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An overview of granular computing in decision-making: Extensions, applications, and challenges

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ABSTRACT

The management of uncertainty in decision-making problems remains a very challenging and timely research issue despite many proposals. An interesting topic in this area in recent years has been Granular Computing. In this paper, we present a coherent framework for various models of information granules, using hierarchies of information for dealing with knowledge acquired and processed at different levels of abstraction. Granular Computing, a paradigm for handling higher types of uncertainty in decision analysis, is considered a new conceptual and algorithmic asset of decision-making studies and, in particular, data-driven decision-making. Most relevant studies have focused on categories of information granules, namely intervals, fuzzy sets, rough sets, and shadowed sets. To better understand this promising area, this study provides a comprehensive overview of Granular Computing for decision-making through literature analysis, enhancing a variety of extensions, applications, and challenges.

1. Introduction

Recently, Granular Computing (GrC) has emerged as a new approach to multiple-criteria decision-making (MCDM) [1]. This paradigm supports a general sequence of phases of data processing depicted as collecting data/experimental evidence, building information granules, and constructing artifacts of decision-making. GrC has been applied to uncertainty handling in decision analysis, using such formal settings as intervals, fuzzy sets, rough sets, and shadowed sets [2–4].

GrC in decision-making has received significant attention from the academic community during the past twenty years. For instance, Pedrycz and Song [5] used GrC for MCDM and group decision-making (GDM) for the first time. Their application concerned data structuring through an optimal allocation of information granularity. Pedrycz [6] extended this approach to a larger class of optimization and decision-making models. Furthermore, Cabrerizo et al. [7] adapted the approach of optimal allocation of information granularity to consensus-reaching in GDM. Additionally, Song and Wang [8] reviewed information granularity in terms of soft computing and discovered the advantages and disadvantages of current approaches. It is acknowledged that GrC has two major characteristics corresponding to solving uncertainty and robustness of decision-making problems, which are as follows: (i) describe and process vague, ambiguous, incomplete, and massive information and (ii) provide a problem-solving method based on a granularity relationship.

To explain the framework of GrC in decision-making, we should first highlight its essence. Currently, no well-established consensus on this topic exists in the academic community [9–11], which limits the understanding of information granules and GrC. According to Pedrycz and Chen [12], GrC embraces of four fundamental frameworks: intervals, fuzzy sets, rough sets, and probability density functions. While, some research [13,14] mainly focuses on rough sets and interval analysis. It is necessary to establish sound relationships among these facets to become familiar with GrC.

In particular, the type-2 fuzzy set, as a representative of fuzzy sets, may be useful for reasoning about information granularity because its most important feature (descriptor) is the centroid, which itself is an interval. At the same time, when the upper and lower membership functions of the type-2 fuzzy sets are the same, type-2 constants boil down to ordinary fuzzy sets, which can establish a mapping relationship with the intervals through their cut sets [15]. Hence, we can establish relationships with other frameworks (the intervals, fuzzy sets,

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rough sets, and probability density functions) in granular decision-making through the concept and operation mechanism of type-2 fuzzy sets. For the rough sets, the uncertainty concept applies the upper and lower approximations, which can be regarded as the upper and lower membership functions of a type-2 fuzzy set, respectively. In a relevant study, Yao [13] proposed a new branch of granular decision-making and found that the rough sets in three-way decision-making are equivalent to intuitionistic fuzzy sets, and the intuitionistic fuzzy sets can be, in turn, mathematically equivalently converted into type-2 fuzzy sets. Therefore, from this point of view, a formal connection exists between type-2 fuzzy sets and rough sets. In addition, the rough set is an essential mathematical tool after probability theory, fuzzy set theory to deal with imprecise and inconsistent information in decision-making, for more details see Refs. [16,17]. Finally, we can establish a connection between probabilistic information and type-2 fuzzy sets through the cloud model, a concept of uncertainty with expectation, entropy, and super entropy as the underlying features [18]. From the cloud model, we determine the connection between the type-2 fuzzy sets and the probability information. In the nutshell, based on this analysis, the type-2 fuzzy set can be regarded as the linkage to analyze other frameworks in GrC.

Data-driven methods have become very popular in decision-making. However, effectively handling data uncertainty has been a critical problem for decision analysis. Obviously, while numerically measuring all kinds of uncertainty is impossible, this approach is suitable for information granules. Meanwhile, to reduce computing overload, we need to consider the problem-solving methods at different levels of granularity, especially for linguistic decisions and computing with words. Thus, we should emphasize the importance of the study of GrC in decision analysis.

Until now, some studies have focused on GrC in decision-making [12,19,20]. Therefore, this paper aims to deliver a comprehensive overview of this area. The overall framework of this research is: first, the basic theoretical knowledge of GrC is presented for the explanation of the essence of GrC. Then, several typical granular models are shown involving coverage-specificity measures based on an optimal allocation of information granularity, multiple granular linguistic decision-making, and other extensions of decision-making models [14,21–25], like cloud models [22,23], three-way decision-making models [14,21], data-driven granular neural network models [26,27], and the up-to-date granular model is shown as well [28–30]. Afterwards, applications of GrC in decision-making based on the statistics of WoS database are exhibited, In the end, we describe the major limitations of GrC in decision-making at present and provide some mathematical implications for its future development. In summary, the contributions of this paper are summarized as follows:

(1) We emphasize the theoretical features of GrC in decision-making by introducing its necessary definitions, principles, and models. Besides, we present the characteristic of the basic element (information granule) of GrC, which is convenient for readers to understand the essence of GrC.
(2) We conduct an in-depth summary and analysis of the respective problems and solutions, as well as relevant applications of GrC. Since the research on GrC in decision-making is still in infancy, this paper has guiding significance for the later research on this area.
(3) We discuss the challenges and future trends of GrC in decision-making including the theoretical support for GrC, corresponding measurement extension, the granulation process improvement, and relationship between GrC and data-driven decision-making as well as combination of GrC with other theory (e.g., game theory).

The paper is organized as follows. Section 2 presents basic definitions, measures, principles, and models related to GrC. Section 3 provides an overview of recent research on GrC in decision-making and discusses extensions of granular decision-making models. Section 4 introduces the corresponding applications in various areas. Next, Section 5 points out the current challenges and future research directions. Finally, Section 6 provides concluding remarks.

2. Granular computing: definitions, measures, principles, and models

In this section, we highlight the major features of three representative granular models [6], as follows: (a) Granular model as a manifestation of transfer knowledge, as shown in Fig. 1(a). The knowledge transfer process for the original model by the input of information granules can be viewed as a more abstract version of the initial model. (b) Granular model of a non-stationary system. Considering the instability of the non-stationary system, we develop the granular model with granular parameters instead of continuously updating the model, as shown in Fig. 1(b). To some extent, this concept of granulation is similar to (a), while the construction of data is different. (c) Granular model using rule-based model reduction, as shown in Fig. 1(c).

First, a model composed by N rules expressed as the following: if condition A_i, the conclusion is B_i $(i = 1, 2, \dots, N)$, and **if** merely P $(P < N)$ rules are selected, **then** input the granular rule-based model, then the condition can be denoted as $\mathbf{G}(A_i)$, and the granular rule is **if** $\mathbf{G}(A_i)$, then B_i $(i = 1, 2, \dots, P)$. Simultaneously, we provide a visual description of the production process of forming information granules in Fig. 2. The three phases are collecting data, forming numeric prototypes, and building information granules. The dataset includes various types of data, such as crisp, internal, fuzzy, and rough numbers, which will be formed into numeric prototypes. Finally, some production rules are adopted to generate granular information. The following subsections review basic definitions, measures, principles, and models of GrC.

2.1. Definitions

GrC exhibits a variety of conceptual definitions in different contexts and applications. Table 1 shows various definitions of GrC provided by scholars. Based on this list, GrC is essentially the construction and decomposition of granulation for problem-solving. These definitions describe the methodology in dealing with different types of data in information granulation involving fuzzy sets [31] and linguistic sets [32,33]. In general, the core of GrC is about the representation, construction, processing, and communication of information granules [34].

2.2. Calculations of coverage and specificity

The measures describing information granules include specificity and coverage. Specificity describes the level of precision of the established information granule. Coverage represents the ability of the information granule to cover the experimental data. Table 2 lists available definitions to enable comparative insights.

Remark 1. The variables in Table 2 are defined as follows: sp is specificity; cov is coverage; $X = \{x_1, x_2, \dots, x_N\}$ comprises numeric data, $x_{\max}$ is the maximum value of x_k; m and b are the left and right boundaries of the information granule, respectively; $A_i (i = 1, 2, \dots, c)$ is the input space; $B = G(y_1, y_2, \dots, y_N, A_1, A_2, \dots, A_c)$ is the output space, $B_i = [b_i^-, b_i^+]$; $Y = [y^-, y^+] = \left[\sum_{i=1}^{c} A_i(\mathbf{x}) b_i^-, \sum_{i=1}^{c} A_i(\mathbf{x}) b_i^+\right]$ is the information granularity, $Y_k = [y_k^-, y_k^+]$, $y = f(x)$, $\mathrm{target}t_k \subset Y_k$; V is the average interval length; G_{ij} is a one-dimensional membership function of x_k; g is a decreasing linear portion of the membership function; ξ is a positive scaling factor; $z_i, i = 1, 2, \dots, N$ is the highest possible coverage of the data; $y_j, j = 1, 2, \dots, M$: the inclusion of inhibitory data; ρ_{ij} is the length of the jth semiaxis of the Y_i; and v_{ij} are the numeric prototypes.

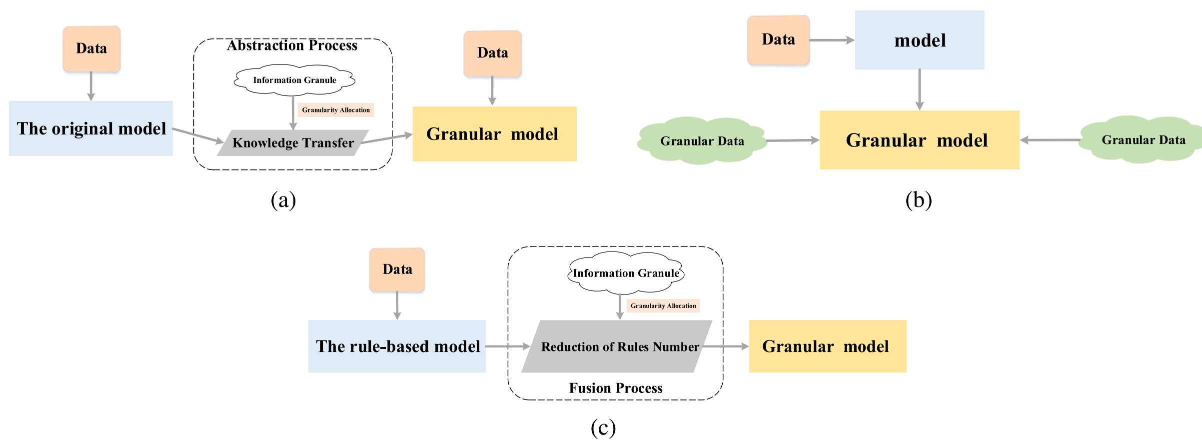

Fig. 1. (a) Granular model in transferring knowledge, (b) granular model in reducing rule-based model structure, and (c) granular model of a non-stationary system.

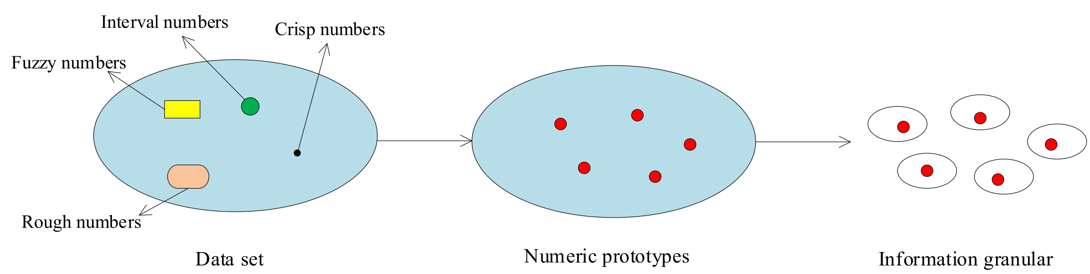

Fig. 2. Construction of information granules.

Table 1
Related definitions of GrC.

Reference	GrC definition
Pedrycz and Bargiela [31]	An approach that supports fuzzy modeling with interpretability and accuracy.
Yao [32]	An umbrella term to cover any theories, methodologies, techniques, and tools that make use of granules in complex problems.
Zadeh [33]	A basis for computing with words, i.e., computation with information described in natural language.
Yao [35]	A new term for the problem-solving paradigm that may be viewed more on a philosophical rather than technical level.
Yao [36]	A triangle of structured thinking in the philosophical perspective, structured problem solving in the methodological perspective, and structured information processing in the computational perspective.
Yao [37]	The semantical transformation of data in the process of granulation and the non-computational verification of information abstractions.
Pedrycz [34]	Solutions from GrC satisfy the non-Boolean specifications.
Pedrycz [38]	A general knowledge model that uses information granules aimed at establishing a coherent methodological and developmental environment.

Table 2
Representation of specificity and coverage.

Papers	Specificity representation	Coverage representation
Pedrycz [6]	$\mathrm{sp} = \frac{1}{N\sum_{k=1}^{N}\lvert y_k^+ - y_k^-\rvert}$	$\mathrm{cov} = \frac{1}{N}\sum_{k=1}^{N}\mathrm{incl}(y_k, Y_k)$, where $Y_k = \sum_{i=1}^{c} A_i \otimes B_i$
Pedrycz [39] and Wang et al.[40]	$\mathrm{sp} = 1 - \frac{b-m}{x_{\max}-m}$	$\mathrm{cov} = \mathrm{card}\left\{x_k \middle\lvert x_k \in [m,b]\right\}$
Pedrycz et al. [41]	$\mathrm{sp} = 1 - \frac{V}{V(\alpha=0)}$, where $V = \frac{1}{N}\sum_{k=1}^{N}\left\lvert y_k^+ - y_k^-\right\rvert$	$\mathrm{cov} = \frac{1}{N}\sum_{k=1}^{N}\mathrm{incl}(y_k, Y_k)$, where $Y_k = \sum_{i=1}^{c} A_i \otimes B_i$
Pedrycz et al. [42]	$\mathrm{sp}(G_i) = \frac{1}{N}\sum_{j=1}^{n}(1 - \frac{\Phi(G_{ij})}{\mathrm{range}_j})$, where $\Phi(G_{ij}) = \int_0^1 \mathrm{length}(\beta)d\beta$	$\mathrm{cov}_i = \frac{1}{N}\sum_{k=1}^{N} G_i(x_k)$, where $G_i(\mathbf{x}_k) = \min_{j=1,2,\ldots,n} G_{ij}(x_{kj})$
Kerr-Wilson and Pedrycz [43]	$\mathrm{sp}(G_i) = \frac{1}{N}\sum_{j=1}^{n}[1 - \exp(-\frac{\mathrm{length}(Y_k)}{\lvert\max - \min\rvert})]$	$\mathrm{cov} = \frac{1}{N}\sum_{k=1}^{N}\chi Y_k(y)$
Wang and Pedrycz [44]	$\mathrm{sp}(A) = \int_0^1 (1 - \frac{\lvert m-b_\alpha\rvert}{\mathrm{range}})d\alpha$, where $\mathrm{range} = \lvert z_{\max} - m\rvert$	$\mathrm{cov}(A) = \max(0, \frac{1}{N}\sum_{z_k\in[m,b]} g(z_k), -\frac{\xi}{M}\sum_{y_k\in[m,b]} g(y_k))$
Zhu et al. [45]	$\mathrm{sp}(\Omega_i) = 1 - \mathrm{volume}\ (Y_i)^{1/n}$, where $\mathrm{volume}(Y_i) = \pi^{\frac{n}{2}} / \Gamma(\frac{n}{2}+1)\prod_{j=1}^{n}\rho_{ij}$	$\mathrm{cov}(Y_i) = \mathrm{card}\{x_k \sum_{j=1}^{n}(\frac{x_{kj}-v_{ij}}{\rho_{ij}})^2 \le 1\}$

The primary reason to present the specificity and coverage formulas in Table 2 is to provide researchers with the relative reference basis in the granular model construction with both considering these two attributes. And the establishment of these mentioned formulas in Table 2 are based on the model requirements, it is difficult to evaluate which formula is the most effective one. While, it is acknowledged that interval-based information granules are the common formalism of information granules in granular models, due to their

Table 3
Comparison of different methods for the optimal allocation of information granularity.

Reference	Objectives	Methods
Pedrycz and Bargiela [31]	Maximum coverage criterion	Fuzzy clustering
Pedrycz [39]	Maximum coverage and degrees of inclusion	Logic descriptors
Pedrycz [46]	Maximum coverage and specificity	AHP model
Pedrycz et al. [52]	Maximum data coverage	A uniform distribution of granularity, symmetric and asymmetric schemes of allocation were discussed
Gacek and Pedrycz [53]	Maximum coverage	A two-phase design
Al-Hmouz et al. [54]	Maximum coverage criterion	A concept of granular representation schemes of time series.
Wang and Pedrycz [55]		A robust granular optimization under hybrid manifold uncertainties
Hu et al. [56]	Maximum specificity	Granular input space
Lu et al. [25]	Maximum coverage and specificity	Sugeno-type granular model
Zhu et al. [57]	Maximum coverage and specificity	Fuzzy C-means method
Shen et al. [58]	Maximum coverage and specificity	Clustering homogeneous information granules model

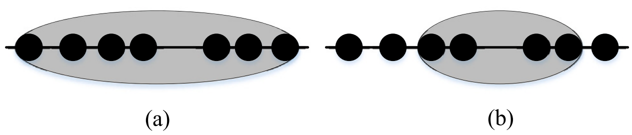

Fig. 3. Visual difference between information granules: (a) High numerical evidence with low specificity and (b) low numerical evidence with high specificity.

simplicity and intuitiveness of mathematical operations [6]. The formulas for the specificity and coverage of interval-based information granules are usually expressed as: $\mathrm{sp} = 1/N\sum_{k=1}^{N}\left|y_k^+ - y_k^-\right|$ and $\mathrm{cov} = 1/N\sum_{k=1}^{N}\mathrm{incl}(y_k, Y_k)$, where sp represents the average length of the intervals and cov denotes the number of cases when Y_k contains y_k, if $y_k \subseteq Y_k$, then the measure returns 1 [6].

2.3. Principles of justifiable granularity

The principle of justifiable granularity lies in producing a meaningful information granule based on available experimental data. The following requirements guide the granule formation [40,41,46–50]:

- The numeric evidence behind the information granule should be as much as possible so that this construct is supported by the experimental data.
- The information granule should be as much specific as possible to describe in detail the experimental data and retain its semantics.

Fig. 3 presents a graphical representation about the difference between information granules [48]. We can observe that the size of information granule in Fig. 3(a) is larger than that in Fig. 3(b), which contains more information and elevate the cardinality, while the specificity of information decreases. In the construction process of granular model, a balance should be established to conform the both rules. Further, Fig. 4 portrays the relationship between the two decision attributes in a more intuitive way. Of note, the values of coverage and specificity rely on the predetermined level of granularity. And once again, it has been confirmed that these two attributes are in conflict and need to be compromised. In addition Pedrycz et al. [42] presented three significant features of the constructing principles of justifiable granularity with regard to the construction of fuzzy sets, and Pedrycz and Wang [51] presented related generalizations:

- A parametric version of the information granule where a certain type of membership function is assumed in advance
- The incorporation of data weights (different levels of data contribution to the realization of information granule) that are the membership grades associated with the data
- The involvement of inhibitory experimental evidence, i.e., some data (inhibitory data) having to be excluded when information granules are constructed

2.4. Granular models based on optimal allocation of information granularity

The core of GrC is to establish an optimal allocation model, given that scholars have presented many models to allocate the information granules, as shown in Table 3. First, Pedrycz [46] developed an optimization problem considering two objectives, maximum coverage and specificity, in which an analytic hierarchy process (AHP) model was established to optimize granularity allocation. Considering the limitation of the existing optimal allocation model, i.e., parameters regarded as information granules rather than numeric entities, Pedrycz et al. [52] optimized the maximum data coverage criterion to produce granules generated by the proposed granular model. Similarly, Pedrycz and Bargiela [31] introduced a concept of granular prototypes that can generalize the numeric representation of the clusters to maximize the coverage criterion, optimizing the granularity allocation.

In another model, Gacek and Pedrycz [53] introduced a two-phase design to allocate the predefined level of granularity to the individual elements of the universe of discourse over which the signals were described. On the other hand, Al-Hmouz et al. [54] built the numeric representation of time series to optimize the coverage criterion. Pedrycz [39] proposed another concept involving logic descriptors, and then two modes of feedforward and feedback design of granular global were established. In other research, Wang and Pedrycz [55] presented a unified structured robust granular optimization to tackle the optimized information granules under hybrid manifold uncertainties. Hu et al. [56] developed an optimization model to allocate information granularity based on granular input space, while Lu et al. [25] proposed a design method of a Sugeno-type granular model to implement an optimal allocation of information granularity. Hence, we review specific algorithms to better understand the allocation of information granularity. When Pedrycz et al. [49] optimized the allocation of information granularity, they used two criteria:

(1) **Coverage criterion**

$$\max_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n} Q \quad s.t.\ \begin{cases}\sum_{i=1}^{n}\varepsilon_i = n\varepsilon \\ \varepsilon_i > 0\end{cases} \tag{1}$$

where $Q = \frac{1}{N}\sum_{k=1}^{N}\mathrm{incl}(\mathrm{target}_k, Y_k)$, $\mathrm{incl}(\mathrm{target}_k, Y_k)$ can qualify an extent to which target_k is included in Y_k; target_k belongs to the output in the input–output data $\{(\mathbf{x}_1, \mathrm{target}_1), (\mathbf{x}_2, \mathrm{target}_2), \ldots, (\mathbf{x}_N, \mathrm{target}_N)\}$; and Y_k is the return value of $\mathbf{x}_k$ on the granular mapping, $Y_k = f(\mathbf{x}_k, \mathbf{A})$, and $\mathbf{A}$ stands for the information granule.

(2) **Specificity criterion**

$$\min_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_P} V \quad s.t.\ \begin{cases}\sum_{i=1}^{P}\varepsilon_i = P\varepsilon \\ \varepsilon_i > 0\end{cases} \tag{2}$$

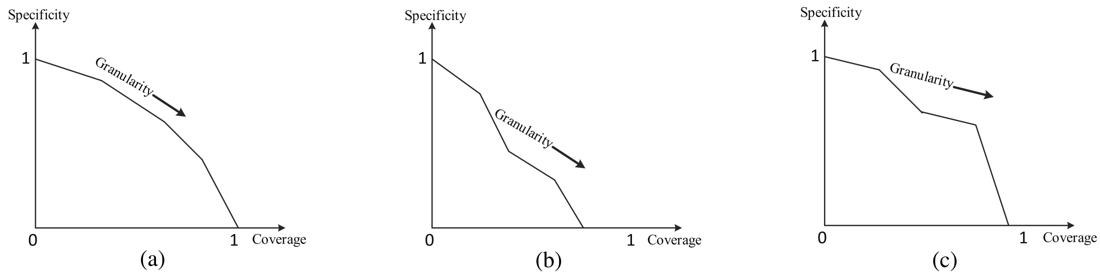

Fig. 4. Competitive effect of information granularity on coverage and specificity: increase of coverage, decreases specificity and vice versa.

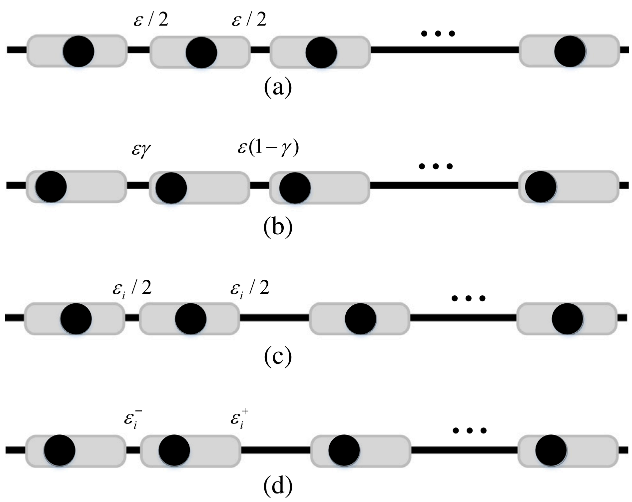

Fig. 5. Classic protocols of allocation of information granularity.

where $V = 1/P\sum_{k=1}^{P}|y_k^+ - y_k^-|$, $|y_k^+ - y_k^-|$ indicates the length of this interval, namely $Y_k = [y_k^-, y_k^+]$.

Furthermore, Pedrycz [6] introduced several protocols for the allocation of information granularity, and their corresponding allocation processes are presented in Fig. 5.

(a) Uniform allocation of information granularity.
(b) Uniform allocation of information granularity with an asymmetric position of intervals around the numeric parameter.
(c) Non-uniform allocation of information granularity with symmetrically distributed intervals of information granules.
(d) Non-uniform allocation of information granularity with asymmetrically distributed intervals of information granules.

In other research, Hu et al. [56] developed an optimization model to allocate information granularity based on granular input space. The author states that the optimization problem can be regarded as the maximization of the specificity with the following conditions:

$$\max_{\text{sp}} \frac{1}{N}\sum_{k=1}^{N}\exp(-|y_k^+ - y_k^-|)$$
$$s.t.\ \begin{cases}\sum_{k=1}^{n}\varepsilon_i^- + \sum_{k=1}^{n}\varepsilon_i^+ = n\varepsilon\\ \varepsilon_i^+, \varepsilon_i^- \in [0,1]\end{cases} \tag{3}$$

where $\exp(-|y_k^+ - y_k^-|)$ denotes a decreasing function of the length of Y_k, $Y_k = [y_k^-, y_k^+]$ corresponds with the granular input $X_k = [x_k^-, x_k^+] = [x_k - \varepsilon_k^- \text{range}_k, x_k + \varepsilon_k^+ \text{range}_k]$, and $k = 1, 2, \ldots, n$, $\text{range}_k = \max(x_k) - \min(x_k)$. Furthermore, $Y_k = \sum_{i=1}^{c}[a_{k,i}^-, a_{k,i}^+] \otimes t[f_i^-(X_k), f_i^+(X_k)]$ is satisfied, where $[f_i^-(X_k), f_i^+(X_k)] = [w_i, w_i] \oplus [a_i^T, a_i^T] \otimes [x_k^- - v_i, x_k^+ - v_i]$. In addition, we have $a_{k,i}^- = \min(a_{k,i}^1, a_{k,i}^2)$ and $a_{k,i}^+ = \max(a_{k,i}^1, a_{k,i}^2)$, where $a_{k,i}^1 = 1/\sum_{j=1}^{c}\left(\frac{\|x_k^- - v_i\|}{\|x_k^- - v_j\|}\right)^{2/(m-1)}$, $a_{k,i}^2 = 1/\sum_{j=1}^{c}\left(\frac{\|x_k^+ - v_i\|}{\|x_k^+ - v_j\|}\right)^{2/(m-1)}$, and v_i is a cluster center (prototype) of the rule positioned in the inputs space.

Based on the above-mentioned basic knowledge, GrC has been applied to solve decision-making problems to exploit its advantages in modeling data. Next, we review current research on GrC in decision-making.

3. Current research on granular computing in decision-making

This section provides a comprehensive overview of GrC in decision-making. After clarifying the fundamentals of decision-making, multi-granular linguistic information in decision-making is first revised, likely because it is the most used form of GrC in decision-making. Consequently, other extensions and models based on GrC are then reviewed.

3.1. MCDM with an optimal allocation of information granularity model

A decision is related to several opinions, i.e., criteria, which usually are multi-dimensional. Therefore, decision-making has been investigated by many scholars in recent years [59,60], especially in terms of multi-attribute utility theory [61,62]. Next, we introduce the basic framework of MCDM. Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives; $C = \{C_1, C_2, \ldots, C_n\}$ is a set of criteria, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is a weighting vector associated with criteria, satisfying the conditions $\lambda_k \in [0,1]$ and $\sum_{k=1}^{n}\lambda_k = 1$. Let $X = (x_{ij})_{m\times n}$ be a decision matrix, where $a_{ij} \in L$ is the evaluation value of alternative $A_i \in A$ with respect to criterion $C_j \in C$. If the criteria are divided into two types – benefit and cost – then normalization needs to be implemented. The general framework showing a transition from data to model is shown in Fig. 6. Considering that GrC is a kind of data-driven method, firstly, we need to construct the granular prototypes according to the characteristic of the original data, then construct the corresponding granular models, typical of which are interval MCDM models, multiple granular linguistic models, and three-way decision-making models. More specifically, Fig. 7 illustrates the detailed decision-making process. The basic decision-making scenario involves a set of decision-makers evaluating a set of attributes with given parameter values. In the process of collecting and processing preferences, the information granules are allocated to granulate these preferences based on the principle of justifiable information granularity. Then, bring these granular preferences into the decision-making methods (e.g., the mathematical methods, interactive methods, and data analysis methods etc.). In the end, the selection, ranking or sorting of attributes is carried out according to the demands of the decision-making problems.

Pedrycz and Song [5] presented an extension of the AHP to increase the consensus level in GDM, in which granular reciprocal matrices were employed to optimize a performance index. Later, Liu et al. [63] established a modified consensus GDM model with the optimal allocation of information granularity based on Pedrycz and Song's model [5]. Furthermore, Castillo et al. [2] designed an interval type-2 fuzzy model, where optimal granularity allocations were adopted through specific membership functions that were associated with experimental data.

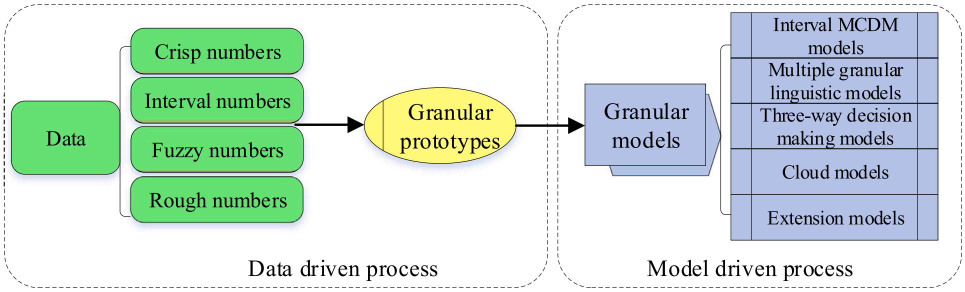

Fig. 6. From data to model: a general scheme.

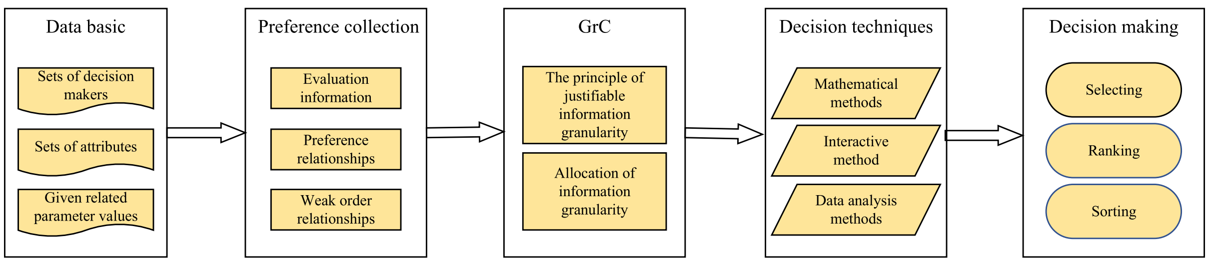

Fig. 7. MCDM process in the setting of GrC.

In related research, Cabrerizo et al. [64] developed a model with regard to an information granulation of the linguistic information in GDM problems defined in heterogeneous contexts. Since consensus-reaching is essential in the GDM process, each pairwise comparison was employed to form the information granule [7]. Afterwards, Cabrerizo et al. [65] introduced the allocation of information granularity to estimate the missing information in GDM problems. Similarly, Pedrycz [6] declared that the information granularity serves as a design asset in system design, further discussing protocols of allocation of information granularity. Zhang et al. [66] constructed a granular model based on granular reciprocal matrices and proposed a consistency improvement method by inputting the level of information granularity. In addition, they applied the adaptive differential evolution algorithm to solve the problem. Additionally, they applied consensus granules and defined the corresponding consensus measurement by considering the coverage and specificity of information granules. Through the corresponding data collection (A retrieval formula with Topic = "granular computing" AND Topic = "decision making") in the Core Collection of the Web of Science (WoS) database on June 29, 2022, the co-citation network shown in Fig. 8 illustrates the highly cited authors among these publications relating to granular decision-making models.

Here, we review the main ideas of Pedrycz and Song [5], which are about the AHP method in GDM with an allocation of information granularity. In addition, the general resolution scheme of the AHP method in GDM problems is displayed in Fig. 9. It represents a general framework of GrC in decision-making situation, which contains the granulation process of evaluation information, the optimal allocation of information granularity, and the process of using the heuristic algorithm to find the suitable results under the requirement (minimization or maximization) of the objective function. Specifically, the optimization model is represented as follows:

$$\min_{R[1],R[2],\ldots,R[c]\in P(R)} Q \qquad s.t.\ c\varepsilon = \sum_{i=1}^{c} \varepsilon_i \tag{4}$$

where $Q = AQ_1 + Q_2$, $Q_1 = \sum_{i=1}^{c}(1-\upsilon_i)\|\mathbf{e}[i]-\hat{\mathbf{e}}\|^2$, $Q_2 = \sum_{i=1}^{c}\upsilon_i$ and $A \geq 0$, $\upsilon_i = \frac{\lambda_{\max}-n}{n-1}$ represents the inconsistent index of the ith matrix, $\mathbf{e}_i$ is the eigenvector of the ith matrix, $\hat{\mathbf{e}} = \frac{\sum_{i=1}^{c}(1-\upsilon_i)\mathbf{e}_i}{\sum_{i=1}^{c}(1-\upsilon_i)}$ is the preference vector, and ε is the level of information granularity. Of note, for the case of $\upsilon_i > 1$, Liu et al. [63] modified the expression of Q_1 as follows :

$$Q_1 = \sum_{i=1}^{c}(1-\frac{\upsilon_i}{\upsilon_a})\|\mathbf{e}[i]-\hat{\mathbf{e}}\|^2 \tag{5}$$

where $\frac{\upsilon_a}{c\cdot \mathrm{RI}} \leq 0.1$ and RI denotes the average υ_i obtained for a large number of randomly generated multiplicative reciprocal matrices. The core granulation formulas of the entries of the pairwise comparison matrix are expressed as:

$$[a,b] = [\max(1/9, r_{ij}-\varepsilon_i(8/9)), \min(1, r_{ij}+\varepsilon_i(8/9))] \tag{6}$$

where r_{ij} is the numeric prototype and $[a,b]$ is the interval formed around r_{ij} by the input of the level of information granularity ε_i.

3.2. Multiple granular linguistic decision-making

Fuzzy linguistic information has been widely and successfully used in decision-making [67–70], but its use has implied the accomplishment of processes of computing with words (CWW) [71–74] to obtain solutions in such decision situations. Fig. 10 presents the highly cited authors in the era of GrC with linguistic information, by a searching strategy (Topic = "granular computing" AND Topic = "decision making" AND Topic = "linguistic information") in the Core Collection of the WoS database on June 29, 2022. Furthermore, the CWW processes follow a computational scheme that states that the input and output information must be linguistic [75,76], as shown in Fig. 11. In the specialized literature about CWW, the underlying idea is that "words mean different things for different people". The two main reasons behind the different meanings of words can be articulated as the following:

(1) The inherent uncertainty of words is one reason for multiple word meanings. For example, "young" can be interpreted differently because it is inherently uncertain. In addition, some understand "young" as a range of 0–25 years and others as 0–35 years. In such a case, the modeling of the meaning of words used type-2 fuzzy sets [74,77–79].
(2) The limited level of knowledge is another perspective. A different meaning of a word comes from the degree of knowledge that one has about the element that is assessed by such a word. For instance, a non-expert on wine can assess a bottle of wine as

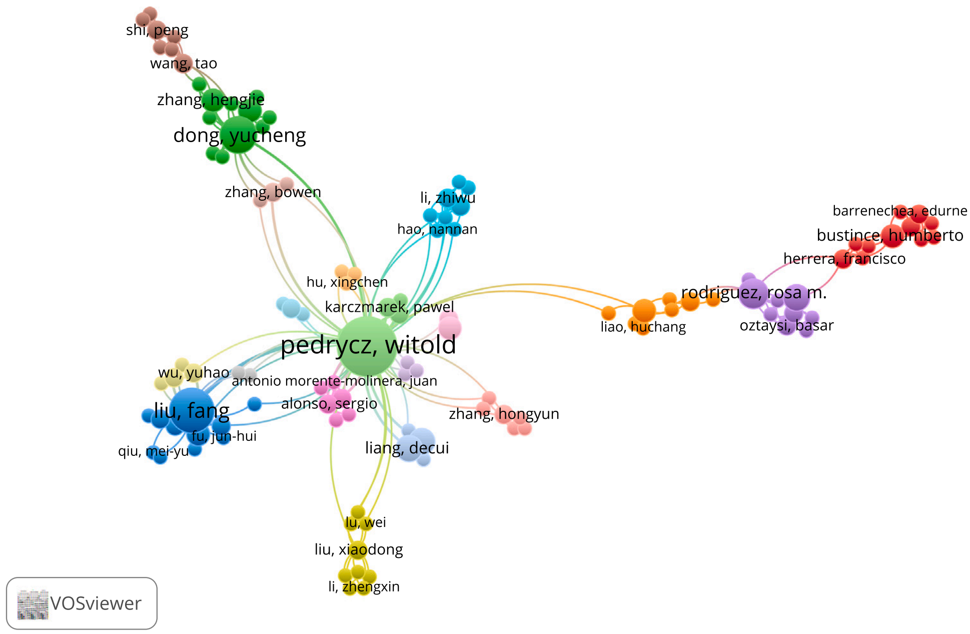

Fig. 8. Author co-citation network regarding decision-making models.

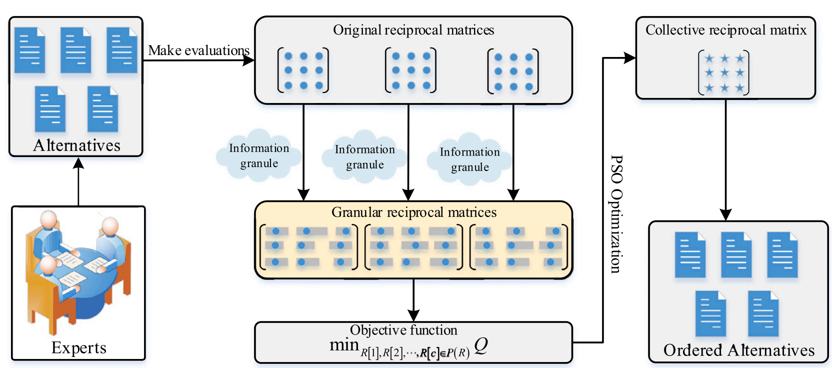

Fig. 9. Route of a granular model in solving a GDM problem.

"good" or "bad", but a wine expert may assess the same bottle of wine as "very poor", "poor", "so and so", "good", "very good", or "excellent". Therefore, the meaning of "good" in each case is different, based on the degree of knowledge one has on an element to be assessed. In this case, the modeling of the meaning of such words was carried out by multiple granules of information, called multi-granular linguistic information [4,80–82].

Our interest in this study is the latter, in which granularity plays a key role. Therefore, we provide a brief review of different multi-granular linguistic frameworks and their computational models:

- *Fusion Approach I for Managing Multi-granular Linguistic Information* [83]
- *Fusion Approach II for Managing Multi-granular Linguistic Information* [84]
- *Linguistic Hierarchies* [85]
- *Extended Linguistic Hierarchies* [86]
- *Hierarchical Tree* [87]

With all the different approaches introduced in the literature, our revision of each involves clarifying the multi-granular framework defined by the approach in which limitations of the allowed granularity can be imposed. Then, the different phases for computing with multi-granular linguistic information are reviewed and exemplified.

3.2.1. Information fusion approaches for managing multi-granular linguistic information

In this section, information fusion approaches I and II for multi-granular linguistic information are management are presented.

Fusion Approach I for managing multi-granular linguistic information : This initial approach for managing multi-granular linguistic information, introduced in [83], deals with linguistic information assessed in linguistic scales with multiple granularities based on fuzzy sets and the extension principle [88].

(1) **Framework**
The framework F_{MS} defines the kind of multiple granularity term sets that can be used. In this approach, no limitation is present regarding the granularity of the linguistic term sets $S_i = \{s_0, s_1, \dots, s_g\}$ that can handle multi-granular linguistic decision-making problems. Therefore, $F_{MS} = \cup S_i$, $i = 0, 1, \dots, g$ and $|S_j| \neq |S_k|$, where $S_j, S_k \in S_i$.

(2) **Unification phase**
Before accomplishing any computation with the multi-granular linguistic information, a unification phase occurs, in which all the linguistic information expressed with different granularity is combined into one linguistic domain, called the basic linguistic

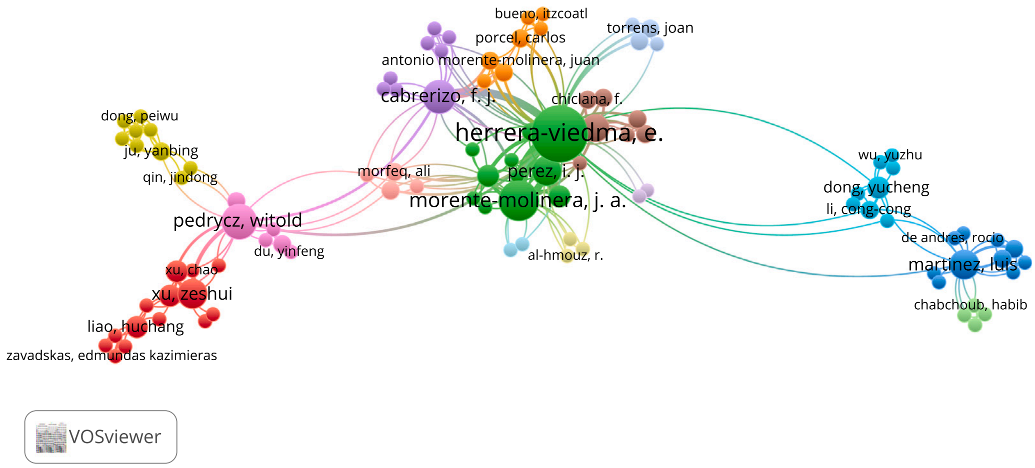

Fig. 10. Author co-citation network for GrC with linguistic information.

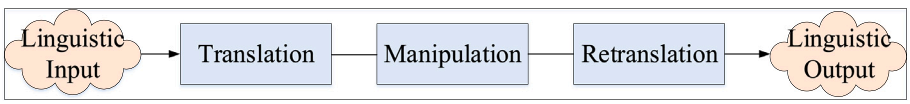

Fig. 11. Computing with words: a general scheme.

term set (BLTS) S_T. The unification process converts the multi-granular linguistic information into fuzzy sets in the BLTS in the following steps:

Step 1. *Election of the BLTS*. The rules to choose S_T, defined in [83], state that the term set with maximum granularity in the framework, F_{MS}, should be the BLTS with the aim of retaining as much information as possible.

Step 2. *Unification process*. Once the BLTS is chosen, the multi-granular linguistic information is converted into the linguistic domain by means of fuzzy sets with the transformation function $\tau_{S_iS_T}$ in Definition 1 below, which can express any linguistic term $s^i_j \in S_i$ as a fuzzy set defined in S_T.

(3) **Computational phase**
This approach takes advantage of the unification domain and fuzzy sets on linguistic term sets and carries out the computational processes by using fuzzy arithmetic [83,89], resulting in fuzzy sets [83]. However, to fulfill the CWW retranslation process, [83] later defined a retranslation function from fuzzy sets to a linguistic 2-tuple [4].

Definition 1 (*[83]*). Let $S_i = \{s^i_0, s^i_1, \dots, s^i_{g_i}\}$ and $S_T = \{s^T_0, s^T_1, \dots, s^T_{g_T}\}$ be two linguistic term sets, such that $g_T > g_i$. Then, a multi-granularity transformation function, $\tau_{S_iS_T}$ is defined as:

$$\tau_{S_iS_T} : S_i \rightarrow F(S_T) \tag{7}$$

$$\tau_{S_iS_T}(s^i_j) = \{(s^T_k, \gamma^j_k) | k \in \{0, 1, \dots, g_T\}\}, \forall s^i_j \in S_i \tag{8}$$

$$\gamma^j_k = \max_y \min\{\mu_{s^i_j}(y), \mu_{s^T_j}(y)\} \tag{9}$$

where $F(S_T)$ is the family of fuzzy sets defined on S_T and $\mu_{s^i_j}$ are the membership functions of the fuzzy sets associated with the terms s^i_j and s^T_k, respectively.

Fusion Approach II for managing multi-granular linguistic information : Another approach to dealing with multi-granular linguistic information that extends from [83] was proposed by Chen and Ben-Arieh in [84]. This approach overcomes the limitation that the granularity of the BLTS must be greater than the other term sets. Additionally, it provides a new method to unify the linguistic information assessed in different scales.

(1) **Framework**
Similar to Fusion Approach I, this approach fixes the framework F_{MS} for the multi-granular linguistic information without any limitation. Therefore, any granularity can be used for eliciting linguistic information.

(2) **Unification phase**
This approach also unifies the multi-granular linguistic information by means of fuzzy sets but using a new transformation function. It allows linguistic terms to transform between any term set of the framework F_{MS} without having to choose a BLTS for the unification process. The new transformation function $\tau'_{S_iS_T}$ for such a unification is defined in Definition 2.

(3) **Computational phase**
Because the unification expression domain results in fuzzy sets on a linguistic term set, the computational model is analogous to the previous approach.

Definition 2 (*[83]*). Let $S_i = \{s^i_0, s^i_1, \dots, s^i_{g_i}\}$ and $S_T = \{s^T_0, s^T_1, \dots, s^T_{g_T}\}$ be two linguistic term sets, such that $g_T \neq g_i$. A multi-granularity transformation function, $\tau_{S_iS_T}$ is defined as:

$$\tau'_{S_iS_T} : S_i \rightarrow F(S_T) \tag{10}$$

$$\tau'_{S_iS_T}(s^i_j) = \{(s^T_k, \mu_{jk}(x)) | k \in \{0, 1, \dots, g_T\}\}, \forall s^i_j \in S_i, j = \{0, 1, \dots g_i\} \tag{11}$$

$$\mu_{jk}(x) = \begin{cases} 1 & k_{\min} < k < k_{\max} \\ \left(\frac{k+1}{g_T+1} - \frac{j}{g_i+1}\right)(g_T+1) & k = k_{\min} \\ \left(\frac{j+1}{g_T+1} - \frac{k}{g_i+1}\right)(g_T+1) & k = k_{\max} \\ 0 & others \end{cases} \tag{12}$$

where $k_{\min}$ and $k_{\max}$ are the minimum and maximum indexes of the linguistic terms, respectively, with nonzero membership functions in the target set S_T. Fig. 12 visualizes this approach.

s_0^i	s_1^i	s_2^i	s_3^i	s_4^i	s_5^i	s_6^i	s_7^i	s_8^i		
s_0^T	s_1^T	s_2^T	s_3^T	s_4^T	s_5^T	s_6^T	s_7^T	s_8^T	s_9^T	s_{10}^T

Fig. 12. Tiers in the Fusion Approach II.

3.2.2. Linguistic hierarchies with their extensions

Although Fusion Approaches I and II can handle multi-granular linguistic information, they present two main flaws: (1) the accuracy of computational processes (because fuzzy arithmetic increases uncertainty) and (2) the domain used to express the computed results, which could be different than that used by the experts in their elicitation. To overcome such drawbacks, Herrera and Martínez [85] introduced a novel symbolic approach that provides accurate linguistic results, called linguistic hierarchy (LH). This approach is based on the 2-tuple linguistic representation model and its symbolic computational model [4]. In what follows, the corresponding framework, unification phase and computation phase of LH and its extensions are discussed successively.

Linguistic hierarchies

(1) **Framework**
Unlike that in the fusion approaches, the LH approach defines a framework in which limitations regarding the permitted granularities are used for the linguistic term sets in the framework F_{MS}. Such limitations are established by rules in the building process of the LH that are revised below.
First, an LH is defined as the union of all levels t : $LH = \cup_t l(t, n(t))$, in which a level t of an LH is a linguistic term set with a granularity of $n(t)$ denoted as $S^{n(t)} = \{s_0^{n(t)}, s_1^{n(t)}, \dots, s_{n(t)-1}^{n(t)}\}$. The building process of an LH introduced in [85] uses the following two LH basic rules:

Rule 1. Preserve all former modal points of the membership functions of each linguistic term from one level to the following one.

Rule 2. Smooth the transitions between successive levels. The aim of this rule is to add a new level $t+1$ to the LH such that the corresponding linguistic term set is $S^{n(t+1)}$. In this level, a new linguistic term is added between each pair of terms of the previous level t. The insertion of these new terms implies the decreased support of the linguistic labels to allow room for the new one in between. Previous rules have an effect regarding the linguistic term sets that can be in the framework F_{MS} of an LH. Because the linguistic term set of level $t+1$ depends on the term set of level t.

$$l(t, n(t)) \to l(t+1, 2 \cdot n(t) - 1) \tag{13}$$

(2) **Unification phase**
The LH can unify the multi-granular linguistic information in any term set of the LH in a precise way without losing information by means of a transformation function $TF_{t'}^{t}$ [85] between any two linguistic levels t and t' of the LH.

(3) **Computational phase**
While the fusion approaches unify the multi-granular linguistic information using fuzzy sets and apply fuzzy arithmetic, the LH employs linguistic 2-tuple values. Therefore, the LH applies the 2-tuple computational model introduced in [4] for the 2-tuple representation model. Remarkably, the results obtained in the LH are linguistic terms, but such results can be retranslated to any linguistic term set in the LH that facilitates the retranslation process to express the results precisely in any of the linguistic term sets of the framework F_{MS} by means of $TF_{t'}^{t}$.

Definition 3 (*[4]*). Let $LH = \cup_t l(t, n(t))$ be a linguistic hierarchy whose linguistic term sets are denoted as $S^{n(t)} = \{s_0^{n(t)}, s_1^{n(t)}, \dots, s_{n(t)-1}^{n(t)}\}$ [85]. Let us also consider the 2-tuple linguistic representation. The transformation function transforming a linguistic label in level t to a label in level t', satisfying the linguistic hierarchy basic rules, is:

$$TF_{t'}^{t}(s_i^{n(t)}, \alpha^{n(t)}) = \Delta\left(\frac{\Delta^{-1}(s_i^{n(t)}, \alpha^{n(t)}) \cdot (n(t') - 1)}{n(t) - 1}\right) \tag{14}$$

Knowing the intuition and motivation behind Eq. (14) aids in understanding. The work in [85] demonstrated that the transformation function $TF_{t'}^{t}$ between linguistic terms in different levels of the LH is a bijective function, and the transformations are carried out without loss of information.

Extended linguistic hierarchies Clearly, the LH approach provides accurate results when dealing with multi-granular linguistic information. However, it presents a key drawback since it limits the linguistic term sets that can be used in the decision problem with multi-granular linguistic information. In contrast, the fusion approaches do not present any limitations, but their results are less accurate. The inability to linguistic term sets with a granularity of five and seven terms in an LH is crucial in problems requiring such granularities. To harness the accuracy of LH and the diversity of granularity of the fusion approaches, the extended linguistic hierarchy (ELH) approach was introduced [86]. ELH extends LH by modifying the building process of the LH, defining new construction rules, and adapting the LH computational model to the new linguistic structure.

(1) **Framework**
Like LHs, ELHs are conformed by a group of linguistic terms sets, in which each term set belongs to a level $l(t, n(t))$ and has a granularity $n(t)$ different from the remaining levels of the ELH. Thus, the framework F_{MS} of the ELH is the union of the levels. However, the ELH does not impose any limitation to the sets that can be used in F_{MS}, and hence, any term set without any limitation can form part of the ELH. Although the building process defines new linguistic hierarchical rules to construct the ELH, its computational symbolic model based on the 2-tuple accomplishes computational processes with multi-granular linguistic information accurately.

$$F_{MS} = ELH \tag{15}$$

As mentioned, ELH handles any linguistic granularity in the framework, so the new building process for an ELH is based on the two new extended hierarchical rules that replace the linguistic hierarchical basic rules. These new rules obliges to keep the former modal points from level t to the next level t':

(a) **Extended Rule 1**: Include a finite number of levels $l(t, n(t))$ with $t = \{1, 2, \dots, m\}$ that defines the multi-granular linguistic context. There is no restriction on the granularity of these m linguistic term sets.
(b) **Extended Rule 2**: Add a final level $l(t^*, n(t^*))$ such that $t^* = m + 1$ keeps all the former modal points of the previous m levels.

(2) **Unification phase**
The ELH also shares the computational scheme with the LH, in which the multi-granular linguistic information is conducted into one linguistic domain. This unification process is carried

out by the transformation function $TF^{t}_{t'}$ for LH. However, with the different hierarchical rules for the construction of the ELH, these transformations do not necessarily achieve accurate results if they are applied to any two term sets because the term set may not keep the former modal points with the next one. Consequently, in ELH, the information transformation is always conducted in the level t^{*} that keeps all the former modal points by using the function TF^{t}_{t*}, where t is any level in $\{1, 2, \dots, m\}$ and $t^{*} = m + 1$.

(3) **Computational phase**
In the unification phase, the information is converted into linguistic 2-tuples values in $S^{n(t*)}$. Hence, the CWW processes are carried out as in the LH by using the 2-tuple linguistic computational model but, in this case, on level t^{*}.

$$\left\{ \left(S_t^{n(t*)}, \alpha\right), \dots, \left(S_t^{n(t*)}, \alpha\right) \right\} \to \Delta\left(\varphi \Delta^{-1}\left(S_t^{n(t*)}, \alpha\right)\right) \to \left(S_t^{n(t*)}, \alpha\right) \tag{16}$$

With the transformation function $TF^{t}_{t'}$, the unified information is expressed in level t^{*}, and its high granularity may make the results difficult to understand. Therefore, to obtain clearer linguistic results in the retranslation phase of CWW, another transformation with the function $TF^{t}_{t'} : \overline{S^{n(t*)}} \to \overline{S^{n(t')}}$ is applied to the linguistic terms at $l(t*, n(t*))$ in ELH into any level $l(t', n(t'))$, $t' \in \{1, 2, \dots, m\}$. This provides results represented in the same linguistic term set used to express their assessments.

$$TF^{t}_{t'} : \overline{S^{n(t*)}} \to \overline{S^{n(t')}} \tag{17}$$

3.2.3. Hierarchical tree

Huynh et al. [87] introduced another important approach for dealing with multi-granular linguistic information. This approach, called hierarchical tree T_{LH}, was a new view of LH in terms of an ordered structure based on the semantics of linguistic terms. The hierarchical tree aims to soften the multi-granular linguistic framework and break its limitation imposed by the LH to accommodate more term sets.

(1) **Framework**
This approach provides a formal definition of the hierarchical tree that fixes the framework.

(2) **Unification phase**
The multi-granular linguistic information elicited in different linguistic levels is unified by the transformation function Φ using Γ_t and Γ_t^{-}.

$$\begin{cases} \Phi^{t}_{t+a} : S^{n(t)} \to 2^{S^{n(t+a)}} \\ (s_i^t) = \bigcup\limits_{\Phi_j^{t+a-1} \in \Phi^{t}_{t+a-1}(s_i^t)} \Gamma_{t+a-1}(s_j^{t+a-1}) \\ \Phi_t^{t+a} : \Gamma_t^{-}, \Gamma_{t+1}^{-}, \dots, \Gamma_{t+a-1}^{-} s_i^{t+a} \end{cases} \tag{18}$$

(3) **Computational phase**
The hierarchical tree approach proposes a computational model for operating with the unified information that uses random preferences, called satisfactory principle, instead of fuzzy sets, supporting the linguistic terms' semantics or the linguistic terms themselves. Therefore, the results obtained are expressed in a numerical domain and do not fulfill the CWW scheme. For further details, see [87].

Definition 4 (*[87]*). A linguistic hierarchy of a linguistic variable X is defined as a hierarchical tree T_X, which has a finite number of levels t with $t = 0, 1, \dots, m$, defined as the following: level $t = 0$ is the root of the tree labeled by X the name of the linguistic variable.

- At each level t, where $t = 1, 2, \dots, m$, is a finite linguistic term set of X, denoted by $S^{n(t)}$, coupled with a total ordering:
  - (a) $n(t) < n(t+1)$, for any $t = 1, 2, \dots, m-1$
  - (b) For each $t = 1, 2, \dots, m-1$, there is just one mapping $\Gamma_t : S^{n(t)} \to 2^{S^{n(t+1)}} - \{\phi\}$ fulfilling $\Gamma_t(s_i^{n(t)}) \bigcap \Gamma_t(s_{i'}^{n(t)}) = \phi$ for any $s_i^{n(t)} \neq s_{i'}^{n(t)}$, $\bigcup_{s_i^{n(t)} \in S^{n(t)}} \Gamma_t(s_i^{n(t)}) = S^{n(t+1)}$
  - (c) If $s_i^{n(t)} < s_{i'}^{n(t)}$ in $S^{n(t)}$, then $s_k^{n(t+1)} < s_{k'}^{n(t+1)}$ in $S^{n(t+1)}$ for any $s_k^{n(t+1)} \in \Gamma_t(s_i^{n(t)})$ and $s_{k'}^{n(t+1)} \in \Gamma_t(s_{i'}^{n(t)})$
- For each $t = 1, 2, \dots, m-1$, the mapping Γ_t performs as a semantic derivation from the term set $S^{n(t)}$ to its refinement $S^{n(t+1)}$. Moreover, for each mapping Γ_t, exists a pseudo-inversion $\Gamma_t^{-} : S^{n(t+1)} \to S^{n(t)}$:
  - (a) $\Gamma_t^{-}(s_i^{n(t+1)}) = s_i^{n(t)}$ such that $s_i^{n(t+1)} \in \Gamma_t(s_i^{n(t)})$;
  - (b) Γ_t and Γ_t^{-} define the transformation functions between levels of the hierarchy.

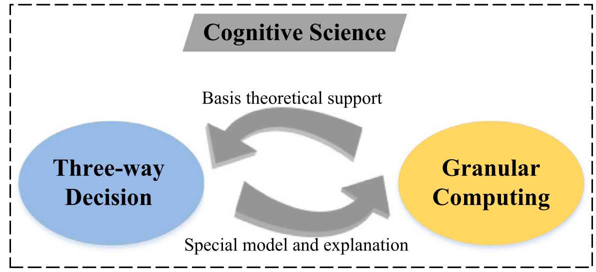

Fig. 13. Relations between GrC and three-way decisions.

3.3. Extensions of granular decision models

In this section, some extended granular decision-making models are presented, which contains the three-way decision-making model, cloud model, data-driven granular neural network model and other granular models.

3.3.1. Three-way granular model

The conception of three-way GrC can be illustrated as we apply the principles of three-way decision-making to granular thinking at multiple levels of granularity and with multiple granules at each level [21]. In addition, the relationship between three-way decision-making and GrC in the cognitive era of computing is described in Fig. 13. GrC provides a basis for three-way decision-making, and three-way decision-making models can explain the granular structures to a certain degree. Generally, three-way decisions have the following key problems with multilevel GrC: (1) the construction of granules, (2) the establishment of the granular level, and (3) the selection of the optimal granularity and the reasonable level of granularity [14]. Fig. 14 shows a relation of control from the top-down reading of the three levels and a relation of support that is bottom-up. In addition, the hierarchical granular structure of the three levels is displayed, in which the levels are ranked by decreasing granularity. In other words, more abstract and larger granules are in the top level, and more specific and smaller granules are in the bottom level.

Three-way decision representations can be classical, probabilistic, or sequential. Their descriptions and decision rules are in Definitions 5–7:

Definition 5 (*[13]*). For given a finite and non-empty set U and an equivalence relation R, the Pawlak approximation space $apr = (U, R)$ is defined on U and R, where a partition of U generated from R, denoted as $[x]_R$ or $[x]$. Therefore, the lower and upper approximations of X for $\forall C \subseteq U$ are defined as follows:

$$\begin{aligned} \underline{apr}(C) &= \{x \mid x \in U, [x] \subseteq C\} = POS(C), \\ \overline{apr}(C) &= \left\{x \mid x \in U, [x] \cap C \neq \emptyset\right\} = POS(C) \cup BND(C). \end{aligned} \tag{19}$$

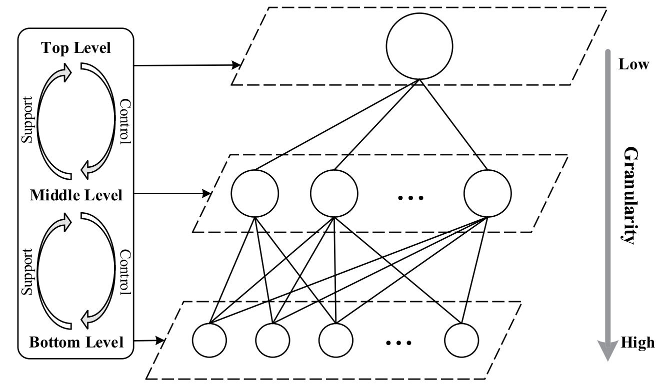

Fig. 14. Architecture of multiple granules in three levels.

The main idea of a three-way decision lies in three disjoint regions, i.e., the positive region $POS(C)$, boundary region $BND(C)$, and negative region $NEG(C)$, produced by the lower and upper approximations:

$$\begin{aligned} &POS(C) = \underline{apr}(C),\\ &BND(C) = \overline{apr}(C) - \underline{apr}(C),\\ &NEG(C) = U - POS(C) \cup BND(C) = U - \overline{apr}(C) = (\overline{apr}(C))^c. \end{aligned} \tag{20}$$

The classical three-way decision rules are:

(1) If $[x] \subseteq POS(C)$, then $x \in C$
(2) If $[x] \subseteq BND(C)$, then it is uncertain x belongs to C
(3) If $[x] \subseteq NEG(C)$, then $x \notin C$

Definition 6 ([13]). Given a decision action set $A = \{a_p, a_N, a_B\}$ and the states set $\Omega = \{S, \overline{S}\}$, $P(S|X)$ and $P(\overline{S}|X)$ represent the conditional probability. The expectation risk loss values of decision actions are defined as follows:

$$R(\alpha_p|X) = \lambda_{pp} P(S|X) + \lambda_{pN} P(\overline{S}|X) \tag{21}$$

$$R(\alpha_N|X) = \lambda_{Np} P(S|X) + \lambda_{NN} P(\overline{S}|X) \tag{22}$$

$$R(\alpha_B|X) = \lambda_{Bp} P(S|X) + \lambda_{BN} P(\overline{S}|X) \tag{23}$$

where λ_{pp}, λ_{Np}, λ_{Bp}, λ_{pN}, λ_{NN}, and λ_{BN} are the loss function values.

The probabilistic three-way decision rules are:

(1) If $\Pr(C|[x]) \geq \alpha$ and $\Pr(C|[x]) \geq \gamma$, decide $x \in POS(C)$
(2) If $\Pr(C|[x]) \leq \alpha$ and $\Pr(C|[x]) \geq \beta$, decide $x \in BND(C)$
(3) If $\Pr(C|[x]) \leq \beta$ and $\Pr(C|[x]) \leq \gamma$, decide $x \in NEG(C)$

Yao and Deng [90] developed sequential three-way decisions based on probabilistic rough sets and totally ordered relations, a GrC perspective on three-way decisions. They demonstrated sequential three-way decisions are superior in the context of multiple levels of information granularity.

Definition 7 ([91]). Given a dataset $M = \{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{N_t}\}$, a decision set $D = \{a_P, a_N, a_B\}$, and a granular feature set $C = \{B^1, B^2, \ldots, B^n\}$ of an image $A \in M$, a series $B^1 \preceq B^2 \preceq, \ldots, \preceq B^n$ is denoted as a sequential three-way decision.

$$\begin{aligned} SD = (SD_1, SD_2, \ldots, SD_l, \ldots, SD_n) = (\phi^*(B^1), \phi^*(B^2), \ldots, \phi^*(B^l), \ldots,\\ \phi^*(B^n)) \end{aligned} \tag{24}$$

where $\phi^*(B^l) = \arg\min_{d \in D} \cos t(d|B^l)$ is the minimum cost decision of lth step and $\cos t(d|B^l)$ is the cost of decision B^l.

The sequential three-way decision rules are:

$$\begin{aligned} &\mathrm{cost}(a_P|B^l) = \lambda_{PP} \Pr(P|B^l) + \lambda_{PN} \Pr(N|B^l),\\ &\mathrm{cost}(a_N|B^l) = \lambda_{NP} \Pr(P|B^l) + \lambda_{NN} \Pr(N|B^l),\\ &\mathrm{cost}(a_B|B^l) = \lambda_{BP} \Pr(P|B^l) + \lambda_{BN} \Pr(N|B^l). \end{aligned} \tag{25}$$

In addition, three-way decision-making has been integrated with MCDM, for instance, in web-based medical decision support systems for three-way medical decision-making [92]. For extended fuzzy sets, Liu and Liu [93] developed an extended three-way decision method, partial fuzzy sets, which pay attention to the uncertain region. Similarly, Xiao et al. [94] investigated three-way decisions of type-2 fuzzy sets based on partially ordered sets.

In terms of GDM, Liang et al. [95] developed three-way decisions with linguistic assessment. Liang et al. [3] developed a novel loss functions determination method and a decision procedure of GDM-based three-way decisions to select a strategic supplier. Furthermore, Sun et al. [96] developed a new approach to GDM under uncertainty by means of a multi-granulation fuzzy decision-theoretic rough set over two universes, denoted by a cost-based method, which could sort among all alternatives of GDM problems. Hu et al. [97] proposed a three-way group decisions method where the weights of experts are unknown, which was applied in the field of medical management.

For classical MCDM, extended methods have been introduced, such as a novel TODIM method-based three-way decision model [98], three-way decisions-based ideal solutions in the Pythagorean fuzzy information system based on TOPSIS [99], and a three-way group decision method based on prospect theory [100]. From the view of GrC, Yao [21] developed a trisecting-acting-outcome (TAO) model. Other three-way decision applications with GrC were reported in [22,23,101].

3.3.2. Cloud model

Cloud modeling can imitate the human cognition process of the selection of optimal granularities automatically [102]. First, we define a cloud model: a transformation model between qualitative concepts and values that can transform natural language into the uncertainty. Then, the existing cloud models with the MCDM method are briefly reviewed.

Definition 8 ([18]). Let T be a qualitative concept over a universe of discourse U, $x \in X \subseteq U$ be a random instantiation of the concept, and $G_T(X) \in [0,1]$ be the certainty degree of $x \in T$, which corresponds to a random number with a steady tendency. Then, the distribution of the domain membership is called a membership cloud, or simply cloud, which maps from U to $[0,1]$. The characteristics of a cloud can be described using the expected value Ex, which is the center value of the qualitative concept domain, the entropy En, which represents the ambiguity of the qualitative concept, and the hyper entropy value He, which reflects the degree of dispersion of droplets and the membership's random changes. The process of producing clouds is described as follows:

(a) **Input**: The quantitative positions of g cloud drops x_i, where $i = 1, 2, \ldots, g$, and the certainty degree that each cloud drop represents a linguistic notion y_i, $i = 1, 2, \ldots, N$.
(b) **Output**: The three numerical parameters Ex, En, and He represent the linguistic notions.

We present a typical example of a Gaussian cloud model to enable a better understanding. Different people may have various interpretations of the concept of "young", but defining a crisp membership degree is difficult. In the Gaussian cloud model, more than one membership degree μ is available for each age x, which can describe the higher-order uncertainty of the concept "young". Simultaneously, the Gaussian cloud has a Gaussian distribution depicted in Fig. 15. Therefore, the basic certainty of uncertainty is also defined.

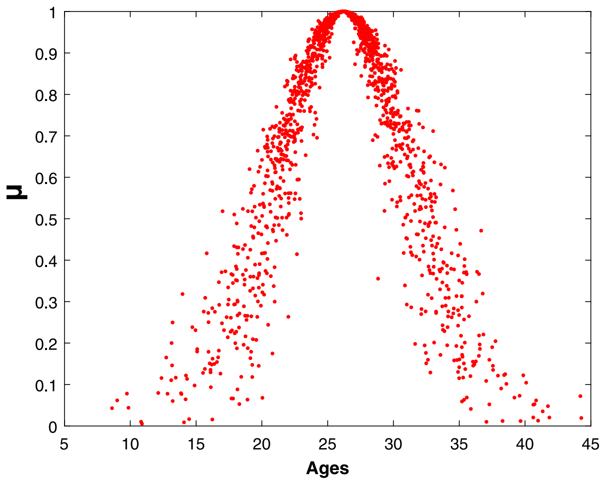

Fig. 15. Description of "young" in a cloud model.

To transform qualitative linguistic information into quantitative values, the cloud model was developed [18] that can accurately represent the uncertainty of evaluation information and objectivity of decision-makers. This theory combines fuzziness and randomness based on fuzzy and probability theories. For MCDM problems, scholars have introduced methods based on the cloud model, proposing various operations on clouds and several aggregation operators and developing the conversation of linguistic information into cloud model [101]. In addition, under the context of an uncertain linguistic environment, the definition and distance of an interval-integrated cloud were proposed for an MCGDM model [22]. Furthermore, Wang et al. [23] presented first operations, a possibility degree, and aggregation operators of trapezium clouds and then developed an Atanassov's interval-valued intuitionistic linguistic MCGDM method based on the trapezium cloud model.

3.3.3. Data-driven granular neural network model

Inspired by the construction of a traditional neural network, Pedrycz [26] introduced the concept of a granular neural network (GNN). Considering that the previous model-driven decision-making models exist limitations in dealing with large-scale complex decision-making problems, the data-driven granular decision-making models are the emerging hotspots in the current research [12]. And the GNN model applied in solving decision-making problems can be view as a representative data-driven decision-making model based on GrC [27]. Based on this, this subsection presents the basic concepts and procedures of GNN model.

Specifically, the construction of a GNN involves two fundamental steps: (1) the input numeric data are condensed in the form of information granules in the well-trained numeric neural network, as shown in Fig. 16(a) and (2) the neural architecture is established with the granular connections (weights) of the neurons, as shown in Fig. 16(b). To expand on this, Song et al. [27] built a GNN model with the optimal allocation of information granularity, and it was formed on the basis of the neural network with granular connections. In addition, Fig. 17 describes the four typical GNN models with their corresponding levels of granularity. In a nutshell, the GNN models are conceptually and practically useful models, built based on existing neural networks.

Remark 2. The first variants of GNN in Fig. 17 is: (i) a high granularity of the structure (connections) with a high granularity of input data. For this case, we encounter numeric quantities, so it belongs to the standard numerically-driven neural networks. Next is (ii) a high granularity of the connections with a low granularity of inputs. The neural network is trained by numeric data (the high granularity data) in advance, and then we input the granular data to the network, which need to confirm that this neural network is capable of operating on non-numeric data. This case can be summarized in the standard neural network with granular inputs. Thirdly, we have (iii) a low granularity of the connections with a high granularity of inputs. The network itself is designed with the use of granular connections, and the input is numeric. The final variant is (iv) a low granularity of the connections with a low granularity of inputs. This is the most flexible case since the network is based on a granular connection with the granular input.

Next, the main procedures of two typical GNN models are introduced, corresponding to Fig. 16(a) and (b), both of which apply interval-base granules.

(1) **Standard neural network with granular input** [24]
Suppose there are M alternatives with N input features. Then, the decision-making matrix can be written as $X = (x_{mn})_{M\times N}$. Also, an instance has one output feature, the output features set $Y = \{y_1, y_2, \dots, y_M\}$. Then, the information granules are allocated around the original input features in the following formula:

$$\mathbf{g}(x_n) = [x_n - \varepsilon_n \times \text{range}_n, x_n + \varepsilon_n \times \text{range}_n] \tag{26}$$

where $\mathbf{g}(x_n)$ $(n = 1, 2, \dots, N)$ is the granular feature, $\varepsilon_n, (\varepsilon_n \in [0, 1])$ represents the level of information granularity allocated on the x_n, and the whole granularity level satisfies the formula: $\sum_{n=1}^{N} \varepsilon_n = N\varepsilon$, $\text{range}_n = \max_{m=1,2,\dots,M} x_{mn} - \min_{m=1,2,\dots,M} x_{mn}$. Then, through a GNN model, for each instance x_n $(n = 1, 2, \dots, N)$, there is a granular corresponding output $\mathbf{G}(y_n) = [y_{nlow}, y_{nup}]$. The two scenarios in the process of optimization of granularities follow:

- ***Only specificity criterion***

$$\max Q_1 = \frac{1}{\mathbf{e}^{\frac{1}{M}\sum_{m=1}^{M}(yg_{nup} - yg_{nlow})}} \quad s.t. \begin{cases} \sum_{n=1}^{N} \varepsilon_n = N\varepsilon \\ \varepsilon_n \in [0, 1] \end{cases} \tag{27}$$

- ***Considering coverage criterion and specificity criterion***

$$Q = \text{coverage} \times Q_1 \quad s.t. \begin{cases} \sum_{i=1}^{n} \varepsilon_j = n\varepsilon \\ \varepsilon_j \in [0, 1] \end{cases} \tag{28}$$

The operation mechanism of the GNN model with granular input is presented in Fig. 18.

(2) **Neural network based on granular connections with numeric input** [27]
For the GNN model with granular connections, the information granularity is allocated around the original weights w_{ji} and biases v_k of the neural network in Fig. 16(b). The specific allocation formula is presented in the following way:

$$\begin{cases} \mathbf{g}(w_{ji}) = [w_{ji} - \varepsilon_i^- |w_{ji}|, w_{ji} - \varepsilon_i^+ |w_{ji}|] \\ \mathbf{g}(v_k) = [v_k - \varepsilon_k^- |v_k|, v_k - \varepsilon_k^+ |v_k|] \end{cases} \tag{29}$$

where $\mathbf{g}(w_{ji})$ and $\mathbf{g}(v_k)$ are the granular weights and biases, respectively, satisfying the overall constraint of granularity, expressed as $\sum \varepsilon_i^- + \varepsilon_i^+ = I\varepsilon$ and $\sum \varepsilon_k^- + \varepsilon_k^+ = K\varepsilon$, where I and K are the numbers of the weights and biases, respectively. Additionally, the two criteria, coverage and specificity, for the optimal allocation of information granularity are as follows:

- ***Coverage criterion***
Assume the corresponding pairs (x_t, target_t) $t = 1, 2, \dots, M$, and $y_t = [y_t^-, y_t^+]$ are the corresponding intervals formed

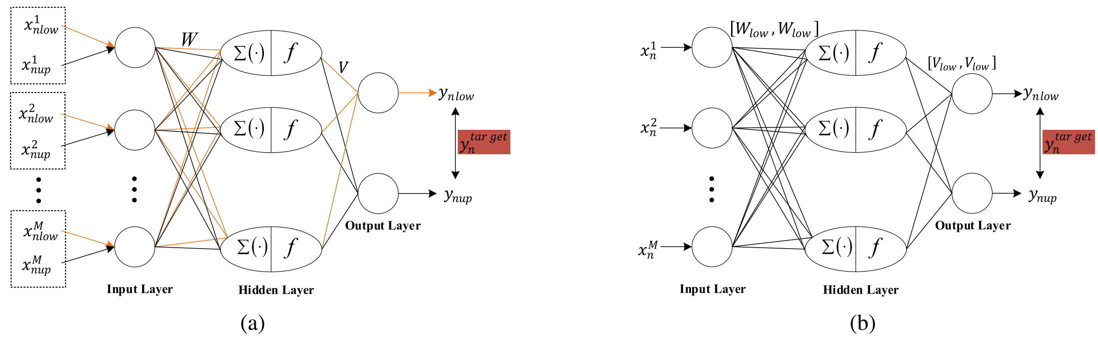

Fig. 16. Construction of GNNs: (a) neural network with granular inputs and (b) neural network with granular connections.

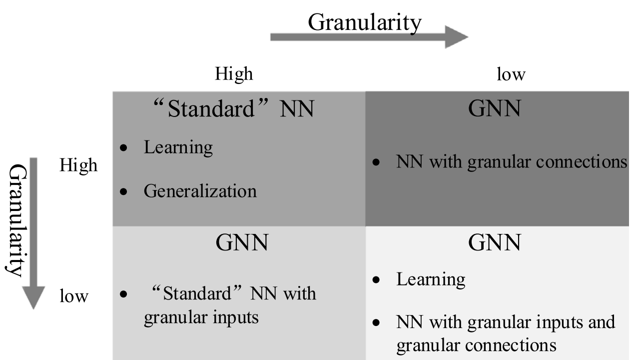

Fig. 17. Four GNN scenarios.

by the GNN model. Then, the objective function can be described as

$$Q_1 = N(\underset{t}{\mathrm{incl}}(\mathrm{target}, y_t))/M \tag{30}$$

where $N(\cdot)$ denotes the counting function and $\mathrm{incl}(x, y)$ is the inclusion degree of x in y. If $x \in y$, then $\mathrm{incl}(x, y) = 1$, otherwise, $\mathrm{incl}(x, y) = 0$.

- ***Specificity criterion***
  For the measurement of the specificity of the information granules, the following formula is established.

$$Q_2 = \sum_{t=1}^{M} |y_t^+ - y_t^-|/M \tag{31}$$

In addition, Fig. 19 presents the granular connections-based GNN model.

3.3.4. Other granular decision models

In addition to the above granular models, extensions of granular decision models have been developed. For example, Kaburlasos and Papadakis [103] introduced an extension of the granular Self-Organizing Map (grSOM), which implemented the structure identification in linguistic system modeling and applications. In other research, Kaburlasos and Papadakis [104] presented a fuzzy-ARTMAP (FAM) neural classifier based on fuzzy lattice reasoning (FLR), which improved the FIR to space F^N and represented statistics of all orders. Additionally, Lu et al. [25] proposed Sugeno-type fuzzy models, applying them to three real-world time-series forecasting scenarios. Zhu, Pedrycz, and Li [105] developed a novel extension model, namely the granular Takagi–Sugeno fuzzy model, which integrated numerical evidence, granular information, and fuzzy subspace clustering with higher accuracy. The optimization model of information granularity was given as follows:

$$\max Q(\varepsilon) = \mathrm{coverage}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_c) \times \mathrm{specificity}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_c)^{\alpha}$$

$$s.t.\begin{cases} \mathrm{coverage}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_c) = \frac{\mathrm{card}\{k=1,2,\dots,N \mid y_k \in Y_k\}}{N} \\ \mathrm{specificity}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_c) = \frac{\sum_{k=1}^{N}(0, 1-\frac{|y_k^- - y_k^+|}{|y_{\max} - y_{\min}|})}{N} \\ \sum_{i=1}^{n} \varepsilon_i = \varepsilon \\ \varepsilon_i \in [0,1] \end{cases} \tag{32}$$

where ε_i is the level of information granularity for the ith numeric prototype. Information granularity $Y_k = f(x_k, G(a))$, as granular output, is matched with x_k. The variable α is a positive weight parameter that reflects the importance of the specificity criterion.

Furthermore, data-driven GrC models deserve research attention as well, which is in accordance with the development of the big data era. GrC occupies an essential place in the construction of data-driven machine learning methods when human centricity is taken into consideration. Simultaneously, it provides a high level of interpretability [106]. The following sub-sections describe big data and GrC [28], data-driven granular cognitive computing (DGCC) [29], and federated learning with information granules [30].

(1) **Big data and Granular Computing**
The three main traits of big data are volume, velocity, and variety (the 3Vs) [107]. Wang et al. [28] emphasized that an essential issue in big data processing is dealing with the 3Vs, i.e., how to reduce the data scale, accommodate rapid growth, and standardize the data with heterogeneity and uncertainty. Furthermore, Skowron et al. [108] proposed using the GrC framework to investigate the control and prediction based on big data. In addition, Dutta and Skowron [109] suggested interactive GrC in a decision support system to deal with complex data. Similarly, Peter and Weber [110] proposed a dynamic granular cube framework to categorize dynamic granular cluster approaches for big data mining. Fig. 20 visualizes the general framework of the theoretical basis of GrC and the operation process of big data processing in a granular mechanism.

(2) **Data-driven granular cognitive computing**
DGCC takes data as a special knowledge type expressed in the lowest granularity level of a multiple granularity space [29,111]. The relationships among data, cognitive computing, and GrC are presented in Fig. 21. In this structure, computation contains efficient models and methods for data processing, cognition focus on intelligent understanding, and the interaction between users and information. Moreover, granulation is involved through multiple granularity thinking and modeling. DGCC features are summarized by [29]: (i) knowledge is the abstraction of data, and data denotes the knowledge in the lowest granularity level; (ii) nodes (concepts) in the same or different granularity layer can contain relationships; and (iii) nodes in different granularity layers can operate jointly and simultaneously in a parallel way.

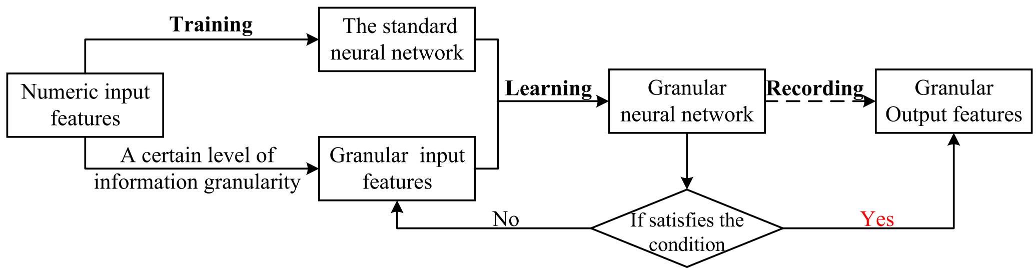

Fig. 18. Major blocks of the GNN model with granular input features.

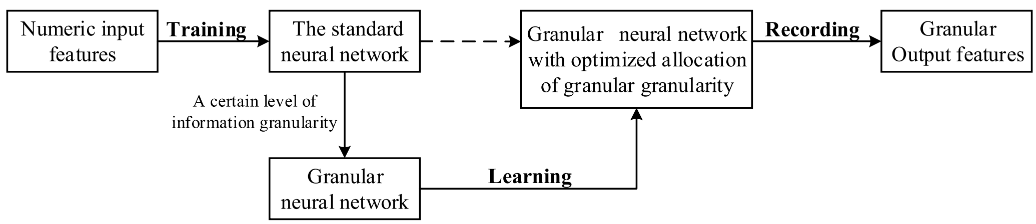

Fig. 19. The major blocks of the GNN model with granular connections.

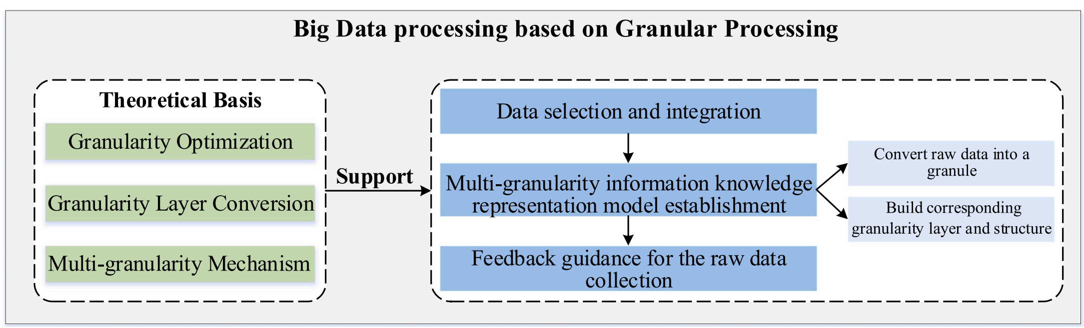

Fig. 20. Big data processing framework based on GrC.

(3) **Federated Learning with information granules**
Here, we present the granular model in [30] to illustrate the process of federated learning with information granules. In the federated learning, there are p datasets (data island): $\{D_1, D_2, \ldots, D_p\}$, the model M with the local database D_i in ith client, the corresponding granular counterpart is expressed as $G(M)|_{D_i}$. The granular model of all the datasets are as follows: $\{G(M)|_{D_1}, G(M)|_{D_2}, \ldots, G(M)|_{D_p}\}$. And $G(M)|_{D_i}$ is associated with the level of information granularity ε_i. Then, the levels of information granularity from these datasets are: $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p$, which are aggregated by some aggregation operator agg to generate ε^* to form the granular model $G(M)$, and deliver the granular results. And the quality of the granular model is formed in the following formula

$$\begin{cases} V = V_1(\varepsilon^*) + V_2(\varepsilon^*) + \cdots + V_p(\varepsilon^*) \\ agg_{opt} = \arg\max_{A,\varepsilon^*} V \end{cases} \tag{33}$$

where agg_{opt} (the optimal agg) is from some family of operation A.

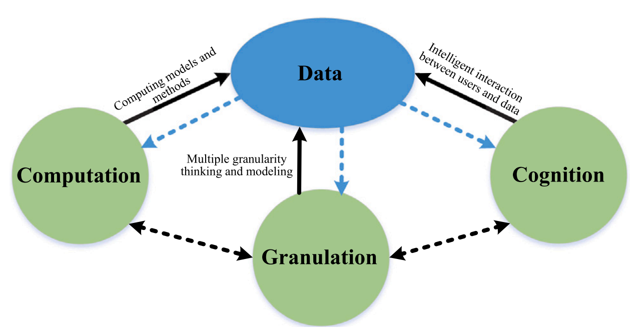

Fig. 21. Relationships in data-driven granular cognitive computing.

4. Applications of GrC in decision-making

In this section, the applications of decision-making with GrC are discussed, as listed in Table 4. Decision-making with GrC comprises an optimal allocation of information granularity model, multiple granular linguistic decision-making, cloud model, and extensions of granular decision models. The proposed optimal allocation of information granularity models is mainly developed to obtain a group consensus. We must first review the importance of multi-granular linguistic information in decision-making and the most effective and widespread approaches to managing this type of information in decision problems. The goal of this section is to provide a comprehensive review of the applications (recently published in the specialized literature) in which multi-granular linguistic information plays a key role in the resolution process. Remarkable, multi-granular linguistic information has been applied to a wide variety of decision-making problems.

We collected 560 publications from the Core Collection of the WoS database through CiteSpace software (citespace.podia.com). Based on

Table 4
Applications of decision-making methods realized with GrC.

Classification	Usage/Methodology	Reference
Models with an optimal allocation of information granularity	Group consistency optimization	[5,63,66]
	Group consensus optimization	[7,64,65,112]
	System modeling	[40,52,56]
	Clustering algorithm	[31,44,48,58,113]
	Others	[25,53,54]
Multiple granular linguistic decision-making	Consensus-reaching processes	[24,80,114]
	Environmental assessment	[115,116]
	GDM	[80–82,117–119]
	Linguistic modeling	[81,120]
	Decision support systems	[121,122]
	Evaluation processes	[123–127]
	Emergency decision-making	[128,129]
	Classification	[130]
	Human resources	[131–133]
Granularity-driven three-way decisions	Cost-sensitive approach	[91,134]
	Rough sets	[14,96,135–139]
	Others	[23,140,141]
Cloud model	Investment management	[22,23,101]
Data-driven GNN	System modeling	[24,26,27]
Extensions of granular decision models	Linguistic system modeling	[103]
	Classification	[104]
	Time-series forecasting	[25]

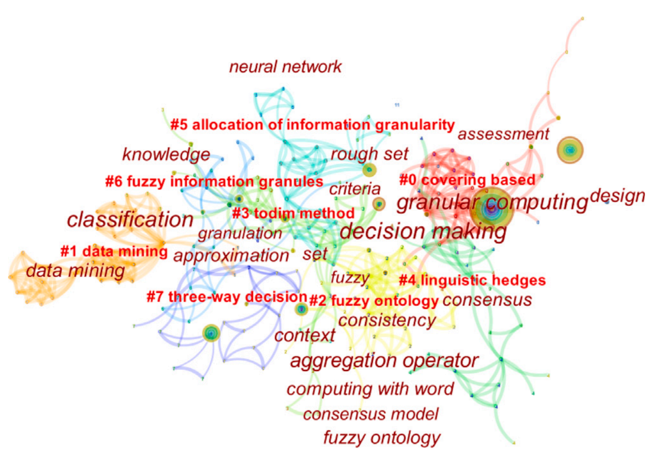

Fig. 22. Clusters of applications of GrC in decision-making.

the retrieval formula: Topic = "granular computing" AND Topic = "decision making" AND Topic = "application", Fig. 22 shows clusters of the major granular decision-making applications, e.g., the research on data mining, three-way decisions, and the allocation of information granularity. In Table 4, the applications range from environmental analysis to human resources, involving GDM, consensus-reaching processes, and so on. Among these, three-way decision-making can be used in various applications, including new product development, medical management, supplier selection, subsidy recipient selection, and numeric experiments. In addition, existing cloud models with MCDM can be employed in investment management. Finally, some extensions of granular decision models are implemented in the field of linguistic system modeling, classification, and time-series forecasting.

Furthermore, to obtain research covering state-of-the-art granular decision-making, we searched the target topic in the WoS database on June 29, 2022. More specifically, we input the keywords "granular decision making" to search the titles of the published papers in journals. As a result, 63 relevant papers were found, which were published in 35 journals. The journals in which more contributions have been published were Applied Soft Computing (6 articles), Expert Systems with Applications (6), Information Sciences (6), Journal of Intelligent Fuzzy Systems (5), Computers Industrial Engineering (3), IEEE Transactions on Fuzzy Systems (3), Knowledge-based Systems (2), European Journal of Operation Research (2), and Fuzzy Sets and Systems (2). Fig. 23 illustrates the distribution of these contributions (see Fig. 24).

In addition, Fig. 25 displays the temporal distribution of the corresponding research, showing that the majority of papers were published between 2018 and 2022, which indicates that granular decision-making has been a hot topic in recent years. Moreover, Table 5 presents the six most highly cited papers.

5. Challenges and trends of GrC in decision-making

In this section, we present the challenges and future development trends of GrC in decision-making area.

5.1. Challenges of GrC in decision-making

Nowadays, GrC is used extensively in data science and computational intelligence. However, related studies in decision-making, especially in data-driven decision-making, are still limited. Based on the previous overview, some challenges exist in this promising and interesting area:

(1) What is the essence of GrC? Even though many studies have discussed this fundamental question over recent decades, no recognized definition of GrC is available. A popular definition of GrC is that it is a general framework of information granules, such as fuzzy sets, rough sets, shadow sets, soft sets, and probability density functions. However, if an information granularity is reduced to a special form, for example, fuzzy sets, we cannot find corresponding theoretical results (e.g., decomposition theorem, representation theorem). In other words, GrC is not just a general form of various uncertainty sets. In addition, no current standardized granular transformation method can fuse the different levels of granularity information. Therefore, real-world decision-making problems may require a non-standard approach.

(2) In the current studies, coverage and specificity are two critical measures that characterize information granules in the granular decision-modeling process. Furthermore, the principle of justifiable information granularity and the development of granular models through an optimal allocation of information granularity are key design tools. The idea is suitable, and the model is easy to

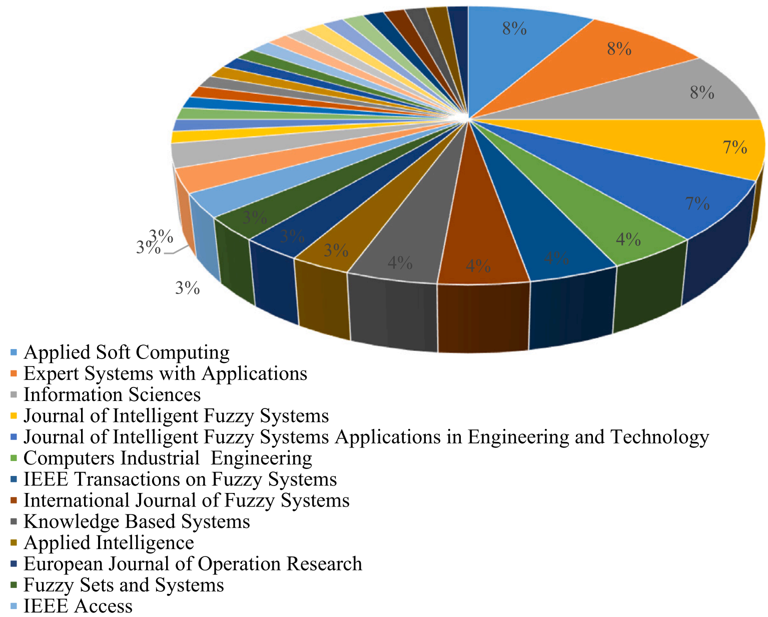

Fig. 23. Journal distribution on the topic of granular decision-making.

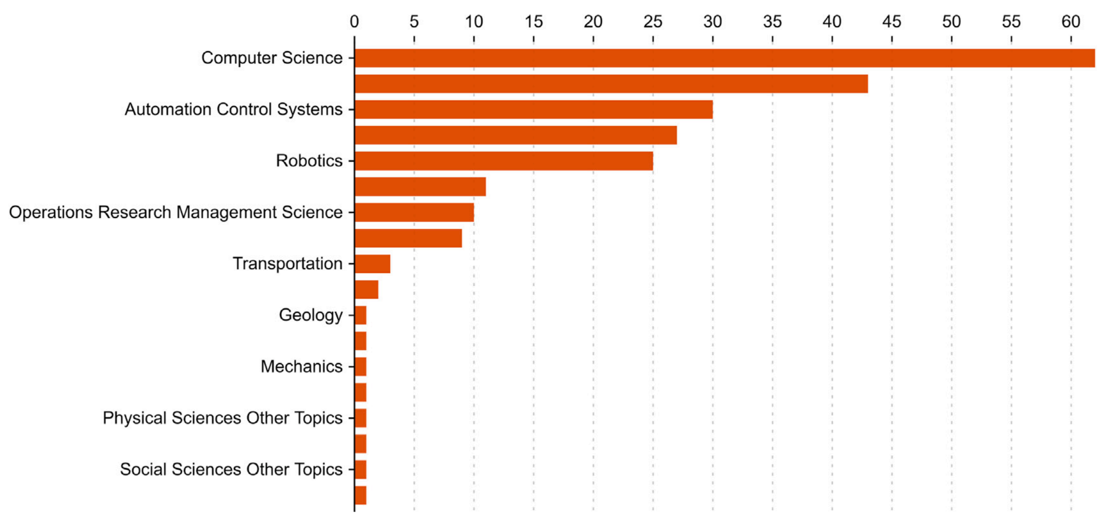

Fig. 24. Detailed distribution of applications of granular decision-making methods.

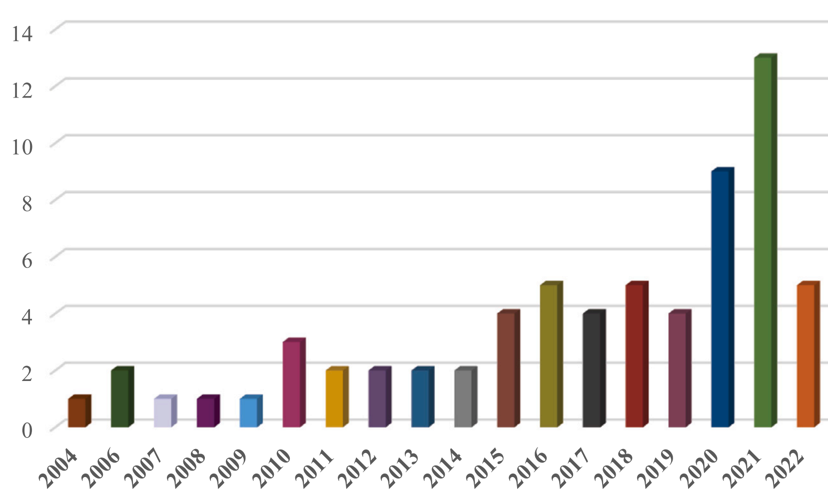

Fig. 25. Temporal distribution of the related research on granular decision-making.

Table 5
Corresponding highly cited papers.

Title	Reference	Journal	No. of citations	Year
On multi-granular fuzzy linguistic modeling in group decision-making problems: A systematic review and future trends	Morente-Molinera et al. [81]	Knowledge-based Systems	206	2015
A method based on PSO and GrC of linguistic information to solve group decision-making problems defined in heterogeneous contexts	Cabrerizo et al. [64]	European Journal of Operation Research	195	2013
Building consensus in group decision making with an allocation of information granularity	Cabrerizo et al. [7]	Fuzzy Sets and Systems	168	2014
Consensus-based group decision making under multi-granular unbalanced 2-tuple linguistic preference relations	Dong et al. [80]	Group Decision and Negotiation	166	2015
Consensus-reaching for group decision making with multi-granular unbalanced linguistic information: A bounded confidence and minimum adjustment-based approach	Zhang et al. [142]	Information Fusion	46	2021
Two-sided matching decision making with multi-granular hesitant fuzzy linguistic term sets and incomplete criteria weight information	Zhang et al. [143]	Expert Systems with Applications	46	2021

handle. However, for a large-scale data-driven decision-making problem, these two measures and a single objective optimization model are not sufficient. Therefore, new measures and extension models are worth studying. In particular, an improved optimal allocation of information granularity and justifiable principle are two valuable research entry points to enrich granular decision-modeling to expand on its flexibility and robustness.

(3) Another challenge is constructing an information granule from data. Recent studies have considered this concern and obtained some valuable results. However, the key issue is that most studies lack strict theoretical justification. In other words, the models and methods are somewhat rough. In some situations of real decision problems, decision-makers (or experts) are unable to fully understand the information granule and how to generate it from data. Clustering is a possible way to construct an information granule, and some advanced clustering analysis methods based on machine learning or deep learning may easily solve this problem.

(4) What is the relationship between GrC and data-driven decision-making? Clearly, this is a crucial question for this overview paper. To some extent, a better understanding of information in decision-making from the perspective of an information granule is needed. From our point of view, the information granule is a key component of knowledge representation and processing, and the granulation level of information particles is critical to the problem description and the solution to the global strategy. Of note, no universal level of granulation exists, the size of which is entirely determined by the problem characteristics and decision-maker preferences. Therefore, when dealing with complex problems, the decision-maker needs to focus on the core issues and flexibly choose a certain level for decision analysis. How to choose such a specific level is a significant challenge facing the current research.

5.2. Future trends of GrC in decision-making

The following points out potential future research directions:

(1) One research option is to establish flexible mechanisms of transformation from data to information granules. In other words, we need to learn how to effectively transform an uncertainty object (e.g., intervals, fuzzy sets, shadowed sets, probability density function) to a general form of information granule. No doubt, this is a fundamental problem in GrC in decision-making. If this problem is addressed, we can find a new way to study the CWW theory in depth.

(2) Another approach is to develop simplified operation rules for GrC. An effective way of operating an information granule is a notable challenge in GrC. Most studies related to this topic are too complex and difficult to use in practice. Therefore, a family of simplified operation rules for GrC needs to be developed. As a result, the type-2 fuzzy computational complexity can be addressed easily.

(3) In current studies in GrC (mainly involving fuzzy sets) with the aid of GDM, most scholars focus on consensus-reaching problems (CRPs), and a variety of models are proposed on this topic. However, little attention has been paid to social networks, behavior factors, and decision-maker utilities. GDM problems with a utility function and new stochastic dominance (say almost stochastic dominance) are worth studying in the near future. Besides, different DMs may adopt various information granules to model the data based on individual habits, which requires to discuss how to obtain personalized information granules and fuse them.

(4) For a large-scale GDM problem, multi-agent game behaviors are present. As a result, we anticipate that the methodologies in this area can be GrC (data-driven) + game theory + cybernetics. Thus, we can establish a granular linguistic dynamics system as a possible way to study the granular decision-making problem in depth.

(5) MCDM requires robust granular optimization, which will be useful in solving an uncertainty decision-making problem. Yet, little attention has been paid to GrC with the aid of MCDM. Data-driven distributed robust optimization is a promising direction in granular decision-making. Furthermore, in the evaluation process of MCDM, alternatives' objective information and experts' subjective preference should be considered simultaneously. There is also necessary to establish the twin data and knowledge-driven granular decision-making model to solve MCDM problems.

(6) In some cases, much data about decision situations are not numerical but ordinal or nominal. Even if ordinal or nominal data are number-coded, they cannot be processed using arithmetic operations. Because this information is very difficult to acquire, the output of these methods is hardly controllable. Additionally, under some complex actual scenes, the data may be incomplete or sparse, which should be discussed to formulate the corresponding information granules. These are open issues in data-driven MCDM problems.

6. Concluding remarks

GrC arises as a sound asset in the area of data-driven decision-making. In this paper, we provide a comprehensive overview of GrC in decision-making. First, basic definitions of GrC are introduced. Then, we review typical decision-making realized with the aid of GrC, such as an optimal allocation of information granularity decision models, multiple granularity linguistic decision models, and three-way decision-making. Finally, open problems and possible directions for future research are formulated. According to the overview, we draw the following conclusions:

(1) The current studies of decision-making with GrC lack a theoretical foundation and normative research paradigm. Most focused on specific uncertainty sets, which do not represent the GrC methodology in decision-making. Therefore, an axiomatization system for GrC to aid decision-making should be established.
(2) Data-driven decision-making presents a coherent framework using concepts like information granules and levels or hierarchies of information for dealing with knowledge acquired and processed at different levels of abstraction. Therefore, we claim that integrating knowledge-based and data-driven paradigms offers a new and prospective research way in GrC.
(3) Several challenges exist in real-time, dynamic, large-scale, and complex decision-making problems. Harnessing the advantages of the GrC principle, along with integrating artificial intelligence, machine learning, and GrC techniques with decision-making methods to solve real problems, is crucial. As a result, some of these challenges in practical decision-making problems can be addressed.

Based on the overview and conclusions, we provide open problems that may be solved in the near future:

(1) Future work can include decomposing a complex decision problem into sub-problems of different levels and designing an information granule by data granulation to obtain multi-level and multi-semantic information granules.
(2) New measures for granular decision-modeling need to be proposed. In other words, besides coverage and specificity, we need to develop other measures to evaluate the rationality of the granular decision model. In addition, advanced and improved granular decision-making models, including algorithms and solutions (especially closed-form analytical solutions), such as an extension of the optimal allocation of information granularity model, are needed.
(3) The development of new GrC decision models based on a robust optimization technique. Since complex decision problems involve various uncertainties, robust optimization can be a powerful solution. Therefore, a systemic and in-depth study of the granular multiple-criteria decision models with robust linear optimization approaches should be developed. Another direction to consider is a distributed granular robust optimization decision model in the dynamic decision problem, which is considered a very promising direction.
(4) GrC in decision-making should proceed from practical problems and adhere to practice domain orientation. Therefore, the fields of application can be extended. Integrating GrC with artificial intelligence, machine learning, and advanced statistics method is not only possible but also necessary to better address more real management decision applications, such as recommender systems, and healthcare, etc.
(5) Finally, another potential problem worth studying is granular-based MCDM with the aid of stochastic dominance and its advanced extensions, such as almost stochastic dominance [134,144,145] and asymptotic stochastic dominance [146]. Moreover, Bayesian statistics decision-making models can also provide a good path to solving dynamic decision-making problems.

We expect this overview paper to inspire further research in the area. Data-driven decision-making with the aid of GrC can be enriched with theories and applications to address challenges in decision-making.

CRediT authorship contribution statement

Jindong Qin: Conceptualization, Methodology and model design, Writing – original draft, Writing – review & editing, Funding acquisition. **Luis Martínez:** Writing – original draft, Writing – review & editing. **Witold Pedrycz:** Writing – review & editing, Supervision. **Xiaoyu Ma:** Conceptualization, Methodology, Data curation, Writing – original draft. **Yingying Liang:** Methodology, Writing – review & editing.

Declaration of competing interest

The authors declared that they have no conflicts of interest to this work.

Data availability

No data was used for the research described in the article.

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