Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 336 (2018) 1–26

FUZZY sets and systems

www.elsevier.com/locate/fss

A comprehensive study of fuzzy covering-based rough set models: Definitions, properties and interrelationships

Lynn D'eer [a,*], Chris Cornelis [a,b]

[a] Department of Applied Mathematics, Computer Science and Statistics, Ghent University, Belgium
[b] Department of Computer Science and Artificial Intelligence, Research Center on Information and Communications Technology (CITIC-UGR), University of Granada, Spain

Received 14 July 2016; received in revised form 6 March 2017; accepted 23 June 2017
Available online 28 June 2017

Abstract

Fuzzy covering-based rough set models are hybrid models using both rough set and fuzzy set theory. The former is often used to deal with uncertain and incomplete information, while the latter is used to describe vague concepts. The study of fuzzy rough set models has provided very good tools for machine learning algorithms such as feature and instance selection. In this article, we discuss different types of dual fuzzy rough set models which all consider fuzzy coverings. In particular, we study two models using non-nested level-based representation of fuzziness. In addition to the study of the theoretical properties for each model, interrelationships between the different models are discussed, resulting in a Hasse diagram of fuzzy covering-based rough set models for a finite fuzzy covering, an IMTL-t-norm and its residual implicator.

Keywords: Approximation operators; Covering; Fuzzy sets; Rough sets

1. Introduction

Rough set theory was introduced by Pawlak in 1982, as a tool to deal with uncertainty caused by indiscernibility and incompleteness in information systems [20]. To discern the elements of a universe U, an equivalence relation on U is considered. Pawlak's definition appeared to have many equivalent formulations which are mutually interpretable [34]. Hence, it is possible to consider the element-based definition using the equivalence relation, the granule-based definition using the partition U/E or the subsystem-based definition using the σ-algebra over U/E.

All equivalent formulations can be generalized. A first generalization of rough sets is obtained by replacing the equivalence relation by a general binary relation or by a neighborhood operator [26,27,33,40]. In this case, the binary relation or the neighborhood operator determines collections of sets which no longer form a partition of U. A second generalization is derived when we substitute the partition obtained by the equivalence relation with a covering,

* Corresponding author.
E-mail addresses: Lynn.Deer@ugent.be (L. D'eer), Chris.Cornelis@ugent.be (C. Cornelis).

http://dx.doi.org/10.1016/j.fss.2017.06.010

i.e., a collection of non-empty sets such that its union is equal to U. Żakowski proposed the first notion of covering-based rough set approximation operators in 1983 [36]. However, his approximation operators are no longer dual as in Pawlak's case. For this reason, Pomykała [21] studied the operators of Żakowski and their dual operators. The pair consisting of the lower approximation operator of Żakowski and its dual upper approximation operator is called the tight pair, while the pair consisting of the upper approximation operator of Żakowski and its dual lower approximation operator is called the loose pair [4]. Finally, to generalize the subsystem-based definition, a closure system over U, i.e., a family of subsets of U that contains U and is closed under set intersection, can be considered [34].

Already in 1965 fuzzy set theory was introduced by Zadeh [35] to describe vague concepts and from early on, it has been clear that rough set theory is complementary rather than competitive with it. The vestiges of fuzzy rough set theory date back to the late 1980s, and originate from work by Fariñas del Cerro and Prade [12], Dubois and Prade [11], Nakamura [19] and Wygralak [30]. From 1990 onwards, research on the hybridization between rough sets and fuzzy sets has flourished. An extensive overview of fuzzy relation-based rough set models can be found in [5].

Analogously as in the crisp case, fuzzy binary relations are closely related with fuzzy neighborhood operators. Therefore, in this paper we will work with fuzzy neighborhood-based rough set models instead of fuzzy relation-based rough sets. In particular, we will focus on models using fuzzy neighborhood operators which are constructed using a fuzzy covering [8,18].

Besides fuzzy extensions of the element-based rough set models, we will discuss fuzzy extensions of the granule-based rough set models. Fuzzy extensions of the tight covering-based approximation operators have been studied by Li et al. [17], Inuiguchi et al. [15,16] and Wu et al. [29] which we resume here. Moreover, in [6], two tight fuzzy covering-based rough set models were presented: one using non-nested representation by levels introduced by Sánchez et al. [25] and one constructed form an intuitive point of view. In addition, we recall the loose fuzzy covering-based rough set model of Li et al. [17] and introduce a new loose fuzzy covering-based rough set model using representation by levels. To our knowledge, the tight and loose fuzzy covering-based rough set models discussed in this article are all the fuzzy granule-based models currently available.

The goal of this article is to provide an overview of the research on fuzzy covering-based rough set models, for which we keep applications in mind. For every model, we study different theoretical properties which are meaningful for machine learning applications. Moreover, we compare different models with respect to each other by discussing the accuracy of approximation operators. For every application, there is a necessary trade-off between the different aspects of a fuzzy rough set model, such as the satisfied properties and the accuracy of the approximation operators. To this aim, the article provides a theoretical background for researchers to use as a starting point in finding a suitable model for their particular application.

The outline of the article is as follows: in Section 2, we discuss some preliminary results. First, concepts of covering-based rough set theory are discussed. Next, we discuss different fuzzy neighborhood operators based on a fuzzy covering. Furthermore, the technique of representation by levels is discussed. To end the preliminaries, we discuss different properties of fuzzy covering-based rough sets. In Section 3, an overview of fuzzy covering-based rough set models is provided. Besides fuzzy neighborhood-based rough set models, we discuss fuzzy extensions of the tight and loose covering-based approximation operators. In Section 4, we study the interrelationships between the different models. We construct a Hasse diagram for a finite fuzzy covering, an IMTL-t-norm and its residual implicator. Conclusions and future work are stated in Section 5.

Finally, note that this paper extends the conference paper [6], where a limited part of the results we obtain was presented.

2. Preliminaries

Throughout this paper we assume that the universe of discourse U is a non-empty, possibly infinite set of objects. We first recall some notions on crisp covering-based rough sets. Furthermore, we discuss fuzzy neighborhood operators based on a fuzzy covering. In addition, we study the technique of representation by levels, a technique constructed to describe fuzzy concepts with non-nested crisp representatives. To end this preliminary section, we discuss some properties a fuzzy covering-based rough set model can satisfy.

2.1. Covering-based rough sets

In 1982, Pawlak introduced the original rough set model [20], in which an equivalence relation E is used to describe the indiscernibility relation between two objects $x, y \in U$. In this model, a subset $A \subseteq U$ can be approximated by a pair of approximation operators $(\underline{\text{apr}}_E, \overline{\text{apr}}_E)$. The lower approximation of A, denoted by $\underline{\text{apr}}_E(A)$, contains all the objects of U certainly belonging to A, while the upper approximation of A, denoted by $\overline{\text{apr}}_E(A)$, contains all the objects of U possibly belonging to A. The pair $(\underline{\text{apr}}_E(A), \overline{\text{apr}}_E(A))$ is defined as follows:

$$\underline{\text{apr}}_E(A) = \{x \in U \mid [x]_E \subseteq A\} = \bigcup\{[x]_E \in U/E \mid [x]_E \subseteq A\}, \tag{1}$$

$$\overline{\text{apr}}_E(A) = \{x \in U \mid [x]_E \cap A \neq \emptyset\} = \bigcup\{[x]_E \in U/E \mid [x]_E \cap A \neq \emptyset\}, \tag{2}$$

where $[x]_E$ represents the equivalence class of $x \in U$ with respect to E. The first equality in Equations (1) and (2) is called the element-based definition, and the second equality is called the granule-based definition of Pawlak's rough set model [34].

By weakening the condition of an equivalence relation, many generalizations of Pawlak's model can be defined. An important generalization can be obtained by replacing the partition U/E with a covering of U.

Definition 1. [38] Let $\mathbb{C} = \{K_i \subseteq U \mid K_i \neq \emptyset, i \in I\}$ be a family of non-empty subsets of U, with I a set of indices. $\mathbb{C}$ is called a covering of U if $\bigcup_{i \in I} K_i = U$. The ordered pair $(U, \mathbb{C})$ is called a covering approximation space.

It is clear that a partition generated by an equivalence relation is a special case of a covering of U. Similarly, equivalence classes can be generalized to neighborhoods.

Definition 2. [34] A neighborhood operator is a mapping $\text{N}: U \to \mathscr{P}(U)$, where $\mathscr{P}(U)$ represents the collection of subsets of U.

In general, it is assumed that a neighborhood operator N is reflexive, i.e., $\forall x \in U: x \in \text{N}(x)$, in order to fulfill the intuitive idea of a *neighborhood*. Furthermore, a neighborhood operator N can be symmetric, i.e., $\forall x, y \in U: x \in \text{N}(y)$ if and only if $y \in \text{N}(x)$, and it can be transitive, i.e., $\forall x, y \in U: x \in \text{N}(y) \Rightarrow \text{N}(x) \subseteq \text{N}(y)$.

The neighborhood of an object $x \in U$ can be regarded as a generalization of the equivalence class $[x]_E$. Therefore, each neighborhood operator N defines an ordered pair $(\underline{\text{apr}}_\text{N}, \overline{\text{apr}}_\text{N})$ of element-based approximation operators defined by, for $A \subseteq U$,

$$\underline{\text{apr}}_\text{N}(A) = \{x \in U \mid \text{N}(x) \subseteq A\}, \tag{3}$$

$$\overline{\text{apr}}_\text{N}(A) = \{x \in U \mid \text{N}(x) \cap A \neq \emptyset\}. \tag{4}$$

Note that neighborhood operators are closely related with binary relations. For example, given a neighborhood operator N, then we can define a binary relation R as follows: $\forall x, y \in U: (x, y) \in R \Leftrightarrow x \in \text{N}(y)$.

Yao and Yao described some neighborhood operators based on a covering $\mathbb{C}$ [34]. For this purpose, they defined the neighborhood system $\mathscr{C}(\mathbb{C}, x)$ of an element $x \in U$ given $\mathbb{C}$ as follows:

$$\mathscr{C}(\mathbb{C}, x) = \{K \in \mathbb{C} \mid x \in K\}. \tag{5}$$

In a neighborhood system $\mathscr{C}(\mathbb{C}, x)$, the minimal and maximal sets that contain an element $x \in U$ are particularly important. The set

$$\text{md}(\mathbb{C}, x) = \{K \in \mathscr{C}(\mathbb{C}, x) \mid (\forall S \in \mathscr{C}(\mathbb{C}, x))(S \subseteq K \Rightarrow K = S)\} \tag{6}$$

is called the minimal description of x [1]. On the other hand, the set

$$\text{MD}(\mathbb{C}, x) = \{K \in \mathscr{C}(\mathbb{C}, x) \mid (\forall S \in \mathscr{C}(\mathbb{C}, x))(S \supseteq K \Rightarrow K = S)\} \tag{7}$$

is called the maximal description of x [39]. The sets $\text{md}(\mathbb{C}, x)$ and $\text{MD}(\mathbb{C}, x)$ are also called the minimal-description and maximal-description neighborhood systems of x [34]. The importance of the minimal and maximal description of x is demonstrated by the following proposition:

Proposition 1. [34] *Let $(U, \mathbb{C})$ be a covering approximation space, $x \in U$ and $K \in \mathscr{C}(\mathbb{C}, x)$.*

(a) *If any descending chain in $\mathbb{C}$ is closed under the infimum, i.e., if for any set $\{K_i \mid i \in I\}$ with $K_{i+1} \subseteq K_i$ it holds that $\inf_{i \in I} K_i = \bigcap_{i \in I} K_i \in \mathbb{C}$, then there exists a set $K_1 \in \mathrm{md}(\mathbb{C}, x)$ with $K_1 \subseteq K$. Moreover, it holds that*

$$\bigcap\{K \in \mathbb{C} \mid K \in \mathrm{md}(\mathbb{C}, x)\} = \bigcap\{K \in \mathbb{C} \mid K \in \mathscr{C}(\mathbb{C}, x)\}.$$

(a) *If any ascending chain in $\mathbb{C}$ is closed under the supremum, i.e., if for any set $\{K_i \mid i \in I\}$ with $K_i \subseteq K_{i+1}$ it holds that $\sup_{i \in I} K_i = \bigcup_{i \in I} K_i \in \mathbb{C}$, then there exists a set $K_2 \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K_2$. Moreover, it holds that*

$$\bigcup\{K \in \mathbb{C} \mid K \in \mathrm{MD}(\mathbb{C}, x)\} = \bigcup\{K \in \mathbb{C} \mid K \in \mathscr{C}(\mathbb{C}, x)\}.$$

Note that Proposition 1 is satisfied when the covering $\mathbb{C}$ is finite. We will always assume that the conditions on $\mathbb{C}$ are satisfied.

Given the three neighborhood systems of $x \in U$, Yao and Yao [34] constructed the following four neighborhood operators based on the covering $\mathbb{C}$:

1. $N_1^{\mathbb{C}}(x) = \bigcap\{K \in \mathbb{C} \mid K \in \mathrm{md}(\mathbb{C}, x)\} = \bigcap \mathscr{C}(\mathbb{C}, x)$,
2. $N_2^{\mathbb{C}}(x) = \bigcup\{K \in \mathbb{C} \mid K \in \mathrm{md}(\mathbb{C}, x)\}$,
3. $N_3^{\mathbb{C}}(x) = \bigcap\{K \in \mathbb{C} \mid K \in \mathrm{MD}(\mathbb{C}, x)\}$,
4. $N_4^{\mathbb{C}}(x) = \bigcup\{K \in \mathbb{C} \mid K \in \mathrm{MD}(\mathbb{C}, x)\} = \bigcup \mathscr{C}(\mathbb{C}, x)$.

Therefore, for each $N_i^{\mathbb{C}}$ with $i = 1, 2, 3, 4$, we have a pair of dual approximation operators $(\underline{\mathrm{apr}}_{N_i^{\mathbb{C}}}, \overline{\mathrm{apr}}_{N_i^{\mathbb{C}}})$ defined in Equations (3) and (4). Note that if $\mathbb{C}$ is a partition U/E, $N_i^{\mathbb{C}}(x) = [x]_E$, for all $x \in U$, for $i = 1, 2, 3, 4$.

Besides generalizations of the element-based definition of Pawlak's rough set model, the granule-based representations can be generalized by considering a covering $\mathbb{C}$ instead of a partition U/E. However, although $(\underline{\mathrm{apr}}, \overline{\mathrm{apr}})$ are dual approximation operators, i.e., $\forall A \subseteq U: \overline{\mathrm{apr}}(A) = \mathrm{co}(\underline{\mathrm{apr}}(\mathrm{co}(A)))$, where co represents the set-theoretic complement, this property is no longer satisfied for the generalizations. Therefore, given a covering $\mathbb{C}$, we have two pairs of dual approximation operators $(\underline{\mathrm{apr}}'_{\mathbb{C}}, \overline{\mathrm{apr}}'_{\mathbb{C}})$ and $(\underline{\mathrm{apr}}''_{\mathbb{C}}, \overline{\mathrm{apr}}''_{\mathbb{C}})$ which are defined, for $A \subseteq U$,

$$\underline{\mathrm{apr}}'_{\mathbb{C}}(A) = \bigcup\{K \in \mathbb{C} \mid K \subseteq A\}, \tag{8}$$

$$\overline{\mathrm{apr}}'_{\mathbb{C}}(A) = \mathrm{co}(\underline{\mathrm{apr}}'_{\mathbb{C}}(\mathrm{co}(A))), \tag{9}$$

$$\underline{\mathrm{apr}}''_{\mathbb{C}}(A) = \mathrm{co}(\overline{\mathrm{apr}}''_{\mathbb{C}}(\mathrm{co}(A))), \tag{10}$$

$$\overline{\mathrm{apr}}''_{\mathbb{C}}(A) = \bigcup\{K \in \mathbb{C} \mid K \cap A \neq \emptyset\}. \tag{11}$$

The pair $(\underline{\mathrm{apr}}'_{\mathbb{C}}, \overline{\mathrm{apr}}'_{\mathbb{C}})$ is called the tight pair of covering-based approximation operators, while $(\underline{\mathrm{apr}}''_{\mathbb{C}}, \overline{\mathrm{apr}}''_{\mathbb{C}})$ is called the loose pair of covering-based approximation operators [4]. This is because of the following property:

$$\forall A \subseteq U: \ \underline{\mathrm{apr}}''_{\mathbb{C}}(A) \subseteq \underline{\mathrm{apr}}'_{\mathbb{C}}(A) \subseteq A \subseteq \overline{\mathrm{apr}}'_{\mathbb{C}}(A) \subseteq \overline{\mathrm{apr}}''_{\mathbb{C}}(A).$$

In addition, note that the granule-based pair $(\underline{\mathrm{apr}}''_{\mathbb{C}}, \overline{\mathrm{apr}}''_{\mathbb{C}})$ is equivalent with the element-based pair $(\underline{\mathrm{apr}}_{N_4^{\mathbb{C}}}, \overline{\mathrm{apr}}_{N_4^{\mathbb{C}}})$ [24].

Besides neighborhood operators based on a covering $\mathbb{C}$, Yao and Yao [34] also considered six coverings derived from an initial covering $\mathbb{C}$:

1. $\mathbb{C}_1 = \bigcup\{\mathrm{md}(\mathbb{C}, x) \mid x \in U\}$,
2. $\mathbb{C}_2 = \bigcup\{\mathrm{MD}(\mathbb{C}, x) \mid x \in U\}$,
3. $\mathbb{C}_3 = \{\bigcap \mathrm{md}(\mathbb{C}, x) \mid x \in U\} = \{\bigcap \mathscr{C}(\mathbb{C}, x) \mid x \in U\}$,
4. $\mathbb{C}_4 = \{\bigcup \mathrm{MD}(\mathbb{C}, x) \mid x \in U\} = \{\bigcup \mathscr{C}(\mathbb{C}, x) \mid x \in U\}$,
5. $\mathbb{C}_\cap = \mathbb{C} \setminus \left\{K \in \mathbb{C} \mid \left(\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}\right)\left(K = \bigcap \mathbb{C}'\right)\right\}$,
6. $\mathbb{C}_\cup = \mathbb{C} \setminus \left\{K \in \mathbb{C} \mid \left(\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}\right)\left(K = \bigcup \mathbb{C}'\right)\right\}$.

The idea behind the first two coverings is similar to the rationale for $N_1^{\mathbb{C}}$, $N_2^{\mathbb{C}}$, $N_3^{\mathbb{C}}$ and $N_4^{\mathbb{C}}$. Given the extreme neighborhood systems $\mathrm{md}(\mathbb{C}, x)$ and $\mathrm{MD}(\mathbb{C}, x)$ for $x \in U$, the union of these systems leads to new coverings. Note that this is not the case when taking the intersection. Coverings $\mathbb{C}_3$ and $\mathbb{C}_4$ are directly related with $N_1^{\mathbb{C}}$ and $N_4^{\mathbb{C}}$. Covering $\mathbb{C}_\cap$ is called the intersection reduct and $\mathbb{C}_\cup$ the union reduct. These reducts eliminate intersection reducible elements, resp. union reducible elements, from the covering, respectively. An intersection reducible element of a covering $\mathbb{C}$ is an element $K \in \mathbb{C}$ such that there exists a subcovering $\mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}$ for which $K = \bigcap \mathbb{C}'$, while a union reducible element of $\mathbb{C}$ is an element $K \in \mathbb{C}$ such that there exists a subcovering $\mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}$ for which $K = \bigcup \mathbb{C}'$. The equality $\mathbb{C}_1 = \mathbb{C}_\cup$ was established in [24], while the other coverings are different in general. Also, note that $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_\cap$ are subcoverings of $\mathbb{C}$, while $\mathbb{C}_3$ and $\mathbb{C}_4$ are not.

2.2. Fuzzy neighborhood operators based on a fuzzy covering

In [8], D'eer et al. extended the four neighborhood operators and six derived coverings discussed in [34] to the fuzzy framework. First, the notion of a fuzzy covering is recalled.

Definition 3. Let $\mathscr{F}(U)$ denote the collection of fuzzy subsets of U and let I be an (infinite) index set. A collection $\mathbb{C} = \{K_i \in \mathscr{F}(U) \mid K_i \neq \emptyset, i \in I\}$ is called a fuzzy covering, if for all $x \in U$ there exists a $K \in \mathbb{C}$ such that $K(x) = 1$. The ordered pair $(U, \mathbb{C})$ is called a fuzzy covering approximation space.

Note that for infinite coverings, this definition guarantees for any $x \in U$ the existence of a set $K \in \mathbb{C}$ to which x fully belongs, which is not the case with the proposals of [10,17].

Moreover, a fuzzy neighborhood operator is defined as follows:

Definition 4. A fuzzy neighborhood operator is a mapping $\mathrm{N}\colon U \to \mathscr{F}(U)$.

This means that a fuzzy neighborhood operator associates a fuzzy set $\mathrm{N}(x)$ to every element $x \in U$. In equivalent terms, fuzzy neighborhood operators N on U are in correspondence with fuzzy binary relations R on U, e.g., by taking $\mathrm{N}(x)(y) = R(x, y)$ for all $x, y \in U$. Analogously to the crisp setting, we will assume in this paper that a fuzzy neighborhood operator is reflexive, i.e., $\mathrm{N}(x)(x) = 1$ for all $x \in U$, that is, an operator arising from a reflexive fuzzy relation. Sometimes we will also be considering fuzzy neighborhood operators satisfying additional properties, that in turn will clearly be in correspondence with analogous properties of the associated fuzzy binary relations. Namely, a fuzzy neighborhood operator is called symmetric if $\mathrm{N}(x)(y) = \mathrm{N}(y)(x)$ for all $x, y \in U$, i.e., the degree in which y belongs to the neighborhood of x equals the degree in which x belongs to the neighborhood of y. Moreover, given a t-norm[1] $\mathscr{T}$, we call a fuzzy neighborhood operator N $\mathscr{T}$-transitive if for all $x, y, z \in U$ holds that

$$\mathscr{T}(\mathrm{N}(x)(y), \mathrm{N}(y)(z)) \leq \mathrm{N}(x)(z).$$

Given a fuzzy covering $\mathbb{C}$ and an element $x \in U$, the fuzzy neighborhood system $\mathscr{C}(\mathbb{C}, x)$, the fuzzy minimal description $\mathrm{md}(\mathbb{C}, x)$ and the fuzzy maximal description $\mathrm{MD}(\mathbb{C}, x)$ of x are defined as follows [8]:

$$\mathscr{C}(\mathbb{C}, x) = \{K \in \mathbb{C} \mid K(x) > 0\}, \tag{12}$$

$$\mathrm{md}(\mathbb{C}, x) = \{K \in \mathscr{C}(\mathbb{C}, x) \mid (\forall S \in \mathscr{C}(\mathbb{C}, x))(S(x) = K(x), S \subseteq K \Rightarrow S = K)\}, \tag{13}$$

$$\mathrm{MD}(\mathbb{C}, x) = \{K \in \mathscr{C}(\mathbb{C}, x) \mid (\forall S \in \mathscr{C}(\mathbb{C}, x))(S(x) = K(x), S \supseteq K \Rightarrow S = K)\}. \tag{14}$$

Therefore, the fuzzy extensions of the four neighborhood operators defined in [34] are given as follows [8]: let $(U, \mathbb{C})$ be a fuzzy covering approximation space, $x, y \in U$, $\mathscr{T}$ a t-norm and $\mathscr{I}$ an implicator[2], then

[1] A t-norm $\mathscr{T}$ is a commutative and associative $[0, 1]^2 - [0, 1]$ mapping which is increasing in both arguments and satisfies the boundary conditions $\mathscr{T}(0, 0) = \mathscr{T}(0, 1) = \mathscr{T}(1, 0) = 0$ and $\mathscr{T}(1, 1) = 1$.

[2] An implicator $\mathscr{I}$ is a $[0, 1]^2 - [0, 1]$ mapping which is decreasing in the first and increasing in the second argument and satisfies the boundary conditions $\mathscr{I}(0, 0) = \mathscr{I}(0, 1) = \mathscr{I}(1, 1) = 1$ and $\mathscr{I}(1, 0) = 0$.

Table 1
Fuzzy neighborhood operators based on fuzzy coverings presented in [8].

Group	Operators	Group	Operators
$A1.$	$N_1^{\mathbb{C}}, N_1^{\mathbb{C}_1}, N_1^{\mathbb{C}_3}, N_1^{\mathbb{C}_\cap}$	$G.$	$N_1^{\mathbb{C}_4}$
$A2.$	$N_2^{\mathbb{C}_3}$	$H1.$	$N_4^{\mathbb{C}}, N_4^{\mathbb{C}_2}, N_4^{\mathbb{C}_\cap}$
$B.$	$N_3^{\mathbb{C}_1}$	$H2.$	$N_2^{\mathbb{C}_2}$
$C.$	$N_3^{\mathbb{C}_3}$	$I.$	$N_2^{\mathbb{C}_4}$
$D.$	$N_4^{\mathbb{C}_3}$	$J.$	$N_3^{\mathbb{C}_4}$
$E.$	$N_2^{\mathbb{C}}, N_2^{\mathbb{C}_1}$	$K.$	$N_4^{\mathbb{C}_4}$
$F1.$	$N_3^{\mathbb{C}}, N_3^{\mathbb{C}_2}, N_3^{\mathbb{C}_\cap}$	$L.$	$N_4^{\mathbb{C}_1}$
$F2.$	$N_1^{\mathbb{C}_2}$	$M.$	$N_2^{\mathbb{C}_\cap}$

$$N_1^{\mathbb{C}}(x)(y) = \inf_{K\in\mathbb{C}} \mathscr{I}(K(x), K(y)), \tag{15}$$

$$N_2^{\mathbb{C}}(x)(y) = \sup_{K\in\mathrm{md}(\mathbb{C},x)} \mathscr{T}(K(x), K(y)), \tag{16}$$

$$N_3^{\mathbb{C}}(x)(y) = \inf_{K\in\mathrm{MD}(\mathbb{C},x)} \mathscr{I}(K(x), K(y)), \tag{17}$$

$$N_4^{\mathbb{C}}(x)(y) = \sup_{K\in\mathbb{C}} \mathscr{T}(K(x), K(y)). \tag{18}$$

In addition, the fuzzy extensions of the six derived coverings of $\mathbb{C}$ are given by

$$\mathbb{C}_1 = \bigcup\{\mathrm{md}(\mathbb{C}, x) \mid x \in U\}, \tag{19}$$

$$\mathbb{C}_2 = \bigcup\{\mathrm{MD}(\mathbb{C}, x) \mid x \in U\}, \tag{20}$$

$$\mathbb{C}_3 = \{N_1^{\mathbb{C}}(x) \mid x \in U\}, \tag{21}$$

$$\mathbb{C}_4 = \{N_4^{\mathbb{C}}(x) \mid x \in U\}, \tag{22}$$

$$\mathbb{C}_\cap = \mathbb{C} \setminus \{K \in \mathbb{C} \mid (\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\})(K = \bigcap \mathbb{C}')\}, \tag{23}$$

$$\mathbb{C}_\cup = \mathbb{C} \setminus \{K \in \mathbb{C} \mid (\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\})(K = \bigcup \mathbb{C}')\}. \tag{24}$$

Note that in order for $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_\cup$ to be a fuzzy covering, the fuzzy covering $\mathbb{C}$ has to be finite. For a finite fuzzy covering $\mathbb{C}$, we have that $\mathbb{C}_1$, $\mathbb{C}_2$, $\mathbb{C}_\cap$ and $\mathbb{C}_\cup$ are subcoverings of $\mathbb{C}$, $\mathbb{C}_2 \subseteq \mathbb{C}_\cap$ and $\mathbb{C}_\cup = \mathbb{C}_1$. The intuition behind the four fuzzy neighborhood operators and the six coverings is similar as in the crisp setting.

In [8], D'eer et al. studied the combinations of the four neighborhood operators ($N_1^{\mathbb{C}}$, $N_2^{\mathbb{C}}$, $N_3^{\mathbb{C}}$, $N_4^{\mathbb{C}}$) and the six coverings ($\mathbb{C}$, $\mathbb{C}_1$, $\mathbb{C}_2$, $\mathbb{C}_3$, $\mathbb{C}_4$, $\mathbb{C}_\cap$) for a finite fuzzy covering $\mathbb{C}$, a left-continuous t-norm $\mathscr{T}$ and its R-implicator $\mathscr{I}$, defined by $\mathscr{I}(a, b) = \sup\{c \in [0, 1] \mid \mathscr{T}(a, c) \leq b\}$, for $a, b \in [0, 1]$. The twenty-four combinations result in 16 different groups of fuzzy neighborhood approximation operators, presented in Table 1. Note that for a crisp covering $\mathbb{C}$ the groups $A1$ and $A2$ coincide, as well as the groups $F1$ and $F2$, and the groups $H1$ and $H2$ [7]. In Section 6 of [7] the authors discussed the intuition behind different combinations of crisp neighborhood operators and coverings, as well as a potential use for these new neighborhood operators in data analysis.

Another fuzzy neighborhood operator based on a fuzzy covering $\mathbb{C}$ was defined by Ma [18]: let $\beta \in (0, 1]$, then the β-fuzzy neighborhood of $x \in U$ is defined by

$$N_{\beta,\mathrm{Ma}}^{\mathbb{C}}(x)(y) = \inf\{K(y) \mid K \in \mathbb{C}, K(x) \geq \beta\}.$$

Note that the 1-fuzzy neighborhood was also described in [13].

2.3. The technique of representation by levels

In literature, rough and fuzzy set theory has been represented using level representation, e.g. [14,32], however, the representations described in these articles are nested. In 2012, Sánchez et al. [25] introduced a non-nested level-based representation of fuzziness. The idea is to describe a fuzzy concept with crisp representatives, each one being a crisp realization under a certain condition [25]. Different levels of restriction are considered, with the levels in [0, 1], where level 1 is the most restrictive level. Level 0 represents no restriction at all, but it will not be taken into account in the representation. Since humans can only distinguish a finite set of levels, for each fuzzy concept A it is assumed that there exists a finite set of levels $\Lambda_A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ with $1 = \alpha_1 > \alpha_2 > \ldots > \alpha_m > \alpha_{m+1} = 0$ and $m \in \mathbb{N} \setminus \{0\}$[3].

A fuzzy concept A is described by a representation by levels (RL) (Λ_A, ρ_A) if Λ_A is a finite set of levels and $\rho_A \colon \Lambda_A \to \mathscr{P}(A)$ is a function which projects each level α onto a crisp subset of A. The set of crisp representatives Ω_A of (Λ_A, ρ_A) is given by $\Omega_A = \{\rho_A(\alpha) \mid \alpha \in \Lambda_A\}$. Note that a fuzzy set A can be seen as a special case of RL, in case there are only finite different membership degrees: let $\Lambda_A = \{A(x) \mid A(x) > 0\} \cup \{1\}$ and $\rho_A(\alpha) = A_\alpha$ for each $\alpha \in \Lambda_A$, where $A_\alpha = \{x \in U \mid A(x) \geq \alpha\}$. Furthermore, the crisp representatives on each level are independent of each other and they are not necessarily nested, i.e., $\alpha > \beta \not\Rightarrow \rho(\alpha) \supseteq \rho(\beta)$.

Although this technique is useful to represent fuzzy information, it is not easy to interpret by humans. Therefore, it is possible to obtain a fuzzy set that summarizes the information given by the RL: let (Λ_A, ρ_A) be an RL associated with a fuzzy concept A, then the fuzzy summary $\upsilon_A \colon U \to [0, 1]$ is given by

$$\begin{aligned}\upsilon_A(x) &= \sum_{\{Y \in \Omega_A \mid x \in Y\}} \left(\sum_{\{\alpha_i \in \Lambda_A \mid Y = \rho_A(\alpha_i)\}} (\alpha_i - \alpha_{i+1}) \right) \\ &= \sum_{\{\alpha_i \in \Lambda_A \mid x \in \rho_A(\alpha_i)\}} (\alpha_i - \alpha_{i+1}),\end{aligned}$$

i.e., we take the summation of the differences $\alpha_i - \alpha_{i+1}$, where x belongs to the crisp representative on level α_i. The fuzzy summary will be used in this article to compare fuzzy covering-based rough set models obtained by this technique with other models. Note that a fuzzy set has a unique representation by levels when using the alpha-cuts, but different RLs can yield the same fuzzy set as fuzzy summary.

Considering operations on fuzzy concepts, this technique will allow to perform the associated crisp operations on each level of the RL. Let $f \colon \mathscr{P}(U)^n \to \mathscr{P}(U)$ be a crisp operation, then f is extended to RLs in the following way: let $(A_1, A_2, \ldots, A_n)$ be fuzzy concepts in U with each $A_i = (\Lambda_{A_i}, \rho_{A_i})$, then $f(A_1, A_2, \ldots, A_n)$ is a fuzzy concept in U represented by $(\Lambda_{f(A_1,A_2,\ldots,A_n)}, \rho_{f(A_1,A_2,\ldots,A_n)})$ where

$$\Lambda_{f(A_1,A_2,\ldots,A_n)} = \bigcup_{1 \leq i \leq n} \Lambda_{A_i}$$

and $\forall \alpha \in \Lambda_{f(A_1,A_2,\ldots,A_n)}$, $\rho_{f(A_1,A_2,\ldots,A_n)}(\alpha) = f(\rho_{A_1}(\alpha), \rho_{A_2}(\alpha), \ldots, \rho_{A_n}(\alpha))$. Examples of such operations are the union, the intersection or the complement. We have the following proposition:

Proposition 2. *[25] Operations on RLs satisfy all the properties of the Boolean logic.*

In other words, all properties using operations of the Boolean logic, e.g., negation, conjunction and disjunction, which hold for a crisp concept will also hold for its fuzzification, when RLs are used. This is the main advantage of non-nested level-based representations.

2.4. On some properties of fuzzy covering-based rough set models

Here we list properties which a fuzzy covering-based rough set model can satisfy. We base ourselves on the properties stated in [5]. Let $(\underline{\text{apr}}, \overline{\text{apr}})$ be a pair of fuzzy approximation operators on U.

[3] It is possible to consider a countable set of levels.

- The pair is called dual (D) with respect to an involutive negator[4] $\mathcal{N}$ if $\overline{\text{apr}}(A) = \text{co}_{\mathcal{N}}(\underline{\text{apr}}(\text{co}_{\mathcal{N}}(A)))$ for $A \in \mathcal{F}(U)$, where $\text{co}_{\mathcal{N}}\colon \mathcal{F}(U) \to \mathcal{F}(U)$ is defined by $(\text{co}_{\mathcal{N}}(A))(x) = \mathcal{N}(A(x))$ for $A \in \mathcal{F}(U)$, $x \in U$.
  The duality property allows us to study only the lower or only the upper approximation operators, as has been done to construct the Hasse diagram in this article. Depending on the application, this property is necessary. For example, in [23], they explicitly assume the operators to be dual to simplify the approach, while in [22] only the lower approximation operator is used, hence, the duality property can be disregarded.
- The pair is called adjoint (A) if $\overline{\text{apr}}(A) \subseteq B \Leftrightarrow A \subseteq \underline{\text{apr}}(B)$ for $A, B \in \mathcal{F}(U)$.
  The adjointness property is motivated by Galois connections. In this framework, we see that if a model satisfies the property (A), it holds that $\forall A \in \mathcal{F}(U)\colon \overline{\text{apr}}(A) \subseteq A \Leftrightarrow A \subseteq \underline{\text{apr}}(A)$. If the inclusion property, which is discussed below, also holds, we know that if the lower approximation of A is exact, the upper approximation of A will be exact and vice versa.
- The pair satisfies the inclusion property (INC) if $\underline{\text{apr}}(A) \subseteq A$ and $A \subseteq \overline{\text{apr}}(A)$ for $A \in \mathcal{F}(U)$.
  The inclusion property is important to state conclusions on the accuracy of a pair of approximation operators. The identity function may be seen as the most accurate pair of approximation operator, although this function is not very meaningful in machine learning purposes. However, we want to use approximation operators which provide approximations close to the approximated set. Intuitively, we expect the lower and upper approximations to be positioned on either side of the approximated set, i.e.,

  $$\forall A \in \mathcal{F}(U)\colon \frac{|\underline{\text{apr}}(A)|}{|\overline{\text{apr}}(A)|} \leq 1.$$

  Without the inclusion property, this is not guaranteed.
- The pair satisfies the set monotonicity property (SM) if $A \subseteq B$ implies $\underline{\text{apr}}(A) \subseteq \underline{\text{apr}}(B)$ and $\overline{\text{apr}}(A) \subseteq \overline{\text{apr}}(B)$ for $A, B \in \mathcal{F}(U)$.
  The set monotonicity is important for applications such as classification problems, as in these applications we want larger decision classes to have larger approximations.
- The pair satisfies the intersection and union property (IU) if $\underline{\text{apr}}(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} \underline{\text{apr}}(A_j)$ and $\overline{\text{apr}}(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} \overline{\text{apr}}(A_j)$ for $A_j \in \mathcal{F}(U)$, $j \in J$.
  This property is very interesting when multiple decision attributes are considered. For example, consider D a finite set of decision attributes. It is sufficient to compute $\underline{\text{apr}}([x]_d)$ for each $[x]_d \in U/d$, $d \in D$ to state conclusions on the approximations of the decision classes $[x]_{D'}$ with $D' \subseteq D$:

  $$\forall x \in U\colon \underline{\text{apr}}([x]_{D'}) = \bigcap_{d \in D'} \underline{\text{apr}}([x]_d).$$

- The pair satisfies the idempotence property (ID) if $\underline{\text{apr}}(\underline{\text{apr}}(A)) = \underline{\text{apr}}(A)$ and $\overline{\text{apr}}(\overline{\text{apr}}(A)) = \overline{\text{apr}}(A)$ for $A \in \mathcal{F}(U)$.
  The property (ID) states that the approximation operators are idempotent.
- The pair satisfies the lower-upper property (LU) if $\overline{\text{apr}}(\underline{\text{apr}}(A)) = \underline{\text{apr}}(A)$ and $\underline{\text{apr}}(\overline{\text{apr}}(A)) = \overline{\text{apr}}(A)$ for $A \in \mathcal{F}(U)$.
  This property is similar to the previous property, and states conclusions ons the interaction between the lower and upper approximation operators.
- The pair satisfies the constant set property (CS) if $\underline{\text{apr}}(\hat{\alpha}) = \hat{\alpha}$ and $\overline{\text{apr}}(\hat{\alpha}) = \hat{\alpha}$ for $\alpha \in (0, 1)$, with $\hat{\alpha} \in \mathcal{F}(U)$ defined by $\hat{\alpha}(x) = \alpha$ for each $x \in U$. If the property holds for $\alpha = 1$ and $\alpha = 0$, i.e., for the fuzzy sets U and $\emptyset$, then the property is denoted with (UE).
  This property is the fuzzification of the (UE) property in the crisp setting, which states conclusions on the boundary conditions of the approximation operators.

Naturally, if a pair of fuzzy approximation operators $(\underline{\text{apr}}, \overline{\text{apr}})$ is an extension of an element- or granule-based rough set model described in Section 2.1, we only need to consider those properties which are satisfied by the crisp

[4] A negator $\mathcal{N}\colon [0, 1] \to [0, 1]$ is a decreasing mapping such that $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 1$. $\mathcal{N}$ is called involutive if $\mathcal{N}(\mathcal{N}(x)) = x$ for each $x \in U$.

model [24]. For fuzzy extensions of the element-based pair $(\underline{\text{apr}}_{\text{N}}, \overline{\text{apr}}_{\text{N}})$, we can consider all properties listed above. For fuzzy extensions of the granule-based pair $(\underline{\text{apr}}'_{\mathbb{C}}, \overline{\text{apr}}'_{\mathbb{C}})$, we will only consider (D), (INC), (SM), (ID), (CS) and (UE). For fuzzy extensions of the granule-based pair $(\underline{\text{apr}}''_{\mathbb{C}}, \overline{\text{apr}}''_{\mathbb{C}})$, we will only consider (D), (A), (INC), (SM), (IU), (CS) and (UE).

In addition, we have two monotonicity properties in the crisp case, depending on whether the approximation operator is element- or granule-based. These monotonicity properties are very necessary for applications: when we consider more conditional attributes, the granulation of the data will become finer, and thus, we want the approximations not to shrink, i.e., we want the approximations to be at least as accurate as for a coarser granulation.

First, let N and N′ be two neighborhood operators such that $\text{N}(x) \subseteq \text{N}'(x)$ for each $x \in U$, then $\underline{\text{apr}}_{\text{N}'}(A) \subseteq \underline{\text{apr}}_{\text{N}}(A)$ and $\overline{\text{apr}}_{\text{N}}(A) \subseteq \overline{\text{apr}}_{\text{N}'}(A)$ for $A \in \mathscr{F}(U)$. This monotonicity property is denoted by (NM).

Second, to describe the monotonicity for the granule-based pairs, denoted by (CM), we define the following partial order relation $\preceq$ on the set of coverings of U, inspired by the partial relation on partitions of the universe [37]: let $\mathbb{C}$ and $\mathbb{C}'$ be coverings of the universe U, then

$$\mathbb{C} \preceq \mathbb{C}' \Leftrightarrow (\forall K \in \mathbb{C})(\exists K' \in \mathbb{C}')(K \subseteq K').$$

Note that this partial order can be naturally extended when $\mathbb{C}$ and $\mathbb{C}'$ are fuzzy coverings of U. When $\mathbb{C} \preceq \mathbb{C}'$ holds for crisp coverings $\mathbb{C}$ and $\mathbb{C}'$, then it is easy to see that $\underline{\text{apr}}''_{\mathbb{C}'}(A) \subseteq \underline{\text{apr}}''_{\mathbb{C}}(A)$ and $\overline{\text{apr}}''_{\mathbb{C}}(A) \subseteq \overline{\text{apr}}''_{\mathbb{C}'}(A)$ for each $A \in \mathscr{F}(U)$. However, for the tight approximation operators $(\underline{\text{apr}}'_{\mathbb{C}}, \overline{\text{apr}}'_{\mathbb{C}})$ no such statement can be made, as illustrated in the next example.

Example 1. Let $U = \{x, y, z\}$. If $\mathbb{C} = \{\{x, y\}, \{y, z\}\}$ and $\mathbb{C}' = \{\{x\}, \{x, y, z\}\}$, then $\mathbb{C} \preceq \mathbb{C}'$. However, on the one hand we have that $\underline{\text{apr}}'_{\mathbb{C}}(\{x\}) = \emptyset \subseteq \{x\} = \underline{\text{apr}}'_{\mathbb{C}'}(\{x\})$. On the other hand, we have that $\underline{\text{apr}}'_{\mathbb{C}}(\{x, y\}) = \{x, y\} \supseteq \{x\} = \underline{\text{apr}}'_{\mathbb{C}'}(\{x, y\})$.

Hence, we will only consider (CM) for models extending the loose approximation operators.

3. An overview of fuzzy covering-based rough set models

In this section, we study different fuzzy covering-based rough set models. First, in Section 3.1 models using a fuzzy neighborhood are discussed. These models are fuzzy extensions of the element-based approximation operators defined in Equations (3) and (4). Moreover, we study models which extend either the tight or the loose granule-based operators described in Equations (8)–(11) in Section 3.2 and Section 3.3 respectively. In addition, we study for each model which properties it satisfies.

3.1. Fuzzy neighborhood-based rough set models

Fuzzy neighborhood operators are closely related to fuzzy relations. For example, let R be a fuzzy relation, then we can easily define a fuzzy neighborhood operator using R, e.g., $\text{N}(x)(y) = R(x, y)$ or $\text{N}(x)(y) = R(y, x)$. Hence, defining a general rough set model using fuzzy neighborhoods is very similar to defining a general fuzzy relation-based rough set model as was done in [5].

Definition 5. Let N be a fuzzy neighborhood operator on U, $\mathscr{I}$ an implicator and $\mathscr{T}$ a t-norm, then the fuzzy neighborhood-based approximation operators $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{T}})$ are defined by, for $A \in \mathscr{F}(U)$ and $x \in U$,

$$(\underline{\text{apr}}_{\text{N},\mathscr{I}}(A))(x) = \inf_{y \in U} \mathscr{I}(N(x)(y), A(y)), \tag{25}$$

$$(\overline{\text{apr}}_{\text{N},\mathscr{T}}(A))(x) = \sup_{y \in U} \mathscr{T}(N(x)(y), A(y)). \tag{26}$$

The neighborhood operators stated in Table 1 can be used to define fuzzy neighborhood-based approximation operators. Other examples of fuzzy neighborhood-based approximation operators are given in [18], where Ma used the β-fuzzy neighborhood operators, the Kleene–Dienes implicator $\mathcal{I}_{KD}(a,b) = \max(1-a,b)$ and the minimum t-norm $\mathcal{T}_{\min}(a,b) = \min(a,b)$ for $a,b \in [0,1]$.

Analogously as for a fuzzy relation-based rough set model, a fuzzy neighborhood-based rough set model satisfies the following properties:

Proposition 3. *[5] Let* N *be a fuzzy neighborhood operator on* U, $\mathcal{I}$ *an implicator and* $\mathcal{T}$ *a t-norm.*

- *The pair* $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{T}})$ *satisfies (D) with respect to the involutive negator* $\mathcal{N}$ *if* $\mathcal{T}$ *is an IMTL-t-norm*[5], $\mathcal{I}$ *is its R-implicator and* $\mathcal{N}$ *equals the negator induced by* $\mathcal{I}$, *i.e.,* $\mathcal{N}(a) = \mathcal{I}(a,0)$ *for all* $a \in [0,1]$.
- *The pair* $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{T}})$ *satisfies (D) with respect to the involutive negator* $\mathcal{N}$ *if* $\mathcal{I}$ *is the S-implicator with respect to the t-conorm* $\mathcal{S}$ *and the negator* $\mathcal{N}$, *i.e.,* $\mathcal{I}(a,b) = \mathcal{S}(\mathcal{N}(a),b)$ *for all* $a,b \in [0,1]$, *where* $\mathcal{S}$ *is the* $\mathcal{N}$*-dual of* $\mathcal{T}$, *i.e.,* $\mathcal{S}(a,b) = \mathcal{N}(\mathcal{T}(\mathcal{N}(a),\mathcal{N}(b)))$ *for all* $a,b \in [0,1]$.
- *The pair* $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{T}})$ *satisfies (A) if* $\mathcal{T}$ *is a left-continuous t-norm and* $\mathcal{I}$ *is its R-implicator and if* N *is symmetric.*
- *The pair* $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{T}})$ *satisfies (INC) if* $\mathcal{I}(1,a) = a$ *for all* $a \in [0,1]$ *and* N *is reflexive.*
- *The pair* $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{T}})$ *satisfies (SM).*
- *The pair* $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{T}})$ *satisfies (IU) if* $\mathcal{I}$ *is right-continuous in the second parameter and* $\mathcal{T}$ *is left-continuous in the second parameter.*
- *The pair* $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{T}})$ *satisfies (ID) if* $\mathcal{T}$ *is a left-continuous t-norm,* $\mathcal{I}$ *is its R-implicator and* N *is reflexive and* $\mathcal{T}$*-transitive.*
- *The pair* $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{T}})$ *satisfies (ID) if* $\mathcal{T}$ *is a left-continuous t-norm,* $\mathcal{I}$ *is the S-implicator with respect to the t-conorm* $\mathcal{S}$ *and the involutive negator* $\mathcal{N}$, *where* $\mathcal{S}$ *is the* $\mathcal{N}$*-dual of* $\mathcal{T}$, *and* N *is reflexive and* $\mathcal{T}$*-transitive.*
- *The pair* $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{T}})$ *satisfies (LU) if* $\mathcal{T}$ *is a left-continuous t-norm,* $\mathcal{I}$ *is its R-implicator and* N *is reflexive, symmetric and* $\mathcal{T}$*-transitive.*
- *The pair* $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{T}})$ *satisfies (CS) and (UE) if* $\mathcal{I}(1,a) = a$ *for all* $a \in [0,1]$ *and* N *is reflexive.*
- *The pair* $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{T}})$ *satisfies (NM).*

3.2. Fuzzy extensions of the tight covering-based approximation operators

Up until now, five models extending the tight covering-based approximation operators have been proposed.

3.2.1. Model of Li et al.

The first fuzzy covering based rough set model we discuss was introduced by Li et al. [17].

Definition 6. [17] Let $(U,\mathbb{C})$ be a fuzzy covering approximation space, $\mathcal{T}$ a t-norm and $\mathcal{I}$ an implicator, then the pair of fuzzy covering-based approximation operators $(\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}, \overline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}})$ is defined as follows: let $A \in \mathcal{F}(U)$, $x \in U$,

$$(\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}(A))(x) = \sup_{K\in\mathbb{C}} \mathcal{T}(K(x), \inf_{y\in U} \mathcal{I}(K(y),A(y))), \tag{27}$$

$$(\overline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}(A))(x) = \inf_{K\in\mathbb{C}} \mathcal{I}(K(x), \sup_{y\in U} \mathcal{T}(K(y),A(y))). \tag{28}$$

This model was proposed by the authors to define a more general model than the models discussed in [2,10], where a fuzzy covering related with a fuzzy relation was used. The properties of this model are given in the following proposition.

Proposition 4. *[6,17] Let* $(U,\mathbb{C})$ *be a fuzzy covering approximation space,* $\mathcal{T}$ *a t-norm and* $\mathcal{I}$ *an implicator.*

[5] An IMTL-t-norm is a left-continuous t-norm such that the negator induced by its R-implicator is involutive.

- *The pair* $(\underline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}})$ *satisfies (D) with respect to the involutive negator* $\mathscr{N}$ *if* $\mathscr{T}$ *is an IMTL-t-norm,* $\mathscr{I}$ *is its R-implicator and* $\mathscr{N}$ *equals the negator induced by* $\mathscr{I}$*.*
- *The pair* $(\underline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}})$ *satisfies (D) with respect to the involutive negator* $\mathscr{N}$ *if* $\mathscr{I}$ *is the S-implicator with respect to the t-conorm* $\mathscr{S}$ *and the negator* $\mathscr{N}$*, where* $\mathscr{S}$ *is the* $\mathscr{N}$*-dual of* $\mathscr{T}$*.*
- *The pair* $(\underline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}})$ *satisfies (INC) if* $\mathscr{T}$ *is left-continuous and* $\mathscr{I}$ *is its R-implicator.*
- *The pair* $(\underline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}})$ *satisfies (SM).*
- *The pair* $(\underline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}})$ *satisfies (ID) if* $\mathscr{T}$ *is left-continuous and* $\mathscr{I}$ *is its R-implicator.*
- *The pair* $(\underline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}})$ *satisfies (CS) and (UE) if* $\mathscr{T}$ *is left-continuous and* $\mathscr{I}$ *is its R-implicator.*

3.2.2. Model of Inuiguchi et al.

Next, we study the model of Inuiguchi et al. [15,16]. They used the following logical connective: let $\mathscr{I}$ be an implicator, then $\xi[\mathscr{I}]\colon [0,1]^2 \to [0,1]$ is defined by

$$\forall a, b \in [0,1]\colon \xi[\mathscr{I}](a,b) = \inf\{c \in [0,1] \mid \mathscr{I}(a,c) \geq b\}.$$

$\xi[\mathscr{I}]$ is a conjunctor[6] if $\forall a \in [0,1[\colon \xi[\mathscr{I}](1,a) < 1$ [16]. Furthermore, note that $\mathscr{I}$ needs to be upper semi-continuous, which is the same as to require that $\mathscr{I}$ is left-continuous in the first parameter and right-continuous in the second, in order to have the following equivalence [15]:

$$\forall a, b, c \in [0,1]\colon \xi[\mathscr{I}](a,b) \leq c \Leftrightarrow \mathscr{I}(a,c) \geq b.$$

The model of Inuiguchi et al. is given in the following definition:

Definition 7. [15,16] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space, $\mathscr{I}$ an upper semi-continuous implicator and $\mathscr{N}$ an involutive negator, then the pair of fuzzy covering-based approximation operators $(\underline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I},\mathscr{N}})$ is defined as follows: let $A \in \mathscr{F}(U)$, $x \in U$,

$$(\underline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I}}(A))(x) = \sup_{K \in \mathbb{C}} \xi[\mathscr{I}](K(x), \inf_{y \in U} \mathscr{I}(K(y), A(y))), \tag{29}$$

$$(\overline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I},\mathscr{N}}(A))(x) = (\mathrm{co}_{\mathscr{N}}(\underline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I}}(\mathrm{co}_{\mathscr{N}}(A))))(x). \tag{30}$$

In [15] and [16], a collection $\mathscr{F} \subseteq \mathscr{F}(U)$ was used to define the operators. However, we will always assume that the collection $\mathscr{F}$ is a covering.

Proposition 5. *[6,15,16] Let* $(U, \mathbb{C})$ *be a fuzzy covering approximation space,* $\mathscr{I}$ *an upper semi-continuous implicator and* $\mathscr{N}$ *an involutive negator.*

- *The pair* $(\underline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I},\mathscr{N}})$ *satisfies (D) with respect to* $\mathscr{N}$*.*
- *The pair* $(\underline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I},\mathscr{N}})$ *satisfies (INC), (SM), (ID) and (UE).*
- *The pair* $(\underline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I},\mathscr{N}})$ *satisfies (CS) if* $\mathscr{I}(1,a) = a$ *for all* $a \in [0,1]$.

3.2.3. Model of Wu et al.

The following model we discuss was introduced by Wu et al. [29]. It is inspired by the use of weak α-level sets for $K \in \mathbb{C}$.

Definition 8. [29] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space, then the pair of fuzzy covering-based approximation operators $(\underline{\mathrm{apr}}'_{\mathbb{C},Wu}, \overline{\mathrm{apr}}'_{\mathbb{C},Wu})$ is defined as follows: let $A \in \mathscr{F}(U)$, $x \in U$,

[6] A conjunctor $\mathscr{C}$ is a $[0,1]^2 - [0,1]$ mapping which is increasing in both arguments and satisfies the boundary conditions $\mathscr{C}(0,0) = \mathscr{C}(0,1) = \mathscr{C}(1,0) = 0$ and $\mathscr{C}(1,1) = 1$.

$$(\underline{\text{apr}}'_{\mathbb{C},Wu}(A))(x) = \sup_{K\in\mathbb{C}} \inf\{A(y) \mid K(y) \geq K(x), y \in U\}, \tag{31}$$

$$(\overline{\text{apr}}'_{\mathbb{C},Wu}(A))(x) = \inf_{K\in\mathbb{C}} \sup\{A(y) \mid K(y) \geq K(x), y \in U\}. \tag{32}$$

Note that this model does not use fuzzy logical connectives. We discuss its properties.

Proposition 6. *[6,29] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space.*

- *The pair $(\underline{\text{apr}}'_{\mathbb{C},Wu}, \overline{\text{apr}}'_{\mathbb{C},Wu})$ satisfies (D) with respect to the standard negator $\mathcal{N}_S$, with $\mathcal{N}_S(a) = 1 - a$ for all $a \in [0, 1]$.*
- *The pair $(\underline{\text{apr}}'_{\mathbb{C},Wu}, \overline{\text{apr}}'_{\mathbb{C},Wu})$ satisfies (INC), (SM), (ID), (CS) and (UE).*

Note that in [29], it is stated that $(\underline{\text{apr}}'_{\mathbb{C},Wu}, \overline{\text{apr}}'_{\mathbb{C},Wu})$ also satisfies (IU). However, this is not correct, as illustrated in the next example. Note that for a t-norm $\mathcal{T}$, the fuzzy $\mathcal{T}$-intersection of two fuzzy sets A and B is defined by $(A \cap_{\mathcal{T}} B)(x) = \mathcal{T}(A(x), B(x))$ for each $x \in U$.

Example 2. Let $U = \{x, y, z\}$ and $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0.7/y + 1/z$ and $K_2 = 0.8/x + 1/y + 1/z$. Let $A = 0.6/x + 0/y + 0.3/z$ and $B = 0.2/x + 0.8/y + 0.4/z$, then $A \cap_{\mathcal{T}_{\min}} B = 0.2/x + 0/y + 0.3/z$. Then we obtain that $(\underline{\text{apr}}'_{\mathbb{C},Wu}(A) \cap_{\mathcal{T}_{\min}} \underline{\text{apr}}'_{\mathbb{C},Wu}(B))(z) = \min(0.3, 0.4) = 0.3$, but $(\underline{\text{apr}}'_{\mathbb{C},Wu}(A \cap_{\mathcal{T}_{\min}} B))(z) = 0.2$.

3.2.4. Model of representation by levels

A possible way to construct a fuzzy extension of the crisp operator $\underline{\text{apr}}'_{\mathbb{C}}$ is to apply the technique of representation by levels stated in Section 2.3. Note that we assume U and $\mathbb{C}$ to be finite, in order to induce a finite set of levels.

Definition 9. [6] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite and $A \in \mathcal{F}(U)$. The fuzzy set $\underline{\text{apr}}'_{\mathbb{C},RBL}(A)$ is represented by the RL $(\Lambda_{\underline{\text{apr}}'_{\mathbb{C},RBL}(A)}, \rho_{\underline{\text{apr}}'_{\mathbb{C},RBL}(A)})$, with

$$\Lambda_{\underline{\text{apr}}'_{\mathbb{C},RBL}(A)} = \Lambda_A \cup \Lambda_{\mathbb{C}} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}, m \in \mathbb{N} \setminus \{0\}$$

$$\rho_{\underline{\text{apr}}'_{\mathbb{C},RBL}(A)}(\alpha) = \bigcup\{K_\alpha \mid K \in \mathbb{C}, K_\alpha \subseteq A_\alpha\},$$

for all $\alpha \in \Lambda_{\underline{\text{apr}}'_{\mathbb{C},RBL}(A)}$. To obtain the membership degree of x in $\underline{\text{apr}}'_{\mathbb{C},RBL}(A)$, we compute the fuzzy summary:

$$(\underline{\text{apr}}'_{\mathbb{C},RBL}(A))(x) = \sum_{\{\alpha_i \in \Lambda_{\underline{\text{apr}}'_{\mathbb{C},RBL}(A)} \mid x \in \rho_{\underline{\text{apr}}'_{\mathbb{C},RBL}(A)}(\alpha_i)\}} (\alpha_i - \alpha_{i+1}),$$

where we have ranked the elements of $\Lambda_{\underline{\text{apr}}'_{\mathbb{C},RBL}(A)}$ as follows: $1 = \alpha_1 > \alpha_2 > \ldots > \alpha_m > \alpha_{m+1} = 0$. The upper approximation operator $\overline{\text{apr}}'_{\mathbb{C},RBL}$ is obtained in a similar way, by taking $\Lambda_{\overline{\text{apr}}'_{\mathbb{C},RBL}(A)} = \Lambda_{\underline{\text{apr}}'_{\mathbb{C},RBL}(A)}$ and

$$\rho_{\overline{\text{apr}}'_{\mathbb{C},RBL}(A)}(\alpha) = \text{co}(\rho_{\underline{\text{apr}}'_{\mathbb{C},RBL}(\text{co}(A))})$$

for each $\alpha \in \Lambda_{\overline{\text{apr}}'_{\mathbb{C},RBL}(A)}$.

It is clear that for a crisp set A and a crisp covering $\mathbb{C}$, the crisp sets $(\underline{\text{apr}}'_{\mathbb{C}}(A), \overline{\text{apr}}'_{\mathbb{C}}(A))$ are obtained. Due to Proposition 2, this model satisfies all properties.

Proposition 7. *[6] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite.*

- *The pair $(\underline{\text{apr}}'_{\mathbb{C},RBL}, \overline{\text{apr}}'_{\mathbb{C},RBL})$ satisfies (D) with respect to an involutive negator $\mathcal{N}$.*
- *The pair $(\underline{\text{apr}}'_{\mathbb{C},RBL}, \overline{\text{apr}}'_{\mathbb{C},RBL})$ satisfies (INC), (SM), (ID), (CS) and (UE).*

3.2.5. Model of intuitive extension

The final model extending the tight covering-based approximation operators we discuss is an intuitive extension of the crisp case. The lower approximation operator is obtained by replacing the union by the supremum, and by taking the membership degrees of x into account.

Definition 10. [6] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space and $\mathcal{N}$ an involutive negator, then the pair of fuzzy covering-based approximation operators $(\underline{\text{apr}}'_{\mathbb{C},InEx}, \overline{\text{apr}}'_{\mathbb{C},InEx,\mathcal{N}})$ is defined as follows: let $A \in \mathscr{F}(U)$, $x \in U$,

$$(\underline{\text{apr}}'_{\mathbb{C},InEx}(A))(x) = \sup_{K\in\mathbb{C}} \{K(x) \mid K \subseteq A\} \tag{33}$$

$$(\overline{\text{apr}}'_{\mathbb{C},InEx,\mathcal{N}}(A))(x) = (\text{co}_{\mathcal{N}}(\underline{\text{apr}}'_{\mathbb{C},InEx}(\text{co}_{\mathcal{N}}(A))))(x). \tag{34}$$

It is clear that for a crisp covering $\mathbb{C}$ and a crisp set A the pair $(\underline{\text{apr}}'_{\mathbb{C}}(A), \overline{\text{apr}}'_{\mathbb{C}}(A))$ is obtained. A drawback of this model is that it is quite extreme: if for each $K \in \mathbb{C}$ holds that $K \nsubseteq A$ for a given $A \in \mathscr{F}(U)$, then $\underline{\text{apr}}'_{\mathbb{C},InEx}(A) = \emptyset$.

To end, we discuss the properties of this model.

Proposition 8. *[6] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space and $\mathcal{N}$ an involutive negator.*

- *The pair $(\underline{\text{apr}}'_{\mathbb{C},InEx}, \overline{\text{apr}}'_{\mathbb{C},InEx,\mathcal{N}})$ satisfies (D) with respect to $\mathcal{N}$.*
- *The pair $(\underline{\text{apr}}'_{\mathbb{C},InEx}, \overline{\text{apr}}'_{\mathbb{C},InEx,\mathcal{N}})$ satisfies (INC), (SM), (ID) and (UE).*

Note that this model does not satisfy (CS) [6].

3.3. Fuzzy extensions of the loose covering-based approximation operators

Before discussing different models extending the loose covering-based approximation operators, we recall that for a crisp covering $\mathbb{C}$ it holds that the pair of operators $(\underline{\text{apr}}''_{\mathbb{C}}, \overline{\text{apr}}''_{\mathbb{C}})$ equals the pair of operators $(\underline{\text{apr}}_{N_4^{\mathbb{C}}}, \overline{\text{apr}}_{N_4^{\mathbb{C}}})$ [24]. This follows from a result concerning crisp approximation operators which has been obtained by Yao in [31]. A similar result for fuzzy approximation operators is given by Wu et al. [28].

Theorem 1. *[28] Let $H : \mathscr{F}(U) \to \mathscr{F}(U)$ be a mapping and $\mathcal{T}$ a left-continuous t-norm. The operator H satisfies the following axioms:*

$$(U1)\ \forall A \in \mathscr{F}(U), \forall \alpha \in [0,1] : H(\hat{\alpha} \cap_{\mathcal{T}} A) = \hat{\alpha} \cap_{\mathcal{T}} H(A),$$

$$(U2)\ \forall A_j \in \mathscr{F}(U), j \in J : H\left(\bigcup_{j\in J} A_j\right) = \bigcup_{j\in J} H(A_j),$$

if and only if there exists a fuzzy relation R on U such that $H = \overline{\text{apr}}_{R,\mathcal{T}}$, where the relation-based upper approximation operator $\overline{\text{apr}}_{R,\mathcal{T}}$ is defined by

$$\forall A \in \mathscr{F}(U), \forall x \in U : (\overline{\text{apr}}_{R,\mathcal{T}}(A))(x) = \sup_{y\in U} \mathcal{T}(R(x,y), A(y)).$$

Hence, if an operator on $\mathscr{F}(U)$ satisfies the axioms $(U1)$ and $(U2)$, then it is equivalent to a fuzzy element-based upper approximation operator on U. Moreover, the fuzzy relation R mentioned in the above theorem is defined by

$$\forall x, y \in U : R(x,y) = (H(1_y))(x),$$

where 1_y is the fuzzy set defined by $1_y(y) = 1$ and $1_y(z) = 0$ for each $z \in U \setminus \{y\}$.

3.3.1. Model of Li et al.

Besides defining a fuzzy extension of the tight covering-based approximation operators, Li et al. defined a fuzzy extension of the loose covering-based approximation operators.

Definition 11. [17] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space, $\mathcal{T}$ a t-norm and $\mathcal{I}$ an implicator, then the pair of fuzzy covering-based approximation operators $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}})$ is defined as follows: let $A \in \mathcal{F}(U)$, $x \in U$,

$$(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}(A))(x) = \inf_{K\in\mathbb{C}} \mathcal{I}(K(x), \inf_{y\in U} \mathcal{I}(K(y), A(y))), \tag{35}$$

$$(\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}}(A))(x) = \sup_{K\in\mathbb{C}} \mathcal{T}(K(x), \sup_{y\in U} \mathcal{T}(K(y), A(y))). \tag{36}$$

We prove that the upper approximation operator of this model is equivalent to an element-based one, when a left-continuous t-norm is taken into consideration.

Proposition 9. *Let $(U, \mathbb{C})$ be a fuzzy covering approximation space and $\mathcal{T}$ a left-continuous t-norm, then*

$$\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}} = \overline{\text{apr}}_{N_4^{\mathbb{C}},\mathcal{T}},$$

where $N_4^{\mathbb{C}}$ is defined with respect to $\mathcal{T}$.

Proof. In [17] it is proven that $\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}}$ satisfies axiom (U2). Moreover, we have for $\alpha \in [0, 1]$, $A \in \mathcal{F}(U)$ and $x \in U$ that

$$\begin{aligned}(\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}}(\hat{\alpha} \cap_{\mathcal{T}} A))(x) &= \sup_{K\in\mathbb{C}} \mathcal{T}(K(x), \sup_{y\in U} \mathcal{T}(K(y), \mathcal{T}(\alpha, A(y))))\\ &= \sup_{K\in\mathbb{C}} \sup_{y\in U} \mathcal{T}(K(x), \mathcal{T}(K(y), \mathcal{T}(\alpha, A(y))))\\ &= \sup_{K\in\mathbb{C}} \sup_{y\in U} \mathcal{T}(\alpha, \mathcal{T}(K(x), \mathcal{T}(K(y), A(y))))\\ &= \sup_{y\in U} \mathcal{T}(\alpha, \sup_{K\in\mathbb{C}} \mathcal{T}(K(x), \mathcal{T}(K(y), A(y))))\\ &= (\hat{\alpha} \cap_{\mathcal{T}} \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}}(A))(x)\end{aligned}$$

hence, $\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}}$ satisfies axiom (U1). By Theorem 1, $\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}}$ is equivalent with a fuzzy relation-based upper approximation operator. This fuzzy relation R is defined by, for $x, y \in U$,

$$\begin{aligned}R(x, y) &= (\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}}(1_y))(x)\\ &= \sup_{K\in\mathbb{C}} \mathcal{T}(K(x), \sup_{z\in U} \mathcal{T}(K(z), 1_y(z)))\\ &= \sup_{K\in\mathbb{C}} \mathcal{T}(K(x), K(y))\\ &= N_4^{\mathbb{C}}(x)(y)\end{aligned}$$

thus, we conclude that $\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}} = \overline{\text{apr}}_{R,\mathcal{T}} = \overline{\text{apr}}_{N_4^{\mathbb{C}},\mathcal{T}}$. □

For a left-continuous t-norm, the properties of $\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}}$ are given by Proposition 3. If $\mathcal{T}$ is an IMTL-t-norm with $\mathcal{I}$ its R-implicator, or $\mathcal{I}$ is defined using the $\mathcal{N}$-dual t-conorm $\mathcal{S}$ of the left-continuous t-norm $\mathcal{T}$, we also obtain that $\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}} = \underline{\text{apr}}_{N_4^{\mathbb{C}},\mathcal{I}}$. In general, the properties of $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}})$ are given in the proposition below.

Proposition 10. *Let $(U, \mathbb{C})$ be a fuzzy covering approximation space, $\mathcal{T}$ a t-norm and $\mathcal{I}$ an implicator.*

- *The pair $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}})$ satisfies (D) with respect to the involutive negator $\mathcal{N}$ if $\mathcal{T}$ is an IMTL-t-norm, $\mathcal{I}$ is its R-implicator and $\mathcal{N}$ equals the negator induced by $\mathcal{I}$.*

- *The pair* $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}})$ *satisfies (D) with respect to the involutive negator* $\mathcal{N}$ *if* $\mathcal{I}$ *is the S-implicator with respect to the t-conorm* $\mathcal{S}$ *and the negator* $\mathcal{N}$*, where* $\mathcal{S}$ *is the* $\mathcal{N}$*-dual of* $\mathcal{T}$*.*
- *The pair* $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}})$ *satisfies (A) if* $\mathcal{T}$ *is a left-continuous t-norm and* $\mathcal{I}$ *is its R-implicator.*
- *The pair* $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}})$ *satisfies (INC) if* $\mathcal{I}(1,a) = a$ *for all* $a \in [0,1]$.
- *The pair* $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}})$ *satisfies (SM).*
- *The pair* $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}})$ *satisfies (IU) if* $\mathcal{I}$ *is right-continuous in the second argument and* $\mathcal{T}$ *is left-continuous in the second argument.*
- *The pair* $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}})$ *satisfies (CS) if* $\mathcal{I}(1,a) = a$ *for all* $a \in [0,1]$.
- *The pair* $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}})$ *satisfies (UE).*
- *The pair* $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}})$ *satisfies (CM).*

Proof. To prove that $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}})$ satisfies (A), let A and B be fuzzy sets, $\mathcal{T}$ a left-continuous t-norm and $\mathcal{I}$ its R-implicator. We have

$$\begin{aligned}
\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}}(A) \subseteq B &\Leftrightarrow \forall x \in U\colon \sup_{K\in\mathbb{C}} \mathcal{T}(K(x), \sup_{y\in U} \mathcal{T}(K(y), A(y))) \leq B(x) \\
&\Leftrightarrow \forall x \in U, \forall K \in \mathbb{C}\colon \mathcal{T}(\sup_{y\in U} \mathcal{T}(K(y), A(y)), K(x)) \leq B(x) \\
&\Leftrightarrow \forall x \in U, \forall K \in \mathbb{C}\colon \sup_{y\in U} \mathcal{T}(K(y), A(y)) \leq \mathcal{I}(K(x), B(x)) \\
&\Leftrightarrow \forall x \in U, \forall K \in \mathbb{C}, \forall y \in U\colon \mathcal{T}(K(y), A(y)) \leq \mathcal{I}(K(x), B(x)) \\
&\Leftrightarrow \forall K \in \mathbb{C}, \forall y \in U\colon \mathcal{T}(A(y), K(y)) \leq \inf_{x\in U} \mathcal{I}(K(x), B(x)) \\
&\Leftrightarrow \forall K \in \mathbb{C}, \forall y \in U\colon A(y) \leq \mathcal{I}(K(y), \inf_{x\in U} \mathcal{I}(K(x), B(x))) \\
&\Leftrightarrow \forall y \in U\colon A(y) \leq \inf_{K\in\mathbb{C}} \mathcal{I}(K(y), \inf_{x\in U} \mathcal{I}(K(x), B(x))) \\
&\Leftrightarrow A \subseteq \underline{\text{apr}}''_{\mathbb{C},Li,\mathcal{I}}(B)
\end{aligned}$$

To prove (CM), let $\mathbb{C}, \mathbb{C}'$ be two fuzzy coverings of U such that $\mathbb{C} \preceq \mathbb{C}'$. For $K \in \mathbb{C}$, denote $L_K \in \mathbb{C}'$ such that $K \subseteq L_K$. We prove the monotonicity for the upper approximation operator, for the lower approximation operator, the proof is similar. Let $A \in \mathcal{F}(U)$ and $x \in U$,

$$\begin{aligned}
(\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}}(A))(x) &= \sup_{K\in\mathbb{C}} \mathcal{T}(K(x), \sup_{y\in U} \mathcal{T}(K(y), A(y))) \\
&\leq \sup_{K\in\mathbb{C}} \mathcal{T}(L_K(x), \sup_{y\in U} \mathcal{T}(L_K(y), A(y))) \\
&\leq \sup_{L\in\mathbb{C}'} \mathcal{T}(L(x), \sup_{y\in U} \mathcal{T}(L(y), A(y))) \\
&= (\overline{\text{apr}}''_{\mathbb{C}',Li,\mathcal{T}}(A))(x)
\end{aligned}$$

The other properties are proven in [17]. □

3.3.2. Model of representation by levels

Another possible fuzzy extension of the loose covering-based approximation operators is constructed using representation by levels.

Definition 12. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite and $A \in \mathcal{F}(U)$. The fuzzy set $\overline{\text{apr}}''_{\mathbb{C},RBL}(A)$ is represented by the RL $(\Lambda_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)}, \rho_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)})$, with

$$\Lambda_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)} = \Lambda_A \cup \Lambda_{\mathbb{C}} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}, m \in \mathbb{N} \setminus \{0\},$$

$$\rho_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)}(\alpha) = \bigcup\{K_\alpha \mid K \in \mathbb{C}, K_\alpha \cap A_\alpha \neq \emptyset\},$$

for all $\alpha \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)}$. To obtain the membership degree of x in $\overline{\text{apr}}''_{\mathbb{C},RBL}(A)$, we compute the fuzzy summary:

$$(\overline{\text{apr}}''_{\mathbb{C},RBL}(A))(x) = \sum_{\{\alpha_i \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)} | x \in \rho_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)}(\alpha_i)\}} (\alpha_i - \alpha_{i+1}),$$

where we have ranked the elements of $\Lambda_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)}$ as follows:

$$1 = \alpha_1 > \alpha_2 > \ldots > \alpha_m > \alpha_{m+1} = 0.$$

The lower approximation operator $\underline{\text{apr}}''_{\mathbb{C},RBL}$ is obtained in a similar way, by taking $\Lambda_{\underline{\text{apr}}''_{\mathbb{C},RBL}(A)} = \Lambda_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)}$ and

$$\rho_{\underline{\text{apr}}''_{\mathbb{C},RBL}(A)}(\alpha) = \text{co}(\rho_{\overline{\text{apr}}''_{\mathbb{C},RBL}(\text{co}(A))}(\alpha))$$

for each $\alpha \in \Lambda_{\underline{\text{apr}}''_{\mathbb{C},RBL}(A)}$.

The upper approximation operator defined above can also be computed in the following way:

Theorem 2. *Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, $A \in \mathscr{F}(U)$ and $x \in U$. Let $1 \leq k \leq m$ such that*

$$\alpha_k = \max\{\alpha \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)} \mid x \in \rho_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)}(\alpha)\},$$

then $(\overline{\text{apr}}''_{\mathbb{C},RBL}(A))(x) = \alpha_k$.

Proof. We first prove that $\overline{\text{apr}}''_{\mathbb{C},RBL}(A)$ is represented by nested levels. Let $\beta, \gamma \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)}$ with $\beta \geq \gamma$ and assume $y \in \rho_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)}(\beta)$, then

$$y \in \bigcup\{K_\beta \mid K \in \mathbb{C}, K_\beta \cap A_\beta \neq \emptyset\}.$$

Let $K \in \mathbb{C}$ be such that $y \in K_\beta$ and $K_\beta \cap A_\beta \neq \emptyset$. Since $K(y) \geq \beta \geq \gamma$, $y \in K_\gamma$. Furthermore, $K_\gamma \cap A_\gamma \supseteq K_\beta \cap A_\beta \neq \emptyset$. Hence, $y \in \rho_{\overline{\text{apr}}''_{\mathbb{C},RBL}(A)}(\gamma)$ and thus, we have a nested level representation of $\overline{\text{apr}}''_{\mathbb{C},RBL}(A)$. Therefore, we obtain for $x \in U$ that

$$(\overline{\text{apr}}''_{\mathbb{C},RBL}(A))(x) = (\alpha_k - \alpha_{k+1}) + (\alpha_{k+1} - \alpha_{k+2}) + \ldots + (\alpha_{m-1} - \alpha_m) + (\alpha_m - 0) = \alpha_k. \qquad \square$$

Similar as with the fuzzy loose upper approximation operator of Li et al., the upper approximation operator of this model is equivalent with an element-based one, when the minimum t-norm is considered.

Proposition 11. *Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite and $\mathscr{T}_{\min}$ the minimum t-norm, then*

$$\overline{\text{apr}}''_{\mathbb{C},RBL} = \overline{\text{apr}}_{N^{\mathbb{C}}_{4,\min},\mathscr{T}_{\min}},$$

where $N^{\mathbb{C}}_{4,\min}$ denotes the fuzzy neighborhood operator $N^{\mathbb{C}}_4$ defined using the minimum t-norm.

Proof. By Proposition 2, $\overline{\text{apr}}''_{\mathbb{C},RBL}$ satisfies (U1) with respect to $\mathscr{T}_{\min}$ and (U2), since for a crisp covering $\mathbb{C}$, $\overline{\text{apr}}''_{\mathbb{C}}$ satisfies the crisp equivalents of (U1) and (U2). Furthermore, by Theorem 2 it holds for $x, y \in U$ that

$$\begin{aligned}
R(x, y) &= \max\{\alpha \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},RBL}(1_y)} \mid x \in \rho_{\overline{\text{apr}}''_{\mathbb{C},RBL}(1_y)}(\alpha)\} \\
&= \max\{\alpha \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},RBL}(1_y)} \mid \exists K \in \mathbb{C}\colon K(x) \geq \alpha \wedge K(y) \geq \alpha\} \\
&= \max\{\alpha \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},RBL}(1_y)} \mid \sup_{K \in \mathbb{C}} \min(K(x), K(y)) \geq \alpha\} \\
&= \sup_{K \in \mathbb{C}} \min(K(x), K(y)) \\
&= N^{\mathbb{C}}_{4,\min}(x)(y)
\end{aligned}$$

Thus, we conclude that $\overline{\text{apr}}''_{\mathbb{C},RBL} = \overline{\text{apr}}_{R,\mathcal{T}_{\min}} = \overline{\text{apr}}_{N^{\mathbb{C}}_{4,\min},\mathcal{T}_{\min}}$. □

Corollary 1. *When the minimum operator is used to define $N_4^{\mathbb{C}}$ and $\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}}$, it holds that $\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}} = \overline{\text{apr}}''_{\mathbb{C},RBL}$. If another left-continuous t-norm $\mathcal{T}$ is used, we obtain that $(\overline{\text{apr}}''_{\mathbb{C},Li,\mathcal{T}}(A))(x) \leq (\overline{\text{apr}}''_{\mathbb{C},RBL}(A))(x)$ for each $A \in \mathcal{F}(U)$ and $x \in U$, as the minimum operator is the largest t-norm.*

Note that $\overline{\text{apr}}''_{\mathbb{C},RBL}$ does not satisfy (U1) with respect to every left-continuous t-norm, as illustrated in the next example.

Example 3. Let $U = \{x, y\}$ and $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0.3/y$ and $K_2 = 0.8/x + 1/y$. Let $A = 0.7/x + 0.8/y$ and $B = 0.8/x + 0.4/y$ be fuzzy sets in U and let $\mathcal{T}_p$ be the product t-norm defined by $\mathcal{T}_p(a,b) = a \cdot b$ for $a, b \in [0,1]$. Then $(\overline{\text{apr}}''_{\mathbb{C},RBL}(A) \cap_{\mathcal{T}_p} \overline{\text{apr}}''_{\mathbb{C},RBL}(B))(x) = 0.8 \cdot 0.8 = 0.64$ and $\overline{\text{apr}}''_{\mathbb{C},RBL}(A \cap_{\mathcal{T}_p} B)(x) = 0.56$.

Let $\mathcal{N}$ be an involutive negator, from the above proposition and Proposition 3, we obtain that $\underline{\text{apr}}''_{\mathbb{C},RBL} = \underline{\text{apr}}_{\text{N},\mathcal{I}}$, with $\text{N} = N^{\mathbb{C}}_{4,\mathcal{T}_{\min}}$ and $\mathcal{I}(a,b) = \mathcal{N}(\min(a, \mathcal{N}(b)))$ for $a, b \in [0,1]$.

To end this section, we discuss the properties of this model.

Proposition 12. *Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite.*

- *The pair $(\underline{\text{apr}}''_{\mathbb{C},RBL}, \overline{\text{apr}}''_{\mathbb{C},RBL})$ satisfies (D) with respect to an involutive negator $\mathcal{N}$.*
- *The pair $(\underline{\text{apr}}''_{\mathbb{C},RBL}, \overline{\text{apr}}''_{\mathbb{C},RBL})$ satisfies (A), (INC), (SM), (IU), (CS), (UE) and (CM).*

Proof. Follows immediately from Proposition 2. □

4. Interrelationships between the different fuzzy covering-based rough set models

In this section, we discuss the interrelationships between the different fuzzy covering-based rough set models discussed in the previous section. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, $\mathcal{T}$ an IMTL-t-norm, $\mathcal{I}$ its R-implicator and $\mathcal{N}$ the induced negator of $\mathcal{I}$. These parameters will be used for all the fuzzy neighborhood operators and fuzzy covering-based approximation operators. Note that for $\mathcal{I}$ it holds that $\mathcal{I}(a,b) = \mathcal{S}(\mathcal{N}(a), b)$ for $a, b \in [0,1]$, with $\mathcal{S}$ the $\mathcal{N}$-dual t-conorm of $\mathcal{T}$. We will use an IMTL-t-norm to guarantee duality. This way, we only need to compare the lower approximation operators. Furthermore, some results presented in this section make use of the properties of an IMTL-t-norm, its R-implicator and the induced negator which is involutive. Some examples of IMTL-t-norms which can be used are the Łukasiewicz t-norm $\mathcal{T}(a,b) = \max(a + b - 1, 0)$ such that $\mathcal{I}(a,b) = \min(1 - a + b, 1)$ and $\mathcal{N}(a) = 1 - a$ for $a, b \in [0,1]$ or the nilpotent minimum

$$\mathcal{T}(a,b) = \begin{cases} \min(a,b) & a + b > 1 \\ 0 & \text{else} \end{cases}$$

such that

$$\mathcal{I}(a,b) = \begin{cases} 1 & a \leq b \\ \max(1 - a, b) & a > b \end{cases}$$

and $\mathcal{N}(a) = 1 - a$ for $a, b \in [0,1]$. As we assume U and $\mathbb{C}$ to be finite, we will use the minimum and maximum operator instead of the infimum and supremum operator.

Given the parameters discussed above, the fuzzy neighborhood operators presented in Table 1 result in 16 pairs of fuzzy neighborhood-based approximation operators $(\underline{\text{apr}}_{\text{N},\mathcal{I}}, \overline{\text{apr}}_{\text{N},\mathcal{T}})$. All operators are different due to the following proposition.

Proposition 13. *Let N and N′ be two fuzzy neighborhood operators on U, then $\text{N} = \text{N}'$ if and only if $\underline{\text{apr}}_{\text{N},\mathcal{I}} = \underline{\text{apr}}_{\text{N}',\mathcal{I}}$ if and only if $\overline{\text{apr}}_{\text{N},\mathcal{T}} = \overline{\text{apr}}_{\text{N}',\mathcal{T}}$.*

Proof. If $\mathrm{N}=\mathrm{N}'$, then $\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}=\underline{\mathrm{apr}}_{\mathrm{N}',\mathscr{I}}$ holds trivially. On the other hand, let $\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}=\underline{\mathrm{apr}}_{\mathrm{N}',\mathscr{I}}$ and let $x,y\in U$, then

$$\begin{aligned}(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}(U\setminus\{y\}))(x)=(\underline{\mathrm{apr}}_{\mathrm{N}',\mathscr{I}}(U\setminus\{y\}))(x)&\Rightarrow\mathscr{I}(\mathrm{N}(x)(y),0)=\mathscr{I}(\mathrm{N}'(x)(y),0)\\&\Rightarrow\mathscr{N}(\mathrm{N}(x)(y))=\mathscr{N}(\mathrm{N}'(x)(y))\\&\Rightarrow\mathrm{N}(x)(y)=\mathrm{N}'(x)(y),\end{aligned}$$

since $\mathscr{N}$ is involutive. The other equivalence follows from duality. □

Furthermore, with each β-fuzzy neighborhood operator $N^{\mathbb{C}}_{\beta,\mathrm{Ma}}$, $\beta\in(0,1]$, a pair of fuzzy neighborhood-based approximation operators is associated. In this section, we only consider the case $\beta=1$, as for $\beta<1$ the operator $N^{\mathbb{C}}_{\beta,\mathrm{Ma}}$ is not reflexive. We will see that the fuzzy neighborhood operator $N^{\mathbb{C}}_{1,\mathrm{Ma}}$ differs from the operators in Table 1. Therefore, we obtain 17 pairs of fuzzy neighborhood-based approximation operators in Table 2.

For the IMTL-t-norm $\mathscr{T}$ and its R-implicator $\mathscr{I}$, we obtain that $\xi[\mathscr{I}]=\mathscr{T}$ and thus,

$$(\underline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}},\overline{\mathrm{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}})=(\underline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I}},\overline{\mathrm{apr}}'_{\mathbb{C},In,\mathscr{I},\mathscr{N}})$$

which was proved in [6]. Hence, we obtain four pairs of fuzzy tight covering-based approximation operators (Numbers 18–21). In addition, by Proposition 9 we obtain that

$$(\underline{\mathrm{apr}}''_{\mathbb{C},Li,\mathscr{I}},\overline{\mathrm{apr}}''_{\mathbb{C},Li,\mathscr{T}})=(\underline{\mathrm{apr}}_{N_4^{\mathbb{C}},\mathscr{I}},\overline{\mathrm{apr}}_{N_4^{\mathbb{C}},\mathscr{T}}),$$

hence, this model coincides with the operators of Number 10. Finally, we add the pair of fuzzy loose covering-based approximation operators by representation by levels to Table 2. As a result, we obtain 22 different pairs of fuzzy covering-based approximation operators.

The main purpose of this section is to study the interrelationships between these 22 models with respect to the partial order relation $\le$, defined as follows: let $(\underline{\mathrm{apr}}_1,\overline{\mathrm{apr}}_1)$ and $(\underline{\mathrm{apr}}_2,\overline{\mathrm{apr}}_2)$ be two pairs of dual approximation operators, then we define

$$(\underline{\mathrm{apr}}_1,\overline{\mathrm{apr}}_1)\le(\underline{\mathrm{apr}}_2,\overline{\mathrm{apr}}_2)\Leftrightarrow\forall A\in\mathscr{F}(U):\underline{\mathrm{apr}}_1(A)\subseteq\underline{\mathrm{apr}}_2(A)\Leftrightarrow\forall A\in\mathscr{F}(U):\overline{\mathrm{apr}}_2(A)\subseteq\overline{\mathrm{apr}}_1(A).$$

When $(\underline{\mathrm{apr}}_1,\overline{\mathrm{apr}}_1)\le(\underline{\mathrm{apr}}_2,\overline{\mathrm{apr}}_2)$, we say that the pair $(\underline{\mathrm{apr}}_2,\overline{\mathrm{apr}}_2)$ is more accurate than the pair $(\underline{\mathrm{apr}}_1,\overline{\mathrm{apr}}_1)$, as the former pair provides approximations closer to the approximated set than the latter pair. Moreover, note that

$$\forall A\in\mathscr{F}(U):\frac{|\underline{\mathrm{apr}}_1(A)|}{|\overline{\mathrm{apr}}_1(A)|}\le\frac{|\underline{\mathrm{apr}}_2(A)|}{|\overline{\mathrm{apr}}_2(A)|}.$$

For fuzzy neighborhood-based approximation operators, we have the following proposition.

Proposition 14. *Let* N *and* N′ *be two fuzzy neighborhood operators on U, then* $\mathrm{N}\preceq\mathrm{N}'$ *if and only if*

$$(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}},\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}})\le(\underline{\mathrm{apr}}_{\mathrm{N}',\mathscr{I}},\overline{\mathrm{apr}}_{\mathrm{N}',\mathscr{T}}),$$

where $\mathrm{N}\preceq\mathrm{N}'$ *is defined by* $\forall x,y\in U:\mathrm{N}(x)(y)\le\mathrm{N}'(x)(y)$.

Proof. Analogously to the proof of Proposition 13. □

Hence, the interrelationships between the models 1–17 can be obtained by studying the interrelationships between the fuzzy neighborhood operators on which they are based. In [8], the Hasse diagram with respect to the partial order relation $\preceq$ of the 16 groups of fuzzy neighborhood operators stated in Table 1 is constructed. It is presented in Fig. 1a. Note that this Hasse diagram also holds for a left-continuous t-norm and its R-implicator.

We now want to add the fuzzy neighborhood operator $N^{\mathbb{C}}_{1,\mathrm{Ma}}$ to the Hasse diagram presented in Fig. 1a. We have the following inclusion relations:

Proposition 15. *Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, then*

Table 2
Fuzzy covering-based rough set models.

Number	N	Pair of approximation operators
1	$A1$	$(\underline{\text{apr}}_{N_1^{\mathbb{C}},\mathscr{I}}, \overline{\text{apr}}_{N_1^{\mathbb{C}},\mathscr{T}})$, $(\underline{\text{apr}}_{N_1^{\mathbb{C}_1},\mathscr{I}}, \overline{\text{apr}}_{N_1^{\mathbb{C}_1},\mathscr{T}})$, $(\underline{\text{apr}}_{N_1^{\mathbb{C}_3},\mathscr{I}}, \overline{\text{apr}}_{N_1^{\mathbb{C}_3},\mathscr{T}})$, $(\underline{\text{apr}}_{N_1^{\mathbb{C}_\cap},\mathscr{I}}, \overline{\text{apr}}_{N_1^{\mathbb{C}_\cap},\mathscr{T}})$
2	$A2$	$(\underline{\text{apr}}_{N_2^{\mathbb{C}_3},\mathscr{I}}, \overline{\text{apr}}_{N_2^{\mathbb{C}_3},\mathscr{T}})$
3	B	$(\underline{\text{apr}}_{N_3^{\mathbb{C}_1},\mathscr{I}}, \overline{\text{apr}}_{N_3^{\mathbb{C}_1},\mathscr{T}})$
4	C	$(\underline{\text{apr}}_{N_3^{\mathbb{C}_3},\mathscr{I}}, \overline{\text{apr}}_{N_3^{\mathbb{C}_3},\mathscr{T}})$
5	D	$(\underline{\text{apr}}_{N_4^{\mathbb{C}_3},\mathscr{I}}, \overline{\text{apr}}_{N_4^{\mathbb{C}_3},\mathscr{T}})$
6	E	$(\underline{\text{apr}}_{N_2^{\mathbb{C}},\mathscr{I}}, \overline{\text{apr}}_{N_2^{\mathbb{C}},\mathscr{T}})$, $(\underline{\text{apr}}_{N_2^{\mathbb{C}_1},\mathscr{I}}, \overline{\text{apr}}_{N_2^{\mathbb{C}_1},\mathscr{T}})$
7	$F1$	$(\underline{\text{apr}}_{N_3^{\mathbb{C}},\mathscr{I}}, \overline{\text{apr}}_{N_3^{\mathbb{C}},\mathscr{T}})$, $(\underline{\text{apr}}_{N_3^{\mathbb{C}_2},\mathscr{I}}, \overline{\text{apr}}_{N_3^{\mathbb{C}_2},\mathscr{T}})$, $(\underline{\text{apr}}_{N_3^{\mathbb{C}_\cap},\mathscr{I}}, \overline{\text{apr}}_{N_3^{\mathbb{C}_\cap},\mathscr{T}})$
8	$F2$	$(\underline{\text{apr}}_{N_1^{\mathbb{C}_2},\mathscr{I}}, \overline{\text{apr}}_{N_1^{\mathbb{C}_2},\mathscr{T}})$
9	G	$(\underline{\text{apr}}_{N_1^{\mathbb{C}_4},\mathscr{I}}, \overline{\text{apr}}_{N_1^{\mathbb{C}_4},\mathscr{T}})$
10	$H1$	$(\underline{\text{apr}}_{N_4^{\mathbb{C}},\mathscr{I}}, \overline{\text{apr}}_{N_4^{\mathbb{C}},\mathscr{T}})$, $(\underline{\text{apr}}_{N_4^{\mathbb{C}_2},\mathscr{I}}, \overline{\text{apr}}_{N_4^{\mathbb{C}_2},\mathscr{T}})$, $(\underline{\text{apr}}_{N_4^{\mathbb{C}_\cap},\mathscr{I}}, \overline{\text{apr}}_{N_4^{\mathbb{C}_\cap},\mathscr{T}})$, $(\underline{\text{apr}}''_{\mathbb{C},Li,\mathscr{I}}, \overline{\text{apr}}''_{\mathbb{C},Li,\mathscr{T}})$
11	$H2$	$(\underline{\text{apr}}_{N_2^{\mathbb{C}_2},\mathscr{I}}, \overline{\text{apr}}_{N_2^{\mathbb{C}_2},\mathscr{T}})$
12	I	$(\underline{\text{apr}}_{N_2^{\mathbb{C}_4},\mathscr{I}}, \overline{\text{apr}}_{N_2^{\mathbb{C}_4},\mathscr{T}})$
13	J	$(\underline{\text{apr}}_{N_3^{\mathbb{C}_4},\mathscr{I}}, \overline{\text{apr}}_{N_3^{\mathbb{C}_4},\mathscr{T}})$
14	K	$(\underline{\text{apr}}_{N_4^{\mathbb{C}_4},\mathscr{I}}, \overline{\text{apr}}_{N_4^{\mathbb{C}_4},\mathscr{T}})$
15	L	$(\underline{\text{apr}}_{N_4^{\mathbb{C}_1},\mathscr{I}}, \overline{\text{apr}}_{N_4^{\mathbb{C}_1},\mathscr{T}})$
16	M	$(\underline{\text{apr}}_{N_2^{\mathbb{C}_\cap},\mathscr{I}}, \overline{\text{apr}}_{N_2^{\mathbb{C}_\cap},\mathscr{T}})$
17	$N_{1,\text{Ma}}^{\mathbb{C}}$	$(\underline{\text{apr}}_{N_{1,\text{Ma}}^{\mathbb{C}},\mathscr{I}}, \overline{\text{apr}}_{N_{1,\text{Ma}}^{\mathbb{C}},\mathscr{T}})$
18		$(\underline{\text{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}}, \overline{\text{apr}}'_{\mathbb{C},Li,\mathscr{T},\mathscr{I}})$, $(\underline{\text{apr}}'_{\mathbb{C},In,\mathscr{I}}, \overline{\text{apr}}'_{\mathbb{C},In,\mathscr{I},\mathscr{N}})$
19		$(\underline{\text{apr}}'_{\mathbb{C},Wu}, \overline{\text{apr}}'_{\mathbb{C},Wu})$
20		$(\underline{\text{apr}}'_{\mathbb{C},RBL}, \overline{\text{apr}}'_{\mathbb{C},RBL})$
21		$(\underline{\text{apr}}'_{\mathbb{C},InEx}, \overline{\text{apr}}'_{\mathbb{C},InEx,\mathscr{N}})$
22		$(\underline{\text{apr}}''_{\mathbb{C},RBL}, \overline{\text{apr}}''_{\mathbb{C},RBL})$

1. $N_1^{\mathbb{C}} \preceq N_{1,Ma}^{\mathbb{C}}$
2. $N_{1,Ma}^{\mathbb{C}} \preceq N_2^{\mathbb{C}}$
3. $N_{1,Ma}^{\mathbb{C}} \preceq N_2^{\mathbb{C}_\cap}$

Proof. Let $x, y \in U$.

1. $N_1^{\mathbb{C}}(x)(y) = \min_{K\in\mathbb{C}} \mathscr{I}(K(x), K(y)) \leq \min_{K\in\mathbb{C}, K(x)=1} \mathscr{I}(K(x), K(y)) = \min_{K\in\mathbb{C}, K(x)=1} K(y) = N_{1,\text{Ma}}^{\mathbb{C}}(x)(y)$.
2.

$$\begin{aligned}
N_2^{\mathbb{C}}(x)(y) &= \max_{K\in\text{md}(\mathbb{C},x)} \mathscr{T}(K(x), K(y)) \\
&\geq \max_{K\in\text{md}(\mathbb{C},x), K(x)=1} \mathscr{T}(K(x), K(y)) \\
&= \max_{K\in\text{md}(\mathbb{C},x), K(x)=1} K(y) \\
&\geq \min_{K\in\text{md}(\mathbb{C},x), K(x)=1} K(y)
\end{aligned}$$

(a) Lattice of the fuzzy neighborhood operators from Table 1

(b) Lattice of the fuzzy neighborhood operators with the operator of Ma added

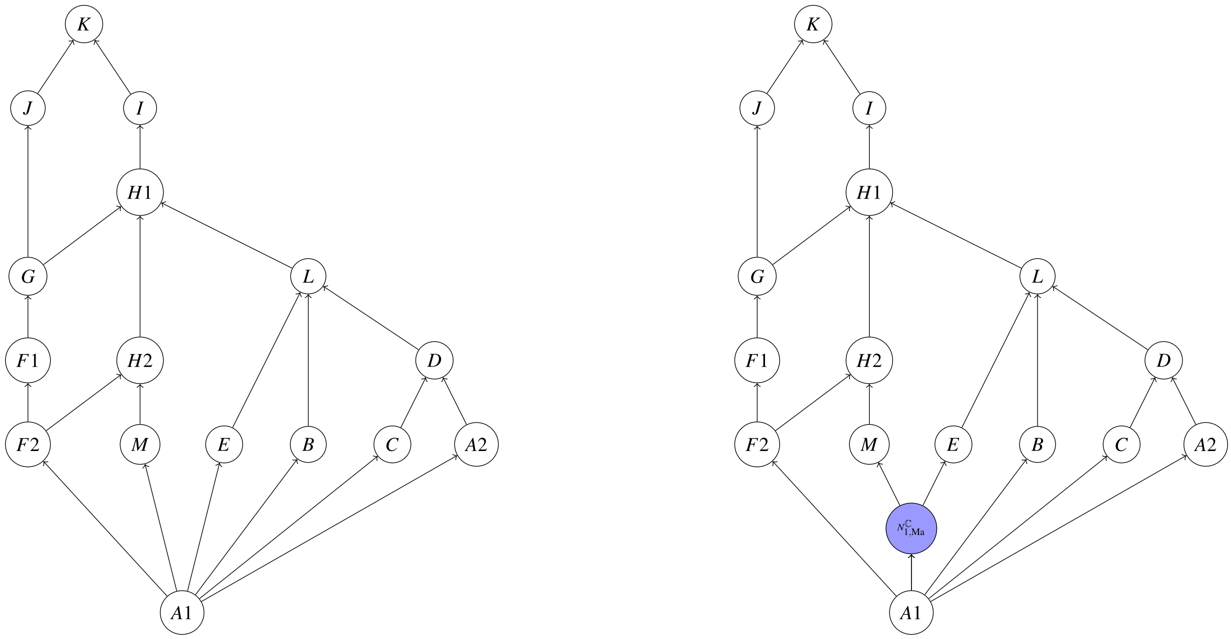

Fig. 1. Lattices of fuzzy neighborhood operators.

$$\geq \min_{K \in \mathbb{C}, K(x)=1} K(y)$$

$$= N_{1,\mathrm{Ma}}^{\mathbb{C}}(x)(y)$$

3. By definition of $\mathbb{C}_\cap$, we have that $N_{1,\mathrm{Ma}}^{\mathbb{C}}(x)(y) = \min\{K(y) \mid K \in \mathbb{C}_\cap, K(x) = 1\}$. By analogy of (2), we obtain that $N_2^{\mathbb{C}_\cap}(x)(y) \geq \min_{K \in \mathbb{C}_\cap, K(x)=1} K(y) = N_{1,\mathrm{Ma}}^{\mathbb{C}}(x)(y)$. □

Remark 1. Proposition 15 also hold for every t-norm $\mathscr{T}$ and every implicator $\mathscr{I}$ satisfying $\mathscr{I}(1, a) = a$ for $a \in [0, 1]$.

As $\preceq$ is a transitive relation, we also have $N_{1,\mathrm{Ma}}^{\mathbb{C}} \preceq \mathrm{N}$ for $\mathrm{N} \in \{H1, H2, I, K, L\}$. There are no other partial order relations as illustrated in the next example.

Example 4. Let $\mathscr{T}$ be the Łukasiewicz t-norm, $\mathscr{I}$ its R-implicator and $U = \{x, y, z\}$. Let $\mathbb{C} = \{K_1, K_2, K_3, K_4, K_5\}$ with $K_1 = 1/x + 0.5/y + 0.8/z$, $K_2 = 1/x + 0.8/y + 0.5/z$, $K_3 = 1/x + 0.8/y + 1/z$, $K_4 = 0.5/x + 0.8/y + 0.5/z$ and $K_5 = 0.5/x + 1/y + 0.5/z$, then

- $N_{1,\mathrm{Ma}}^{\mathbb{C}}(x)(y) = 0.5 < 0.8 = \mathrm{N}(x)(y)$ for $\mathrm{N} \in \{B, E, F1, G, H1, H2, I, J, K, L, M\}$,
- $N_{1,\mathrm{Ma}}^{\mathbb{C}}(x)(y) = 0.5 < 0.7 = \mathrm{N}(x)(y)$ for $\mathrm{N} \in \{C, D\}$,
- $N_{1,\mathrm{Ma}}^{\mathbb{C}}(y)(x) = 0.5 < 0.7 = \mathrm{N}(y)(x)$ for $\mathrm{N} \in \{A2\}$,
- $N_{1,\mathrm{Ma}}^{\mathbb{C}}(z)(y) = 0.8 > 0.7 = \mathrm{N}(z)(y)$ for $\mathrm{N} \in \{A1, A2, B, C, D, F1, F2\}$.

On the other hand, let $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 1/y + 0/z$ and $K_2 = 0.8/x + 0/y + 1/z$, then

- $N_{1,\mathrm{Ma}}^{\mathbb{C}}(x)(y) = 1 > 0.2 = \mathrm{N}(x)(y)$ for $\mathrm{N} \in \{G, J\}$.

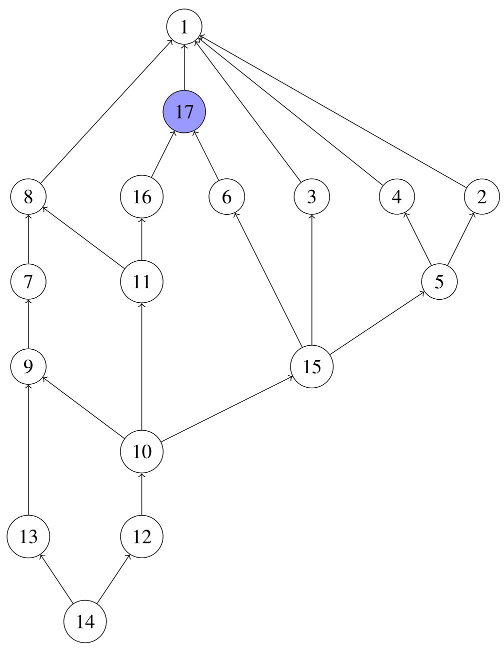

Fig. 2. Hasse diagram of the first 17 lower approximation operators from Table 2.

Finally, let $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0/y + 0/z$ and $K_2 = 1/x + 1/y + 1/z$, then

- $N^{\mathbb{C}}_{1,\mathrm{Ma}}(x)(y) = 0 < 1 = \mathrm{N}(x)(y)$ for $\mathrm{N} \in \{F2\}$.

Hence, the lattice of the 17 fuzzy neighborhood operators is given in Fig. 1b. By Propositions 14 and 15, the Hasse diagram with respect to $\leq$ of the first 17 fuzzy covering-based lower approximation operators is given in Fig. 2.

We now want to add the four fuzzy-covering based tight lower approximation operators to Fig. 2. First, note that in [6], the following partial order relations were proved:

Proposition 16. *[6] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, then*

- $\underline{\mathrm{apr}}'_{\mathbb{C},InEx} \leq \underline{\mathrm{apr}}'_{\mathbb{C},RBL}$*,*
- $\underline{\mathrm{apr}}'_{\mathbb{C},RBL} \leq \underline{\mathrm{apr}}'_{\mathbb{C},Wu}$*,*
- $\underline{\mathrm{apr}}'_{\mathbb{C},InEx} \leq \underline{\mathrm{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}$ *for a left-continuous t-norm and its R-implicator.*

Hence, all three inclusion relations hold for the assumed parameters in this section. Moreover, we can prove the following partial order relation:

Proposition 17. *Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, then $\underline{\mathrm{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}} \leq \underline{\mathrm{apr}}'_{\mathbb{C},Wu}$.*

Proof. Let $K \in \mathbb{C}$ and note that $K_{K(x)} = \{y \in U \mid K(y) \leq K(x)\}$. We have that

$$\begin{aligned}
\mathcal{T}(K(x), \min_{y \in U} \mathcal{I}(K(y), A(y))) &\leq \mathcal{T}(K(x), \min_{y \in K_{K(x)}} \mathcal{I}(K(y), A(y))) \\
&\leq \mathcal{T}(K(x), \min_{y \in K_{K(x)}} \mathcal{I}(K(x), A(y))) \\
&= \mathcal{T}(K(x), \mathcal{I}(K(x), \min_{y \in K_{K(x)}} A(y)))
\end{aligned}$$

$$\leq \min_{y \in K_{K(x)}} A(y),$$

where we have used various properties which hold for a left-continuous t-norm and its R-implicator. As this holds for every $K \in \mathbb{C}$, we obtain that $\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}} \leq \underline{\text{apr}}'_{\mathbb{C},Wu}$. □

Remark 2. Proposition 17 also holds for a left-continuous t-norm $\mathcal{T}$ and its R-implicator $\mathcal{I}$.

Propositions 16 and 17 contain the only partial order relations which hold between these four fuzzy covering-based lower approximation operators as illustrated in the next example.

Example 5. Let $U = \{x, y, z\}$, let $\mathcal{T}$ be the Łukasiewicz t-norm and $\mathcal{I}$ its R-implicator. Let $\mathbb{C} = \{K_1, K_2, K_3\}$ with $K_1 = 0.6/x + 0.6/y + 0.6/z$, $K_2 = 1/x + 1/y + 1/z$ and $K_3 = 0.9/x + 0.8/y + 0.6/z$. Let $A = 0.9/x + 0.7/y + 0.6/z$, then

- $\underline{\text{apr}}'_{\mathbb{C},InEx}(A) = 0.6/x + 0.6/y + 0.6/z$,
- $\underline{\text{apr}}'_{\mathbb{C},RBL}(A) = \underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}(A) = 0.8/x + 0.7/y + 0.6/z$,
- $\underline{\text{apr}}'_{\mathbb{C},Wu}(A) = 0.9/x + 0.7/y + 0.6/z$.

To illustrate that the model of Li et al. and the model of representation by levels are incomparable with each other, let $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0/y + 0.6/z$ and $K_2 = 0/x + 1/y + 1/z$, and $A = 1/x + 0/y + 0.4/z$, then $(\underline{\text{apr}}'_{\mathbb{C},RBL}(A))(x) = 0.8$ and $(\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}(A))(x) = 0.4$, when the nilpotent minimum and its R-implicator is used. On the other hand, let $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0.7/y + 0.7/z$ and $K_2 = 0/x + 1/y + 1/z$, and $A = 0.5/x + 0.5/y + 0.4/z$, then $(\underline{\text{apr}}'_{\mathbb{C},RBL}(A))(x) = 0.4$ and $(\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}(A))(x) = 0.5$, when the Łukasiewicz t-norm and implicator are used.

In order to add the four fuzzy covering-based tight approximation operators to Fig. 2, we discuss their comparability with respect to $\leq$ to the 17 fuzzy neighborhood-based lower approximation operators. However, note that for a crisp covering $\mathbb{C}$, the approximation operator $\underline{\text{apr}}'_{\mathbb{C}}$ is incomparable to the approximation operators $\underline{\text{apr}}_{\text{N}}$ with $\text{N} \in \{B, C, D, F1, F2, G, J\}$ [7], hence, the four fuzzy covering-based tight approximation operators are incomparable with these seven fuzzy neighborhood-based approximation operators. Moreover, we only need to discuss the partial order relations which hold for a crisp covering. Thus, we need to study the following partial order relations for $\underline{\text{apr}} \in \{\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}, \underline{\text{apr}}'_{\mathbb{C},Wu}, \underline{\text{apr}}'_{\mathbb{C},RBL}, \underline{\text{apr}}'_{\mathbb{C},InEx}\}$:

- $\underline{\text{apr}} \leq \underline{\text{apr}}_{\text{N},\mathcal{I}}$ for $\text{N} \in \{A1, A2, N_{1,\text{Ma}}^{\mathbb{C}}\}$,
- $\underline{\text{apr}}_{\text{N},\mathcal{I}} \leq \underline{\text{apr}}$ for $\text{N} \in \{E, H1, H2, I, K, L, M, N_{1,\text{Ma}}^{\mathbb{C}}\}$.

The following partial order relations hold:

Proposition 18. *Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, then*

1. $\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}} \leq \underline{\text{apr}}_{A1,\mathcal{I}}$,
2. $\underline{\text{apr}}_{N_{1,Ma}^{\mathbb{C}},\mathcal{I}} \leq \underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}$.

Proof. Let $A \in \mathcal{F}(U)$ and $x \in U$.

1. First note that $\forall a, b, c \in [0, 1]$: $\mathcal{T}(a, \mathcal{I}(b, c)) \leq \mathcal{I}(\mathcal{I}(a, b), c)$ by the fact that

$$\mathcal{I}(a, \mathcal{I}(\mathcal{I}(a, b), c)) = \mathcal{I}(\mathcal{T}(a, \mathcal{I}(a, b)), c) \geq \mathcal{I}(b, c).$$

$$(\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}(A))(x) = \max_{K \in \mathbb{C}} \mathcal{T}(K(x), \min_{y \in U} \mathcal{I}(K(y), A(y)))$$

$$= \max_{K\in\mathbb{C}} \min_{y\in U} \mathcal{T}(K(x), \mathcal{I}(K(y), A(y)))$$
$$\leq \max_{K\in\mathbb{C}} \min_{y\in U} \mathcal{I}(\mathcal{I}(K(x), K(y)), A(y))$$
$$= \max_{K\in\mathbb{C}} \mathcal{I}\left(\min_{y\in U} \mathcal{I}(K(x), K(y)), A(y)\right)$$
$$= (\underline{\text{apr}}_{A1,\mathcal{I}}(A))(x)$$

2.

$$(\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}(A))(x) \geq \max_{K\in\mathbb{C},K(x)=1} \mathcal{T}(K(x), \min_{y\in U} \mathcal{I}(K(y), A(y)))$$
$$= \max_{K\in\mathbb{C},K(x)=1} \min_{y\in U} \mathcal{I}(K(y), A(y))$$
$$= \min_{y\in U} \mathcal{I}\left(\min_{K\in\mathbb{C},K(x)=1} K(y), A(y)\right)$$
$$= \min_{y\in U} \mathcal{I}(N^{\mathbb{C}}_{1,\text{Ma}}(x)(y), A(y))$$
$$= (\underline{\text{apr}}_{N^{\mathbb{C}}_{1,\text{Ma}},\mathcal{I}}(A))(x) \qquad \square$$

Remark 3. Proposition 18 also holds for a left-continuous t-norm $\mathcal{T}$ and its R-implicator $\mathcal{I}$.

By transitivity of the partial order relations $\leq$ we also have the following results:

Corollary 2. *Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, then*

1. $\underline{\text{apr}}'_{\mathbb{C},InEx} \leq \underline{\text{apr}}_{\text{N},\mathcal{I}}$ *for* $\text{N} \in \{A1\}$,
2. $\underline{\text{apr}}_{\text{N},\mathcal{I}} \leq \underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}$ *for* $\text{N} \in \{E, H1, H2, I, K, L, M\}$,
3. $\underline{\text{apr}}_{\text{N},\mathcal{I}} \leq \underline{\text{apr}}'_{\mathbb{C},Wu}$ *for* $\text{N} \in \{E, H1, H2, I, K, L, M, N^{\mathbb{C}}_{1,Ma}\}$.

In the following example we illustrate that no other partial order relations hold.

Example 6. Let $U = \{x, y, z\}$ and let $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0/y + 0.5/z$ and $K_2 = 0/x + 1/y + 1/z$. Let $\mathcal{T}$ be the Łukasiewicz t-norm and $\mathcal{I}$ the Łukasiewicz implicator.

- Let $A = 1/x + 0.5/y + 0/z$, then $(\underline{\text{apr}}'_{\mathbb{C},InEx}(A))(x) = 0$, $(\underline{\text{apr}}'_{\mathbb{C},Wu}(A))(x) = 1$ and $(\underline{\text{apr}}_{\text{N},\mathcal{I}}(A))(x) = 0.5$ for $\text{N} \in \{A1, A2, E, H1, H2, I, K, L, M, N^{\mathbb{C}}_{1,\text{Ma}}\}$. Hence, $\underline{\text{apr}}_{\text{N},\mathcal{I}} \leq \underline{\text{apr}}'_{\mathbb{C},InEx}$ for $\text{N} \in \{E, H1, H2, I, K, L, M, N^{\mathbb{C}}_{1,\text{Ma}}\}$ and $\underline{\text{apr}}'_{\mathbb{C},Wu} \leq \underline{\text{apr}}_{\text{N},\mathcal{I}}$ for $\text{N} \in \{A1, A2, N^{\mathbb{C}}_{1,\text{Ma}}\}$ do not hold.
- Let $A = 1/x + 0/y + 0.7/z$, then it holds that $(\underline{\text{apr}}'_{\mathbb{C},InEx}(A))(z) = (\underline{\text{apr}}'_{\mathbb{C},RBL}(A))(z) = (\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}(A))(z) = 0.5$ and $(\underline{\text{apr}}_{\text{N},\mathcal{I}}(A))(z) = 0$ for $\text{N} \in \{N^{\mathbb{C}}_{1,\text{Ma}}\}$. Hence, $\underline{\text{apr}}'_{\mathbb{C},InEx} \leq \underline{\text{apr}}_{\text{N},\mathcal{I}}$, $\underline{\text{apr}}'_{\mathbb{C},RBL} \leq \underline{\text{apr}}_{\text{N},\mathcal{I}}$ and $\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}} \leq \underline{\text{apr}}_{\text{N},\mathcal{I}}$ do not hold for $\text{N} \in \{N^{\mathbb{C}}_{1,\text{Ma}}\}$.

Furthermore, let $\mathbb{C} = \{K_1, K_2, K_3\}$ with $K_1 = 1/x + 0.8/y + 0.6/z$, $K_2 = 0.2/x + 1/y + 0.6/z$ and $K_3 = 0.2/x + 0.8/y + 1/z$ and let $\mathcal{T}$ be the Łukasiewicz t-norm and $\mathcal{I}$ the Łukasiewicz implicator.

- Let $A = 0.1/x + 0.5/y + 0.4/z$, then we have that $(\underline{\text{apr}}'_{\mathbb{C},RBL}(A))(z) = 0.3$ and $(\underline{\text{apr}}_{\text{N},\mathcal{I}}(A))(z) = 0.4$ for $\text{N} \in \{E, H1, H2, I, K, L, M, N^{\mathbb{C}}_{1,\text{Ma}}\}$. Hence, $\underline{\text{apr}}_{\text{N},\mathcal{I}} \leq \underline{\text{apr}}'_{\mathbb{C},RBL}$ does not hold for $\text{N} \in \{E, H1, H2, I, K, L, M, N^{\mathbb{C}}_{1,\text{Ma}}\}$.

- Let $A = 1/x + 1/y + 0.7/z$, then it holds that $(\underline{\text{apr}}'_{\mathbb{C},InEx}(A))(y) = (\underline{\text{apr}}'_{\mathbb{C},RBL}(A))(y) = (\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}}(A))(y) = 1$ and $(\underline{\text{apr}}_{\text{N},\mathcal{I}}(A))(y) = 0.9$ for $\text{N} \in \{A2\}$. Hence, $\underline{\text{apr}}'_{\mathbb{C},InEx} \leq \underline{\text{apr}}_{\text{N},\mathcal{I}}$, $\underline{\text{apr}}'_{\mathbb{C},RBL} \leq \underline{\text{apr}}_{\text{N},\mathcal{I}}$ and $\underline{\text{apr}}'_{\mathbb{C},Li,\mathcal{T},\mathcal{I}} \leq \underline{\text{apr}}_{\text{N},\mathcal{I}}$ do not hold for $\text{N} \in \{A2\}$.

Finally, let $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0/y + 0.6/z$ and $K_2 = 0/x + 1/y + 1/z$. Let $\mathcal{T}$ be the nilpotent minimum and $\mathcal{I}$ its R-implicator.

- Let $A = 1/x + 0/y + 0.4/z$, then it holds that $(\underline{\text{apr}}'_{\mathbb{C},RBL}(A))(x) = 0.8$ and $(\underline{\text{apr}}_{\text{N},\mathcal{I}}(A))(x) = 0.4$ for $\text{N} \in \{A1\}$. Hence, $\underline{\text{apr}}'_{\mathbb{C},RBL} \leq \underline{\text{apr}}_{\text{N},\mathcal{I}}$ does not hold for $\text{N} \in \{A1\}$.

To end, we only need to discuss the fuzzy covering-based loose lower approximation operator $\underline{\text{apr}}''_{\mathbb{C},RBL}$. Since for a crisp covering $\mathbb{C}$ it holds that $\underline{\text{apr}}''_{\mathbb{C}} \leq \underline{\text{apr}}'_{\mathbb{C}}$, we have the following property:

Proposition 19. *Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, then* $\underline{\text{apr}}''_{\mathbb{C},RBL} \leq \underline{\text{apr}}'_{\mathbb{C},RBL}$.

Proof. Let $A \in \mathcal{F}(U)$, then $\Lambda_{\underline{\text{apr}}''_{\mathbb{C},RBL}(A)} = \Lambda_{\underline{\text{apr}}'_{\mathbb{C},RBL}(A)}$ and for all $\alpha \in \Lambda_{\underline{\text{apr}}''_{\mathbb{C},RBL}(A)}$ it holds that

$$\rho_{\underline{\text{apr}}''_{\mathbb{C},RBL}(A)}(\alpha) \subseteq \rho_{\underline{\text{apr}}'_{\mathbb{C},RBL}(A)}(\alpha). \qquad \square$$

By the transitivity of $\leq$, we also have:

Corollary 3. *Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, then* $\underline{\text{apr}}''_{\mathbb{C},RBL} \leq \underline{\text{apr}}'_{\mathbb{C},Wu}$.

As illustrated in the next example, these are the only partial order relations which hold:

Example 7. Let $U = \{x, y, z\}$ and let $\mathbb{C} = \{K_1, K_2, K_3\}$ with $K_1 = 1/x + 0.8/y + 0.6/z$, $K_2 = 0.2/x + 1/y + 0.6/z$ and $K_3 = 0.2/x + 0.8/y + 1/z$ and let $\mathcal{T}$ be the Łukasiewicz t-norm and $\mathcal{I}$ the Łukasiewicz implicator.

- Let $A = 0.1/x + 0.5/y + 0.4/z$, then we have that $(\underline{\text{apr}}''_{\mathbb{C},RBL}(A))(y) = 0.1$ and
$$(\underline{\text{apr}}'_{\mathbb{C},RBL}(A))(y) = (\underline{\text{apr}}_{N_4^{\mathbb{C}_4},\mathcal{I}}(A))(y) = 0.3.$$
Hence, $\underline{\text{apr}} \leq \underline{\text{apr}}''_{\mathbb{C},RBL}$ does not hold for any of the first 20 lower approximation operators of Table 2.
- Let $A = 1/x + 1/y + 0.7/z$, then it holds that $\underline{\text{apr}}'_{\mathbb{C},InEx}(A) = 1/x + 1/y + 0.6/z$ and $\underline{\text{apr}}''_{\mathbb{C},RBL} = 1/x + 0.9/y + 0.7/z$. Hence, $\underline{\text{apr}}'_{\mathbb{C},InEx}$ and $\underline{\text{apr}}''_{\mathbb{C},RBL}$ are incomparable.

Furthermore, let $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0/y + 0.6/z$ and $K_2 = 0/x + 1/y + 1/z$. Let $\mathcal{T}$ be the nilpotent minimum and $\mathcal{I}$ its R-implicator.

- Let $A = 1/x + 0/y + 0.4/z$, then it holds that $(\underline{\text{apr}}_{N_1^{\mathbb{C}},\mathcal{I}}(A))(x) = 0.4$ and $(\underline{\text{apr}}''_{\mathbb{C},RBL}(A))(x) = 0.8$. Hence, $\underline{\text{apr}}''_{\mathbb{C},RBL} \leq \underline{\text{apr}}$ does not hold for any of the first 18 lower approximation operators of Table 2.

Hence, the Hasse diagram representing all fuzzy covering-based lower approximation operators stated in Table 2 is given in Fig. 3. Minimal elements of the Hasse diagram are given by the operators $\underline{\text{apr}}_{N_4^{\mathbb{C}_4},\mathcal{I}}$, $\underline{\text{apr}}'_{\mathbb{C},InEx}$ and $\underline{\text{apr}}''_{\mathbb{C},RBL}$. The approximation operators $\underline{\text{apr}}_{N_1^{\mathbb{C}},\mathcal{I}}$ and $\underline{\text{apr}}'_{\mathbb{C},Wu}$ are maximal elements of the Hasse diagram. Therefore, the pairs $(\underline{\text{apr}}_{N_1^{\mathbb{C}},\mathcal{I}}, \overline{\text{apr}}_{N_1^{\mathbb{C}},\mathcal{T}})$ and $(\underline{\text{apr}}'_{\mathbb{C},Wu}, \overline{\text{apr}}'_{\mathbb{C},Wu})$ provide the most accurate approximations.

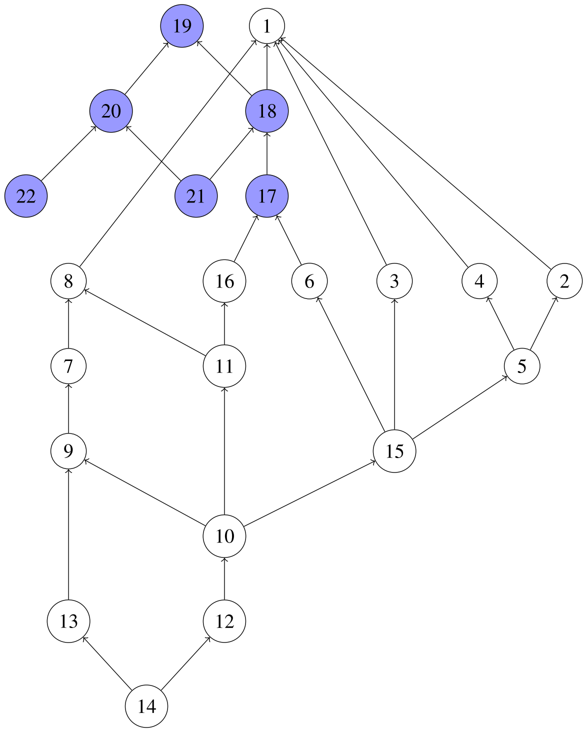

Fig. 3. Hasse diagram of the lower approximation operators from Table 2.

5. Conclusions and future work

In this article, we have discussed different types of dual fuzzy covering-based rough set models. Fuzzy neighborhood-based rough set models can be seen as generalizations of fuzzy relation-based rough set models. Moreover, fuzzy covering-based rough approximation operators extending the tight and loose rough set approximation operators are discussed. Note that we have introducted three new granule-based models, of which two use non-nested level-based representation. Of each model considered in this article we have studied its properties.

In addition, we have discussed the interrelationships of 17 fuzzy neighborhood-based rough set models, four tight fuzzy covering-based rough set models and one loose fuzzy covering-based rough set models for an IMTL-t-norm and its residual implicator, in a finite setting. The tight model of Wu et al. and the fuzzy neighborhood-based model using neighborhood operator $N_1^{\mathbb{C}}$ appear to have the largest lower approximation operator, resulting in more accurate approximations.

Future research directions include the study of new fuzzy covering-based rough set models and the integration of other models into the framework presented in this work. Moreover, we are interested in the applicability of fuzzy covering-based rough set models in feature selection [3]. Another future research direction is the study of semantical interpretations of fuzzy covering-based approximation operators, as has been done by the authors for the crisp setting in [9].

Acknowledgements

Lynn D'eer has been supported by the Ghent University Special Research Fund. Chris Cornelis was partially supported by the Spanish Ministry of Science and Technology under the Project TIN2014-57251-P and the Andalusian Research Plans P10-TIC-6858, P11-TIC-7765 and P12-TIC-2958, and by Project PYR-2014-8 of the Genil Program of CEI BioTic GRANADA.

References

[1] Z. Bonikowski, E. Brynarski, Extensions and intensions in rough set theory, Inf. Sci. 107 (1998) 149–167.
[2] C. Cornelis, M. De Cock, A.M. Radzikowska, Fuzzy rough sets: from theory into practice, in: W. Pedrycz, A. Skowron, V. Kreinovich (Eds.), Handbook of Granular Computing, Wiley, 2008, pp. 533–552.
[3] C. Cornelis, R. Jensen, G. Hurtado, D. Ślęzak, Attribute selection with fuzzy decision reducts, Inf. Sci. 180 (16) (2010) 3006–3020.
[4] I. Couso, D. Dubois, Rough sets, coverings and incomplete information, Fundam. Inform. 108 (2011) 223–247.
[5] L. D'eer, N. Verbiest, C. Cornelis, L. Godo, A comprehensive study of implicator–conjunctor based and noise-tolerant fuzzy rough sets: definitions, properties and robustness analysis, Fuzzy Sets Syst. 275 (2015) 1–38.
[6] L. D'eer, C. Cornelis, D. Sánchez, Fuzzy covering based rough sets revisited, in: Proceedings of the 2015 Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology, IFSA–EUSFLAT 2015, in: AISR, ISSN 1951-6851, vol. 89, 2015, pp. 651–658.
[7] L. D'eer, M. Restrepo, C. Cornelis, J. Gómez, Neighborhood operators for covering based rough sets, Inf. Sci. 336 (2016) 21–44.
[8] L. D'eer, C. Cornelis, L. Godo, Fuzzy neighborhood operators based on fuzzy coverings, Fuzzy Sets Syst. 312 (2017) 17–35.
[9] L. D'eer, C. Cornelis, Y. Yao, A semantically sound approach to Pawlak rough sets and covering-based rough sets, Int. J. Approx. Reason. 78 (2016) 62–72.
[10] T. Deng, Y. Chen, W. Xu, Q. Dai, A novel approach to fuzzy rough sets based on a fuzzy covering, Inf. Sci. 177 (11) (2007) 2308–2326.
[11] D. Dubois, H. Prade, Twofold fuzzy sets and rough sets – some issues in knowledge representation, Fuzzy Sets Syst. 23 (1987) 3–18.
[12] L. Fariñas del Cerro, H. Prade, Rough sets, twofold fuzzy sets and modal logic – fuzziness in indiscernibility and partial information, in: A. Di Nola, A.G.S. Ventre (Eds.), The Mathematics of Fuzzy Systems, Verlag TUV, Rheinland, Köln, 1986, pp. 103–120.
[13] T. Feng, S.P. Zhang, J.S. Mi, The reduction and fusion of fuzzy covering systems based on the evidence theory, Int. J. Approx. Reason. 53 (2012) 87–103.
[14] J.A. Herencia, Graded sets and points: a stratified approach to fuzzy sets and points, Fuzzy Sets Syst. 77 (1996) 191–202.
[15] M. Inuiguchi, T. Tanino, New fuzzy rough sets based on certainty qualification, in: K. Pal, et al. (Eds.), Rough-Neural Computing, Springer, Berlin, Heidelberg, 2003, pp. 278–296.
[16] M. Inuiguchi, Classification and approximation oriented fuzzy rough sets, in: B. Bouchon-Meunier, et al. (Eds.), Proceedings of Information Processing and Managment of Uncertainty in Knowledge-Based Systems, 2004, SD-ROM.
[17] T.J. Li, Y. Leung, W.X. Zhang, Generalized fuzzy rough approximation operators based on fuzzy coverings, Int. J. Approx. Reason. 48 (2008) 836–856.
[18] L. Ma, Two fuzzy covering rough set models and their generalizations over fuzzy lattices, Fuzzy Sets Syst. 294 (2016) 1–17.
[19] A. Nakamura, Fuzzy rough sets, Notes Mult.-Val. Log. Jpn. 9 (1988) 1–8.
[20] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (1982) 341–356.
[21] J.A. Pomykala, Approximation operations in approximation space, Bull. Acad. Pol. Sci. 35 (9–10) (1987) 653–662.
[22] P. Prasad, C. Rao, IQuickReduct: an improvement to QuickReduct algorithm, in: Rough Sets, Fuzzy Sets, Data Mining and Granular Computing, Springer, 2009, pp. 152–159.
[23] E. Ramentol, S. Vluymans, N. Verbiest, Y. Caballero, R. Bello, C. Cornelis, F. Herrera, IFROWANN: imbalanced fuzzy-rough ordered weighted average nearest neighbor classification, IEEE Trans. Fuzzy Syst. 23 (5) (2015) 1622–1637.
[24] M. Restrepo, C. Cornelis, J. Gómez, Partial order relation for approximation operators in covering-based rough sets, Inf. Sci. 284 (2014) 44–59.
[25] D. Sánchez, M. Delgado, M.A. Vila, J. Chamorro-Martínez, On a non-nested level-based representation of fuzziness, Fuzzy Sets Syst. 192 (2012) 159–175.
[26] R. Słowinski, D. Vanderpooten, A generalized definition of rough approximation based on similarity, IEEE Trans. Knowl. Data Eng. 12 (2000) 331–336.
[27] W.Z. Wu, W.X. Zhang, Neighborhood operators systems and approximations, Inf. Sci. 144 (2002) 201–207.
[28] W.Z. Wu, Y. Leung, J.S. Mi, On characterizations of (I, T)-fuzzy rough approximation operators, Fuzzy Sets Syst. 154 (2005) 76–102.
[29] M.F. Wu, H.H. Han, Y.F. Si, Properties and axiomatization of fuzzy rough sets based on fuzzy coverings, in: Proceedings of the 2012 International Conference on Machine Learning and Cybernetics, 2012, pp. 184–189.
[30] M. Wygralak, Rough sets and fuzzy sets – some remarks on interrelations, Fuzzy Sets Syst. 29 (2) (1989) 241–243.
[31] Y.Y. Yao, Two views of the theory of rough sets in finite universes, Int. J. Approx. Reason. 15 (4) (1996) 291–317.
[32] Y.Y. Yao, Combination of rough and fuzzy sets based on alpha-level sets, in: Rough Sets and Data Mining: Analysis for Imprecise Data, 1997, pp. 301–321.
[33] Y.Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Inf. Sci. 101 (1998) 21–47.
[34] Y.Y. Yao, B. Yao, Covering based rough sets approximations, Inf. Sci. 200 (2012) 91–107.
[35] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353.
[36] W. Żakowski, Approximations in the space (u, π), Demonstr. Math. 16 (1983) 761–769.
[37] P. Zhu, Q. Wen, Information-theoretic measures associated with rough set approximations, Inf. Sci. 212 (2012) 33–43.
[38] W. Zhu, Properties of the first type of covering-based rough sets, in: Sixth IEEE International Conference on Data Mining – Workshops IEEE, 2006.
[39] W. Zhu, F. Wang, On three types of covering based rough sets, IEEE Trans. Knowl. Data Eng. 8 (2007).
[40] W. Zhu, Relationship between generalized rough sets based on binary relation and covering, Inf. Sci. 179 (2009) 210–225.