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On the lattice of fuzzy rough sets

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ABSTRACT

By the means of lower and upper fuzzy approximations we define quasiorders. Their properties are used to prove our main results. First, we characterize the pairs of fuzzy sets that form fuzzy rough sets w.r.t. a t-similarity relation θ on U, for certain t-norms and implicators. If U is finite or the range of θ and of the fuzzy sets is a fixed finite chain, we establish conditions under which fuzzy rough sets form lattices. We show that this is the case for the min t-norm and any S-implicator defined by an involutive negator and the max co-norm.

1. Introduction

Rough sets were introduced by Zdzisław Pawlak [23], by defining the lower and upper approximations of a (crisp) set based on a so-called indiscernibility relation of the elements. An overview of the first decade of development of rough sets can be found in [24], and some further research directions and applications are mentioned in [26]. Originally, Pawlak assumed that this relation is an equivalence, but later several other types of relations were also examined (see [28], [32], [15], [13] or [14], [12], [16], [17], [37]). For a relation $\varrho \subseteq U \times U$ and any element $u \in U$, denote $\varrho(u) := \{x \in U \mid (u,x) \in \varrho\}$, called the *$\varrho$-neighbourhood* of u, according to [28]. Now, for any subset $A \subseteq U$, the *lower approximation* of A is defined as

$$A_\varrho := \{x \in U \mid \varrho(x) \subseteq A\},$$

and the *upper approximation* of A is given by

$$A^\varrho := \{x \in U \mid \varrho(x) \cap A \neq \emptyset\}.$$

If ϱ is reflexive and transitive, i.e. it is a *quasiorder*, then the properties $A_\varrho \subseteq A \subseteq A^\varrho$ and $(A_\varrho)_\varrho = A_\varrho$, $(A^\varrho)^\varrho = A^\varrho$ hold for all $A \subseteq U$.

The rough sets induced by ϱ can be ordered w.r.t. the component-wise inclusion, and for an equivalence, or more generally, for a quasiorder ϱ, they form a complete distributive lattice with several particular properties, see e.g. [29] or [14].

If ϱ is reflexive and symmetric, i.e. it is a *tolerance*, then the rough sets induced by ϱ do not form a lattice in general (see [15]). The case when they form complete distributive lattices is characterized in [13]. We note that lower and upper approximations induced by tolerances were discussed first in [28], and rough sets defined by them were investigated in [32] and [15]. Another generalization of tolerance-based rough sets was proposed by Pawlak in [25], where the approximations are defined to be unions of tolerance

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neighbourhoods. A comparison of these two approaches can be found in [33], where the authors also examined a method using maximal tolerance cliques.

The notion of a fuzzy set was introduced by Lotfi Zadeh [38]. A fuzzy set is defined by a mapping $f : U \to [0,1]$. We say that f has a *finite range*, whenever the (crisp) set $\{f(x) \mid x \in U\}$ is finite. The collection of all fuzzy sets on U is denoted by $\mathcal{F}(U)$. Ordering any elements $f, g \in \mathcal{F}(U)$ as follows

$$f \le g \Leftrightarrow f(x) \le g(x), \text{ for all } x \in U,$$

we obtain a completely distributive (complete) lattice $\mathcal{F}(U)$. For any system $f_i \in \mathcal{F}(U)$, $i \in I$, its infimum and supremum are given by the formulas

$$\left(\bigwedge_{i\in I} f_i\right)(x) = \bigwedge_{i\in I} f_i(x); \left(\bigvee_{i\in I} f_i\right)(x) = \bigvee_{i\in I} f_i(x), \tag{1}$$

where $\bigwedge$ and $\bigvee$ denote the infimum and the supremum, respectively, in the complete lattice $([0,1], \le)$.

The first step to integrating the two main theories relates to the works of del Cerro and Prade [2], Nakamura [22] and Dubois and Prade [5]. In [5] the fuzzy rough sets are defined as pairs $(\underline{f}, \overline{f}) \in \mathcal{F}(U) \times \mathcal{F}(U)$ of lower and upper approximations of the fuzzy sets $f \in \mathcal{F}(U)$. These fuzzy approximations were defined using a similarity relation, the t-norm min and t-conorm max. Their approach was generalized in several papers, like [3,8,10,11,27,30,35,36] and [4], where fuzzy rough sets are defined on the basis of different t-norms (or conjunctors) and related implicators. A detailed study of these approximation operators was developed in [4], [30] and in [19], [31], where the structure of the lower and upper approximations of L-fuzzy sets is also investigated. An axiomatic approach to these properties was elaborated e.g. in [1], [18], [20], and [21]. In [9] it was shown that for crisp reference sets (i.e. for $f(x) \in \{0,1\}$, $\forall x \in U$) the fuzzy rough sets defined by a t-similarity relation θ with a well-ordered spectrum form a completely distributive lattice.

The goal of the present paper is to find conditions under which fuzzy rough sets form lattices. With this purpose, by the means of lower and upper fuzzy approximations, we define (crisp) quasiorders on U. The properties of these quasiorders and of the equivalences determined by them are discussed in Sections 3, 4 and 6. These properties will be used to prove our main results, Theorems 5.1 and 7.4. Section 2 contains the essential prerequisites of our study. In Section 5, by using singleton equivalence classes, we characterize the pairs of fuzzy sets that form a fuzzy rough set with respect to a t-similarity relation θ for certain t-norms and related implicators. In Section 7, we establish conditions under which fuzzy rough sets with a finite range form lattices. We consider the case when U is finite or the range of θ and of the fuzzy sets is a fixed finite chain $L \subseteq [0,1]$, and the rough fuzzy sets are defined by the means of the min t-norm and of an S-implicator based on the max t-conorm and an involutive negator. We show that under these conditions the rough fuzzy sets form a complete lattice.

2. Preliminaries

2.1. T-norms, implicators and fuzzy relations

A triangular norm $\odot$ (*t-norm* for short) is a commutative, associative and monotone increasing binary operation $\odot$ defined on $[0,1]$ satisfying $1 \odot x = x \odot 1 = x$, for all $x \in [0,1]$ (see e.g. [30]). The t-norm $\odot$ is called *(left) continuous* (see [6]), if it is (left) continuous as a function $\odot : [0,1]^2 \to [0,1]$ in the usual interval topology on $[0,1]^2$. Every t-norm $\odot$ satisfies $x \odot 0 = 0 \odot x = 0$, for all $x \in L$. The most known t-norms are:

- the *standard min operator*: $x \odot y := \min(x, y)$;
- the *arithmetical product*: $x \odot y := x \cdot y$;
- the *Łukasiewicz t-norm* $x \odot y := \max(0, x + y - 1)$.

A *negator* is a decreasing map $n : [0,1] \to [0,1]$ with $n(0) = 1$ and $n(1) = 0$. n is called *involutive* if $n(n(x)) = x$, for all $x \in [0,1]$ (see e.g. [6]). The so-called *standard negator* $n(x) := 1 - x$, $x \in [0,1]$ is an involutive negator.

A triangular conorm $\oplus$ (shortly *t-conorm*, see [30]) is a commutative, associative and monotone increasing binary operation $\oplus$ defined on $[0,1]$, that satisfies $0 \oplus x = x \oplus 0 = x$, for all $x \in [0,1]$. The t-conorm $\oplus$ is *(left) continuous* [6], if it is (left) continuous as a function $\oplus : [0,1]^2 \to [0,1]$ in the usual topology.

Given an involutive negator n, a t-norm $\odot$ and a t-conorm $\oplus$, we say that $\odot$ and $\oplus$ form an *n-dual* pair [30] if for all $x, y \in [0,1]$

$$n(x \oplus y) = n(x) \odot n(y).$$

Clearly, this identity also implies the identity

$$n(x \odot y) = n(x) \oplus n(y).$$

For instance, $\min(x, y)$, $\max(x, y)$ form a well-known n-dual pair w.r.t. any involutive negator on $[0,1]$.

An *implicator* is a binary operation (mapping) $\rhd : [0,1]^2 \to [0,1]$ that is decreasing in the first and increasing in the second argument and that satisfies the boundary conditions

$$0 \rhd 0 = 0 \rhd 1 = 1 \rhd 1 = 1 \text{ and } 1 \rhd 0 = 0.$$

$\rhd$ is called a *border implicator* if $1 \rhd x = x$ holds for all $x \in [0,1]$. There are two important classes of border implicators (see e.g. [4]). One of them is the *R-implicator* based on a t-norm $\odot$ is defined by

$$x \rhd y := \bigvee \{z \in [0,1] \mid x \odot z \leq y\}, \text{ for all } x, y \in [0,1].$$

If $\odot$ is a continuous t-norm, then the algebra $([0,1], \vee, \wedge, \odot, \rhd, 0, 1)$ is a so-called *commutative (integral) residuated lattice* (see [7]). Let $\oplus$ be a t-conorm and n a negator on $[0,1]$, then the *S-implicator* based on them is defined by

$$x \rhd y := n(x) \oplus y$$

The *Łukasiewicz implicator* $\rhd_L$ is both an R-implicator and an S-implicator defined by $x \rhd_L y := \min(1, 1 - x + y)$, $\forall x \in [0,1]$. The *Kleene-Dienes (KD) implicator* $\rhd_{KD}$ is an S-implicator given by $x \rhd_{KD} y := \max(1 - x, y)$, $\forall x \in [0,1]$. If $\rhd$ is an implicator, then a corresponding negator is defined by $n(x) = x \rhd 0$. If n is an involutive negator and $\rhd$ is an R-implicator defined by a left-continuous t-norm, then $\rhd$ is called an *ITML-implicator* (see [4]).

A *fuzzy binary relation* on U is a fuzzy set $\theta: U \times U \to [0,1]$ (see e.g. [5]). The pair (U, θ) is usually called a *fuzzy approximation space*. θ is called *reflexive* if $\theta(x,x) = 1$ for all $x \in U$, and it is called *symmetric* if for all $x, y \in U$, $\theta(x,y) = \theta(y,x)$. Given a t-norm $\odot$, the relation θ is called $\odot$-*transitive* if

$$\theta(x,y) \odot \theta(y,z) \leq \theta(x,z)$$

holds for every $x, y, z \in U$. If a relation θ is reflexive and $\odot$-transitive, then it is called a (fuzzy) $\odot$-*quasiorder*. A symmetric $\odot$-quasiorder θ is called a (fuzzy) $\odot$-*similarity relation*. When $x \odot y = \min(x,y)$, then θ is simply named a *similarity relation*. Since the minimum t-norm is the largest t-norm, a similarity relation is always $\odot$-transitive for any t-norm $\odot$. We say that θ is of a *finite range*, if the (crisp) set $\{\theta(x,y) \mid x, y \in U\}$ of its values is finite.

2.2. Fuzzy rough sets

Let (U, θ) be a fuzzy approximation space with a relation $\theta: U \times U \to [0,1]$. The precise notion of a fuzzy rough set was introduced by D. Dubois and H. Prade in [5]. For any fuzzy set $f \in \mathcal{F}(U)$ they defined its *lower approximation* $\underline{\theta}(f)$ and its *upper approximations* $\overline{\theta}(f)$ *relative to a similarity relation* θ by the formulas

$$\underline{\theta}(f)(x) := \bigwedge \{\max(1 - \theta(x,y), f(y)) \mid y \in U\}, \text{ for all } x \in U;$$
$$\overline{\theta}(f)(x) := \bigvee \{\min(\theta(x,y), f(y)) \mid y \in U\}, \text{ for all } x \in U.$$

The *fuzzy rough set of* f is identified by the pair $(\underline{\theta}(f), \overline{\theta}(f)) \in \mathcal{F}(U) \times \mathcal{F}(U)$ (see [5]). This definition was generalized in several papers. Here we will use the approach based on implicators and t-norms from [4] and [30]. Hence, in what follows, let $\odot$ be a t-norm and $\rhd$ a border implicator on $[0,1]$.

Definition 2.1. If (U, θ) is a fuzzy approximation space, then for any fuzzy set $f \in \mathcal{F}(U)$ its *fuzzy lower approximation* $\underline{\theta}(f)$ and its *fuzzy upper approximation* $\overline{\theta}(f)$ are defined as follows:

$$\underline{\theta}(f)(x) := \bigwedge \{\theta(x,y) \rhd f(y) \mid y \in U\}, \text{ for all } x \in U. \tag{2}$$
$$\overline{\theta}(f)(x) := \bigvee \{\theta(x,y) \odot f(y) \mid y \in U\}, \text{ for all } x \in U. \tag{2'}$$

The pair $(\underline{\theta}(f), \overline{\theta}(f)) \in \mathcal{F}(U) \times \mathcal{F}(U)$ is called a *fuzzy rough set in* (U, θ).

This definition also includes the one of Dubois and Prade, where $\odot$ is the min t-norm and $\rhd$ is the Kleene-Dienes implicator $x \rhd_{KD} y = \max(1 - x, y)$.

Notice that $\underline{\theta}$ and $\overline{\theta}$ are *order-preserving* operators, i.e. $f \leq g$ implies $\underline{\theta}(f) \leq \underline{\theta}(g)$ and $\overline{\theta}(f) \leq \overline{\theta}(g)$. In addition, if θ is a reflexive fuzzy relation, then $\underline{\theta}(f) \leq f \leq \overline{\theta}(f)$ holds for all $f \in \mathcal{F}(U)$ (see e.g. [4] or [30]) The following properties will have a special importance in our proofs:

(D) Let $\odot$ be such a left-continuous t-norm that its induced R-implicator $\rhd$ is an ITML implicator, i.e. $n(x) := x \rhd 0$, $x \in U$ is an involutive negator, or let n be an involutive negator, $\oplus$ a t-conorm n-dual to $\odot$ and $\rhd$ the S-implicator defined by them (i.e. $x \rhd y = n(x) \oplus y$). Then $n(\overline{\theta}(f)) = \underline{\theta}(n(f))$ and $n(\underline{\theta}(f)) = \overline{\theta}(n(f))$ (see e.g. [4], [19] or [30]).

(ID) Let $\odot$ be a left-continuous t-norm and $\rhd$ the R-implicator induced by it, or n an involutive negator, $\oplus$ a t-conorm n-dual to $\odot$ and $\rhd$ the S-implicator corresponding to them. If θ is $\odot$-transitive, then for any $f, g \in \mathcal{F}(U)$ we have $\overline{\theta}(\overline{\theta}(f)) = \overline{\theta}(f)$ and $\underline{\theta}(\underline{\theta}(g)) = \underline{\theta}(g)$ (see [4], [19], [30]). In other words, for $F = \overline{\theta}(f)$ and $G = \underline{\theta}(g)$ we have $F = \overline{\theta}(F)$ and $G = \underline{\theta}(G)$.

Lemma 2.2. *Let (U, θ) be a fuzzy approximation space such that the relation θ is of a finite range. If $f \in \mathcal{F}(U)$ has a finite range, then the fuzzy sets $\underline{\theta}(f)$ and $\overline{\theta}(f)$ are also of a finite range.*

Proof. Since $\{\theta(x,y) \mid x,y \in U\}$ and $\{f(y) \mid y \in U\}$ are finite sets, their Cartesian product $\{(\theta(x,y), f(y)) \mid x,y \in U\}$ is finite, hence the sets $\mathcal{C} = \{\theta(x,y) \odot f(y) \mid x,y \in U\}$ and $\mathcal{I} = \{\theta(x,y) \rhd f(y) \mid x,y \in U\}$ are also finite. In particular, this means that the sets $\mathcal{C}$ and $\mathcal{I}$ have finitely many (different) subsets of the form $\{\theta(x,y) \rhd f(y) \mid y \in U\}$ and $\{\theta(x,y) \odot f(y) \mid y \in U\}$, and this immediately implies that both $\underline{\theta}(f)$ and $\overline{\theta}(f)$ have finitely many different values, i.e. they have finite ranges. □

3. Quasiorders induced by lower and upper approximations

In what follows, suppose that conditions in (ID) hold and $n(x) := x \rhd 0$. For any $f, g \in \mathcal{F}(U)$, denote $F = \overline{\theta}(f)$ and $G = \underline{\theta}(g)$. Using F and G we define two binary relations $R(F)$ and $\varrho(G)$ on U as follows:

Definition 3.1. Let (U, θ) be a fuzzy approximation space, $a, b \in U$ and $F = \overline{\theta}(f)$, $G = \underline{\theta}(g)$. Then
(i) $(a,b) \in R(F) \Leftrightarrow F(a) = \theta(a,b) \odot F(b)$;
(ii) $(a,b) \in \varrho(G) \Leftrightarrow G(a) = \theta(a,b) \rhd G(b)$.

Proposition 3.2. *(i) $(a,b) \in R(F)$ implies $F(a) \le \theta(a,b)$, and $(a,b) \in \varrho(G)$ implies $G(a) \ge n(\theta(a,b))$.*
(ii) If θ is reflexive, then for any $f, g \in \mathcal{F}(U)$, $R(F)$ and $\varrho(G)$ are reflexive.
(iii) If θ is a $\odot$-quasiorder, then $R(F)$, $\varrho(G)$ are crisp quasiorders, and
$F(a) \ge \theta(a,y) \odot F(y)$, $G(a) \le \theta(a,y) \rhd G(y)$, for any $a, y \in U$.
(iv) If $n(x)$ is involutive, then $R(F) = \varrho(n(F))$, and $\varrho(G) = R(n(G))$.
(v) Let $\odot$ be the minimum t-norm, n an involutive negator, and $x \rhd y := \max(n(x), y)$. If θ is a similarity relation, then $(a,b) \in R(F) \Leftrightarrow F(a) \le \theta(a,b)$ and $(a,b) \in \varrho(G) \Leftrightarrow G(a) \ge n(\theta(a,b))$.

Proof. (i) By definition $(a,b) \in R(F)$ implies $F(a) = \theta(a,b) \odot F(b) \le \theta(a,b)$ and $(a,b) \in \varrho(G)$ yields $G(a) = \theta(a,b) \rhd G(b) \ge \theta(a,b) \rhd 0 = n(\theta(a,b))$.

(ii) If θ is reflexive, then $\theta(a,a) = 1$ implies $F(a) = \theta(a,a) \odot F(a)$ and $G(a) = \theta(a,a) \rhd G(a)$, i.e. $(a,a) \in R(F)$ and $(a,a) \in \varrho(G)$ hold for all $a \in U$. Thus $R(F)$ and $\varrho(G)$ are reflexive.

(iii) Let θ be a $\odot$-quasiorder. Then $R(F)$, $\varrho(G)$ are reflexive, property (ID) holds, and hence $F(a) = \overline{\theta}(F)(a)$, $G(a) = \underline{\theta}(G)(a)$ imply $F(a) = \bigvee\{\theta(x,y) \odot F(y) \mid y \in U\} \ge \theta(a,y) \odot F(y)$ and $G(a) = \bigwedge\{\theta(x,y) \rhd G(y) \mid y \in U\} \le \theta(a,y) \rhd G(y)$, $\forall y \in U$. Take $a, b, c \in U$ with $(a,b), (b,c) \in R(F)$. Then $F(a) = \theta(a,b) \odot F(b)$ and $F(b) = \theta(b,c) \odot F(c)$ imply $F(a) = (\theta(a,b) \odot \theta(b,c)) \odot F(c) \le \theta(a,c) \odot F(c)$, because θ is $\odot$-transitive. Now $F(a) \ge \theta(a,c) \odot F(c)$ yields $F(a) = \theta(a,c) \odot F(c)$, i.e. $(a,c) \in R(F)$. Thus $R(F)$ is also transitive, hence it is a quasiorder.

Let $(a,b), (b,c) \in \varrho(G)$. Then $G(a) = \theta(a,b) \rhd G(b)$ and $G(b) = \theta(b,c) \rhd G(c)$ imply $G(a) = \theta(a,b) \rhd (\theta(b,c) \rhd G(c))$. If $\rhd$ is an R-implicator, then $\theta(a,b) \rhd (\theta(b,c) \rhd G(c)) = (\theta(a,b) \odot \theta(b,c)) \rhd G(c)$. If $\rhd$ is an S-implicator $x \rhd y = n(x) \oplus y$, then $\theta(a,b) \rhd (\theta(b,c) \rhd G(c)) = n(\theta(a,b)) \oplus (n(\theta(b,c)) \oplus G(c)) = (n(\theta(a,b)) \oplus n(\theta(b,c)) \oplus G(c) = n(\theta(a,b) \odot \theta(b,c)) \oplus G(c) = (\theta(a,b) \odot \theta(b,c)) \rhd G(c)$. Hence in both cases $G(a) = (\theta(a,b) \odot \theta(b,c)) \rhd G(c)$. Because θ is $\odot$-transitive (and $\rhd$ is decreasing in the first variable) we get $G(a) \ge \theta(a,c) \rhd G(c)$. Then $G(a) \le \theta(a,c) \rhd G(c)$ yields $G(a) = \theta(a,c) \rhd G(c)$, i.e. $(a,c) \in \varrho(G)$. Thus $\varrho(G)$ is a $\odot$-quasiorder.

(iv) Observe, that in this case property (D) holds, i.e., $n(\overline{\theta}(f)) = \underline{\theta}(n(f))$. This yields $n(F) = \underline{\theta}(n(f))$, and $(a,b) \in R(F)$ means $F(a) = \theta(a,b) \odot F(b)$. As n is involutive, this is equivalent to $n(F(a)) = n(\theta(a,b) \odot F(b))$. If $\rhd$ is an ITML implicator, then $n(\theta(a,b) \odot F(b)) = (\theta(a,b) \odot F(b)) \rhd 0 = \theta(a,b) \rhd (F(b) \rhd 0) = \theta(a,b) \rhd n(F(b))$. If $\rhd$ is an S-implicator, then $n(\theta(a,b) \odot F(b)) = n(\theta(a,b)) \oplus n(F(b)) = \theta(a,b) \rhd n(F(b))$. Hence in both cases $n(F(a)) = n(\theta(a,b) \odot F(b)) \Leftrightarrow n(F)(a) = \theta(a,b) \rhd n(F)(b)$. The right side means $(a,b) \in \varrho(n(F))$. Thus we get $(a,b) \in R(F) \Leftrightarrow (a,b) \in \varrho(n(F))$, proving $R(F) = \varrho(n(F))$.

Let $g = n(h)$ for some $h \in \mathcal{F}(U)$. Then $n(G) = n(\underline{\theta}(g)) = \overline{\theta}(n(g)) = \overline{\theta}(h)$, and $G = n(\overline{\theta}(h))$. Hence $(a,b) \in \varrho(G) \Leftrightarrow (a,b) \in \varrho(n(\overline{\theta}(h))) \Leftrightarrow (a,b) \in R(\overline{\theta}(h)) = R(n(G))$, and this proves $\varrho(G) = R(n(G))$.

(v) In view of (i) $(a,b) \in R(F)$ yields $F(a) \le \theta(a,b)$ and $(a,b) \in \varrho(G)$ implies $G(a) \ge n(\theta(a,b))$. We need only to prove the converse implications. Let $F(a) \le \theta(a,b)$. Since θ is also a $\odot$-quasiorder, in view of (iii) $F(b) \ge \theta(b,a) \odot F(a) = \min(\theta(b,a), F(a)) = \min(\theta(a,b), F(a)) = F(a)$. Hence $F(a) \le \min(\theta(a,b), F(b))$. As $F(a) \ge \theta(a,b) \odot F(b) = \min(\theta(a,b), F(b))$ also holds, we get $F(a) = \min(\theta(a,b), F(b))$, i.e. $(a,b) \in R(F)$.

Now let $G(a) \ge n(\theta(a,b))$. As $\theta(b,a) = \theta(a,b)$, and by (iii), $G(b) \le \theta(b,a) \rhd G(a) = \max(n(\theta(a,b)), G(a)) = G(a)$, we get $G(a) \ge \max(n(\theta(a,b), G(b))$. Since $G(a) \le \theta(a,b) \rhd G(b) = \max(n(\theta(a,b)), G(b))$ also holds, we obtain $G(a) = \max(n(\theta(a,b)), G(b)) = \theta(a,b) \rhd G(b)$, i.e. $(a,b) \in \varrho(G)$. □

Corollary 3.3. *If the conditions in Proposition 3.2(v) are satisfied, then $(a,b) \notin R(F) \Leftrightarrow F(a) > \theta(a,b)$ and $(a,b) \notin \varrho(G) \Leftrightarrow G(a) < n(\theta(a,b))$.*

Proposition 3.4. *Let θ be a $\odot$-quasiorder and $F = \overline{\theta}(f)$, $G = \underline{\theta}(g)$, for some $f, g \in \mathcal{F}(U)$, and $a, b \in U$. The following hold true:*
(i) If $(a,b) \in R(F)$, then for any $h \in \mathcal{F}(U)$ with $h \le F$, $h(b) = F(b)$ implies $\overline{\theta}(h)(a) = F(a)$.
(ii) If $(a,b) \in \varrho(G)$, then for any $h \in \mathcal{F}(U)$ with $h \ge G$, $h(b) = G(b)$ implies $\underline{\theta}(h)(a) = G(a)$.

Proof. (i) By definition, we have $F(a) = \theta(a,b) \odot F(b)$. Hence we get:

$$\overline{\theta}(h)(a) = \bigvee \{\theta(a,y) \odot h(y) \mid y \in U\} \geq \theta(a,b) \odot h(b) = \theta(a,b) \odot F(b) = F(a)$$

On the other hand, $\overline{\theta}(h) \leq \overline{\theta}(F) = F$ implies $\overline{\theta}(h)(a) \leq F(a)$. Thus we obtain $\overline{\theta}(h)(a) = F(a)$.

(ii) Now, analogously we have $G(a) = \theta(a,b) \rhd G(b)$. Therefore, we get:

$$\underline{\theta}(h)(a) = \bigwedge \{\theta(a,y) \rhd h(y) \mid y \in U\} \leq \theta(a,b) \rhd h(b) = \theta(a,b) \rhd G(b) = G(a).$$

Now $\underline{\theta}(h) \geq \underline{\theta}(G) = G$ yields $\underline{\theta}(h)(a) \geq G(a)$, whence $\underline{\theta}(h)(a) = G(a)$. □

4. The equivalences induced by the quasiorders $R(F)$ and $\varrho(G)$

In this section we assume that conditions in (ID) are satisfied, i.e. that $\odot$ is a left-continuous t-norm, $\rhd$ is the R-implicator induced by it and $n(x) = x \rhd 0$, or n is an involutive negator, $\oplus$ is the t-conorm n-dual to $\odot$ and $\rhd$ is the S-implicator defined by them. We also suppose that (U,θ) is an approximation space with a $\odot$-similarity relation θ and $F = \overline{\theta}(f)$, $G = \underline{\theta}(g)$, for some $f, g \in \mathcal{F}(U)$. Now, by Proposition 3.2(iii), $R(F)$, $\varrho(G) \subseteq U \times U$ are (crisp) quasiorders. It is known that for any quasiorder $q \subseteq U \times U$ the relation $\varepsilon_q := q \cap q^{-1}$ is an equivalence and q induces a *natural partial order* $\leq_q$ on the factor-set U/ε_q as follows: for any equivalence classes $A, B \in U/\varepsilon_q$ we say that $A \leq_q B$, whenever there exists $a \in A$ and $b \in B$ with $(a,b) \in q$. This is equivalent to the fact that $(x,y) \in q$ holds for all $x \in A$ and $y \in B$.

Thus we can introduce two equivalence relations $E(F)$ and $\varepsilon(G)$ as follows:

$$E(F) := R(F) \cap R(F)^{-1} \text{and } \varepsilon(G) := \varrho(G) \cap \varrho(G)^{-1}.$$

The corresponding (natural) partial orders on the factor-sets $U/E(F)$ and $U/\varepsilon(G)$ can be defined as follows:

For any $E_1, E_2 \in U/E(F)$, we have $E_1 \leq_{R(F)} E_2 \Leftrightarrow (a_1,a_2) \in R(F)$ for some $a_1 \in E_1$ and $a_2 \in E_2$, and for any $\mathcal{E}_1, \mathcal{E}_2 \in U/\varepsilon(G)$ we have $\mathcal{E}_1 \leq_{\varrho(G)} \mathcal{E}_2 \Leftrightarrow (b_1,b_2) \in \varrho(G)$ for some $b_1 \in \mathcal{E}_1$ and $b_2 \in \mathcal{E}_2$.

E is called a *maximal* $E(F)$ *class*, if it is a maximal element of the poset $(U/E(F), \leq_{R(F)})$ and $\mathcal{E}$ is a *maximal* $\varepsilon(G)$ *class* if it is maximal in $(U/\varepsilon(G), \leq_{\varrho(G)})$. The $E(F)$ and $\varepsilon(G)$ class of an $a \in U$ is denoted by $[a]_{E(F)}$ and $[a]_{\varepsilon(G)}$, respectively. In this section, we prove several properties of these classes used to characterize pairs of fuzzy sets that, together, form fuzzy rough sets.

Lemma 4.1. *The following assertions hold true:*
(i) If $E \subseteq U$ is an $E(F)$ class, then $F(a) = F(b) \leq \theta(a,b)$, for all $a, b \in E$;
(ii) If $\mathcal{E} \subseteq U$ is an $\varepsilon(G)$ class, then $G(a) = G(b) \geq n(\theta(a,b))$, for all $a, b \in \mathcal{E}$;
(iii) If $E \subseteq U$ is a maximal $E(F)$ class, then $\theta(a,z) \odot F(z) < F(a) = F(b) \leq \theta(a,b)$ and $\theta(a,z) < \theta(a,b)$, for all $a, b \in E$ and $z \notin E$;
(iv) If $\mathcal{E} \subseteq U$ is a maximal $\varepsilon(G)$ class, then $n(\theta(a,b)) \leq G(a) = G(b) < \theta(a,z) \rhd G(z)$ and $\theta(a,z) < \theta(a,b)$, for all $a, b \in \mathcal{E}$ and $z \notin \mathcal{E}$;
(v) Assume that $n(x) = x \rhd 0$ is involutive. Then the $E(F)$ classes and the $\varepsilon(n(F))$ classes are the same, and $E \subseteq U$ is a maximal $E(F)$ class if and only if it is also a maximal $\varepsilon(n(F))$ class.

Proof. (i) If $E \subseteq U$ is an $E(F)$ class, then $(a,b),(b,a) \in R(F)$ holds for any $a, b \in E$. Therefore, $F(a) = \theta(a,b) \odot F(b) \leq \theta(a,b)$, $F(b)$ and $F(b) = \theta(b,a) \odot F(a) \leq F(a)$. Thus we obtain $F(b) = F(a) \leq \theta(a,b)$.

(ii) If $\mathcal{E} \subseteq U$ is an $\varepsilon(G)$ class, then $(a,b),(b,a) \in \varrho(G)$ imply $G(a) = \theta(a,b) \rhd G(b) \geq 1 \rhd G(b) = G(b)$ and $G(b) \geq \theta(b,a) \rhd G(a) \geq 1 \rhd G(a) = G(a)$, for any $a, b \in \mathcal{E}$. Hence $G(a) = G(b)$. By Proposition 3.2(i) we obtain $G(a) = G(b) \geq n(\theta(a,b))$, for all $a, b \in \mathcal{E}$.

(iii) Let E be a maximal $E(F)$ class. Then $F(a) = \theta(a,b) \odot F(b)$, $F(b) = \theta(a,b) \odot F(a)$, and in view of (i), $F(a) = F(b) \leq \theta(a,b)$. We also have $F(a) \geq \theta(a,z) \odot F(z)$, according to Proposition 3.2(iii). As $E \nleq [z]_{E(F)}$ implies $(a,z) \notin R(F)$ for each $z \notin E$, we obtain $F(a) > \theta(a,z) \odot F(z)$, for all $z \notin E$. Now, suppose that $\theta(a,c) \geq \theta(a,b)$, for some $c \notin E$. Then $F(c) \geq \theta(c,a) \odot F(a) = \theta(a,c) \odot F(b) \geq \theta(a,b) \odot F(b) = F(a)$. This further yields $F(a) > \theta(a,c) \odot F(c) \geq \theta(a,b) \odot F(a) = F(b)$, a contradiction. Thus $\theta(a,z) < \theta(a,b)$, for each $z \notin E$.

(iv) If $\mathcal{E} \subseteq U$ is a maximal $\varepsilon(G)$ class, then $G(a) = \theta(a,b) \rhd G(b)$, $G(b) = \theta(a,b) \rhd G(a)$, and in view of (i), $n(\theta(a,b)) \leq G(a) = G(b)$, for all $a, b \in \mathcal{E}$ and $\mathcal{E} \nleq [z]_{\varepsilon(G)}$, for any $z \notin \mathcal{E}$. Now, by Proposition 3.2(iii), $G(a) \leq \theta(a,z) \rhd G(z)$, hence $(a,z) \notin \varrho(G)$ yields $G(a) < \theta(a,z) \rhd G(z)$. By way of contradiction, assume $\theta(a,c) \geq \theta(a,b)$, for some $c \notin E$. Then $G(c) \leq \theta(a,c) \rhd G(a) \leq \theta(a,b) \rhd G(b) = G(a)$. This further yields $G(a) < \theta(a,c) \rhd G(c) \leq \theta(a,b) \rhd G(a) = G(b)$, a contradiction again.

(v) If $n(x) = x \rhd 0$ is involutive, then property (D) means that $n(F) = n(\overline{\theta}(f)) = \underline{\theta}(n(f))$. Hence, relation $\varrho(n(F)) = R(F)$ is well defined, and $E(F) = R(F) \cap R(F)^{-1} = \varrho(n(F)) \cap \varrho(n(F))^{-1} = \varepsilon(n(F))$. Thus the equivalence classes of $E(F)$ and $\varepsilon(n(F))$ coincide. $R(F) = \varrho(n(F))$ also yields $\leq_{R(F)} = \leq_{\varrho(n(F))}$, i.e. the posets $(U/E(F), \leq_{R(F)})$ and $(U/\varepsilon(n(F)), \leq_{\varrho(n(F))})$ are the same. Therefore, the maximal $E(F)$ and $\varepsilon(n(F))$ classes coincide. □

Corollary 4.2. *Let E be an $E(F)$ class and $\mathcal{E}$ be an $\varepsilon(G)$ class such that $E \cap \mathcal{E} \neq \emptyset$. Then the following assertions hold:*
(i) If $E \subseteq U$ is a maximal $E(F)$ class and $\mathcal{E} \subseteq U$ is a maximal $\varepsilon(G)$ class, then $E \subseteq \mathcal{E}$ or $\mathcal{E} \subseteq E$ holds.
(ii) If θ is a similarity relation, then $(x,y) \in R(F)$ or $(y,x) \in \varrho(G)$ holds for all $x \in E$ and $y \in \mathcal{E}$.

Proof. Let $a \in E \cap \mathcal{E}$. (i) Assume that neither $E \subseteq \mathcal{E}$ nor $\mathcal{E} \subseteq E$ hold. Then there exist elements $b \in E \setminus \mathcal{E}$, $c \in \mathcal{E} \setminus E$. As $a, b \in E$ but $c \notin E$, in view of Lemma 4.1(iii) we have $\theta(a,c) < \theta(a,b)$. Similarly, $a, c \in \mathcal{E}$ and $b \notin \mathcal{E}$ imply $\theta(a,b) < \theta(a,c)$, a contradiction to the previous result.

(ii) If $\mathcal{E} \subseteq E$ or $E \subseteq \mathcal{E}$ then (ii) is clearly satisfied. Hence we may assume $\mathcal{E} \setminus E \neq \emptyset$ and $E \setminus \mathcal{E} \neq \emptyset$. Suppose that there exist $x \in E$ and $y \in \mathcal{E}$ with $(x,y) \notin R(F)$. We claim that $(y,x) \in \varrho(G)$. Assume by contradiction $(y,x) \notin \varrho(G)$. Since $x, a \in E$, $y \notin E$ and $y, a \in \mathcal{E}$, $x \notin \mathcal{E}$, in view of Lemma 4.1(iii) and (iv) we get $\theta(x,y) < \theta(x,a)$ and $\theta(x,y) = \theta(y,x) < \theta(y,a) = \theta(a,y)$. Thus we obtain $\theta(x,y) < \min(\theta(x,a), \theta(a,y)) \leq \theta(x,y)$, a contradiction. This proves $(y,x) \in \varrho(G)$. □

Proposition 4.3. *(i) If E_1, E_2 are different $E(F)$ classes with $E_1 \leq_{R(F)} E_2$, then for any $a_1 \in E_1$ and $a_2 \in E_2$ we have $F(a_1) < F(a_2)$.*
(ii) If $\mathcal{E}_1, \mathcal{E}_2$ are different $\varepsilon(G)$ classes with $\mathcal{E}_1 \leq_{\varrho(G)} \mathcal{E}_2$ then for any $b_1 \in \mathcal{E}_1$ and $b_2 \in \mathcal{E}_2$ we have $G(b_1) > G(b_2)$.

Proof. (i) Assume $E_1 \leq_{R(F)} E_2$. Then for any $a_1 \in E_1$ and $a_2 \in E_2$ we have $(a_1, a_2) \in R(F)$, i.e. $F(a_1) = \theta(a_1, a_2) \odot F(a_2) \leq F(a_2)$. Observe that $F(a_2) \neq F(a_1)$. Indeed, $F(a_2) = F(a_1)$ would imply $F(a_2) = \theta(a_1, a_2) \odot F(a_1) = \theta(a_2, a_1) \odot F(a_1)$, i.e. $(a_2, a_1) \in R(F)$, which means $E_2 \leq_{R(F)} E_1$. As $\leq_{R(F)}$ is a partial order, this would yield $E_1 = E_2$, a contradiction. Thus we deduce $F(a_1) < F(a_2)$.

(ii) Let $\mathcal{E}_1 \leq_{\varrho(G)} \mathcal{E}_2$. Then for any $b_1 \in \mathcal{E}_1$, $b_2 \in \mathcal{E}_2$ we have $(b_1, b_2) \in \varrho(G)$, which gives $G(b_1) = \theta(b_1, b_2) \rhd G(b_2) \geq G(b_2)$. We claim $G(b_1) > G(b_2)$. Indeed, $G(b_2) = G(b_1)$ would imply $G(b_2) = \theta(b_1, b_2) \rhd G(b_1) = \theta(b_2, b_1) \rhd G(b_1)$, i.e. $(b_2, b_1) \in \varrho(G)$, which would yield $\mathcal{E}_1 = \mathcal{E}_2$, a contradiction. □

Clearly, if each chain in the posets $(U/E(F), \leq_{R(F)})$ and $(U/\varepsilon(G), \leq_{\varrho(G)})$ is finite, then any element of them is less than or equal to a maximal element in the corresponding poset. By using this observation we deduce

Corollary 4.4. *Assume that the relation θ and the fuzzy sets $f, g \in \mathcal{F}(U)$ have a finite range, and let $F = \overline{\theta}(f)$, $G = \underline{\theta}(g)$. Then for any $E(F)$ class E, there exists a maximal $E(F)$ class E_M such that $E \leq_{R(F)} E_M$, and for any $\varepsilon(G)$ class $\mathcal{E}$, there is a maximal $\varepsilon(G)$ class $\mathcal{E}_M$ with $\mathcal{E} \leq_{\varrho(G)} \mathcal{E}_M$.*

Proof. If the above conditions hold, then the fuzzy sets F and G also have a finite range. Now let $\{E_i \mid i \in I\}$ be an arbitrary (nonempty) chain of $E(F)$ classes. In view of Proposition 4.3, for any $a_i \in E_i$, $i \in I$, the values $\{F(a_i) \mid i \in I\}$ also form a chain, and for $E_i \leq_{R(F)} E_j$, $E_i \neq E_j$ we have $F(a_i) < F(a_j)$, and vice versa. This means that the chains $\{E_i \mid i \in I\}$ and $\{F(a_i) \mid i \in I\}$ are order-isomorphic. Since F has a finite range, the chain $\{F(a_i) \mid i \in I\}$ has a finite length. Hence the chain $\{E_i \mid i \in I\}$ is also finite. As every chain in the poset $(U/E(F), \leq_{R(F)})$ is finite, any element E of it is less than or equal to a maximal element E_M of it, i.e. $E \leq_{R(F)} E_M$. The second statement is proved analogously. □

The importance of maximal classes in this case is shown by the following:

Proposition 4.5. *Suppose that θ and the fuzzy sets $f, g \in \mathcal{F}(U)$ have a finite range, and let $F = \overline{\theta}(f)$, $G = \underline{\theta}(g)$. Then the following assertions hold:*
(i) If $h \leq F$ for some $h \in \mathcal{F}(U)$ and for any maximal $E(F)$ class E_M there exists an element $u \in E_M$ with $h(u) = F(u)$, then $\overline{\theta}(h) = F$.
(ii) If $h \geq G$ for some $h \in \mathcal{F}(U)$ and for any maximal $\varepsilon(G)$ class $\mathcal{E}_M$ there exists an element $v \in \mathcal{E}_M$ with $h(v) = G(v)$, then $\underline{\theta}(h) = G$.

Proof. (i) Let $x \in U$ be arbitrary. As θ and f have finite ranges, in view of Corollary 4.4, there exists a maximal $E(F)$ class E_M such that $[x]_{E(F)} \leq_{R(F)} E_M$. Then $(x,y) \in R(F)$ for all $y \in E_M$. By assumption, there exists an element $u \in E_M$ with $h(u) = F(u)$. Since $h \leq F$ and $(x,u) \in R(F)$, in view of Proposition 3.4(i) we obtain $\overline{\theta}(h)(x) = F(x)$. This proves $\overline{\theta}(h) = F$.

(ii) is proved dually, by using Corollary 4.4 and Proposition 3.4(ii). □

Proposition 4.6. *Suppose that the relation θ and the fuzzy sets $f, g \in \mathcal{F}(U)$ have a finite range, and let $F = \overline{\theta}(f)$, $G = \underline{\theta}(g)$.*
(i) If E is a maximal $E(F)$ class, then for any $a \in E$ we have

$$F(a) = max\{f(y) \mid y \in E\}.$$

(ii) If $\mathcal{E}$ is a maximal $\varepsilon(G)$ class, then for any $a \in \mathcal{E}$ we have

$$G(a) = min\{g(y) \mid y \in \mathcal{E}\}.$$

Proof. (i) By definition, $F(a) = \overline{\theta}(f)(a) = \bigvee\{\theta(a,y) \odot f(y) \mid y \in U\}$. If $y \notin E$, then $(a,y) \notin R(F)$, because E is a maximal $E(F)$ class. This means that $F(a) = \theta(a,y) \odot F(y)$ is not possible, and hence $F(a) > \theta(a,y) \odot F(y)$, according to Proposition 3.2(iii). Since $f \leq \overline{\theta}(f) = F$, we obtain $F(a) > \theta(a,y) \odot f(y)$, for all $y \in U \setminus E$. As θ and f are of a finite range, the set $\{\theta(a,y) \odot f(y) \mid y \in U \setminus E\}$ has finitely many different elements, and hence $\bigvee\{\theta(a,y) \odot f(y) \mid y \in U \setminus E\} < F(a)$. This implies

$$F(a) = (\bigvee\{\theta(a,y) \odot f(y) \mid y \in E\}) \vee \left(\bigvee\{\theta(a,y) \odot f(y) \mid y \in U \setminus E\}\right) = \bigvee\{\theta(a,y) \odot f(y) \mid y \in E\}.$$

If $y \in E$, then $F(y) = F(a)$. As $\theta(a,y) \odot f(y) \leq f(y) \leq F(y) = F(a)$, we obtain:

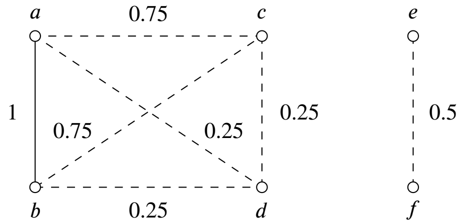

Fig. 1. The fuzzy similarity relation θ of Example 4.9.

Table 1
The fuzzy set h of Example 4.9.

u	a	b	c	d	e	f
$h(u)$	0	1	0.25	0.5	0.5	0.75
$F(u)$	1	1	0.75	0.5	0.5	0.75
$G(u)$	0	0	0.25	0.5	0.5	0.5

$$F(a) = \bigvee\{\theta(a,y) \odot f(y) \mid y \in E\} \leq \bigvee\{f(y) \mid y \in E\} \leq F(a).$$

This implies $F(a) = \bigvee\{f(y) \mid y \in E\}$. Because f has a finite range, the set $\{f(y) \mid y \in E\}$ is finite, and hence we can write $F(a) = \max\{f(y) \mid y \in E\}$.

(ii) By definition $G(a) = \underline{\theta}(g)(a) = \bigwedge\{\theta(a,y) \rhd g(y) \mid y \in U\}$. If $y \notin \mathcal{E}$, then $(a,y) \notin \varrho(G)$, because $\mathcal{E}$ is a maximal $\varepsilon(G)$ class, and hence $G(a) \neq \theta(a,y) \rhd G(y)$. Thus we have $G(a) < \theta(a,y) \rhd G(y)$, according to Proposition 3.2(iii). Since $G = \underline{\theta}(g) \leq g$, we obtain $G(a) < \theta(a,y) \rhd g(y)$, for all $y \in U \setminus \mathcal{E}$. As θ and g are of a finite range, the set $\{\theta(a,y) \rhd g(y) \mid y \in U \setminus E\}$ is finite, whence we get $G(a) < \bigwedge\{\theta(a,y) \odot g(y) \mid y \in U \setminus E\}$. This yields

$$G(a) = (\bigwedge\{\theta(a,y) \rhd g(y) \mid y \in \mathcal{E}\}) \wedge (\bigwedge\{\theta(a,y) \rhd g(y) \mid y \in U \setminus \mathcal{E}\}) = \bigwedge\{\theta(a,y) \rhd g(y) \mid y \in \mathcal{E}\}.$$

If $y \in \mathcal{E}$, then $G(y) = G(a)$. Since $\theta(a,y) \rhd g(y) \geq 1 \rhd g(y) = g(y) \geq G(y) = G(a)$, we obtain

$$G(a) = \bigwedge\{\theta(a,y) \rhd g(y) \mid y \in \mathcal{E}\} \geq \bigwedge\{g(y) \mid y \in \mathcal{E}\} \geq G(a),$$

and this implies $G(a) = \bigwedge\{g(y) \mid y \in \mathcal{E}\}$. Since $\{g(y) \mid y \in \mathcal{E}\}$ is a finite set, we can write: $G(a) = \min\{g(y) \mid y \in \mathcal{E}\}$. □

The following corollary is immediate:

Corollary 4.7. *Assume that θ and $f, g \in \mathcal{F}(U)$ are of a finite range, and let $a \in U$ and $F = \overline{\theta}(f)$, $G = \underline{\theta}(g)$.*
(i) If $\{a\}$ is a maximal $E(F)$ class, then $F(a) = f(a)$.
(ii) If $\{a\}$ is a maximal $\varepsilon(G)$ class, then $G(a) = g(a)$.

Corollary 4.8. *Assume that θ and $f \in \mathcal{F}(U)$ are of a finite range, $F = \overline{\theta}(f)$, $G = \underline{\theta}(f)$, and let E be a maximal $E(F)$ class and $\mathcal{E}$ a maximal $\varepsilon(G)$ class.*
(i) If every $\{x\} \subseteq E$ is a maximal $\varepsilon(G)$ class, then there exists a $u \in E$ with $F(u) = G(u)$.
(ii) If every $\{x\} \subseteq \mathcal{E}$ is a maximal $E(F)$ class, then there exists a $v \in \mathcal{E}$ with $F(v) = G(v)$.

Proof. (i) In view of Proposition 4.6(i), for each $x \in E$ we have $F(x) = \max\{f(y) \mid y \in E\}$, i.e. $F(x) = f(u)$, for some $u \in E$. As $\{u\}$ is a maximal $\varepsilon(G)$ class, by applying Corollary 4.7(ii) with $g := f$ we get $G(u) = f(u)$. Hence $F(u) = G(u)$. (ii) is proved dually. □

Example 4.9. Let us consider the similarity relation θ, a fuzzy set h and its approximations $F = \overline{\theta}(h)$, $G = \underline{\theta}(h)$ given on Fig. 1 and Table 1.

The quasiorders $R(F)$ and $\varrho(G)$, their $E(F)$ and $\varepsilon(G)$ equivalence classes and the partial orders induced on the factor-sets are given in Fig. 2.

Loops are not drawn for any relation. As θ is symmetric, its edges are undirected, and those with $\theta(x,y) = 0$ are not shown either. The maximal $\varepsilon(G)$ classes are $\mathcal{E}_1$, $\mathcal{E}_4$, and the maximal $E(F)$ classes are E_1 and E_5. In all our examples the approximations are defined by a min t-norm and the KD-implicator $\max(1-x, y)$. Clearly, all statements in 4.1, 4.2, 4.8 hold.

5. A characterization of fuzzy rough sets

In case of an equivalence $\varrho \subseteq U \times U$, the sets $\{X \subseteq U \mid X_\varrho = X\}$ and $\{X \subseteq U \mid X^\varrho = X\}$ coincide and their members are called ϱ-*definable* subsets of U. They can be described as those subsets of U which are the union of some ϱ-equivalence classes, and their

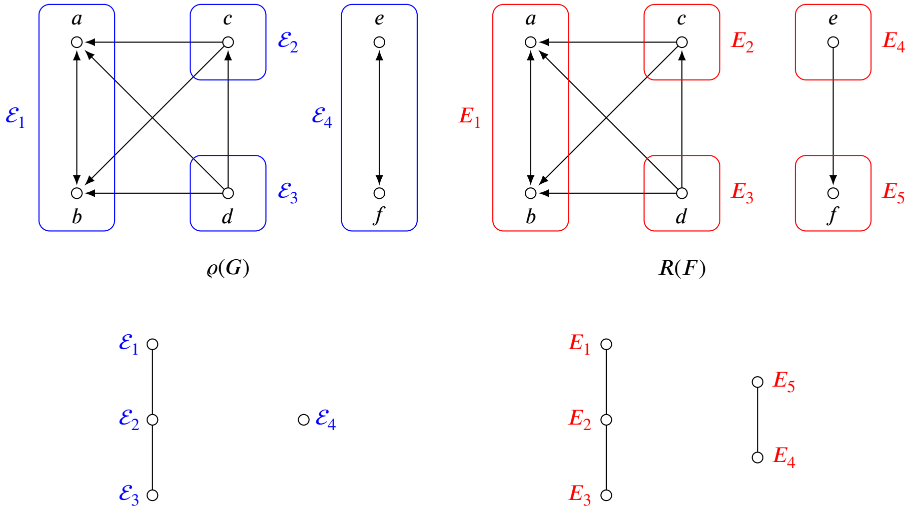

Fig. 2. The quasiorders, the factor-sets and their Hasse-diagrams for Example 4.9.

set is denoted by $\mathrm{Def}(U, \varrho)$. The rough sets induced by an equivalence relation $\varrho \subseteq U \times U$ can be characterized by using the *set of* its *singletons* $S = \{s \in U \mid \varrho(s) = \{s\}\}$, as follows:

(A, B) is rough set of ϱ, if and only if $(A, B) \in \mathrm{Def}(U, \varrho) \times \mathrm{Def}(U, \varrho)$, $A \subseteq B$ and $A \cap S = B \cap S$ (see e.g. [12]).

In this section we will derive an analogous characterization for the fuzzy rough sets with finite ranges and satisfying conditions in (ID). For a fuzzy approximation space (U, θ) we will introduce the notations:

$$\mathrm{Fix}(\underline{\theta}) = \{f \in \mathcal{F}(U) \mid \underline{\theta}(f) = f\},\ \mathrm{Fix}(\overline{\theta}) = \{f \in \mathcal{F}(U) \mid \overline{\theta}(f) = f\}.$$

Unfortunately, in case of a $\odot$-similarity relation $\mathrm{Fix}(\underline{\theta})$ and $\mathrm{Fix}(\overline{\theta})$ coincide only for a left-continuous t-norm $\odot$ and the R-implicator $\triangleright$ induced by it.

Theorem 5.1. *Assume that conditions in (ID) are satisfied and let (U, θ) be a fuzzy approximation space with a $\odot$-similarity relation θ of a finite range, and $F, G \in \mathcal{F}(U)$. Then (F, G) is a fuzzy rough set induced by a fuzzy set with a finite range, if and only if the following conditions hold:*
(1) $G \in \mathrm{Fix}(\underline{\theta})$, $F \in \mathrm{Fix}(\overline{\theta})$, $G \le F$, and F and G have finite ranges;
(2) If $\mathcal{E}$ is a maximal $\varepsilon(G)$ class such that each $\{a\} \subseteq \mathcal{E}$ is a maximal $E(F)$ class, then there exists an element $u \in \mathcal{E}$ such that $G(u) = F(u)$;
(3) If E is a maximal $E(F)$ class such that each $\{a\} \subseteq E$ is a maximal $\varepsilon(G)$ class, then there exists an element $v \in E$ such that $G(v) = F(v)$.

Proof. By definition, (F, G) is a fuzzy rough set if there exists a map $f \in \mathcal{F}(U)$ such that $F = \overline{\theta}(f)$, $G = \underline{\theta}(f)$. Suppose that f has a finite range. We prove that the conditions of Theorem 5.1 are satisfied. Indeed,

(1) Property (ID) implies $\overline{\theta}(F) = F$, $\underline{\theta}(G) = G$, hence $F \in \mathrm{Fix}(\overline{\theta})$ and $G \in \mathrm{Fix}(\underline{\theta})$. Clearly, $G = \underline{\theta}(f) \le \overline{\theta}(f) = F$. Because f has a finite range, in view of Lemma 2.2, F and G also have finite ranges.

In view of Corollary 4.8, conditions (2) and (3) are also satisfied.

Conversely, suppose that conditions (1), (2) and (3) are satisfied by F and G. In order to prove that (F, G) is a fuzzy rough set, we will construct a fuzzy set $f \in \mathcal{F}(U)$ with $F = \overline{\theta}(f)$, $G = \underline{\theta}(f)$.

Since F and G are of a finite range, in view of Corollary 4.4, for each $E(F)$ class E there exists a maximal $E(F)$ class E_M such that $E \le_{R(F)} E_M$, and for any $\varepsilon(G)$ class $\mathcal{E}$ there is a maximal $\varepsilon(G)$ class $\mathcal{E}_M$ with $\mathcal{E} \le_{\varrho(G)} \mathcal{E}_M$. Denote the family of maximal $\varepsilon(G)$ classes by $\{\mathcal{E}_t \mid t \in T\}$. As a first step, from each class $\mathcal{E}_t$, $t \in T$ we select exactly one element $a_t \in \mathcal{E}_t$ as follows:
1) If $\mathcal{E}_t$ contains an element p_t which does not belong to any maximal $E(F)$ class, then we select it and set $a_t := p_t$.
2) If in $\mathcal{E}_t$ there is no element of type 1), however there exists an element $q_t \in \mathcal{E}_t$ with $G(q_t) = F(q_t)$, then we select it and set $a_t := q_t$.
3) If there are no elements of type 1) or 2) in $\mathcal{E}_t$, then we select an element $r_t \in \mathcal{E}_t$ such that $\{r_t\}$ is not a maximal $E(F)$ class, and we set $a_t := r_t$.

First, we show that we can always effectuate such a selection: assume by contradiction that in some class $\mathcal{E}_t$ there are no elements of type 1), 2) or 3). This means that for each $a_t \in \mathcal{E}_t$ the set $\{a_t\}$ is a maximal $E(F)$ class. Then by Corollary 4.8(ii) there exists an element $v \in \mathcal{E}_t$ with $F(v) = G(v)$. Since this means that $v \in \mathcal{E}_t$ is of type 2), this is a contradiction.

As next step, we construct a fuzzy set $f \in \mathcal{F}(U)$ as follows:

$$f(x) = \begin{cases} G(x), \text{ if } x \in \{a_t \mid t \in T\}; \\ F(x), \text{ if } x \in U \setminus \{a_t \mid t \in T\} \end{cases} \tag{3}$$

By its construction, f also has a finite range. Now, we prove that in any maximal $E(F)$ class E there exists an element $u \in E$ with $f(u) = F(u)$. By our construction, this would mean that any such class E contains an element $x_0 \in U \setminus \{a_t \mid t \in T\}$ or an element $a_t = q_t \in E$ of type 2) with $f(q_t) = G(q_t) = F(q_t)$.

By way of contradiction, assume that there is a maximal $E(F)$ class E_M with $E_M \subseteq \{a_t \mid t \in T\}$ and $F(x) \neq f(x) = G(x)$, for all $x \in E_M$. Then $E_M = \{a_s \mid s \in S\}$, for some nonempty $S \subseteq T$. Observe that in this case E_M cannot contain elements of type 1) and 2). Hence, by our construction, for any element a_s, $s \in S$ the set $\{a_s\}$ is not a maximal $E(F)$ class. Thus E_M is not a one-element set, i.e. $|S| \geq 2$. Observe also, that we can exclude the case when each element $a_s, s \in S$ belongs to an $\mathcal{E}_t$ class with a single element. Indeed, as in such case each $\{a_s\} \subseteq E_M$ would be a maximal $\varepsilon(G)$ class, and by Corollary 4.8(i) we would obtain $G(a_{s_0}) = F(a_{s_0})$, for some $a_{s_0} \in E_M$, contrary to our assumption. Hence there exists an element $a_{s^*} \in E_M$ which was chosen from a maximal $\varepsilon(G)$ class $\mathcal{E}_{s^*}$ with $|\mathcal{E}_{s^*}| \geq 2$. Since $a_{s^*} \in E_M \cap \mathcal{E}_{s^*}$, in view of Corollary 4.2(i), we have $E_M \subseteq \mathcal{E}_{s^*}$ or $\mathcal{E}_{s^*} \subseteq E_M$. Since both E_M and $\mathcal{E}_{s^*}$ have at least two elements, both cases would imply that from the class $\mathcal{E}_{s^*}$ at least two elements had been inserted into the set $\{a_t \mid t \in T\}$, in contradiction to our construction for $\{a_t \mid t \in T\}$.

Thus we proved that in any maximal $E(F)$ class E there is an element $u \in E$ with $f(u) = F(u)$. It is also clear, that by our construction from each maximal $\varepsilon(G)$ class $\mathcal{E}_t$, $t \in T$ an element $v = a_t \in \mathcal{E}_t$ had been selected with $f(v) = G(v)$. Since by definition $G \leq f \leq F$, applying Proposition 4.5 we obtain $F = \overline{\theta}(f)$, $G = \underline{\theta}(f)$, and our proof is completed. □

6. Further properties of $E(F)$ and $\varepsilon(G)$ classes

In this section we deduce some additional properties of $E(F)$ and $\varepsilon(G)$ classes which will be used to prove our main Theorem 7.4. In the whole section we assume that for all $x, y \in [0, 1]$ condition

$$x \odot y = \min(x, y),\ x \rhd y := \max(n(x), y),\ n \text{ is an involutive negator} \tag{C}$$

holds, and that θ is a similarity relation. Then $\rhd$ is an S-implicator, and for $n(x) = 1 - x$, we re-obtain the Kleene-Dienes implicator, therefore our $\rhd$ is an extension of it. Clearly, if (C) holds then (D) and (ID) are also satisfied.

Proposition 6.1. *(i) $E \subseteq U$ is a maximal $E(F)$ class if and only if $\theta(a, z) < F(a) = F(b) \leq \theta(a, b)$, for all $a, b \in E$ and $z \notin E$;*
(ii) $\mathcal{E} \subseteq U$ is a maximal $\varepsilon(G)$ class, if and only if $n(\theta(a, b)) \leq G(a) = G(b) < n(\theta(a, z))$, for all $a, b \in \mathcal{E}$ and $z \notin \mathcal{E}$.

Proof. (i) If $E \subseteq U$ is a maximal $E(F)$ class, then by Lemma 4.1(i) and (iii), we have $F(a) = F(b) \leq \theta(a, b)$ and $\theta(a, z) < F(a)$, for all $a, b \in E$ and $z \notin E$.

Conversely, let $E \subseteq U$ and assume that all the relations from (i) are satisfied. Then, $F(a) = \min(\theta(a, b), F(b))$, for all $a, b \in E$, i.e. we get $(a, b) \in R(F)$ for all $a, b \in E$, and in view of Corollary 3.3, we have $(a, z) \notin R(F)$ for all $z \notin E$. Hence $(a, b) \in R(F) \cap R(F)^{-1} = E(F)$ holds for all $a, b \in E$, and $(a, z) \notin E(F)$ for each $z \notin E$. This means that E is an $E(F)$ class. We also get $E \nsubseteq [z]_{E(F)}$ for all $z \notin E$, because $(a, z) \notin R(F)$. Thus E is a maximal $E(F)$ class.

(ii) If $\mathcal{E} \subseteq U$ is a maximal $\varepsilon(G)$ class, then in view of Lemma 4.1(iv) $n(\theta(a, b)) \leq G(a) = G(b)$, for all $a, b \in \mathcal{E}$ and we have $\mathcal{E} \nsubseteq [z]_{\varepsilon(G)}$, for any $z \notin \mathcal{E}$. Then $(a, z) \notin \varrho(G)$ implies $G(a) < n(\theta(a, z))$, according to Lemma 4.1(iv). The converse implication is proved analogously as in (i). □

Lemma 6.2. *Let $F = \overline{\theta}(f)$, $G = \underline{\theta}(g)$, for some $f, g \in \mathcal{F}(U)$, and let E be an $E(F)$ class and $\mathcal{E}$ be an $\varepsilon(G)$ class such that $E \cap \mathcal{E} \neq \emptyset$. Then the following assertions hold:*
(i) If $(x, y) \notin R(F)$, for some $x \in E$ and $y \in \mathcal{E}$, then $(x, z) \in R(F)$ implies $(y, z) \in \varrho(G)$, for all $z \notin E \cup \mathcal{E}$.
(ii) If $(x, y) \notin \varrho(G)$, for some $x \in \mathcal{E}$ and $y \in E$, then $(x, z) \in \varrho(G)$ implies $(y, z) \in R(F)$, for all $z \notin E \cup \mathcal{E}$.

Proof. Let $a \in E \cap \mathcal{E}$. (i) Observe that the relations $(x, y) \notin R(F)$ and $(x, a) \in R(F)$ exclude $(a, y) \in R(F)$. Thus $(a, y) \notin R(F)$ yields $F(a) > \theta(a, y)$, by Corollary 3.3. Now, let $(x, z) \in R(F)$ and assume by contradiction $(y, z) \notin \varrho(G)$. Then $a, y \in \mathcal{E}$ and $z \notin \mathcal{E}$ imply $\theta(a, z) < \theta(a, y)$. On the other hand, $(a, x) \in R(F)$ and $(x, z) \in R(F)$ imply $(a, z) \in R(F)$. Hence, Proposition 6.1(i) yields $F(a) \leq \theta(a, z) < \theta(a, y)$, a contradiction to $F(a) > \theta(a, y)$. This proves $(y, z) \in \varrho(G)$.

(ii) By Proposition 3.2(iv) we have $\varrho(G) = R(n(G))$, $R(F) = \varrho(n(F))$, and by Lemma 4.1 (v), $\mathcal{E}$ is an $E(n(G))$ class, and E is an $\varepsilon(n(F))$ class. Hence $(x, y) \notin \varrho(G)$ for some $x \in \mathcal{E}$ and $y \in E$ and $(x, z) \in \varrho(G)$ is equivalent to $(x, y) \notin R(n(G))$ and $(x, z) \in R(n(G))$, therefore, $n(G) = n(\underline{\theta}(g)) = \overline{\theta}(n(g))$ and $n(F) = n(\overline{\theta}(f)) = \underline{\theta}(n(f))$ form a pair that replaces in the context of (ii) the pair (F, G) from (i). Thus $(y, z) \in \varrho(n(F)) = R(F)$, in view of (i). □

Corollary 6.3. *Let $F = \overline{\theta}(f)$, $G = \underline{\theta}(g)$, for some $f, g \in \mathcal{F}(U)$ and let E be an $E(F)$ class and $\mathcal{E}$ an $\varepsilon(G)$ class such that $E \cap \mathcal{E} \neq \emptyset$.*
(i) If $\mathcal{E}$ is a maximal $\varepsilon(G)$ class, then $(x, y) \in R(F)$ for all $x \in E$ and $y \in \mathcal{E}$ or $E \subseteq \mathcal{E}$ and there is no $t \in E$ and $z \notin \mathcal{E}$ with $(t, z) \in R(F)$.
(ii) If $\mathcal{E}$ is a maximal $\varepsilon(G)$ class with $E \subsetneq \mathcal{E}$ and there is no element $x \in E$ and $y \in \mathcal{E} \setminus E$ with $(x, y) \in R(F)$, then E is a maximal $E(F)$ class.
(iii) If E is a maximal $E(F)$ class, then $(x, y) \in \varrho(G)$ for all $x \in \mathcal{E}$ and $y \in E$ or $\mathcal{E} \subseteq E$ and there is no $t \in \mathcal{E}$ and $z \notin E$ with $(t, z) \in \varrho(G)$.
(iv) If E is a maximal $E(F)$ class such that $\mathcal{E} \subsetneq E$ and there is no element $x \in \mathcal{E}$ and $y \in E \setminus \mathcal{E}$ with $(x, y) \in \varrho(G)$, then $\mathcal{E}$ is a maximal $\varepsilon(G)$ class.

Proof. (i) Let $\mathcal{E}$ be a maximal $\varepsilon(G)$ class and assume $E \nsubseteq \mathcal{E}$. Then there exists $a \in E \setminus \mathcal{E}$, and for all $y \in \mathcal{E}$, $(y,a) \notin \varrho(G)$ by maximality of $\mathcal{E}$. Hence, in view of Corollary 4.2(ii) we have $(a,y) \in R(F)$, and because $(x,a) \in R(F)$ for each $x \in E$, we get $(x,y) \in R(F)$ for all $x \in E$ and $y \in \mathcal{E}$. Consider now the case when $E \subseteq \mathcal{E}$ and there are $x \in E$, $y \in \mathcal{E}$ with $(x,y) \notin R(F)$. Then $E \cup \mathcal{E} = \mathcal{E}$. Assume that there exist some elements $t \in E$ and $z \notin \mathcal{E}$ with $(t,z) \in R(F)$. Then $(x,t) \in R(F)$ also yields $(x,z) \in R(F)$. Now, applying Lemma 6.2(i) we obtain $(y,z) \in \varrho(G)$. Since $\mathcal{E}$ is a maximal $\varepsilon(G)$ class and $z \notin \mathcal{E}$, this is not possible, and this means that the second part of (i) holds.

(ii) Suppose that for all $x \in E$ and $y \in \mathcal{E} \setminus E$ we have $(x,y) \notin R(F)$, and let $z \notin \mathcal{E}$. In view of Lemma 6.2(i), $(x,z) \in R(F)$ for some $x \in E$ would imply $(y,z) \in \varrho(G)$, for all $y \in \mathcal{E}$ - in contradiction to the fact that $\mathcal{E}$ is a maximal $\varepsilon(G)$ class. Thus we deduce $(x,z) \notin R(F)$, for all $x \in E$ and $z \notin E$. This means that E is a maximal $E(F)$ class.

The proofs of (iii) and (iv) are duals of the proofs of (i) and (ii). □

Proposition 6.4. *Let θ be a similarity relation with a finite range, and $F = \overline{\theta}(f)$, $G = \underline{\theta}(g)$, for some $f, g \in \mathcal{F}(U)$.*
(i) If $\{a\} \subseteq U$ is a maximal $E(F)$ class, then for any $h \in \mathcal{F}(U)$ with $\overline{\theta}(h)(a) \geq F(a)$, we have $\overline{\theta}(h)(a) = h(a)$.
(ii) If $\{b\} \subseteq U$ is a maximal $\varepsilon(G)$ class, then for any $h \in \mathcal{F}(U)$ with $\underline{\theta}((h)(b) \leq G(b)$, we have $\underline{\theta}(h)(b) = h(b)$.

Proof. (i) If $\{a\} \subseteq U$ is a maximal $E(F)$ class, then $F(a) > \theta(a,y)$, for all $y \in U$, $y \neq a$, according to Corollary 3.3. Now, we can write:

$$\overline{\theta}(h)(a) = \bigvee \{\min(\theta(a,y), h(y)) \mid y \in U\} = h(a) \vee \left(\bigvee \{\min(\theta(a,y), h(y)) \mid y \in U \setminus \{a\}\} \right),$$

and $\bigvee \{\min(\theta(a,y), h(y)) \mid y \in U \setminus \{a\}\} \leq \bigvee \{\theta(a,y) \mid y \in U \setminus \{a\}\} < F(a) \leq \overline{\theta}(h)(a)$, because θ is of a finite range. This implies $\overline{\theta}(h)(a) = h(a)$.

(ii) If $\{b\} \subseteq U$ is a maximal $\varepsilon(G)$ class, then $G(b) < n(\theta(b,y))$, for all $y \in U$, $y \neq b$, according to Corollary 3.3. We can write:

$$\underline{\theta}((h)(b) = \bigwedge \{\max(n(\theta(b,y)), h(y)) \mid y \in U\} = h(b) \wedge \left(\bigwedge \{\max(n(\theta(b,y)), h(y)) \mid y \in U \setminus \{b\}\} \right),$$

and $\bigwedge \{\max(n(\theta(b,y)), h(y)) \mid y \in U \setminus \{b\}\} \geq \bigwedge \{n(\theta(b,y)) \mid y \in U \setminus \{b\}\} > G(b) \geq \underline{\theta}(h)(b)$, since θ is of a finite range. This yields $\underline{\theta}(h)(b) = h(b)$. □

7. The lattice of fuzzy rough sets

Clearly, fuzzy rough sets corresponding to an approximation space (U,θ), can be ordered as follows:

$$\left(\underline{\theta}(f), \overline{\theta}(f)\right) \leq \left(\underline{\theta}(g), \overline{\theta}(g)\right) \Leftrightarrow \underline{\theta}(f) \leq \underline{\theta}(g) \text{ and } \overline{\theta}(f) \leq \overline{\theta}(g), \tag{4}$$

obtaining a poset $(\mathcal{FR}(U,\theta), \leq)$. If θ is reflexive, then $(\mathbf{0},\mathbf{0})$ and $(\mathbf{1},\mathbf{1})$ are its least and greatest elements. If conditions in (D) hold, $n(\overline{\theta}(f)) = \underline{\theta}(n(f))$ and $n(\underline{\theta}(f)) = \overline{\theta}(n(f))$ imply $(n(\overline{\theta}(f), n(\underline{\theta}(f)) \in \mathcal{FR}(U,\theta)$, for all $f \in \mathcal{F}(U)$. As n is an involutive negator, $\Phi : \mathcal{FR}(U,\theta) \to \mathcal{FR}(U,\theta)$, $\Phi(\left(\underline{\theta}(f), \overline{\theta}(f)\right)) = (n(\overline{\theta}(f), n(\underline{\theta}(f))$ is an involution, i.e. $\Phi(\Phi\left(\underline{\theta}(f), \overline{\theta}(f)\right)) = \left(\underline{\theta}(f), \overline{\theta}(f)\right)$. Since $\left(\underline{\theta}(f), \overline{\theta}(f)\right) \leq \left(\underline{\theta}(g), \overline{\theta}(g)\right) \Leftrightarrow (n(\overline{\theta}(g), n(\underline{\theta}(g)) \leq (n(\overline{\theta}(f), n(\underline{\theta}(f))$, we have

$$\left(\underline{\theta}(f), \overline{\theta}(f)\right) \leq \left(\underline{\theta}(g), \overline{\theta}(g)\right) \Leftrightarrow \Phi\left(\underline{\theta}(g), \overline{\theta}(g)\right) \leq \Phi\left(\underline{\theta}(f), \overline{\theta}(f)\right),$$

meaning that Φ is a dual order-isomorphism. Thus $(\mathcal{FR}(U,\theta), \leq)$ is a self-dual poset, whenever conditions in (D) hold. In this section, we will deduce some conditions under which $(\mathcal{FR}(U,\theta), \leq)$ is a lattice.

Now let L be a complete sublattice of $[0,1]$, and let $\mathcal{F}(U,L)$ stand for the family of all fuzzy sets $f : U \to L$. The system of all $f \in \mathcal{F}(U,L)$ with a finite range is denoted by $\mathcal{F}_{fr}(U,L)$. If $L = [0,1]$, then we write simply $\mathcal{F}_{fr}(U)$. As $0, 1 \in L$, we have $\mathbf{0}, \mathbf{1} \in \mathcal{F}_{fr}(U,L)$. It is obvious that for any $f_1, f_2 \in \mathcal{F}_{fr}(U,L)$, $f_1 \vee f_2 = \max(f_1, f_2)$ and $f_1 \wedge f_2 = \min(f_1, f_2)$ are of a finite range and their values are in L, hence $(\mathcal{F}_{fr}(U,L), \leq)$ is a bounded distributive lattice. Clearly, for any $f \in \mathcal{F}(U,L)$ with a finite range and any negator n, the fuzzy set $n(f)$ also has a finite range, i.e. $n(f) \in \mathcal{F}_{fr}(U,L)$. Further, if relation θ has a finite range, then in view of Lemma 2.2, for any $f \in \mathcal{F}_{fr}(U)$: $\underline{\theta}(f), \overline{\theta}(f) \in \mathcal{F}_{fr}(U)$. In all that follows, suppose that condition (C) holds with $n(L) \subseteq L$, and $\theta : U \times U \to L$ is a similarity relation. Then

$$\overline{\theta}(f)(x) = \bigvee \{\min(\theta(x,y), f(y)) \mid y \in U\} \text{ and } \underline{\theta}(f)(x) = \bigwedge \{\max(n(\theta(x,y)), f(y)) \mid y \in U\},$$

for all $x \in U$. As L is closed w.r.t. arbitrary joins and meets, and $n(L) \subseteq L$, we get that $\underline{\theta}(f), \overline{\theta}(f) \in \mathcal{F}_{fr}(U,L)$. Now consider the poset defined on

$$\mathcal{H} := \{\left(\underline{\theta}(f), \overline{\theta}(f)\right) \mid f \in \mathcal{F}_{fr}(U,L)\}.$$

We will prove that $(\mathcal{H}, \leq)$ is a lattice, moreover if U or L is finite, then it is a complete lattice. This approach is motivated by the following examples:

1) If U is a finite set, then θ and all $f \in \mathcal{F}(U)$ have finite ranges. Hence for $L=[0,1]$ we have $\mathcal{F}_{fr}(U,L)=\mathcal{F}(U)$, and $(\mathcal{H},\leq)$ equals to $(\mathcal{FR}(U,\theta),\leq)$.

2) If L is a finite chain with $0,1\in L$, then any $f\in\mathcal{F}(U,L)$ has a finite range, hence $\mathcal{F}_{fr}(U,L)=\mathcal{F}(U,L)$, and $(\mathcal{H},\leq)$ is the same as $(\mathcal{FR}(U,L),\leq)$.

Remark 7.1. (a) The relations $\underline{\theta}(f_1)\wedge\underline{\theta}(f_1)=\underline{\theta}\left(f_1\wedge f_2\right)$ and $\overline{\theta}(f_1)\vee\overline{\theta}(f_2)=\overline{\theta}(f_1\vee f_2)$ always hold (see e.g. [4]) for any $f_1,f_2\in\mathcal{F}(U)$. Assume now that condition (C) holds, or $\odot$ is a left continuous t-norm and $\rhd$ is its R-implicator. It is known (see e.g. [19]) that in this case the equalities

$$\bigwedge\{\underline{\theta}(f_i)\mid i\in I\}=\underline{\theta}\left(\bigwedge\{f_i\mid i\in I\}\right),\ \bigvee\{\overline{\theta}(f_i)\mid i\in I\}=\overline{\theta}\left(\bigvee\{f_i\mid i\in I\}\right)$$

also hold for any (nonempty) system $f_i\in\mathcal{F}(U)$, $i\in I$.

(b) If now $L\subseteq[0,1]$ is a complete lattice and $\theta:\ U\times U\to L$, then clearly, for any $f_i\in\mathcal{F}(U,L)$, $i\in I$ we get $\bigwedge\{f_i\mid i\in I\},\bigvee\{f_i\mid i\in I\}\in\mathcal{F}(U,L)$ and $\bigwedge\{\ \underline{\theta}(f_i)\mid i\in I\}=\underline{\theta}\left(\bigwedge\{f_i\mid i\in I\}\right)\in\mathcal{F}(U,L)$, $\overline{\theta}\left(\bigvee\{f_i\mid i\in I\}\right)\in\mathcal{F}(U,L)$.

(c) As in this case conditions from (ID) also hold, in view of [4], for a $\odot$-similarity relation θ, $f\mapsto\underline{\theta}(f)$, $f\in\mathcal{F}(U,L)$ is an interior operator, and the map $f\mapsto\overline{\theta}(f)$, $f\in\mathcal{F}(U,L)$ is a closure operator. Hence $\left(\mathrm{Fix}_L\left(\underline{\theta}\right),\leq\right)$ and $\left(\mathrm{Fix}_L\left(\overline{\theta}\right),\leq\right)$ are complete lattices, where $\mathrm{Fix}_L\left(\underline{\theta}\right):=\{f\in\mathcal{F}(U,L)\mid\underline{\theta}(f)=f\}$ and $\mathrm{Fix}_L\left(\overline{\theta}\right):=\{f\in\mathcal{F}(U,L)\mid\overline{\theta}(f)=f\}$.

Proposition 7.2. *Assume that conditions in (ID) are satisfied, and let $L\subseteq[0,1]$ be a complete lattice and $\theta:\ U\times U\to L$ be a $\odot$-similarity relation. Then $\left(\mathrm{Fix}_L\left(\overline{\theta}\right),\leq\right)$ and $\left(\mathrm{Fix}_L\left(\underline{\theta}\right),\leq\right)$ are complete sublattices of $\mathcal{F}(U,L)$.*

Proof. Let $f_i\in\mathrm{Fix}_L\left(\overline{\theta}\right)$, $i\in I$ arbitrary. Then, in view of Remark 7.1, $\bigvee\{f_i\mid i\in I\}\in\mathcal{F}(U,L)$, and $\overline{\theta}\left(\bigvee\{f_i\mid i\in I\}\right)=\bigvee\{\overline{\theta}(f_i)\mid i\in I\}=\bigvee\{f_i\mid i\in I\}$. Hence $\bigvee\{f_i\mid i\in I\}\in\mathrm{Fix}_L\left(\overline{\theta}\right)$. As $\mathrm{Fix}_L\left(\overline{\theta}\right)$ is the system of closed sets of the operator $f\mapsto\overline{\theta}(f)$ and $f_i\in\mathrm{Fix}_L\left(\overline{\theta}\right)$, $i\in I$, we also have $\bigwedge_{i\in I}f_i\in\mathrm{Fix}_L\left(\overline{\theta}\right)$. Hence $\left(\mathrm{Fix}_L\left(\overline{\theta}\right),\leq\right)$ is a complete sublattice of $(\mathcal{F}(U,L),\leq)$. The claim that $\left(\mathrm{Fix}_L\left(\underline{\theta}\right),\leq\right)$ is complete sublattice of $(\mathcal{F}(U,L),\leq)$ is proved dually. □

Corollary 7.3. *Let $\theta:\ U\times U\to L$ be a similarity relation with a finite range on U, $f_i\in\mathcal{F}(U,L)$, $i\in I$, $F=\bigwedge\{\overline{\theta}(f_i)\mid i\in I\}$ and let $\{a\}\subseteq U$ be a maximal $E(F)$ class. Then $F(a)=\bigwedge\{f_i(a)\mid i\in I\}$.*

Proof. In view of Proposition 7.2 we have $F=\bigwedge\{\overline{\theta}(f_i)\mid i\in I\}\in\mathrm{Fix}_L(\overline{\theta})$, i.e. $F=\overline{\theta}(F)$. Since $\overline{\theta}(f_i)(a)\geq F(a)$, $i\in I$, by using Proposition 6.4(i) we obtain $\overline{\theta}(f_i)(a)=f_i(a)$, for all $i\in I$. This yields $F(a)=\bigwedge\{f_i(a)\mid i\in I\}$. □

Theorem 7.4. *Let $\theta:\ U\times U\to L$ be a similarity relation of a finite range, and assume that condition (C) holds with a negator satisfying $n(L)\subseteq L$.*

(i) If the fuzzy sets $\bigwedge_{i\in I}f_i$, $\bigwedge_{i\in I}\overline{\theta}(f_i)$, $f_i\in\mathcal{F}(U,L)$, $i\in I$ have finite ranges, then the infimum of fuzzy rough sets $\left(\underline{\theta}(f_i),\overline{\theta}(f_i)\right)$, $i\in I$ exists in $(\mathcal{FR}(U,L),\leq)$ and its components have finite ranges.

(ii) $(\mathcal{H},\leq)=(\{\left(\underline{\theta}(f),\overline{\theta}(f)\right)\mid f\in\mathcal{F}_{fr}(U,L)\},\leq)$ is a lattice.

(iii) If U or L is finite, then $(\mathcal{FR}(U,L),\leq)$ is a complete lattice.

Proof. (i) Denote $G=\underline{\theta}(\bigwedge_{i\in I}f_i)$ and $F=\bigwedge_{i\in I}\overline{\theta}(f_i)$. Then $G,F\in\mathcal{F}(U,L)$, by Remark 7.1(b), and we have $\underline{\theta}(G)=G$ and $\overline{\theta}(\overline{\theta}(f_i))=\overline{\theta}(f_i)$, $i\in I$, according to Remark 7.1(c). Thus $G\in\mathrm{Fix}_L(\underline{\theta})$. Since θ and $\bigwedge_{i\in I}f_i$ have finite ranges, G also has a finite range. As $\overline{\theta}(f_i)\in\mathrm{Fix}_L(\overline{\theta})$, Proposition 7.2 gives $F\in\mathrm{Fix}_L(\overline{\theta})$, and by assumption F has a finite range. Clearly, $G=\underline{\theta}(\bigwedge_{i\in I}f_i)\leq\overline{\theta}(f_i)$, for all $i\in I$, whence $G\leq F$. Using G and F we will construct a fuzzy set $f\in\mathcal{F}(U,L)$ such that $\left(\underline{\theta}(f),\overline{\theta}(f)\right)$ equals to $\inf\{(\underline{\theta}(f_i),\overline{\theta}(f_i))\mid i\in I\}$.

First, from each maximal $\varepsilon(G)$ class $\mathcal{E}_t$, $t\in T$ we select exactly one element $b_t\in\mathcal{E}_t$ as follows:

1) If $\mathcal{E}_t$ contains an element $q_t\in\mathcal{E}_t$ with $G(q_t)=F(q_t)$, then we set $b_t:=q_t$.

2) If there are no such elements in $\mathcal{E}_t$, however there exists an $s_t\in\mathcal{E}_t$ such that $\{s_t\}$ is not an $E(F)$ class, then we choose it and set $b_t:=s_t$.

3) If there are no elements of type 1) or 2) in $\mathcal{E}_t$, then we select an element $r_t\in\mathcal{E}_t$ such that $\{r_t\}$ is not a maximal $E(F)$ class, and we set $b_t:=r_t$.

Now we show that we can always manage such a selection. Indeed, assume by contradiction that in some class $\mathcal{E}_z$ there are no elements of type 1), 2) and 3). This means that for each $x\in\mathcal{E}_z$ the set $\{x\}$ is a maximal $E(F)$ class. Then in view of Corollary 7.3, we have $F(x)=\bigwedge_{i\in I} f_i(x)$, for each $x\in\mathcal{E}_z$. As $\bigwedge_{i\in I} f_i(x)$ has a finite range, by Proposition 4.6(ii) we get $G(y)=\min\{\bigwedge_{i\in I} f_i(x)\mid x\in\mathcal{E}_z\}$, for all $y\in\mathcal{E}_z$, because $G=\underline{\theta}(\bigwedge_{i\in I} f_i)$. Hence, there exists an element $v\in\mathcal{E}_z$ such that $G(v)=\bigwedge_{i\in I} f_i(v)=F(v)$. Since this result means that v is an element of type 1) in $\mathcal{E}_z$, this is a contradiction.

As next step, we construct a fuzzy set $f\in\mathcal{F}(U,L)$ as follows:

$$f(x)=\begin{cases} G(x), \text{ if } x\in\{b_t\mid t\in T\};\\ F(x), \text{ if } x\in U\setminus\{b_t\mid t\in T\}\end{cases} \tag{5}$$

As $G,F\in\mathcal{F}(U,L)$, we have $f\in\mathcal{F}(U,L)$. Since F and G have finite ranges, f also has a finite range. As from each maximal $\varepsilon(G)$ class $\mathcal{E}_t$, $t\in T$ an element $b_t\in\mathcal{E}_t$ was selected and $f(b_t)=G(b_t)$, $f\geq G$ hold, by Proposition 4.5(ii) we have $\underline{\theta}(f)=G=\underline{\theta}(\bigwedge_{i\in I} f_i)$. We prove that $\left(\underline{\theta}(f),\overline{\theta}(f)\right)$ is the infimum of the system $(\underline{\theta}(f_i),\overline{\theta}(f_i)), i\in I$. Thus we are going to show that $\left(\underline{\theta}(f),\overline{\theta}(f)\right)$ is a lower bound of $(\underline{\theta}(f_i),\overline{\theta}(f_i)), i\in I$ and for any $h\in\mathcal{F}(U,L)$ with $(\underline{\theta}(h),\overline{\theta}(h))\leq(\underline{\theta}(f_i),\overline{\theta}(f_i))$, $i\in I$ we have $(\underline{\theta}(h),\overline{\theta}(h))\leq\left(\underline{\theta}(f),\overline{\theta}(f)\right)$. As by definition $f\leq F$, we also have $\overline{\theta}(f)\leq\overline{\theta}(F)=F\leq\overline{\theta}(f_i)$, $i\in I$. Since $\underline{\theta}(f)=\underline{\theta}(\bigwedge_{i\in I} f_i)\leq\underline{\theta}(f_i)$, $i\in I$, now $\left(\underline{\theta}(f),\overline{\theta}(f)\right)$ is a lower bound of $(\underline{\theta}(f_i),\overline{\theta}(f_i)), i\in I$ and condition $\overline{\theta}(h)\leq\overline{\theta}(f_i)$, $i\in I$ is equivalent to $\overline{\theta}(h)\leq\bigwedge_{i\in I}\overline{\theta}(f_i)=F$. Since $\underline{\theta}(f)=\underline{\theta}(\bigwedge_{i\in I} f_i)=\bigwedge_{i\in I}\underline{\theta}(f_i)$, we also have

$$\underline{\theta}(h)\leq\underline{\theta}(f_i),\ i\in I\Longleftrightarrow\underline{\theta}(h)\leq\bigwedge_{i\in I}\underline{\theta}(f_i)=\underline{\theta}(f)=G.$$

Hence to prove $(\underline{\theta}(h),\overline{\theta}(h))\leq\left(\underline{\theta}(f),\overline{\theta}(f)\right)$, for all sets $h\in\mathcal{F}(U,L)$ with $(\underline{\theta}(h),\overline{\theta}(h))\leq(\underline{\theta}(f_i),\overline{\theta}(f_i))$, $i\in I$, it is enough to show that $\overline{\theta}(h)\leq\overline{\theta}(f)$ holds for any $h\in\mathcal{F}(U,L)$ with $\underline{\theta}(h)\leq G$ and $\overline{\theta}(h)\leq F$.

Take any h with this property and any $x\in U$. If $x\in U\setminus\{b_t\mid t\in T\}$ or $x=b_{t_0}$ for some $t_0\in T$ with $G(b_{t_0})=F(b_{t_0})$, then $f(x)=F(x)$, hence $h(x)\leq\overline{\theta}(h)(x)\leq F(x)=f(x)\leq\overline{\theta}(f)(x)$.

Let $x=b_{t_0}$, for some $t_0\in T$ such that $G(b_{t_0})\neq F(b_{t_0})$. Then $f(b_{t_0})=G(b_{t_0})$, by our construction. If $\{b_{t_0}\}$ is a maximal $\varepsilon(G)$ class, then in view of Proposition 6.4(ii), $\underline{\theta}(h)(b_{t_0})\leq G(b_{t_0})$ implies $h(b_{t_0})=\underline{\theta}(h)(b_{t_0})\leq G(b_{t_0})$, i.e. we obtain $h(x)\leq G(x)=f(x)\leq\overline{\theta}(f)(x)$.

Assume now that $\mathcal{E}_{t_0}$, the maximal $\varepsilon(G)$ class containing b_{t_0}, has at least two elements. Denote the $E(F)$ class containing b_{t_0} by E_0.

If $E_0\nsubseteq\mathcal{E}_{t_0}$, then there exists a $z_0\in E_0\setminus\mathcal{E}_{t_0}$, and we have $(y,z_0)\notin\varrho(G)$ for each $y\in\mathcal{E}_{t_0}$, because $\mathcal{E}_{t_0}$ is a maximal $\varepsilon(G)$ class. Hence, by Corollary 4.2(ii), we get $(z,y)\in R(F)$ for all $z\in E_0$ and $y\in\mathcal{E}_{t_0}$. Thus $(b_{t_0},c)\in R(F)$ for any $c\in\mathcal{E}_{t_0}$, $c\neq b_{t_0}$. Clearly, $c\notin\{b_t\mid t\in T\}$, because only a single element b_{t_0} was selected from $\mathcal{E}_{t_0}$, and hence $f(c)=F(c)$. Since $\left(b_{t_0},c\right)\in R(F)$ and $f\leq F$, by applying Proposition 3.4(i) we get $\overline{\theta}(f)(b_{t_0})=F(b_{t_0})$. Thus we obtain $h(b_{t_0})\leq\overline{\theta}(h)(b_{t_0})\leq F(b_{t_0})=\overline{\theta}(f)(b_{t_0})$, i.e. $h(x)\leq\overline{\theta}(f)(x)$.

If $E_0\subseteq\mathcal{E}_{t_0}$, then we claim that $(b_{t_0},e)\in R(F)$ for some element $e\in\mathcal{E}_{t_0}\setminus\{b_{t_0}\}$ (such an element exists, because $|\mathcal{E}_{t_0}|\geq 2$). Clearly, if E_0 has at least two elements, then e can be chosen as any element from $E_0\setminus\{b_{t_0}\}$. If $E_0=\{b_{t_0}\}$, then in view of our construction, the element b_{t_0} is of type 3), i.e. $\{b_{t_0}\}$ is an $E(F)$ class which is not maximal. However, if $(b_{t_0},e)\notin R(F)$ would hold for each $e\in\mathcal{E}_{t_0}\setminus\{b_{t_0}\}$, then in view of Corollary 6.3(ii), $\{b_{t_0}\}$ would be a maximal $E(F)$ class, contrary to our hypothesis. As no element different from b_{t_0} was selected from $\mathcal{E}_{t_0}$, we have $e\notin\{b_t\mid t\in T\}$, and hence $f(e)=F(e)$. Since $(b_{t_0},e)\in R(F)$, repeating now the previous argument, we obtain again $h(b_{t_0})\leq\overline{\theta}(f)(b_{t_0})$, i.e. $h(x)\leq\overline{\theta}(f)(x)$.

Hence for each $x\in U$ we obtained $h(x)\leq\overline{\theta}(f)(x)$. Thus $h\leq\overline{\theta}(f)$. In view of [4], this implies $\overline{\theta}(h)\leq\overline{\theta}\left(\overline{\theta}(f)\right)=\overline{\theta}(f)$. Thus $\left(\underline{\theta}(f),\overline{\theta}(f)\right)$ is the infimum of $(\underline{\theta}(f_i),\overline{\theta}(f_i))$, $i\in I$. Since $f\in\mathcal{F}(U,L)$ has a finite range, $\underline{\theta}(f),\overline{\theta}(f)\in\mathcal{F}(U,L)$ also have finite ranges.

(ii) For any $f_1,f_2\in\mathcal{F}_{fr}(U,L)$, $f_1\wedge f_2$ has a finite range. By Lemma 2.2, as $\overline{\theta}(f_1),\overline{\theta}(f_2)\in\mathcal{F}_{fr}(U,L)$, $\overline{\theta}(f_1)\wedge\overline{\theta}(f_2)$ also has a finite range. Applying now (i) with $I=\{1,2\}$, we get that $(\mathcal{H},\leq)$ is a $\wedge$-semilattice. Since condition (C) implies property (D), $(\mathcal{H},\leq)$ is self-dual, and hence it is a lattice.

(iii) If U or L is finite, then θ and each $f\in\mathcal{F}(U,L)$ have finite ranges, i.e. $\mathcal{F}(U,L)=\mathcal{F}_{fr}(U,L)$. As for any $f_i\in\mathcal{F}(U,L)$, $i\in I$ we have $\bigwedge_{i\in I} f_i$, $\bigwedge_{i\in I}\overline{\theta}(f_i)\in\mathcal{F}(U,L)$, the fuzzy sets $\bigwedge_{i\in I} f_i$ and $\bigwedge_{i\in I}\overline{\theta}(f_i)$ also have finite ranges. Hence, in view of (i), $\inf\{(\underline{\theta}(f_i),\overline{\theta}(f_i))\mid i\in I\}$ always exists, i.e. $(\mathcal{H},\leq)$ is a complete $\wedge$-semilattice. Since $(\mathcal{H},\leq)$ is self-dual, it is a complete lattice. □

Remark 7.5. If for a system $f_i\in\mathcal{F}(U,L)$, $i\in I$ we have $\left(\underline{\theta}(f),\overline{\theta}(f)\right)=\left(\bigwedge\{\underline{\theta}\left(f_i\right)\mid i\in I\},\bigwedge\{\overline{\theta}\left(f_i\right)\mid i\in I\}\right)$, for an $f\in\mathcal{F}(U,L)$, then $\left(\underline{\theta}(f),\overline{\theta}(f)\right)$ equals to the infimum of $\left(\underline{\theta}(f_i),\overline{\theta}(f_i)\right), i\in I$. Indeed, for any $h\in\mathcal{F}(U,L)$ with $(\underline{\theta}(h),\overline{\theta}(h))\leq(\underline{\theta}(f_i),\overline{\theta}(f_i))$, $i\in I$ we get $(\underline{\theta}(h),\overline{\theta}(h))\leq\left(\underline{\theta}(f),\overline{\theta}(f)\right)$, meaning that $\left(\underline{\theta}(f),\overline{\theta}(f)\right)$ is the infimum of $\left(\underline{\theta}(f_i),\overline{\theta}(f_i)\right), i\in I$. Analogously, $\left(\bigvee\{\underline{\theta}\left(f_i\right)\mid i\in I\},\bigvee\{\overline{\theta}\left(f_i\right)\mid i\in I\}\right)$ is the supremum of $\left(\underline{\theta}(f_i),\overline{\theta}(f_i)\right), i\in I$ whenever $\left(\bigvee\{\underline{\theta}\left(f_i\right)\mid i\in I\},\bigvee\{\overline{\theta}\left(f_i\right)\mid i\in I\}\right)\in\mathcal{FR}(U,L)$.

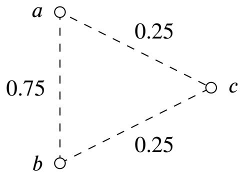

Fig. 3. The fuzzy similarity relation θ of Example 7.6.

Table 2
The fuzzy sets f_1 and f_2 of Example 7.6 and their approximations.

u	a	b	c
$f_1(u)$	1	0.1	0.5
$\overline{\theta}(f_1)(u)$	1	0.75	0.5
$\underline{\theta}(f_1)(u)$	0.25	0.1	0.5

u	a	b	c
$f_2(u)$	0.1	1	0.5
$\overline{\theta}(f_2)(u)$	0.75	1	0.5
$\underline{\theta}(f_2)(u)$	0.1	0.25	0.5

Fig. 4. The quasiorders $\varrho(G)$ and $R(F)$.

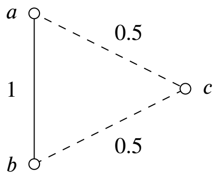

Fig. 5. The similarity relation θ of Example 7.7.

Example 7.6. Here we show how a meet $\left(\underline{\theta}(f_1), \overline{\theta}(f_1)\right) \wedge \left(\underline{\theta}(f_2), \overline{\theta}(f_2)\right)$ can be calculated by using construction (5) in the proof of Theorem 7.4. The similarity relation θ is given on Fig. 3, and $L = \{0, 0.1, 0.25, 0.5, 0.75, 1\}$. The fuzzy sets f_1, f_2 and their approximations are given in Table 2.

The corresponding fuzzy rough sets are represented in the form $\alpha_1 = \begin{pmatrix} 1 & 0.75 & 0.5 \\ 0.25 & 0.1 & 0.5 \end{pmatrix}$ and $\alpha_2 = \begin{pmatrix} 0.75 & 1 & 0.5 \\ 0.1 & 0.25 & 0.5 \end{pmatrix}$, where the first row stands for the upper approximations and the second row shows their lower approximations. Computing the meets $F = \overline{\theta}(f_1) \wedge \overline{\theta}(f_2)$ and $G = \underline{\theta}(f_1) \wedge \underline{\theta}(f_2)$, we obtain the pair $\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} 0.75 & 0.75 & 0.5 \\ 0.1 & 0.1 & 0.5 \end{pmatrix}$, which is not a fuzzy rough set. The quasiorders induced by F and G are given in Fig. 4.

Observe that each element is a maximal $\varepsilon(G)$ class. Hence, applying formula (5) from the proof of Theorem 7.4, we obtain the reference set $f := G$, and as a corresponding fuzzy rough set $\begin{pmatrix} 0.25 & 0.25 & 0.5 \\ 0.1 & 0.1 & 0.5 \end{pmatrix}$.

Example 7.7. Let us consider the similarity relation θ on Fig. 5, and set $L = \{0, 0.5, 1\}$. The lattice $(\mathcal{H}, \leq)$ of fuzzy rough sets is shown on Fig. 6.

8. Conclusions

The properties of the poset formed by fuzzy rough sets depend strongly both on the framework in which the approximations are defined (t-norm $\odot$ - implicator $\rhd$), and on the properties of the approximation space (U, θ).

The majority of our arguments work only under certain finiteness conditions imposed on the domain or range of the fuzzy reference sets and of the relation θ. We hope that these conditions can be replaced with weaker ones (see e.g. [34]) or with conditions related to some topology defined on U.

In the case of a finite universe or finite range set L, by using property (D), we were able to show that $(\mathcal{FR}(U, L), \leq)$ forms a lattice only for a similarity relation θ in a given particular framework. It would be interesting to check if the proof can be extended for fuzzy quasiorders or other types of relations. Theorem 5.1 seems to suggest that such a result can be obtained even in a general context (of a t-norm and a related implicator) for a t-similarity relation θ with some (strong) particular properties, even in the absence of the property (D). This can serve as a further research goal.

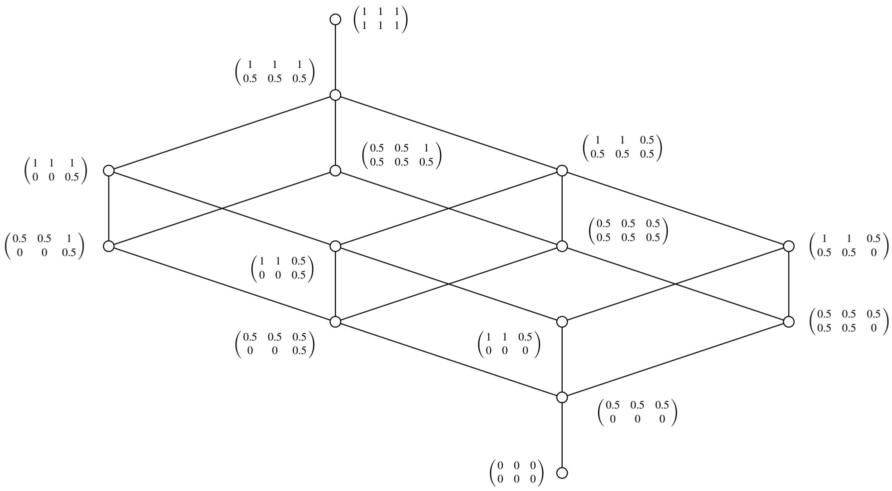

Fig. 6. The lattice of fuzzy rough sets for Example 7.7.

Even under conditions of Theorem 7.4, the lattices formed by fuzzy rough sets are not distributive in general - this is shown in Example 8.1 below. Hence an interesting question could be whether these lattices have any characteristic common property. We can see that for some particular approximation spaces as in Example 7.6, we obtain a particular distributive lattice (a so-called double Stone lattice). Therefore, it makes sense to ask under what conditions imposed on (U, θ) will we obtain a distributive lattice $\mathcal{FR}(U, L)$.

Example 8.1. Let U, L, the similarity relation θ be as in Example 7.7, and let us consider the fuzzy rough sets α_1, α_2 from 7.7 and $c = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}$. We prove that $(\alpha_1 \wedge \alpha_2) \vee c \neq (\alpha_1 \vee c) \wedge (\alpha_2 \vee c)$:

Indeed, by Example 7.7, $\alpha_1 \wedge \alpha_2 = \begin{pmatrix} 0.25 & 0.25 & 0.5 \\ 0.1 & 0.1 & 0.5 \end{pmatrix} < c$, and hence $(\alpha_1 \wedge \alpha_2) \vee c = c$. In view of Remark 7.5 we have $\alpha_1 \vee c = \begin{pmatrix} 1 & 0.75 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}$ and $\alpha_2 \vee c = \begin{pmatrix} 0.75 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}$, because $\alpha_1 \vee c = \begin{pmatrix} \overline{\theta}(h_1) \\ \underline{\theta}(h_1) \end{pmatrix}$ and $\alpha_2 \vee c = \begin{pmatrix} \overline{\theta}(h_2) \\ \underline{\theta}(h_2) \end{pmatrix}$, where $h_1 = 1/a + 0.5/b + 0.5/c$ and $h_2 = 0.5/a + 1/b + 0.5/c$.

Now, observe that $\begin{pmatrix} \overline{\theta}(h_1) \wedge \overline{\theta}(h_2) \\ \underline{\theta}(h_1) \wedge \underline{\theta}(h_2) \end{pmatrix} = \begin{pmatrix} 0.75 & 0.75 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}$ is a fuzzy rough set induced by the fuzzy set $m = 0.75/a + 0.5/b + 0.5/c$. In view of Remark 7.5 this means that $(\alpha_1 \vee c) \wedge (\alpha_2 \vee c) = \begin{pmatrix} 0.75 & 0.75 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix} \neq c$.

CRediT authorship contribution statement

Dávid Gégény: Formal analysis, Funding acquisition, Investigation, Project administration, Validation, Visualization, Writing – original draft, Writing – review & editing. **Sándor Radeleczki:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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