_2)}) \subseteq \overline{R}_{(\alpha_1,\alpha_2)}(A_{[\alpha_1,\alpha_2]}) \subseteq \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]}) \subseteq [\overline{R}(A)]_{[\alpha_1,\alpha_2]}$,
(2) $[\overline{R}(A)]_{(\alpha_1,\alpha_2)} \subseteq \overline{R}_{(\alpha_1,\alpha_2)}(A_{(\alpha_1,\alpha_2)}) \subseteq \overline{R}_{[\alpha_1,\alpha_2]}(A_{(\alpha_1,\alpha_2)}) \subseteq \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]}) \subseteq [\overline{R}(A)]_{[\alpha_1,\alpha_2]}$.

Proof. For $\forall x \in U$, Since

$$\begin{aligned}
\bigvee_{[\alpha_1,\alpha_2]\in[I]}([\alpha_1,\alpha_2] \wedge \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})(x)) &= \sup\{[\alpha_1,\alpha_2] : x \in \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})\} \\
&= \sup\{[\alpha_1,\alpha_2] : [x]_{R_{[\alpha_1,\alpha_2]}} \cap A_{[\alpha_1,\alpha_2]} \neq \emptyset\} \\
&= \sup\{[\alpha_1,\alpha_2] : \exists y \in U, y \in [x]_{R_{[\alpha_1,\alpha_2]}}, y \in A_{[\alpha_1,\alpha_2]}\} \\
&= \sup\{[\alpha_1,\alpha_2] : \exists y \in U, R(x,y) \geqslant [\alpha_1,\alpha_2], A(y) \geqslant [\alpha_1,\alpha_2]\} \\
&= \bigvee\{A(y) \wedge R(x,y) : y \in U\} \\
&= \overline{R}(A)(x).
\end{aligned}$$

For any arbitrary $x (x \in U)$, we have

$$\overline{R}(A) = \bigvee_{[\alpha_1,\alpha_2]\in[I]} ([\alpha_1, \alpha_2] \wedge \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})).$$

Then, the other equations can be proved in a similar way. Furthermore, we could easily prove that (1) and (2) hold by the representation theorem of the interval-valued fuzzy set and the binary nested set. □

Theorem 3.10. *Let (U, R) be an interval-valued fuzzy information system, A the interval-valued fuzzy set of the universe U, and $A_{[\alpha_1,\alpha_2]}$ and $A_{(\alpha_1,\alpha_2)}([\alpha_1, \alpha_2] \in [I])$ the $[\alpha_1, \alpha_2]$-level set and strong $[\alpha_1, \alpha_2]$-level set of A, respectively. Then*

$$\begin{aligned}
\underline{R}(A) &= \bigvee_{[\alpha_1,\alpha_2]\in[I]} ([\alpha_1, \alpha_2] \wedge \underline{R}_{[\alpha_1,\alpha_2]^c}(A_{[\alpha_1,\alpha_2]})) \\
&= \bigvee_{[\alpha_1,\alpha_2]\in[I]} ([\alpha_1, \alpha_2] \wedge \underline{R}_{[\alpha_1,\alpha_2]^c}(A_{(\alpha_1,\alpha_2)})) \\
&= \bigvee_{[\alpha_1,\alpha_2]\in[I]} ([\alpha_1, \alpha_2] \wedge \underline{R}_{(\alpha_1,\alpha_2)^c}(A_{[\alpha_1,\alpha_2]})) \\
&= \bigvee_{[\alpha_1,\alpha_2]\in[I]} ([\alpha_1, \alpha_2] \wedge \underline{R}_{(\alpha_1,\alpha_2)^c}(A_{(\alpha_1,\alpha_2)})).
\end{aligned}$$

Furthermore,

(1) $[\underline{R}(A)]_{(\alpha_1,\alpha_2)} \subseteq \overline{R}_{(\alpha_1,\alpha_2)^c}(A_{(\alpha_1,\alpha_2)}) \subseteq \underline{R}_{(\alpha_1,\alpha_2)^c}(A_{(\alpha_1,\alpha_2)}) \subseteq \underline{R}_{(\alpha_1,\alpha_2)^c}(A_{[\alpha_1,\alpha_2]}) \subseteq [\underline{R}(A)]_{[\alpha_1,\alpha_2]}$,
(2) $[\underline{R}(A)]_{(\alpha_1,\alpha_2)} \subseteq \underline{R}_{[\alpha_1,\alpha_2]^c}(A_{(\alpha_1,\alpha_2)}) \subseteq \underline{R}_{[\alpha_1,\alpha_2]^c}(A_{[\alpha_1,\alpha_2]}) \subseteq \underline{R}_{(\alpha_1,\alpha_2)^c}(A_{[\alpha_1,\alpha_2]}) \subseteq [\underline{R}(A)]_{[\alpha_1,\alpha_2]}$.

Proof. For $\forall x \in U$,

$$\begin{aligned}
\bigvee_{[\alpha_1,\alpha_2]\in[I]} ([\alpha_1, \alpha_2] \wedge \underline{R}_{[\alpha_1,\alpha_2]^c}(A_{[\alpha_1,\alpha_2]})(x)) &= \sup\{[\alpha_1, \alpha_2] \in [I] : x \in \underline{R}_{[\alpha_1,\alpha_2]^c}(A_{[\alpha_1,\alpha_2]})\} \\
&= \sup\{[\alpha_1, \alpha_2] \in [I] : [x]_{R_{[\alpha_1,\alpha_2]^c}} \subseteq A_{[\alpha_1,\alpha_2]}\} \\
&= \sup\{[\alpha_1, \alpha_2] \in [I] : \forall y \in U, y \in [x]_{R_{[\alpha_1,\alpha_2]^c}}, y \in A_{[\alpha_1,\alpha_2]}\} \\
&= \sup\{[\alpha_1, \alpha_2] \in [I] : \forall y \in U, R(x, y) \geqslant \bar{1} - [\alpha_1, \alpha_2], A(y) \geqslant [\alpha_1, \alpha_2]\} \\
&= \bigvee_{[\alpha_1,\alpha_2]\in[I]} \{A(y) \wedge R(x, y), \forall y \in U\} \\
&= \bigwedge_{[\alpha_1,\alpha_2]\in[I]} \{A(y) \vee (\bar{1} - R(x, y)), \forall y \in U\} \\
&= \underline{R}(A)(x).
\end{aligned}$$

For any arbitrary $x (x \in U)$, we have

$$\bigvee_{[\alpha_1,\alpha_2]\in[I]} ([\alpha_1, \alpha_2] \wedge \underline{R}_{[\alpha_1,\alpha_2]^c}(A_{[\alpha_1,\alpha_2]})) = \underline{R}(A).$$

Therefore, the other equations can be proved in a similar way. Furthermore, we could easily prove that (1) and (2) hold by the representation theorem of the interval-valued fuzzy set and the binary nested set. □

Theorem 3.11. *Let (U, R) be an interval-valued fuzzy information system and A the interval-valued fuzzy sets of the universe U. Then*

$$\begin{aligned}
\overline{R}(A)(x) &= \sup\{[\alpha_1, \alpha_2] : x \in \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})\} \\
&= \sup\{[\alpha_1, \alpha_2] : x \in \overline{R}_{[\alpha_1,\alpha_2]}(A_{(\alpha_1,\alpha_2)})\} \\
&= \sup\{[\alpha_1, \alpha_2] : x \in \overline{R}_{(\alpha_1,\alpha_2)}(A_{[\alpha_1,\alpha_2]})\} \\
&= \sup\{[\alpha_1, \alpha_2] : x \in \overline{R}_{(\alpha_1,\alpha_2)}(A_{(\alpha_1,\alpha_2)})\},
\end{aligned}$$

$$\begin{aligned}\underline{R}(A)(x) &= \sup\{[\alpha_1,\alpha_2] : x \in \underline{R}_{[\alpha_1,\alpha_2]^c}(A_{[\alpha_1,\alpha_2]})\}\\ &= \sup\{[\alpha_1,\alpha_2] : x \in \underline{R}_{[\alpha_1,\alpha_2]^c}(A_{(\alpha_1,\alpha_2)})\}\\ &= \sup\{[\alpha_1,\alpha_2] : x \in \underline{R}_{(\alpha_1,\alpha_2)^c}(A_{[\alpha_1,\alpha_2]})\}\\ &= \sup\{[\alpha_1,\alpha_2] : x \in \underline{R}_{(\alpha_1,\alpha_2)^c}(A_{(\alpha_1,\alpha_2)})\}\end{aligned}$$

for any $x \in U$, *and* $[\alpha_1,\alpha_2] \in [I]$.

Proof. According to the Theorem 3.9, we have

$$\overline{R}(A) = \bigvee_{[\alpha_1,\alpha_2]\in[I]} ([\alpha_1,\alpha_2] \wedge \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})).$$

Then

$$\begin{aligned}\overline{R}(A) &= \left(\bigvee_{[\alpha_1,\alpha_2]\in[I]} ([\alpha_1,\alpha_2] \wedge \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]}))\right)(x)\\ &= \left(\bigvee_{\overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})(x)\geqslant[\alpha_1,\alpha_2]} ([\alpha_1,\alpha_2] \wedge \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})(x))\right)\\ &\vee \left(\bigvee_{\overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})(x)<[\alpha_1,\alpha_2]} ([\alpha_1,\alpha_2] \wedge \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})(x))\right)\\ &= \left(\bigvee_{\overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})(x)\geqslant[\alpha_1,\alpha_2]} ([\alpha_1,\alpha_2] \wedge \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})(x))\right) \vee 0\\ &= \sup\{[\alpha_1,\alpha_2] : x \in \overline{R}_{[\alpha_1,\alpha_2]}(A_{[\alpha_1,\alpha_2]})\}.\end{aligned}$$

The other equations can be proved in a similar way. □

4. Comparison of the interval-valued fuzzy rough set model with the other rough set models

In this section, we will establish the relationships between the interval-valued fuzzy rough set with the other classical Pawlak rough set model by modifying the relations and the subsets of the universe U. It is easy to prove that the interval-valued fuzzy rough set model which defined in Section 3 is an extension of the classical Pawlak rough set model.

(1) If $A \in F(U), R \in F^{(i)}(U \times U)$, i.e., A is the ordinary fuzzy sets of U, R is the interval-valued fuzzy relation of U. For any $a \in [0,1]$ write $\bar{a} = [a,a] \in [I]$, then

$$\begin{aligned}\underline{R}(A)(x) &= \bigwedge_{y\in U}(A(y) \vee (\bar{1} - R(x,y)))\\ &= \bigwedge_{y\in U}([A(y),A(y)] \vee (\bar{1} - R(x,y)))\\ &= \left[\bigwedge_{y\in U}(A(y) \vee (1 - R^+(x,y))),\ \bigwedge_{y\in U}(A(y) \vee (1 - R^-(x,y)))\right],\\ \overline{R}(A)(x) &= \bigvee_{y\in U}(A(y) \wedge R(x,y))\\ &= \bigvee_{y\in U}(A(y) \wedge [R^-(x,y), R^+(x,y)])\end{aligned}$$

$$= \bigvee_{y\in U}([A(y),A(y)] \wedge [R^-(x,y),R^+(x,y)])$$

$$= \left[\bigvee_{y\in U}(A(y) \wedge R^-(x,y)),\ \bigvee_{y\in U}(A(y) \wedge R^+(x,y))\right]$$

for any $x \in U$. This is the approximation in an ordinary fuzzy set of the interval-valued fuzzy information system.

(2) If $A \in F^{(i)}(U), R \in F(U \times U)$, i.e., A is an interval-valued fuzzy set of U, R is an ordinary fuzzy relation of U. It is similar to what we discuss in point (1). We obtain the following relations for any $x \in U$:

$$\underline{R}(A)(x) = \left[\bigwedge_{y\in U}(A^-(y) \vee (1 - R(x,y))), \bigwedge_{y\in U}(A^+(y) \vee (1 - R(x,y)))\right],$$

$$\overline{R}(A)(x) = \left[\bigvee_{y\in U}(A^-(y) \wedge (1 - R(x,y))), \bigvee_{y\in U}(A^+(y) \wedge (1 - R(x,y)))\right].$$

This is the approximation of the interval-valued fuzzy set in the ordinary fuzzy information system.

(3) If $A \in F^{(i)}(U), R \subseteq U \times U$, i.e., A is the interval-valued fuzzy set of U, R is the classical equivalence relation of U. For any $y \in U$, we have $y \in [x]_R$, then $R(x,y) = 1$. Therefore, for any $x \in U$

$$\begin{aligned}\underline{R}(A)(x) &= \left[\bigwedge_{y\in U}(A^-(y) \vee (1-1)), \bigwedge_{y\in U}(A^+(y) \vee (1-1)\right]\\ &= \wedge_{y\in U}[A^-(y), A^+(y)]\\ &= [\wedge_{y\in U}A^-(y), \wedge_{y\in U}A^+(y)],\\ \overline{R}(A)(x) &= \left[\bigvee_{y\in U}(A^-(y) \wedge 1), \bigvee_{y\in U}(A^+(y) \wedge 1)\right]\\ &= \bigvee_{y\in U}[A^-(y), A^+(y)]\\ &= [\vee_{y\in U}A^-(y), \vee_{y\in U}A^+(y)].\end{aligned}$$

This is the interval-valued rough fuzzy set model.

(4) If $A \subseteq U, R \in F^{(i)}(U \times U)$, i.e., A is the crisp of U, R is the interval-valued fuzzy relation of U: Then for any $x \in U$

$$\begin{aligned}\underline{R}(A)(x) &= \min\{A(y) \vee (\bar{1} - R(x,y)) : y \in U\}\\ &= \min_{y\notin A}(\bar{1} - R(x,y))\\ &= [\min_{y\notin A}(\bar{1} - R^+(x,y)), \min_{y\notin A}(\bar{1} - R^-(x,y))],\\ \overline{R}(A)(x) &= \max\{A(y) \wedge R(x,y) : y \in U\}\\ &= \max_{y\in A} R(x,y)\\ &= [\max_{y\in A} R^-(x,y), \max_{y\in A} R^+(x,y)].\end{aligned}$$

This is the rough set model of the interval-valued fuzzy information system. That is, the approximation of the crisp set of U in the interval-valued fuzzy information systems (U,R).

(5) If $A \in F(U), R \in F(U \times U)$, i.e., A is the ordinary fuzzy set of U, R is the ordinary fuzzy relation of U. Then for any $x \in U$

$$\begin{aligned}\underline{R}(A)(x) &= \min\{A(y) \vee (\bar{1} - R(x,y)) : y \in U\}\\ &= \min\{A(y) \vee (1 - R(x,y)) : y \in U\},\\ \overline{R}(A)(x) &= \max\{A(y) \wedge R(x,y) : y \in U\}.\end{aligned}$$

This is the rough approximation of the ordinary fuzzy set in the ordinary fuzzy information system.

(6) If $A \in F(U), R \in U \times U$, i.e., A is the ordinary fuzzy set of U, R is the classical equivalence relation of U. Then for any $x \in U$

$$\begin{aligned}\underline{R}(A)(x) &= \min\{A(y) \vee (\bar{1} - R(x,y)) : y \in U\}\\ &= \min\{A(y) \vee (1 - R(x,y)) : y \in U\}\\ &= \min\{A(y) : (x,y) \in R\}\\ &= \min\{A(y) : y \in [x]_R\},\\ \overline{R}(A)(x) &= \max\{A(y) \wedge R(x,y) : y \in U\}\\ &= \max\{A(y) : (x,y) \in R\}\\ &= \max\{A(y) : y \in [x]_R\}.\end{aligned}$$

This is the classical rough fuzzy sets model.

(7) If $A \subseteq U, R \in F(U \times U)$, i.e., A is the crisp set of U, R is the ordinary fuzzy relation of U. Then for any $x \in U$

$$\begin{aligned}\underline{R}(A)(x) &= \min\{A(y) \vee (\bar{1} - R(x,y)) : y \in U\}\\ &= \min\{A(y) \vee (1 - R(x,y)) : y \in U\}\\ &= \min_{y \notin A}(1 - R(x,y)),\\ \overline{R}(A)(x) &= \max\{A(y) \wedge R(x,y) : y \in U\}\\ &= \max_{y \in A} R(x,y).\end{aligned}$$

This is the rough approximation of any crisp set in the classical information system.

(8) If $A \subseteq U, R \subseteq U \times U$, i.e., A is the crisp set of U, R is the classical equivalence relation of U. For any $x \in U$,

$$\begin{aligned}\underline{R}(A)(x) = 1 &\iff \forall y \in U, \text{ then there is } A(y) \vee (\bar{1} - R(x,y)) = A(y) \vee (1 - R(x,y)) = 1\\ &\iff \forall y \in U,\ y \notin A \text{ implicates the } (x,y) \notin R\\ &\iff \forall y \notin A \text{ implicates the } y \notin [x]_R\\ &\iff [x]_R \subseteq A,\\ \overline{R}(A)(x) = 1 &\iff \exists y \in U, \text{ So } A(y) = 1 \text{ and } R(x,y) = 1 \text{ is holding.}\\ &\iff A \cap [x]_R \neq \emptyset.\end{aligned}$$

This is the classical Pawlak rough set model.

(9) If $A \in F^{(i)}(U), R \subseteq P(U \times U)$, i.e., A is the interval-valued fuzzy relation of U, R is the general relation of U. That is, $R(x,y) = R_s(x) = \{y \in U : (x,y) \in R\}$. Then for any $y \in U$. If $y \in R_s(x)$, then $R(x,y) = 1$. Therefore, for any $x \in U$

$$\begin{aligned}\underline{R}(A)(x) &= \min\{A(y) \vee (\bar{1} - R(x,y)) : y \in R_s(x)\}\\ &= \min\{A(y) \vee (\bar{1} - 1) : y \in R_s(x)\}\\ &= \left[\bigwedge_{y\in U}(A^-(y) \vee (1-1)), \bigwedge_{y\in U}(A^+(y) \vee (1-1))\right]\\ &= \bigwedge_{y\in U}[A^-(y), A^+(y)] = [\wedge_{y\in U}A^-(y), \wedge_{y\in U}A^+(y)]\\ &= [\min\{A^-(y) : y \in R_s(x)\}, \min\{A^+(y) : y \in R_s(x)\}] = \underline{apr}(A)(x)\\ \overline{R}(A)(x) &= \max\{A(y) \wedge R(x,y) : y \in U\}\\ &= \max\{A(y) \wedge R(x,y) : y \in R_s(x)\}\\ &= \max\{A(y) \wedge \bar{1} : y \in R_s(x)\}\end{aligned}$$

$$= \left[\bigvee_{y\in U}(A^-(y)\wedge 1), \bigvee_{y\in U}(A^+(y)\wedge 1)\right]$$
$$= \bigvee_{y\in U}[A^-(y), A^+(y)] = [\vee_{y\in U}A^-(y), \vee_{y\in U}A^+(y)]$$
$$= [\max\{A^-(y) : y\in R_s(x)\}, \max\{A^+(y) : y\in R_s(x)\}] = \overline{apr}(A)(x).$$

This is the generalized interval-valued rough fuzzy set model.

(10) If $A\in F(U), R\subseteq P(U\times U)$, i.e., A is the interval-valued fuzzy set of U, R is the general relation of U. That is $R(x,y) = R_s(x) = \{y\in U : (x,y)\in R\}$. Then for $\forall y\in U$. If $y\in R_s(x)$, then $R(x,y) = 1$, too. Therefore, for any $x\in U$

$$\begin{aligned}\underline{R}(A)(x) &= \min\{A(y)\vee(\bar{1}-R(x,y)) : y\in U\}\\ &= \min\{A(y)\vee(1-1) : y\in U\}\\ &= \min\{A(y) : (x,y)\in R\}\\ &= \min\{A(y) : y\in R_s(x)\} = \underline{apr}(A)(x),\\ \overline{R}(A)(x) &= \max\{A(y)\wedge R(x,y) : y\in U\}\\ &= \max\{A(y) : (x,y)\in R\}\\ &= \max\{A(y) : y\in R_s(x)\} = \overline{apr}(A)(x).\end{aligned}$$

This is the rough set model based on the general relation.

(11) If $A\in P(U\times U), R\subseteq P(U\times U)$, i.e., A is the crisp set of U, R is the general relation of U. That is, $R(x,y) = R_s(x) = \{y\in U : (x,y)\in R\}$, then for any $y\in U$. If $y\in R_s(x)$, then $R(x,y) = 1$. Therefore, for any $x\in U$,

$$\begin{aligned}\underline{R}(A)(x) = 1 &\Longleftrightarrow \forall y\in U,\ \text{we have } A(y)\vee(\bar{1}-R(x,y)) = A(y)\vee(1-R(x,y)) = 1\\ &\Longleftrightarrow \forall y\in U, y\notin A \text{ implicates } (x,y)\notin R\\ &\Longleftrightarrow \forall y\notin A \text{ implicates } y\notin R_s(x)\\ &\Longleftrightarrow R_s(x)\subseteq A\\ &\Longleftrightarrow \{x\in U : R_s(x)\subseteq A\} = \underline{apr}(A)(x).\end{aligned}$$

And for $\overline{R}(A)(x) = 1 \Longleftrightarrow \exists y\in U$ such that $A(y) = 1$, and $R(x,y) = 1$,

$$\Longleftrightarrow A\cap R_s(x)\neq\emptyset, \forall x\in U.$$
$$\Longleftrightarrow \{x\in U : A\cap R_s(x)\neq\emptyset, \forall x\in U\} = \overline{apr}A(x).$$

This is the classical Pawlak rough set model based on the general relation.In terms of the above discussion, we know that there are all of the new rough set models except the models (5), (6) and (8). Consequently, it is the most extension rough set models that defined in Section 3.

5. The knowledge reduction of the interval-valued fuzzy information system

In general, one of the central problems of the rough set theory is classification analysis. The whole approach is inspired by the notion of inadequacy of available information to perform complete classification of objects belonging to a specified category. One fundamental aspect of the rough set theory involves the search for some particular subset of condition attributes. By one such subset the information for classification purpose provided is the same as the condition attribute set. Such subsets are called reductions. To acquire a brief decision rule from consistent or inconsistent systems, knowledge reduction is needed.

In this section, we will deal with approaches to knowledge reduction based on the interval-valued fuzzy rough set models of the interval-valued fuzzy information system.

5.1. The interval-valued fuzzy reduction of the classical information system

Let $(U, \mathbb{A}, F)$ be an information or database system. Here U is the set of objects, i.e., $U = \{x_1, x_2, \ldots, x_n\}$. Every element $x_i (i \leqslant n)$ in U is called an object, and $\mathbb{A}$ is the attribute set, i.e., $\mathbb{A} = \{a_1, a_2, \ldots, a_m\}$. Every element $a_j (j \leqslant m)$ in $\mathbb{A}$ is an attribute, F is the relation set of U and $\mathbb{A}$, i.e., $F = \{f_j : j \leqslant m\}$, $(f_j : U \to V_j, (j \leqslant m),)$ and V_j is the domain of the attribute a_j.

Definition 5.1. Let $(U, \mathbb{A}, F)$ be a classical information system, for any subset $B (B \subseteq \mathbb{A})$. B is called the interval-valued fuzzy reduction of the classical information system $(U, \mathbb{A}, F)$, if B is the minimum set in the inclusion set which satisfies the following relations:

$$\underline{R}_{\mathbb{A}}(X)(x) = \underline{R}_B(X)(x), \quad \overline{R}_{\mathbb{A}}(X)(x) = \overline{R}_B(X)(x)$$

for any $X \in F^{(i)}(U)$ $\forall x \in U$, where $\underline{R}_{\mathbb{A}}(X)(x)$, $\underline{R}_B(X)(x)$, $\overline{R}_{\mathbb{A}}(X)(x)$, $\overline{R}_B(X)(x)$ are defined as the interval-valued rough fuzzy sets.

B is called the interval-valued fuzzy lower approximation reduction of the classical information system $(U, \mathbb{A}, F)$ if B is the minimum set that satisfies the following relations:

$$\underline{R}_{\mathbb{A}}(X)(x) = \underline{R}_B(X)(x)$$

for any $X \in F^{(i)}(U),\ x \in U$.

B is called the interval-valued fuzzy upper approximation reduction of the classical information system $(U, \mathbb{A}, F)$ if B is the minimum set that satisfies the following relations:

$$\overline{R}_{\mathbb{A}}(X)(x) = \overline{R}_B(X)(x)$$

for any $X \in F^{(i)}(U),\ x \in U$.

Obviously, the interval-valued fuzzy reduction is both the interval-valued fuzzy lower approximation reduction and the interval-valued fuzzy upper approximation reduction.

Remark 5.1. If X is the ordinary fuzzy set of the universe U, i.e., $X \in F(U)$, then the set B which satisfied the conditions of Definition 5.1 is the fuzzy reduction of the classical information systems $(U, \mathbb{A}, F)$.

Remark 5.2. If X is the crisp set of U, i.e., $X \in P(U)$, then the set B which satisfied the conditions of Definition 5.1 is the reduction of the classical information system $(U, \mathbb{A}, F)$ [22].

Definition 5.2. Let $(U, \mathbb{A}, F)$ be a classical information system. Then the interval-valued fuzzy reduction exists.

5.2. The knowledge reduction of the interval-valued fuzzy information system

We call $(U, \mathbb{A}, F, D, G)$ an information system or decision table, where $(U, \mathbb{A}, F)$ is the classical information system, $\mathbb{A}$ is the condition attribute set and D the decision attribute set, i.e., $D = \{d_1, d_2, \ldots, d_p\}$. G is the relation set of the U and D, $G = \{g_j : j \leqslant p\}$ (where $g_j : U \to V'_j, (j \leqslant p)$), V'_j is the domain of the decision attribute d_j.

Let $(U, \mathbb{A}, F, D, G)$ be the information system. If $R_{\mathbb{A}} \subseteq R_D$, i.e., $U/R_{\mathbb{A}} \leqslant U/R_D$, (or $\forall x \in U$, for any $[x]_{\mathbb{A}}$, there exists $[x]_D$ such that the $[x]_{\mathbb{A}} \subseteq [x]_D$,) then the information system is called a consistent information systems, or called an inconsistent information system.

In this paper, we only consider consistent information systems.

$(U, \mathbb{A}, F, \widetilde{\mathbf{D}}, G)$ is called an interval-valued fuzzy information system, where $(U, \mathbb{A}, F)$ is the classical information system, $\widetilde{\mathbf{D}} = \{\widetilde{D}_k : k = 1, 2, \ldots, n\}$,$\widetilde{D}_k$ is the interval-valued fuzzy sets of U, and G the relation set of U and $\widetilde{\mathbf{D}}$.

Definition 5.3. Let $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ be the interval-valued fuzzy information system, for any $B \subseteq \mathbb{A}$, if the following relations hold:

$$\underline{R}_B(\widetilde{D}_i)(x) > \underline{R}_B(\widetilde{D}_j)(x) \Longleftrightarrow \underline{R}_{\mathbb{A}}(\widetilde{D}_i)(x) > \underline{R}_{\mathbb{A}}(\widetilde{D}_j)(x) \quad (i \neq j).$$

B is called the consistent set of $\mathbb{A}$. If B is the minimum consistent set of $\mathbb{A}$ in the inclusion set, then B is called the reduction of the interval-valued fuzzy information system $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$.

In the following section, we present the knowledge reduction of the interval-valued fuzzy information system by introducing the discernibility matrix.

Let $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ be the interval-valued fuzzy information system, $R_{\mathbb{A}}$ be the equivalence classes which induced by the condition attribute set $\mathbb{A}$, and the universe is divided by $R_{\mathbb{A}}$ as following: $U/R_{\mathbb{A}} = \{X_1, X_2, \ldots, X_k\}$, denoted as

$$\underline{R}_{\mathbb{A}}(\widetilde{\mathbf{D}})(X_i) = (\underline{R}_{\mathbb{A}}(\widetilde{D}_1)(X_i), \underline{R}_{\mathbb{A}}(\widetilde{D}_2)(X_i), \ldots, \underline{R}_{\mathbb{A}}(\widetilde{D}_r)(X_i)). \tag{$*$}$$

Definition 5.4. Let $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ be interval-valued fuzzy information system.

$$D_{ij} = \begin{cases} \{a_k \in \mathbb{A} : f_l(X_i) \neq f_l(X_j)\}, & g_{X_i}(\widetilde{D}_k) \neq g_{X_j}(\widetilde{D}_k), \\ \mathbb{A}, & g_{X_i}(\widetilde{D}_k) = g_{X_j}(\widetilde{D}_k), \end{cases}$$

is called the discernibility matrix of $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ (where $g_{X_i}(\widetilde{D}_k)$ denotes the maximum value of $\underline{R}_{\mathbb{A}}(\widetilde{\mathbf{D}})(X_i)$ at the line of k, i.e., the rows i and j of Eq. $(*)$).

Theorem 5.1. *Let $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ be the interval-valued fuzzy information system. If there exists a subset $B \subseteq \mathbb{A}$ such that $B \cap D_{ij} \neq \emptyset (i, j \leqslant k)$, then B is the consistent set of $\mathbb{A}$.*

Proof. If $g_{X_i}(\widetilde{D}_k) \neq g_{X_j}(\widetilde{D}_k)$, then $B \cap D_{ij} \neq \emptyset$ $(i, j \leqslant k)$. Therefore, there is $a_l \in B$, such that $f_l(X_i) \neq f_l(X_j)$. i.e., $X_i \cap X_j = \emptyset$. That is, $\widetilde{D}_{k_i}$ and $\widetilde{D}_{k_j}$ can be discerned by B (where k_i and k_j denote the value of the interval-valued fuzzy set $\widetilde{D}_k$ at the classes of i and j). $\square$

Example 5.1. Table 5.1 gives an interval-valued fuzzy information system, where the universe is $U = \{x_1, x_2, \ldots, x_{10}\}$, the condition attribute set is $\mathbb{A} = \{a_1, a_2, a_3\}$, and the decision attribute set is $\widetilde{\mathbf{D}} = \{\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3\}$, where $\widetilde{D}_i \in F^{(i)}(U)$, $(i = 1, 2, 3)$.

Obviously, the universe U can be divided into five basic classes according to the conditional attribute set $A = \{a_1, a_2, a_3\}$ (Table 5.2).

$$U/R_A = \{X_1, X_2, X_3, X_4, X_5\} = \{\{x_1, x_3, x_9\}, \{x_2, x_7, x_{10}\}, \{x_4\}, \{x_5, x_8\}, \{x_6\}\}.$$

Let $B = \{a_1, a_2\}$, then we can obtain the approximation value given in Table 5.3.

It is clear that B satisfies the Definition 5.2, i.e., B is the consistent set of A. Moreover, it can easily be tested that B is a minimum consistent set. Therefore, B is the reduction.

Table 5.4 gives the discernibility matrix of the Example 5.1 in terms of the Definition 5.4.

Table 5.1
A interval-valued fuzzy information system

U	a_1	a_2	a_3	$\widetilde{D}_1$	$\widetilde{D}_2$	$\widetilde{D}_3$
x_1	2	1	3	[0.7, 0.9]	[0.15, 0.2]	[0.4, 0.5]
x_2	3	2	1	[0.3, 0.5]	[0.5, 0.7]	[0.35, 0.4]
x_3	2	1	3	[0.7, 0.8]	[0.3, 0.4]	[0.1, 0.2]
x_4	2	2	3	[0.15, 0.2]	[0.5, 0.8]	[0.2, 0.3]
x_5	1	1	4	[0.05, 0.1]	[0.2, 0.3]	[0.65, 0.9]
x_6	1	1	2	[0.1, 0.2]	[0.35, 0.5]	[1.0, 1.0]
x_7	3	2	1	[0.25, 0.4]	[1.0, 1.0]	[0.3, 0.4]
x_8	1	1	4	[0.1, 0.2]	[0.25, 0.4]	[0.5, 0.6]
x_9	2	1	3	[0.45, 0.6]	[0.25, 0.3]	[0.2, 0.3]
x_{10}	3	2	1	[0.05, 0.1]	[0.8, 0.9]	[0.05, 0.2]

Table 5.2
The approximation of the interval-valued fuzzy objection

$U/R_{\mathbb{A}}$	$\underline{R}_{\mathbb{A}}(\widetilde{D}_1)$	$\underline{R}_{\mathbb{A}}(\widetilde{D}_2)$	$\underline{R}_{\mathbb{A}}(\widetilde{D}_3)$
X_1	$[0.45, 0.6]$	$[0.15, 0.2]$	$[0.1, 0.2]$
X_2	$[0.05, 0.1]$	$[0.5, 0.7]$	$[0.05, 0.2]$
X_3	$[0.15, 0.2]$	$[0.5, 0.8]$	$[0.2, 0.3]$
X_4	$[0.05, 0.1]$	$[0.2, 0.3]$	$[0.65, 0.9]$
X_5	$[0.1, 0.2]$	$[0.35, 0.5]$	$[1.00, 1.00]$

Table 5.3
The approximation of the interval-valued fuzzy objection

$U/R_{\mathbb{A}}$	$\underline{R}_{\mathbb{A}}(\widetilde{D}_1)$	$\underline{R}_{\mathbb{A}}(\widetilde{D}_2)$	$\underline{R}_{\mathbb{A}}(\widetilde{D}_3)$
X_1	$[0.45, 0.6]$	$[0.15, 0.2]$	$[0.1, 0.2]$
X_2	$[0.05, 0.1]$	$[0.5, 0.7]$	$[0.05, 0.2]$
X_3	$[0.15, 0.2]$	$[0.5, 0.8]$	$[0.2, 0.3]$
$X_4 \cup X_5$	$[0.05, 0.1]$	$[0.2, 0.3]$	$[0.65, 0.9]$

Table 5.4
The discernibility matrix of the interval-valued fuzzy objection

$U/R_{\mathbb{A}}$	X_1	X_2	X_3	X_4	X_5
X_1	$\mathbb{A}$				
X_2	$\mathbb{A}$	$\mathbb{A}$			
X_3	$\{a_2\}$	$\mathbb{A}$	$\mathbb{A}$		
X_4	$\{a_1, a_2\}$	$\mathbb{A}$	$\mathbb{A}$	$\mathbb{A}$	
X_5	$\{a_1, a_3\}$	$\mathbb{A}$	$\mathbb{A}$	$\mathbb{A}$	$\mathbb{A}$

From the above table and Theorem 5.1, we know that both $B = \{a_2, a_3\}$ and $B = \{a_1, a_2\}$ are consistent sets, and they are the reduction.

Definition 5.5. Let $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ be interval-valued fuzzy information system. Denoted as

$$\underline{L}_B(x) = (\underline{R}_B(\widetilde{D}_1)(x), \underline{R}_B(\widetilde{D}_2)(x), \ldots, \underline{R}_B(\widetilde{D}_r)(x)),$$
$$\overline{L}_B(x) = (\overline{R}_B(\widetilde{D}_1)(x), \overline{R}_B(\widetilde{D}_2)(x), \ldots, \overline{R}_B(\widetilde{D}_r)(x)),$$
$$\underline{L}_B^{(m)}(x) = \{\widetilde{D}_j : \underline{R}_B(\widetilde{D}_j)(x) = \max_{k \leqslant r} \underline{R}_B(\widetilde{D}_k)(x)\},$$
$$= \{\widetilde{D}_j : \overline{R}_B(\widetilde{D}_j)(x) = [\max_{k \leqslant r} \underline{R}_B^-(\widetilde{D}_k)(x), \max_{k \leqslant r} \underline{R}_B^+(\widetilde{D}_k)(x)]\},$$
$$\overline{L}_B^{(+)}(x) = \{\widetilde{D}_j : \overline{R}_B(\widetilde{D}_j)(x) > \bar{0}\}.$$

If $\underline{L}_B(x) = \underline{L}_{\mathbb{A}}(x)$ or $\overline{L}_B(x) = \overline{L}_{\mathbb{A}}(x)$, for any $x \in U$, then B call the lower(or upper) approximation consistent set of $\mathbb{A}$.

If $\underline{L}_B^{(m)}(x) = \underline{L}_{\mathbb{A}}^{(m)}(x), \forall x \in U$, then B is called the maximum lower approximation consistent set of $\mathbb{A}$.

If $\overline{L}_B^{(+)}(x) = \overline{L}_{\mathbb{A}}^{(+)}(x)$, for any $x \in U$, then B is called the non-negative upper approximation consistent set of $\mathbb{A}$.

If $B \subseteq \mathbb{A}$ is the lower approximation consistent set of $\mathbb{A}$, and any subset of B are not the lower approximation consistent sets of $\mathbb{A}$, then B is called the lower approximation reduction of $\mathbb{A}$.

It is similar to define the upper approximation reduction, the maximum lower approximation reduction and the non-negative upper approximation reduction.

Theorem 5.2 (The knowledge reduction theorem of the interval-valued fuzzy information system I). *Let $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ be the interval-valued fuzzy information system. For $\forall x, y \in U$,*

(1) *B is the lower approximation consistent set of* $\mathbb{A}$ *iff* $\underline{L}_{\mathbb{A}}(x) \neq \underline{L}_{\mathbb{A}}(y)$, *and* $[x]_B \cap [y]_B = \emptyset$;
(2) *B is the upper approximation consistent set of* $\mathbb{A}$ *iff* $\overline{L}_{\mathbb{A}}(x) \neq \overline{L}_{\mathbb{A}}(y)$, *and* $[x]_B \cap [y]_B = \emptyset$;
(3) *B is the maximum lower approximation consistent set of* $\mathbb{A}$ *iff* $\underline{L}^{(m)}_{\mathbb{A}}(x) \neq \underline{L}^{(m)}_{\mathbb{A}}(y)$, *and* $[x]_B \cap [y]_B = \emptyset$;
(4) *B is the non-negative upper approximation consistent set of A iff* $\underline{L}^{(+)}_{\mathbb{A}}(x) \neq \underline{L}^{(+)}_{\mathbb{A}}(y)$, *and* $[x]_B \cap [y]_B = \emptyset$.

Proof. (1) Since B is the lower approximation consistent set of A, i.e., $\underline{L}_B(x) = \underline{L}_{\mathbb{A}}(x)$, for any $x \in U$, $j \leqslant r$, we have $\underline{R}_B(\widetilde{D}_j)(x) = \underline{R}_{\mathbb{A}}(\widetilde{D}_j)(x), \forall x \in U$. Then $[x]_B = [y]_B$ is satisfied while $[x]_B \cap [y]_B \neq \emptyset$. Therefore, we obtain that $\underline{R}_B(\widetilde{D}_j)(x) = \underline{R}_B(\widetilde{D}_j)(y)$ for any $j \leqslant r$ according to the definition of the lower approximation.

Since B is the lower approximation consistent set of A for any $j \leqslant r$, we have $\underline{R}_B(\widetilde{D}_j)(x) = \underline{R}_{\mathbb{A}}(\widetilde{D}_j)(x)$ and $\underline{R}_B(\widetilde{D}_j)(y) = \underline{R}_{\mathbb{A}}(\widetilde{D}_j)(y)$. Therefore, $\underline{R}_{\mathbb{A}}(\widetilde{D}_j)(x) = \underline{R}_{\mathbb{A}}(\widetilde{D}_j)(y)$, $\forall j \leqslant r$, i.e., $\underline{L}_{\mathbb{A}}(x) = \underline{L}_{\mathbb{A}}(y)$. Then, there must be $[x]_B \cap [y]_B = \emptyset$ hold for $\underline{L}_{\mathbb{A}}(x) \neq \underline{L}_{\mathbb{A}}(y)$.

Conversely, if $\underline{L}_{\mathbb{A}}(x) \neq \underline{L}_{\mathbb{A}}(y)$, so is the $[x]_B \cap [y]_B = \emptyset$. If $[x]_B \cap [y]_B \neq \emptyset$, then $\underline{L}_{\mathbb{A}}(x) = \underline{L}_{\mathbb{A}}(y)$. Since $\mathfrak{R}_{\mathbb{A}}([x]_B) = \{[z]_{\mathbb{A}} : [z]_{\mathbb{A}} \subseteq [x]_B\}$ forms a partition of $[x]_B$, then there is $[x]_B = \bigcup\{[z]_{\mathbb{A}} : z \in [x]_B\} = \bigcup\{[z]_{\mathbb{A}} : [z]_{\mathbb{A}} \in \mathfrak{R}_{\mathbb{A}}([x]_B)\}$. Meanwhile, for $y \in [x]_B$, there is $[x]_B = [y]_B$. Therefore, $\underline{L}_{\mathbb{A}}(x) = \underline{L}_{\mathbb{A}}(y)$, i.e.,

$$\begin{aligned}
\underline{R}_B(\widetilde{D}_j)(x) &= \min_{y \in [x]_B} \widetilde{D}_j(y) \\
&= [\min_{y \in [x]_B} \widetilde{D}^-_j(y), \min_{y \in [x]_B} \widetilde{D}^+_j(y)] \\
&= \min\{[\widetilde{D}^-_j(y), \widetilde{D}^+_j(y)] : y \in [z]_{\mathbb{A}}, [z]_{\mathbb{A}} \in \mathfrak{R}_{\mathbb{A}}([x]_B)\} \\
&= \min_{[z]_{\mathbb{A}} \in \mathfrak{R}_{\mathbb{A}}([x]_B)} [\min_{y \in [z]_{\mathbb{A}}} \widetilde{D}^-_j(y), \min_{y \in [z]_{\mathbb{A}}} \widetilde{D}^+_j(y)] \\
&= \min_{[z]_{\mathbb{A}} \in \mathfrak{R}_{\mathbb{A}}([x]_B)} [\underline{R}^-_{\mathbb{A}}(\widetilde{D}_j)(z), \underline{R}^+_{\mathbb{A}}(\widetilde{D}_j)(z)] = \underline{R}_{\mathbb{A}}(\widetilde{D}_j)(x).
\end{aligned}$$

This proves that B is the lower approximation consistent set of $\mathbb{A}$.

That (2)–(4) can be proved in a similar way. □

Definition 5.6. Let $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ be the interval-valued fuzzy information system. Denoted as

$$\underline{D}(x,y) = \begin{cases} \{a_l \in \mathbb{A} : f_l(x) \neq f_l(y)\}, & \underline{L}_{\mathbb{A}}(x) \neq \underline{L}_{\mathbb{A}}(y), \\ \mathbb{A}, & \underline{L}_{\mathbb{A}}(x) = \underline{L}_{\mathbb{A}}(y), \end{cases}$$

$$\overline{D}(x,y) = \begin{cases} \{a_l \in \mathbb{A} : f_l(x) \neq f_l(y)\}, & \overline{L}_{\mathbb{A}}(x) \neq \overline{L}_{\mathbb{A}}(y), \\ \mathbb{A}, & \overline{L}_{\mathbb{A}}(x) = \overline{L}_{\mathbb{A}}(y), \end{cases}$$

$$\underline{D}^{(m)}(x,y) = \begin{cases} \{a_l \in \mathbb{A} : f_l(x) \neq f_l(y)\}, & \underline{L}^{(m)}_{\mathbb{A}}(x) \neq \underline{L}^{(m)}_{\mathbb{A}}(y), \\ \mathbb{A}, & \underline{L}^{(m)}_{\mathbb{A}}(x) = \underline{L}^{(m)}_{\mathbb{A}}(y), \end{cases}$$

$$\overline{D}^{(+)}(x,y) = \begin{cases} \{a_l \in \mathbb{A} : f_l(x) \neq f_l(y)\}, & \overline{L}^{(+)}_{\mathbb{A}}(x) \neq \overline{L}^{(+)}_{\mathbb{A}}(y), \\ \mathbb{A}, & \overline{L}^{(+)}_{\mathbb{A}}(x) = \overline{L}^{(+)}_{\mathbb{A}}(y), \end{cases}$$

$\underline{D}(x,y), \overline{D}(x,y), \underline{D}^{(m)}(x,y), \overline{D}^{(+)}(x,y)$ are called the lower approximation, maximum lower approximation and non-negative upper approximation attribute set of discernibility of the objects x and y in the interval-valued fuzzy information system $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$.

Theorem 5.3 (The knowledge reduction theorem of the interval-valued fuzzy information system II). *Let* $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ *be the interval-valued fuzzy information system. For any* $B \subseteq \mathbb{A}$, *then*

(1) *B is the lower approximation consistent set of* $\mathbb{A}$ *iff* $B \cap \underline{D}(x,y) \neq \emptyset$;
(2) *B is the upper approximation consistent set of* $\mathbb{A}$ *iff* $B \cap \overline{D}(x,y) \neq \emptyset$;
(3) *B is the maximum lower approximation consistent set of* $\mathbb{A}$ *iff* $B \cap \underline{D}^{(m)}(x,y) \neq \emptyset$;
(4) *B is the non-negative upper approximation consistent set of* $\mathbb{A}$ *iff* $B \cap \overline{D}^{(+)}(x,y) \neq \emptyset$.

Proof. (1) Let B be the lower approximation consistent set of $\mathbb{A}$ for any $x, y \in U$, $x \in [y]_{\mathbb{A}}$ or $y \in [x]_{\mathbb{A}}$. Then x and y are indiscernible, and $B \cap \underline{D}(x,y) \neq \emptyset$ (Since $\underline{D}(x,y) = \mathbb{A}$). Then $[x]_{\mathbb{A}}, [y]_{\mathbb{A}} \in U/R_{\mathbb{A}}$, and $[x]_{\mathbb{A}} \cap [y]_{\mathbb{A}} = \emptyset$. According to the definition of $\underline{L}_{\mathbb{A}}(x)$ and $\underline{L}_{\mathbb{A}}(y)$, there is $\underline{L}_{\mathbb{A}}(x) \neq \underline{L}_{\mathbb{A}}(y)$. Therefore, we have $[x]_B \cap [y]_B = \emptyset$. So, there exists $a_k \in B$ such that $f_k(x) \neq f_k(y)$. Then $a_k \in \underline{D}(x,y)$, and $B \cap \underline{D}(x,y) \neq \emptyset$.

Conversely, for any $x, y \in U, B \cap \underline{D}(x,y) \neq \emptyset$, there is $a_l \in B$ and $a_l \in \underline{D}(x,y)$. Then, $f_l(x) \neq f_l(y)$, so $[x]_B \cap [y]_B = \emptyset$, and $B \subseteq \mathbb{A}$. Obviously, there is $[x]_{\mathbb{A}} \subseteq [x]_B$ and $[y]_{\mathbb{A}} \subseteq [y]_B$ hold. Therefore, according to the arbitrary of x, y, we have $[x]_{\mathbb{A}} = [x]_B, [y]_{\mathbb{A}} = [y]_B$.

This prove $U/R_{\mathbb{A}} = U/R_B$, i.e.,$\underline{L}_{\mathbb{A}}(x) \neq \underline{L}_B(x)$. That is, B is the lower approximation consistent set of $\mathbb{A}$.

That (2)–(4) can be proved in a similar way. □

Corollary 5.1. *Let $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ be the interval-valued fuzzy information system. For any $B \subseteq \mathbb{A}$, the lower approximation consistent set must be the maximum lower approximation consistent set of $\mathbb{A}$; the upper approximation consistent set must be the non-negative upper approximation consistent set of $\mathbb{A}$.*

Let $[a_i, b_i] \in [I], \forall a_i, b_i \in [0.1]$, we convention the following facts:

$$m([a_i, b_i]) = b_i - a_i \in [0, 1],$$

where $m([a_i, b_i])$ denote the measure of the interval $[a_i, b_i]$.

In particular, if $[a_i, b_i]$ is the interval of the real line, then $m([a_i, b_i])$ is the length of the interval.

Definition 5.7. We call D the degree of the inclusion on the $(F_{\bar{0}}^{(i)}(U), \subseteq)$ (where $F_{\bar{0}}^{(i)}(U)$ be the $\bar{0}$ level set of U). That is, for any $A_1, A_2 \in F_{\bar{0}}^{(i)}(U)$, there exists a real number $D(A_2/A_1)$, and it satisfies the following conditions:

(1) $0 \leqslant D(A_2/A_1) \leqslant 1$,
(2) $A_1 \subseteq A_2 \Rightarrow D(A_2/A_1) = 1$,
(3) $A_1 \subseteq A_2 \subseteq A_3 \Rightarrow D(A_1/A_3) \leqslant D(A_1/A_2)$.

Definition 5.8. Let $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ be the interval-valued fuzzy information system. For any $B \subseteq A$, Let

$$D(A_2/A_1) = \frac{\sum_{x \in U} |A_1(x) \wedge A_2(x)|}{\sum_{x \in U} |A_1(x)|} \quad \forall A_1, A_2 \in F_{\bar{0}}^{(i)}(U).$$

Denoted as $M_B(x) = (D(\widetilde{D}_1/[x]_B), D(\widetilde{D}_2/[x]_B), \ldots, D(\widetilde{D}_r/[x]_B),)$, where $D_i \in F_{\bar{0}}^{(i)}(U)$.

Then $M_B(x)$ is called the distribution function of the decision of the object x about the condition B in interval-valued fuzzy information system $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$.

In general, for the interval-valued fuzzy information system $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$, for any $B \subseteq \mathbb{A}$ and $x \in U$, if $M_B(x) = M_{\mathbb{A}}(x)$, then B is called the distribution consistent set of the information systems. If B is the distribution consistent set, and any subset of B is not the distribution consistent set, then we call B the distribution reduction of $\mathbb{A}$.

Denoted as

$$D(x,y) = \begin{cases} \{a_l \in \mathbb{A} : f_l(x) \neq f_l(y)\}, & M_{\mathbb{A}}(x) \neq M_{\mathbb{A}}(y), \\ \mathbb{A}, & M_{\mathbb{A}}(x) = M_{\mathbb{A}}(y). \end{cases}$$

In view of the above definition, the following facts are clear.

Theorem 5.4. *Let $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ be the interval-valued fuzzy information system. For any $B \subseteq \mathbb{A}$, then B is the distribution consistent set iff for any $x, y \in U$, $M_{\mathbb{A}}(x) \neq M_{\mathbb{A}}(y)$, and $[x]_B \cap [y]_B = \emptyset$.*

Theorem 5.5. *Let $(U, \mathbb{A}, F, \widetilde{\mathbf{D}})$ be the interval-valued fuzzy information system. For any $B \subseteq \mathbb{A}$, then B is the distribution consistent set iff $B \cap D(x,y) \neq \emptyset (\forall x, y \in U)$.*

The proof are similar to Theorems 5.3 and 5.4.

Using Example 5.1, one can discover the corresponding reduction that is defined in Definitions 5.4, 5.5 and 5.6. The conclusions of Theorems 5.4, 5.5 and Theorem 5.6, can also be tested.

6. Conclusions

In this paper, we define an interval-valued fuzzy relation in the universe U, and the interval-valued fuzzy information system is built. Then we give a rough approximation of every interval-valued fuzzy set in the interval-valued fuzzy information system and discuss the relationship of the model defined in this paper and the other rough set models. Finally, the knowledge reduction of the interval-valued fuzzy information system is investigated and some knowledge reduction theorems are also presented.

Throughout the paper, we only discuss the consistent and complete interval-valued fuzzy information systems in the finite universe. In fact, the inconsistent and incomplete interval-valued fuzzy information systems are more important in practice. Since inconsistent and incomplete interval-valued fuzzy information system are more complicated than consistent and complete information systems, further research of the knowledge reduction in inconsistent and incomplete interval-valued fuzzy information system is needed. In further research, we will develop proposed approaches to those various interval-valued information system.

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