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Parameterized fuzzy β-covering relations-based fuzzy rough set models and their applications to three-way decision and attribute reduction

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ABSTRACT

In the existing research on fuzzy β-covering rough set (Fβ-CRS) models, it is often required that the parameter β of each covering be equal, which seriously restricts the flexibility and practical efficiency of the model in applications. Additionally, the existing fuzzy β-neighborhood operators (Fβ-NOs) have limited expressive power, as most of them fail to satisfy reflexivity or symmetry. Therefore, it is necessary to study a more flexible and general model. To address this issue, this paper constructs some novel Fβ-NOs and Fβ-CRS models within a new variable-scale fuzzy β-covering approximation space (VSFβ-CAS), and explores their properties. First, based on overlap functions and grouping functions, four types of Fβ-NOs satisfying reflexivity and symmetry are proposed in VSFβ-CAS, which include parameterized fuzzy β-neighborhoods and parameterized fuzzy complementary β-neighborhoods. Their properties are demonstrated and it is proven that all conform to the structural requirements of fuzzy β-covering relations. On this theoretical basis, four Fβ-CRS models with upper approximation-inclusion lower approximation relations are further constructed. The basic properties of the models and the interrelationships among different models are discussed. Finally, based on the four constructed models, we designed experiments for both samples and attributes, i.e., three-way decision and attribute reduction. First, we applied the four proposed models to three-way decision tasks: we defined both fuzzy and crisp three-way regions, and performed comparative experiments on public datasets using six measurements. Meanwhile, we introduced fuzzy dependency functions and fuzzy composite functions, and developed some variable-scale attribute reduction algorithms. We then conducted in-depth analyses on how different β values and function configurations affect experimental results, and carried out comparative experiments with existing models on public datasets. The experimental results indicate that our models not only demonstrate advantages and rationality in addressing both sample and attribute-related issues, but also exhibit higher flexibility—further validating the strong generalization ability of the proposed models.

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1. Introduction

Pawlak first introduced rough set theory in 1982 [1,2]. This theory can characterize imprecise, incomplete, or unknown knowledge using raw data without requiring a priori knowledge or additional information. This renders it a valuable instrument for the study of data mining, generalization and knowledge representation. In 1983, Zakowski [3] introduced the concept of covering-based rough sets, replacing the partition utilized in classical rough sets with a covering. On the basis of their study, Zhu and Wang [4] investigated the characteristics of covering generalized rough sets and made a comparison between these sets and Pawlak's rough sets.

Fuzzy theory [5] is a mathematical tool for dealing with fuzziness and uncertainty. It quantifies fuzzy concepts via membership functions, enables reasoning and decision-making through fuzzy logic [6–8], and thus compensates for the limitations of classical theories in characterizing the fuzziness and uncertainty inherent in the real world [9,10]. In practical applications, both covering rough sets and classical rough sets are primarily used for handling discrete data. However, the application of these necessitates the discretization of continuous data, a process that may result in the loss of original information. This has been addressed by Dubois et al. [11] who introduced the concept of fuzzy rough sets, integrating fuzzy sets [5] and rough sets. Fuzzy rough sets represent a significant mathematical tool for the handling of uncertain and incomplete information, with the capacity to minimize information loss in data and enhance the model's classification capability. Combining the fuzzy rough set and the covering rough set is the fuzzy covering rough set (FCRS), thereby enriching the theory of fuzzy rough sets and providing a method for solving many practical application problems. Li et al. [12] concentrated on the generalization of covering-based rough set models through the employment of the fuzzy covering concept.

In light of the excessively strict conditions of fuzzy covering, Ma [13] proposed the fuzzy β-covering, wherein 1 is substituted by β. Furthermore, Ma put forward two novel FCRS models by leveraging Fβ-NOs. Soon after, Yang and Hu did an in-depth exploration based on the research of Ma. On the one hand [14], they studied the characterizations of these Fβ-CRS models and extended the concept to propose the fuzzy complementary β-neighborhood. On the other hand [15,16], the definitions of fuzzy β-maximal and fuzzy β-minimal descriptions laid the theoretical foundations for later studies of Fβ-NOs. In 2020, Zhang and Wang [17] conducted further research into key problems, along with novel concepts and properties in the fuzzy β-covering approximation space (Fβ-CAS).

Fβ-CRS models proposed in existing studies provide an important reference for the theoretical development and application exploration in this field. However, there are still certain limitations in this model system. Specifically, these Fβ-NOs fail to satisfy the reflexivity and symmetry. Meanwhile, some models also have the theoretical defect that the relation that upper approximations contain lower approximations is not satisfied. As one of the core properties of the rough set theory, the absence of the inclusion relation undermines the logical integrity of the model. For that matter, Zhang et al. [18] constructed a reflexive fuzzy α-neighborhood operator, and based on this, they proposed a FCRS model for the study of multi-criteria decision-making. Zhang and Dai [19] defined the parameterized fuzzy β-neighborhood and parameterized fuzzy complementary β-neighborhood to filter out partially noisy data in Fβ-CAS, and established the Fβ-CRS models to satisfy the inclusion relationship. In 2025, Jiang and Hu [20] proposed indiscernible fuzzy β-neighborhood and indiscernible fuzzy β-co-neighborhood satisfying the reflexivity, and constructed Fβ-CRS models based on them that conform to the inclusion relation. Zhang et al. [21] developed new models using a combination of Fβ-NOs and R-implication operators, and explored the Fβ-CRS approach through eight different types of operators. Wang et al. [22] constructed Fβ-CRS models by means of generalized fuzzy β-neighborhoods, and introduced a feature selection method on the basis of the relative discernibility relation. Additionally, Zou et al. [23] constructed a β-neighborhood that satisfies reflexivity in the multi-Fβ-CAS, and defined several monotonic uncertainty measures of fuzzy β-coverings based on the proposed model.

Compared with traditional operators, the overlap functions and grouping functions defined by Bustince et al. [24,25] are non-associative. They can effectively reflect the correlation between samples or attributes and show superiority in modeling interdependencies compared to conventional operators [26], which enables them to play a pivotal role in various applications such as information fusion, multi-attribute decision making, and uncertainty reasoning. In recent years, because of their good properties, many scholars have also applied overlap and grouping functions in constructing Fβ-CRS models. Dai et al. [27] constructed some Fβ-NOs with reflexivity and symmetry by t-norms and established a new Fβ-CRS framework using fuzzy β-covering relation. Fan et al. [28] further advanced the field by developing overlap functions-based Fβ-CRS models and exploring their application in attribute reduction (ATR). Unlike them, Qi et al. [29] used another approach to construct four fuzzy neighborhood operators based on overlap functions and their implicators. Based on these operators, two types of Fβ-CRS models were proposed. Recently, Wen et al. [30] proposed the (I,SO)-FCRS model using the semi-overlap function within a variable-scale fuzzy β-covering group approximation space (VSFβ-CGAS) and extended the traditional ATR to δ-approximate reduction.

Although some of the existing studies can compensate for the limitations of Fβ-CRS in applications, there are still some open problems. The motivation for this study is described in more detail in the following section:

- In most of the current Fβ-CRS models, β is the same value in all coverings, which is contrary to the original purpose of introducing fuzzy β-covering, i.e., to provide a flexible threshold to satisfy the preferences of different decision makers in a wide range of applications. But a single threshold cannot satisfy the requirements of multiple features or attributes groups. To address this problem, this paper proposes a VSFβ-CAS that allows different β parameters to be chosen for different coverings within the same framework.
- Some of the existing Fβ-NOs fail to satisfy reflexivity and symmetry. To address this problem, we propose four types of Fβ-NOs satisfying reflexivity and symmetry in VSFβ-CAS by using overlap functions and grouping functions, and these Fβ-NOs can also filter out the noise effectively.

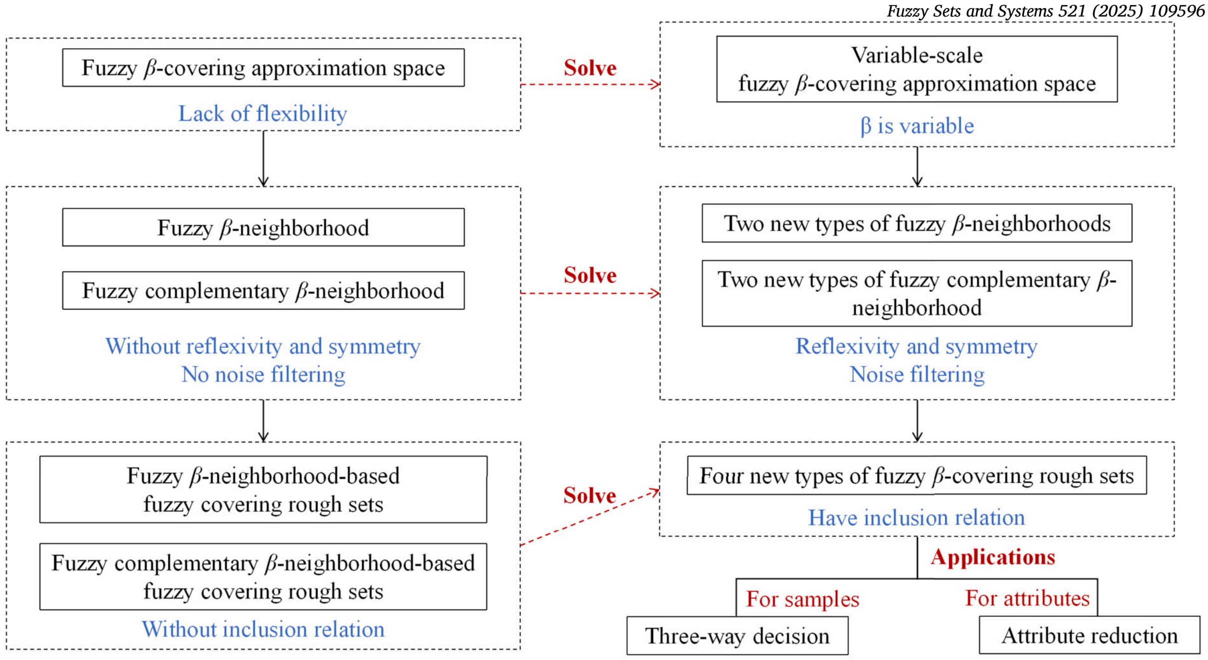

Fig. 1. Motivation and main work of this paper. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

- Aiming at the problem that the existing models do not satisfy the relationship that the upper approximation contains the lower approximation, it is proven that our Fβ-NOs conform to the structural requirements of fuzzy β-covering relations. Then, this paper constructs four types of Fβ-CRS models with inclusion relationship by using the fuzzy β-covering relations, and discusses the basic properties and interrelationships among different models.
- Three-way decision (TWD) has been one of the research hotspots in Fβ-CRS in recent years, but the above issues lead to the existing TWD algorithms lacking strong flexibility. Therefore, we use the four proposed models to define fuzzy and crisp three-way regions, and design a TWD method based on this, which has stronger flexibility and the ability to describe uncertain information.
- ATR based on the Fβ-CRS theory has been developed in recent years. However, in these studies, the improper choice of the parameter β often leads to abnormal interruptions of the algorithm, which is mainly manifested in the sudden decrease of the size or classification accuracy of the obtained attribute subsets, and this bottleneck seriously restricts the practical efficacy and flexibility of ATR. To address this problem, two monotonic uncertainty measures are defined based on the proposed models, named fuzzy dependency functions and fuzzy composite functions, and new variable-scale ATR methods are designed using them respectively.

The research motivation and main work of this paper can be illustrated by the following figure (see Fig. 1).

The paper is organized as follows: In Section 2, we present an overview of key concepts relevant to this paper. Section 3 introduces several new Fβ-NOs. In Section 4, we prove these Fβ-NOs conform to the structural requirements of fuzzy β-covering relations, propose the new Fβ-CRS models, and discuss their basic properties and interrelationships among different models. Section 5 defines fuzzy and crisp three-way regions using the four proposed models, and designs a TWD method based on them. Subsequently, comparative experiments are performed on public datasets using six measurements. Section 6 defines two monotonic uncertainty measures based on the proposed models, and new variable-scale ATR methods are designed using each measure respectively. Moreover, a series of experiments are conducted, as well as the effects of different β values and functions on the performance of these algorithms are analyzed. In Section 7, we summarize this paper and suggest directions for future research.

2. Preliminaries

In this section, we will briefly review some basic definitions and concepts that will be used throughout this paper.

2.1. *Fβ-NOs and Fβ-CRSs in Fβ-CAS*

The concepts of fuzzy β-covering, fuzzy β-neighborhood and fuzzy complementary β-neighborhood have been proposed in [13,14].

Definition 2.1 *([13])*. Suppose U is a universe, and $F(U)$ denotes the set of all fuzzy sets on U. A family of fuzzy sets $\mathbb{C} = \{C_1, C_2, \cdots, C_n\}$, where $C_i \in F(U)$ for $i = 1, 2, \cdots, n$, is defined as a fuzzy β-covering of U, if $\left(\bigcup_{i=1}^{n} C_i\right)(u) \geq \beta$ $(\forall \beta \in (0, 1],\ u \in U)$. Furthermore, denoted as an Fβ-CAS is $(U, \mathbb{C})$.

Table 1
$N^{\beta}_{\mathbb{C}}(u_i)(u_j)(i,j=1,2,3,4,5,6)$.

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	0.63	0.37	0.39	0.55	0.39	0.36
u_2	0.43	0.62	0.37	0.27	0.37	0.53
u_3	0.51	0.46	0.64	0.32	0.64	0.36
u_4	0.63	0.37	0.39	0.67	0.39	0.36
u_5	0.51	0.46	0.64	0.32	0.64	0.36
u_6	0.43	0.62	0.37	0.27	0.37	0.72

Table 2
$\widehat{N}^{\beta}_{\mathbb{C}}(u_i)(u_j)(i,j=1,2,3,4,5,6)$.

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	1	0.37	0.39	0.55	0.39	0.36
u_2	0.43	1	0.37	0.27	0.37	0.53
u_3	0.51	0.46	1	0.32	1	0.36
u_4	0.63	0.37	0.39	1	0.39	0.36
u_5	0.51	0.46	1	0.32	1	0.36
u_6	0.43	0.62	0.37	0.27	0.37	1

When $\beta = 1$, fuzzy β-covering degenerates to fuzzy covering, so we usually consider fuzzy β-covering as a generalization of fuzzy covering.

Definition 2.2 *([13])*. Suppose $(U, \mathbb{C})$ is an Fβ-CAS. For each $u \in U$, we define the fuzzy β-neighborhood $N^{\beta}_{\mathbb{C}}(u)$ of u as:

$$N^{\beta}_{\mathbb{C}}(u) = \cap \left\{ C_i | C_i(u) \geq \beta, C_i \in \mathbb{C} \right\}. \tag{1}$$

Definition 2.3 *([14])*. Suppose $(U, \mathbb{C})$ is an Fβ-CAS. For each $u \in U$, we define the fuzzy complementary β-neighborhood $M^{\beta}_{\mathbb{C}}(u)$ of u as:

$$M^{\beta}_{\mathbb{C}}(u)(v) = N^{\beta}_{\mathbb{C}}(v)(u),\ \forall v \in U \tag{2}$$

In [20], Jiang and Hu proposed indiscernible fuzzy β-neighborhood, and constructed Fβ-CRS model based on it.

Definition 2.4 *([20])*. Suppose $(U, \mathbb{C})$ is an Fβ-CAS, $\mathbb{C} = \{C_1, C_2, \cdots, C_n\}$ is a fuzzy β-covering of U. For each $u \in U$, we define the indiscernible fuzzy β-neighborhood $\widehat{N}^{\beta}_{\mathbb{C}}(u)$ of u as:

$$\widehat{N}^{\beta}_{\mathbb{C}}(u)(v) = \begin{cases} 1, & if\ C_i(u) = C_i(v), \\ N^{\beta}_{\mathbb{C}}(u)(v), & otherwise, \end{cases} \tag{3}$$

Example 2.5 *([20])*. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $\mathbb{C} = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ be a family of fuzzy sets of U, where

$$C_1 = \frac{0.82}{u_1} + \frac{0.51}{u_2} + \frac{0.42}{u_3} + \frac{0.55}{u_4} + \frac{0.42}{u_5} + \frac{0.48}{u_6},$$
$$C_2 = \frac{0.43}{u_1} + \frac{0.62}{u_2} + \frac{0.37}{u_3} + \frac{0.27}{u_4} + \frac{0.37}{u_5} + \frac{0.72}{u_6},$$
$$C_3 = \frac{0.51}{u_1} + \frac{0.74}{u_2} + \frac{0.64}{u_3} + \frac{0.32}{u_4} + \frac{0.64}{u_5} + \frac{0.53}{u_6},$$
$$C_4 = \frac{0.63}{u_1} + \frac{0.46}{u_2} + \frac{0.92}{u_3} + \frac{0.67}{u_4} + \frac{0.92}{u_5} + \frac{0.36}{u_6},$$
$$C_5 = \frac{0.74}{u_1} + \frac{0.37}{u_2} + \frac{0.54}{u_3} + \frac{0.83}{u_4} + \frac{0.54}{u_5} + \frac{0.45}{u_6},$$
$$C_6 = \frac{0.65}{u_1} + \frac{0.83}{u_2} + \frac{0.39}{u_3} + \frac{0.72}{u_4} + \frac{0.39}{u_5} + \frac{0.82}{u_6}.$$

Let $\beta = 0.6$. Then the Fβ-NOs $N^{\beta}_{\mathbb{C}}(u_i)(u_j)$ and $\widehat{N}^{\beta}_{\mathbb{C}}(u_i)(u_j)$ $(i,j = 1,2,3,4,5,6)$ are listed in Table 1 and 2, respectively.

Definition 2.6 *([20])*. Suppose $(U, \mathbb{C})$ is an Fβ-CAS, $\beta \in (0, 1]$. For each $u \in U$ and $E \in F(U)$, we define the fuzzy lower and upper approximations of E as:

$$\underline{Appr}(E)(u) = \bigwedge_{v\in U} \left\{ \left(1 - [u]^{\beta}_{\mathbb{C}}(v)\right) \vee E(v) \right\}, \tag{4}$$

$$\overline{Appr}(E)(u) = \bigvee_{v\in U} \left\{ [u]^{\beta}_{\mathbb{C}}(v) \wedge E(v) \right\}, \tag{5}$$

where $[u]^{\beta}_{\mathbb{C}} = \widehat{N}^{\beta}_{\mathbb{C}}(u)$, then E is the Fβ-CRS; otherwise it is definable.

In [27], fuzzy β-covering relation is proposed to characterize the similarity between samples and Fβ-CRS model is constructed based on it.

Definition 2.7 *([27]).* Suppose $(U, \mathbb{C})$ is an Fβ-CAS, $\mathbb{B} \subseteq \mathbb{C}$. For each $u, v \in U$, we define the fuzzy β-covering relation $R^{\beta}_{\mathbb{B}}$ as:
(1) $R^{\beta}_{\mathbb{B}}$ is reflexive, that is, $R^{\beta}_{\mathbb{B}}(u,u) = 1$;
(2) $R^{\beta}_{\mathbb{B}}$ is symmetric, that is, $R^{\beta}_{\mathbb{B}}(u,v) = R^{\beta}_{\mathbb{B}}(v,u)$.

The fuzzy similarity class $[u]^{\beta}_{\mathbb{B}}$ of u with respect to $\mathbb{B}$ is defined as a fuzzy set on U. $[u]^{\beta}_{\mathbb{B}}$ is interpreted as a fuzzy neighborhood associated with $\mathbb{B}$ when $R^{\beta}_{\mathbb{B}}$ constitutes a fuzzy β-covering relation. Specifically, for any $v \in U$, it is given by $[u]^{\beta}_{\mathbb{B}}(v) = R^{\beta}_{\mathbb{B}}(u,v)$.

Definition 2.8 *([27]).* Suppose $(U, \mathbb{C})$ is an Fβ-CAS, $\mathbb{B} \subseteq \mathbb{C}$. For each $u \in U$, $E \in F(U)$, we define the lower and upper approximation of E as:

$$\underline{R^{\beta}_{\mathbb{B}}}(E)(u) = \bigwedge_{v\in U} \left\{ \left(1 - R^{\beta}_{\mathbb{B}}(u,v)\right) \vee E(v) \right\}, \tag{6}$$

$$\overline{R^{\beta}_{\mathbb{B}}}(E)(u) = \bigvee_{v\in U} \left\{ R^{\beta}_{\mathbb{B}}(u,v) \wedge E(v) \right\}. \tag{7}$$

E is called definable set when $\overline{R^{\beta}_{\mathbb{B}}}(E) = \underline{R^{\beta}_{\mathbb{B}}}(E)$; Otherwise, if these two approximation operators are not equal, E is called a Fβ-CRS. Under such circumstances, the ordered pair $(\overline{R^{\beta}_{\mathbb{B}}}(E), \underline{R^{\beta}_{\mathbb{B}}}(E))$ is called a Fβ-CRS model of E.

2.2. Overlap function and grouping function

Next, the basic definitions of overlap functions and grouping functions are introduced, and examples of commonly used ones are given.

Definition 2.9 *([24,31]).* Suppose $O: [0,1]^2 \to [0,1]$ is a binary function. For each $u, v, w \in [0,1]$, we define the overlap function as:
(O1) $O(u,v) = O(v,u)$;
(O2) $O(u,v) = 0 \Leftrightarrow uv = 0$;
(O3) $O(u,v) = 1 \Leftrightarrow uv = 1$;
(O4) $v \le w \Rightarrow O(u,v) \le O(u,w)$;
(O5) O is continuous.

Definition 2.10 *([25]).* Suppose $G: [0,1]^2 \to [0,1]$ is a binary function. For each $u, v, w \in [0,1]$, we define the grouping function as:
(G1) $G(u,v) = G(v,u)$;
(G2) $G(u,v) = 0 \Leftrightarrow u = v = 0$;
(G3) $G(u,v) = 1 \Leftrightarrow u = 1$ or $v = 1$;
(G4) $v \le w \Rightarrow G(u,v) \le G(u,w)$;
(G5) G is continuous.

Below, we introduce several functions that fit the above definition. They will be used in the examples and experiments in this paper.

Example 2.11 *([24,31,32]).* For each $u, v \in [0,1]$, examples of several overlap functions:
(1) $O_{mp}(u,v) = \min\{u^p, v^p\}, p > 0$;
(2) $O_{mM}(u,v) = \min\{u,v\} \max\left\{u^2, v^2\right\}$;
(3) $O_p(u,v) = u^p v^p,\ p > 0$;
(4) $O_{Mid}(u,v) = uv\frac{u+v}{2}$;
(5) $O_{DB}(u,v) = \begin{cases} \frac{2uv}{u+v} & \text{if } u+v \neq 0, \\ 0 & \text{if } u+v = 0. \end{cases}$

Example 2.12 *([25,33])*. For each $u, v \in [0,1]$, examples of several grouping functions:
(1) $G_{M_p}(u,v) = \max\{u^p, v^p\}, p > 0$;
(2) $G_p(u,v) = 1-(1-u)^p(1-v)^p,\ p>0$;
(3) $G_{mM}(u,v) = 1-\min\{1-u, 1-v\}\max\{(1-u)^2,(1-v)^2\}$;
(4) $G_{DB}(u,v) = \begin{cases} \frac{u+v-2uv}{2-u-v} & \text{if } u+v \neq 2, \\ 1 & \text{if } u+v = 2. \end{cases}$

3. Some novel Fβ-NOs

3.1. Indiscernible fuzzy complementary β-neighborhood in Fβ-CAS

Due to the defect that fuzzy complementary β-neighborhood in Fβ-CAS are not reflexive, an indiscernible fuzzy complementary β-neighborhood is proposed and some of its basic properties are explored.

Definition 3.1. Suppose $(U,\mathbb{C})$ is an Fβ-CAS, $\mathbb{C}=\{C_1, C_2, \cdots, C_n\}$ is a fuzzy β-covering of U. For each $u \in U$, we define the indiscernible fuzzy complementary β-neighborhood $\widehat{W}_{\mathbb{C}}^{\beta}(u)$ of u as:

$$\widehat{W}_{\mathbb{C}}^{\beta}(u)(v) = \widehat{N}_{\mathbb{C}}^{\beta}(v)(u) = \begin{cases} 1, & if\ C_i(u)=C_i(v), \\ N_{\mathbb{C}}^{\beta}(v)(u), & otherwise, \end{cases} = \begin{cases} 1, & if\ C_i(u)=C_i(v), \\ M_{\mathbb{C}}^{\beta}(u)(v), & otherwise. \end{cases} \quad (8)$$

Remark 3.2. If the membership degree of the element u for all fuzzy sets $C_i \in U$ is equal to that of the element v, it can be understood that there is no difference between the elements u and v. In other words, these two elements belong to each other's neighborhoods. Similarly, they also belong to each other's complementary neighborhoods. Therefore, the membership degree of v in the fuzzy complementary β-neighborhood and fuzzy β-neighborhood of u is both 1, that is, $\widehat{W}_{\mathbb{C}}^{\beta}(u)(v) = \widehat{N}_{\mathbb{C}}^{\beta}(v)(u) = 1$.

The indiscernible fuzzy complementary β-neighborhood can be viewed as an extension of indiscernible relation in Fβ-CAS, due to the fact that it describes the indistinguishability between samples with the same description in a fuzzy β-covering.

Proposition 3.3. *Suppose $(U,\mathbb{C})$ is an Fβ-CAS. For each $u\in U$, $\widehat{W}_{\mathbb{C}}^{\beta}(u)$ is reflexive, i.e. $\widehat{W}_{\mathbb{C}}^{\beta}(u)(u)=1$.*

Proof. According to Definition 3.1, for each $u \in U$, we have $C_i(u)=C_i(u) (i=1,2,\cdots,n)$, then $\widehat{W}_{\mathbb{C}}^{\beta}(u)(u)=1$. □

Proposition 3.4. *Suppose $(U,\mathbb{C})$ is an Fβ-CAS. For each $u\in U$, if $\beta_1 \le \beta_2$, then $\widehat{W}_{\mathbb{C}}^{\beta_1}(u) \subseteq \widehat{W}_{\mathbb{C}}^{\beta_2}(u)$.*

Proof. For any given $u \in U$, two cases arise.

Case 1: According to Definition 3.1, for each $v\in U$, if $C_i(u)=C_i(v)(i=1,2,\cdots,n)$, then $\widehat{W}_{\mathbb{C}}^{\beta_1}(u)(v)=1$ and $\widehat{W}_{\mathbb{C}}^{\beta_2}(u)(v)=1$. Thus, $\widehat{W}_{\mathbb{C}}^{\beta_1}(u)=\widehat{W}_{\mathbb{C}}^{\beta_2}(u)$.

Case 2: According to Definition 3.1, for each $v\in U$, $C_{i_0}(u)\neq C_{i_0}(v)$ if there exists a fuzzy set $C_{i_0}\in\mathbb{C}$, then $\widehat{W}_{\mathbb{C}}^{\beta_1}(u)(v) = M_{\mathbb{C}}^{\beta_1}(u)(v) = N_{\mathbb{C}}^{\beta_1}(v)(u)$ and $\widehat{W}_{\mathbb{C}}^{\beta_2}(u)(v) = M_{\mathbb{C}}^{\beta_2}(u)(v) = N_{\mathbb{C}}^{\beta_2}(v)(u)$. Furthermore, $\beta_1 \le \beta_2$ means that $N_{\mathbb{C}}^{\beta_1}(v) \subseteq N_{\mathbb{C}}^{\beta_2}(v)$, i.e. $M_{\mathbb{C}}^{\beta_1}(u) \subseteq M_{\mathbb{C}}^{\beta_2}(u)$. Thus, we have

$$\widehat{W}_{\mathbb{C}}^{\beta_1}(u)(v) = M_{\mathbb{C}}^{\beta_1}(u)(v) \le M_{\mathbb{C}}^{\beta_2}(u)(v) = \widehat{W}_{\mathbb{C}}^{\beta_2}(u)(v).$$

That is $\widehat{W}_{\mathbb{C}}^{\beta_1}(u) \subseteq \widehat{W}_{\mathbb{C}}^{\beta_2}(u)$.

Combining Cases 1 and 2, we can conclude that $\widehat{W}_{\mathbb{C}}^{\beta_1}(u) \subseteq \widehat{W}_{\mathbb{C}}^{\beta_2}(u)$ if $\beta_1 \le \beta_2$. □

Proposition 3.5. *Suppose $(U,\mathbb{C})$ is an Fβ-CAS. For each $u\in U$, $\widehat{W}_{\mathbb{C}}^{\beta}(u) \supseteq M_{\mathbb{C}}^{\beta}(u)$.*

Proof. For any given $u \in U$, two cases arise.

Case 1: According to Definition 3.1, for each $v\in U$, if $C_i(u)=C_i(v)(i=1,2,\cdots,n)$, then $\widehat{W}_{\mathbb{C}}^{\beta}(u)(v)=1$. Based on this, if $u=v$, then $\beta \le M_{\mathbb{C}}^{\beta}(u)(u) \le 1 = \widehat{W}_{\mathbb{C}}^{\beta}(u)(u)$; else, $0 \le M_{\mathbb{C}}^{\beta}(u)(v) \le 1 = \widehat{W}_{\mathbb{C}}^{\beta}(u)(v)$. Thus, $\widehat{W}_{\mathbb{C}}^{\beta}(u) \supseteq M_{\mathbb{C}}^{\beta}(u)$.

Case 2: According to Definition 3.1, for each $v\in U$, $C_{i_0}(u)\neq C_{i_0}(v)$ if there exists a fuzzy set $C_{i_0}\in\mathbb{C}$, then $\widehat{W}_{\mathbb{C}}^{\beta}(u)(v) = M_{\mathbb{C}}^{\beta}(u)(v)$.

Combining Cases 1 and 2, we can conclude that $\widehat{W}_{\mathbb{C}}^{\beta}(u) \supseteq M_{\mathbb{C}}^{\beta}(u)$. □

An example is given here to illustrate the difference between the indiscernible fuzzy complementary β-neighborhood and fuzzy complementary β-neighborhood.

Table 3
$M^{\beta}_{\mathbb{C}}(u_i)(u_j)(i,j=1,2,3,4,5,6)$.

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	0.63	0.43	0.51	0.63	0.51	0.43
u_2	0.37	0.62	0.46	0.37	0.46	0.62
u_3	0.39	0.37	0.64	0.39	0.64	0.37
u_4	0.55	0.27	0.32	0.67	0.32	0.27
u_5	0.39	0.37	0.64	0.39	0.64	0.37
u_6	0.36	0.53	0.36	0.36	0.36	0.72

Table 4
$\widehat{W}^{\beta}_{\mathbb{C}}(u_i)(u_j)(i,j=1,2,3,4,5,6)$.

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	1	0.43	0.51	0.63	0.51	0.43
u_2	0.37	1	0.46	0.37	0.46	0.62
u_3	0.39	0.37	1	0.39	1	0.37
u_4	0.55	0.27	0.32	1	0.32	0.27
u_5	0.39	0.37	1	0.39	1	0.37
u_6	0.36	0.53	0.36	0.36	0.36	1

Example 3.6. (Following Example 2.5) Assume that $\beta=0.6$. Then the Fβ-NOs $M^{\beta}_{\mathbb{C}}(u_i)(u_j)$, $\widehat{W}^{\beta}_{\mathbb{C}}(u_i)(u_j)$ $(i,j=1,2,3,4,5,6)$ are listed in Table 3 and 4, respectively.

It can be easily found from Tables 3 and 4 that for each $u_i\in U$, $\widehat{W}^{\beta}_{\mathbb{C}}(u_i)(u_i)=1$. However, in Table 3, all $M^{\beta}_{\mathbb{C}}(u_i)$ $(i=1,2,3,4,5,6)$ are not equal to 1. At the same time, for all $C_i\in C$, $C_i(u_3)=C_i(u_5)$, but $M^{\beta}_{\mathbb{C}}(u_3)(u_5)=M^{\beta}_{\mathbb{C}}(u_5)(u_3)=0.64$. On the contrary, $\widehat{W}^{\beta}_{\mathbb{C}}(u_3)(u_5)=\widehat{W}^{\beta}_{\mathbb{C}}(u_5)(u_3)=1$, which is consistent with the membership degrees of u_3 and u_5 under different fuzzy sets in $\mathbb{C}$ are the same.

Based on the existing indiscernible fuzzy β-neighborhood [20] and the indiscernible fuzzy complementary β-neighborhood defined by us, we further explore how to determine the indiscernible β-neighborhood and indiscernible complementary β-neighborhood of each object in the Fβ-CAS, and study the related properties.

Definition 3.7. Suppose $(U,\mathbb{C})$ is an Fβ-CAS. For each $u\in U$, we define the indiscernible β-neighborhood $\overline{N}^{\beta}_{\mathbb{C}}(u)$ of u as:

$$\overline{N}^{\beta}_{\mathbb{C}}(u)=\left\{v\in U\,|\,\widehat{N}^{\beta}_{\mathbb{C}}(u)(v)\geq\beta\right\} \tag{9}$$

For each $u\in U$, we define the indiscernible complementary β-neighborhood $\overline{W}^{\beta}_{\mathbb{C}}(u)$ of u as:

$$\overline{W}^{\beta}_{\mathbb{C}}(u)=\left\{v\in U\,|\,\widehat{W}^{\beta}_{\mathbb{C}}(u)(v)\geq\beta\right\} \tag{10}$$

Remark 3.8. Unlike Definitions 2.4 and 3.1, the indiscernible β-neighborhood and indiscernible complementary β-neighborhood can be used to determine classical neighborhood class of each object in an Fβ-CAS. The following example will illustrate this.

Example 3.9. (Following Examples 2.5 and 3.6) The indiscernible 0.6-neighborhood $\overline{N}^{0.6}_{\mathbb{C}}(u)$ and indiscernible complementary 0.6-neighborhood $\overline{W}^{0.6}_{\mathbb{C}}(u)$ of all u_n are calculated as follows:

$$\overline{N}^{0.6}_{\mathbb{C}}(u_1)=\{u_1\},\overline{N}^{0.6}_{\mathbb{C}}(u_2)=\{u_2\},\overline{N}^{0.6}_{\mathbb{C}}(u_3)=\{u_3,u_5\},\overline{N}^{0.6}_{\mathbb{C}}(u_4)=\{u_1,u_4\},\overline{N}^{0.6}_{\mathbb{C}}(u_5)=\{u_3,u_5\},\overline{N}^{0.6}_{\mathbb{C}}(u_6)=\{u_2,u_6\},$$
$$\overline{W}^{0.6}_{\mathbb{C}}(u_1)=\{u_1,u_4\},\overline{W}^{0.6}_{\mathbb{C}}(u_2)=\{u_2,u_6\},\overline{W}^{0.6}_{\mathbb{C}}(u_3)=\{u_3,u_5\},\overline{W}^{0.6}_{\mathbb{C}}(u_4)=\{u_4\},\overline{W}^{0.6}_{\mathbb{C}}(u_5)=\{u_3,u_5\},\overline{W}^{0.6}_{\mathbb{C}}(u_6)=\{u_6\}.$$

Next, we will discuss the properties of the indiscernible β-neighborhood and indiscernible complementary β-neighborhood.

Proposition 3.10. *Suppose $(U,\mathbb{C})$ is an Fβ-CAS. For each $u,v\in U$, the following conclusions are correct:*
(1) $u\in\overline{N}^{\beta}_{\mathbb{C}}(u)$ and $u\in\overline{W}^{\beta}_{\mathbb{C}}(u)$;
(2) $u\in\overline{N}^{\beta}_{\mathbb{C}}(v)\Leftrightarrow v\in\overline{W}^{\beta}_{\mathbb{C}}(u)$;
(3) $u\in\overline{N}^{\beta}_{\mathbb{C}}(v)\Leftrightarrow\overline{N}^{\beta}_{\mathbb{C}}(u)\subseteq\overline{N}^{\beta}_{\mathbb{C}}(v)$ and $u\in\overline{W}^{\beta}_{\mathbb{C}}(v)\Leftrightarrow\overline{W}^{\beta}_{\mathbb{C}}(u)\subseteq\overline{W}^{\beta}_{\mathbb{C}}(v)$;
(4) $\overline{N}^{\beta}_{\mathbb{C}}(u)=\left\{v\in U\,|\,u\in\overline{W}^{\beta}_{\mathbb{C}}(v)\right\}$ and $\overline{W}^{\beta}_{\mathbb{C}}(u)=\left\{v\in U\,|\,u\in\overline{N}^{\beta}_{\mathbb{C}}(v)\right\}$.

Proof. According to Definition 3.7, the proposition is obvious. □

3.2. Some new parameterized Fβ-NOs in VSFβ-CAS

In this subsection, four Fβ-NOs satisfying reflexivity and symmetry will be defined based on two indiscernible fuzzy neighborhoods using overlap functions and grouping functions, and their properties as well as interrelationships will be investigated.

Definition 3.11. Suppose U is universe and Λ denotes a nonempty index set. If $\mathbb{C}' = \bigcup_{j\in\Lambda}\left\{\mathbb{C}_j^{\beta_j}\right\}$ is a fuzzy β-covering for $\beta_j \in (0,1]$, where $\mathbb{C}_j^{\beta_j} = \left\{C_1^{\beta_j}, C_2^{\beta_j}, \ldots, C_m^{\beta_j}\right\}$ $(m \le n)$ is a fuzzy β_j-covering, then $(U,\mathbb{C}')$ is variable-scale fuzzy β-covering approximation space (VSFβ-CAS).

Remark 3.12. Unlike the definition of VSFβ-CAS, different fuzzy sets C have the same β_j in Fβ-CAS $(U,\mathbb{C})$, or $\widetilde{\mathbb{C}} = \left\{\mathbb{C}_j^{\beta_j}\right\}_{j\in\Lambda}$ is a family fuzzy β-covering in VSFβ-CGA $(U,\widetilde{\mathbb{C}})$ [30]. Therefore, VSFβ-CAS is a completely different concept from Fβ-CAS or VSFβ-CGA.

For readability, in the following, $\mathbb{C}_j^{\beta_j}$ will simply be written as $\mathbb{C}^{\beta_j}$. That is, in the following, $\mathbb{C}^{\beta_1}$ denotes $\mathbb{C}_1^{\beta_1}$, and $\mathbb{B}_1^{\beta_j} \subseteq \mathbb{B}_2^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$ denotes $\mathbb{B}_{j1}^{\beta_j} \subseteq \mathbb{B}_{j2}^{\beta_j} \subseteq \mathbb{C}_j^{\beta_j}$, which will not be described later.

Definition 3.13. Suppose $(U,\mathbb{C}')$ is a VSFβ-CAS, O is an overlap function and G is a grouping function. For each $u \in U$, we define the parameterized fuzzy β-neighborhoods $\widehat{ON}_{\mathbb{C}^{\beta_j}}^{\beta_j,\lambda}(u)$ and $\widehat{GN}_{\mathbb{C}^{\beta_j}}^{\beta_j,\lambda}(u)$ of u as:

$$\widehat{ON}_{\mathbb{C}^{\beta_j}}^{\beta_j,\lambda}(u)(v) = \begin{cases} 0, & \widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v) \vee \widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(v)(u) < \lambda, \\ O\left(\widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v), \widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(v)(u)\right), & otherwise, \end{cases} \tag{11}$$

$$\widehat{GN}_{\mathbb{C}^{\beta_j}}^{\beta_j,\lambda}(u)(v) = \begin{cases} 0, & \widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v) \vee \widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(v)(u) < \lambda, \\ G\left(\widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v), \widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(v)(u)\right), & otherwise, \end{cases} \tag{12}$$

where $\lambda \in [0,1]$ and $\lambda \le \beta_j$, $\forall j \in \Lambda$.

Definition 3.14. Suppose $(U,\mathbb{C}')$ is a VSFβ-CAS, O is an overlap function and G is a grouping function. For each $u \in U$, we define the parameterized fuzzy complementary β-neighborhoods $\widehat{OW}_{\mathbb{C}^{\beta_j}}^{\beta_j,\lambda}(u)$ and $\widehat{GW}_{\mathbb{C}^{\beta_j}}^{\beta_j,\lambda}(u)$ of u as:

$$\widehat{OW}_{\mathbb{C}^{\beta_j}}^{\beta_j,\lambda}(u)(v) = \begin{cases} 0, & \widehat{W}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v) \vee \widehat{W}_{\mathbb{C}^{\beta_j}}^{\beta_j}(v)(u) < \lambda, \\ O\left(\widehat{W}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v), \widehat{W}_{\mathbb{C}^{\beta_j}}^{\beta_j}(v)(u)\right), & otherwise, \end{cases} \tag{13}$$

$$\widehat{GW}_{\mathbb{C}^{\beta_j}}^{\beta_j,\lambda}(u)(v) = \begin{cases} 0, & \widehat{W}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v) \vee \widehat{W}_{\mathbb{C}^{\beta_j}}^{\beta_j}(v)(u) < \lambda, \\ G\left(\widehat{W}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v), \widehat{W}_{\mathbb{C}^{\beta_j}}^{\beta_j}(v)(u)\right), & otherwise, \end{cases} \tag{14}$$

where $\lambda \in [0,1]$ and $\lambda \le \beta_j$, $\forall j \in \Lambda$.

Remark 3.15. The original Fβ-NOs are constructed directly based on the affiliations in the data, but the actual data may contain noise that may blur the real associations between the objects and thus show a low membership degrees. More robust parameterized fuzzy β-neighborhood and parameterized fuzzy complementary β-neighborhood can be introduced by threshold filtering, which is both intuitively consistent with fuzzy set theory and parameterized to give the model tunability.

Proposition 3.16. *Suppose $(U,\mathbb{C}')$ is a VSFβ-CAS. For each $u, v \in U$, the following conclusions are correct:*

(1) $$\widehat{ON}_{\mathbb{C}^{\beta_j}}^{\beta_j,\lambda}(u)(v) = \begin{cases} 0, & \widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v) \vee \widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(v)(u) < \lambda, \\ O\left(\widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v), \widehat{W}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v)\right), & otherwise; \end{cases}$$

(2) $$\widehat{GN}_{\mathbb{C}^{\beta_j}}^{\beta_j,\lambda}(u)(v) = \begin{cases} 0, & \widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v) \vee \widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(v)(u) < \lambda, \\ G\left(\widehat{N}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v), \widehat{W}_{\mathbb{C}^{\beta_j}}^{\beta_j}(u)(v)\right), & otherwise; \end{cases}$$

(3) $\widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = \begin{cases} 0, & \widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \vee \widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u) < \lambda, \\ O\left(\widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v), \hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v)\right), & otherwise; \end{cases}$

(4) $\widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = \begin{cases} 0, & \widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \vee \widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u) < \lambda, \\ G\left(\widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v), \hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v)\right), & otherwise. \end{cases}$

Proof. According to Definitions 3.1, 3.13 and 3.14, the proposition is obvious. □

Remark 3.17. According to the symmetric of overlap function and grouping function (Definitions 2.9(O1) and 2.10(G1)), and Proposition 3.16, the following conclusions are obvious:

(1) $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = \begin{cases} 0, & \hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \vee \hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u) < \lambda, \\ O\left(\widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v), \hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v)\right), & otherwise; \end{cases}$

(2) $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = \begin{cases} 0, & \hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \vee \hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u) < \lambda, \\ G\left(\widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v), \hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v)\right), & otherwise; \end{cases}$

(3) $\widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = \begin{cases} 0, & \widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \vee \widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u) < \lambda, \\ O\left(\hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v), \widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v)\right), & otherwise; \end{cases}$

(4) $\widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = \begin{cases} 0, & \widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \vee \widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u) < \lambda, \\ G\left(\hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v), \widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v)\right), & otherwise. \end{cases}$

This conclusion shows that the two pairs of parameterized Fβ-NOs proposed in this paper can be obtained from two indiscernible fuzzy neighborhoods together.

Proposition 3.18. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. For each $u, v \in U$, $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = \widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(v)(u)$ and $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = \widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(v)(u)$.*

Proof. According to Definitions 3.1, 3.13 and 3.14, it is obvious. □

Proposition 3.19. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. For each $u \in U$, the following conclusions are correct:*
(1) $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$, $\widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$, $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ and $\widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ are reflexive;
(2) $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$, $\widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$, $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ and $\widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ are symmetric.

Proof. (1) According to Definitions 2.4, 2.9(O3) and 3.13, $\forall u \in U$, we have

$$\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(u) = O\left(\hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(u), \hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(u)\right) = O(1,1) = 1,$$

thus, $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ is reflexive. Similarly, according to Definitions 2.4, 2.9(O3), 2.10(G3), 3.1, 3.13 and 3.14, $\widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$, $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ and $\widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ are reflexive.

(2) According to Definitions 2.9(O1) and 3.13, $\forall u, v \in U$, if $\hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \vee \hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u) < \lambda$, obviously $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(v)(u) = 0$; else

$$\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = O\left(\hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v), \hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u)\right) = O\left(\hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u), \hat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v)\right) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(v)(u),$$

thus, $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ is symmetric. Similarly, according to Definitions 2.9(O1), 2.10(G1), 3.13 and 3.14, $\widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$, $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ and $\widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ are symmetric. □

Remark 3.20. Symmetry ensures bi-directional consistency between objects and avoids counter-intuitive conclusions such as “A looks like B to the extent of 0.8, while B looks like A to the extent of 0.3”. On the other hand, when calculating the contribution of an attribute to sample similarity, symmetry requires that the attribute has the same effect on “A is similar to B” and “B is similar to A”, which avoids biasing the results due to the assumption of attribute directionality.

Table 5
$\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u_i)(u_j)$ $\left(or\ \widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u_i)(u_j)\right)$ $(i,j=1,2,3,4,5,6)$.

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	1	0.37	0.39	0.55	0.39	0.36
u_2	0.37	1	0.37	0.27	0.37	0.53
u_3	0.39	0.37	1	0.32	1	0.36
u_4	0.55	0.27	0.32	1	0.32	0
u_5	0.39	0.37	1	0.32	1	0.36
u_6	0.36	0.53	0.36	0	0.36	1

Table 6
$\widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u_i)(u_j)$ $\left(or\ \widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u_i)(u_j)\right)$ $(i,j=1,2,3,4,5,6)$.

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	1	0.43	0.51	0.63	0.51	0.43
u_2	0.43	1	0.46	0.37	0.46	0.62
u_3	0.51	0.46	1	0.39	1	0.37
u_4	0.63	0.37	0.39	1	0.39	0
u_5	0.51	0.46	1	0.39	1	0.37
u_6	0.43	0.62	0.37	0	0.37	1

Proposition 3.21. *Suppose $(U,\mathbb{C}')$ is a VSFβ-CAS. For each $u,v\in U$, $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v)=\widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v)$ and $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v)=\widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v)$.*

Proof. According to Propositions 3.18 and 3.19, the proposition is obvious. □

Example 3.22. Assume that $U=\{u_1,u_2,u_3,u_4,u_5,u_6\}$ and $\mathbb{C}'=\{C_1,C_2,C_3,C_4,C_5,C_6\}$ is a family of fuzzy sets of U, where $\mathbb{C}'$ is given in Example 2.5.

Let $\beta=0.6$, $\lambda=0.37$, $O=O_{mp}$ and $G=G_{Mp}(p=1)$. Then the Fβ-NOs $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u_i)(u_j)$, $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u_i)(u_j)$ (or $\widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u_i)(u_j)$, $\widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u_i)(u_j)$) $(i,j=1,2,3,4,5,6)$ are listed in Table 5 and 6, respectively.

Below we explore the monotonicity of two pairs of parameterized Fβ-NOs with respect to the parameters λ, β and subsets, respectively.

Proposition 3.23. *Suppose $(U,\mathbb{C}')$ is a VSFβ-CAS. For each $u\in U$, if $\lambda_1\le\lambda_2$, then $\widehat{ON}^{\beta_j,\lambda_1}_{\mathbb{C}^{\beta_j}}(u)\supseteq\widehat{ON}^{\beta_j,\lambda_2}_{\mathbb{C}^{\beta_j}}(u)$, $\widehat{GN}^{\beta_j,\lambda_1}_{\mathbb{C}^{\beta_j}}(u)\supseteq\widehat{GN}^{\beta_j,\lambda_2}_{\mathbb{C}^{\beta_j}}(u)$.*

Proof. If $\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v)\vee\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u)<\lambda_2$, then $\widehat{ON}^{\beta_j,\lambda_2}_{\mathbb{C}^{\beta_j}}(u)(v)=0$, that is $\widehat{ON}^{\beta_j,\lambda_1}_{\mathbb{C}^{\beta_j}}(u)(v)\ge\widehat{ON}^{\beta_j,\lambda_2}_{\mathbb{C}^{\beta_j}}(u)(v)$; else, $\widehat{ON}^{\beta_j,\lambda_2}_{\mathbb{C}^{\beta_j}}(u)(v)=O\left(\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v),\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u)\right)$, and $\widehat{ON}^{\beta_j,\lambda_1}_{\mathbb{C}^{\beta_j}}(u)(v)=O\left(\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v),\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u)\right)$ by $\lambda_1\le\lambda_2$, i.e. $\widehat{ON}^{\beta_j,\lambda_1}_{\mathbb{C}^{\beta_j}}(u)(v)=\widehat{ON}^{\beta_j,\lambda_2}_{\mathbb{C}^{\beta_j}}(u)(v)$. Thus, $\widehat{ON}^{\beta_j,\lambda_1}_{\mathbb{C}^{\beta_j}}(u)\supseteq\widehat{ON}^{\beta_j,\lambda_2}_{\mathbb{C}^{\beta_j}}(u)$. Similarly, we have $\widehat{GN}^{\beta_j,\lambda_1}_{\mathbb{C}^{\beta_j}}(u)\supseteq\widehat{GN}^{\beta_j,\lambda_2}_{\mathbb{C}^{\beta_j}}(u)$. □

Proposition 3.24. *Suppose $(U,\mathbb{C}')$ is a VSFβ-CAS. For each $u\in U$, if $\beta_1\le\beta_2$, then $\widehat{ON}^{\beta_1,\lambda}_{\mathbb{C}^{\beta_1}}(u)\subseteq\widehat{ON}^{\beta_2,\lambda}_{\mathbb{C}^{\beta_2}}(u)$, $\widehat{GN}^{\beta_1,\lambda}_{\mathbb{C}^{\beta_1}}(u)\subseteq\widehat{GN}^{\beta_2,\lambda}_{\mathbb{C}^{\beta_2}}(u)$.*

Proof. For any given $u\in U$, two cases arise.

Case 1: According to Definition 2.4, for each $v\in U$, if $C_i(u)=C_i(v)(i=1,2,\cdots,n)$, then $\widehat{N}^{\beta_1}_{\mathbb{C}^{\beta_1}}(u)(v)=1$ and $\widehat{N}^{\beta_2}_{\mathbb{C}^{\beta_2}}(u)(v)=1$, that is $\widehat{ON}^{\beta_1,\lambda}_{\mathbb{C}^{\beta_1}}(u)(v)=1$ and $\widehat{ON}^{\beta_2,\lambda}_{\mathbb{C}^{\beta_2}}(u)(v)=1$ by Definition 3.13. Thus, $\widehat{ON}^{\beta_1,\lambda}_{\mathbb{C}^{\beta_1}}(u)=\widehat{ON}^{\beta_2,\lambda}_{\mathbb{C}^{\beta_2}}(u)$.

Case 2: According to Definition 2.4, for each $v\in U$, $C_{i_0}(u)\ne C_{i_0}(v)$ if there exists a fuzzy set $C_{i_0}\in\mathbb{C}'$, then $\widehat{N}^{\beta_1}_{\mathbb{C}^{\beta_1}}(u)(v)=N^{\beta_1}_{\mathbb{C}^{\beta_1}}(u)(v)$ and $\widehat{N}^{\beta_2}_{\mathbb{C}^{\beta_2}}(u)(v)=N^{\beta_2}_{\mathbb{C}^{\beta_2}}(u)(v)$. Furthermore, $\beta_1\le\beta_2$ means that $N^{\beta_1}_{\mathbb{C}^{\beta_1}}(u)\subseteq N^{\beta_2}_{\mathbb{C}^{\beta_2}}(u)$. Thus, according to Definition 3.13, if $\widehat{N}^{\beta_2}_{\mathbb{C}^{\beta_2}}(u)(v)\vee\widehat{N}^{\beta_2}_{\mathbb{C}^{\beta_2}}(v)(u)<\lambda$, then $\widehat{ON}^{\beta_2,\lambda}_{\mathbb{C}^{\beta_2}}(u)(v)=0$; moreover, $\widehat{N}^{\beta_1}_{\mathbb{C}^{\beta_1}}(u)(v)\vee\widehat{N}^{\beta_1}_{\mathbb{C}^{\beta_1}}(v)(u)<\lambda$ must hold, that is $\widehat{ON}^{\beta_1,\lambda}_{\mathbb{C}^{\beta_1}}(u)(v)=\widehat{ON}^{\beta_2,\lambda}_{\mathbb{C}^{\beta_2}}(u)(v)=0$. Else

$$
\begin{aligned}
0\le\widehat{ON}^{\beta_1,\lambda}_{\mathbb{C}^{\beta_1}}(u)(v)&=O\left(\widehat{N}^{\beta_1}_{\mathbb{C}^{\beta_1}}(u)(v),\widehat{N}^{\beta_1}_{\mathbb{C}^{\beta_1}}(v)(u)\right)=O\left(N^{\beta_1}_{\mathbb{C}^{\beta_1}}(u)(v),N^{\beta_1}_{\mathbb{C}^{\beta_1}}(v)(u)\right)\\
&\le O\left(N^{\beta_2}_{\mathbb{C}^{\beta_2}}(u)(v),N^{\beta_2}_{\mathbb{C}^{\beta_2}}(v)(u)\right)=O\left(\widehat{N}^{\beta_2}_{\mathbb{C}^{\beta_2}}(u)(v),\widehat{N}^{\beta_2}_{\mathbb{C}^{\beta_2}}(v)(u)\right)=\widehat{ON}^{\beta_2,\lambda}_{\mathbb{C}^{\beta_2}}(u)(v).
\end{aligned}
$$

That is $\widehat{ON}^{\beta_1,\lambda}_{\mathbb{C}^{\beta_1}}(u)\subseteq\widehat{ON}^{\beta_2,\lambda}_{\mathbb{C}^{\beta_2}}(u)$.

Combining Cases 1 and 2, we can conclude that $\widehat{ON}^{\beta_1,\lambda}_{\mathbb{C}^{\beta_1}}(u) \subseteq \widehat{ON}^{\beta_2,\lambda}_{\mathbb{C}^{\beta_2}}(u)$ if $\beta_1 \le \beta_2$. Similarly, we have $\widehat{GN}^{\beta_1,\lambda}_{\mathbb{C}^{\beta_1}}(u) \subseteq \widehat{GN}^{\beta_2,\lambda}_{\mathbb{C}^{\beta_2}}(u)$. □

Proposition 3.25. *Suppose $(U,\mathbb{C}')$ is a VSFβ-CAS. For each $u \in U$, if $\mathbb{B}_1^{\beta_j} \subseteq \mathbb{B}_2^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$, then $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j}}(u) \supseteq \widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j}}(u)$, $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j}}(u) \supseteq \widehat{GN}^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j}}(u)$.*

Proof. For any given $u \in U$, two cases arise.

Case 1: According to Definition 2.4, for each $v \in U$, if $C_i(u) = C_i(v)(i = 1,2,\cdots,n)$, then $\widehat{N}^{\beta_j}_{\mathbb{B}_1^{\beta_j}}(u)(v) = 1$ and $\widehat{N}^{\beta_j}_{\mathbb{B}_2^{\beta_j}}(u)(v) = 1$, that is $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j}}(u)(v) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j}}(u)(v) = 1$ by Definition 3.13. Thus, $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j}}(u) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j}}(u)$.

Case 2: According to Definition 2.4, for each $v \in U$, $C_{i_0}(u) \neq C_{i_0}(v)$ if there exists a fuzzy set $C_{i_0} \in \mathbb{C}'$, then $\widehat{N}^{\beta_j}_{\mathbb{B}_1^{\beta_j}}(u)(v) = N^{\beta_j}_{\mathbb{B}_1^{\beta_j}}(u)(v)$ and $\widehat{N}^{\beta_j}_{\mathbb{B}_2^{\beta_j}}(u)(v) = N^{\beta_j}_{\mathbb{B}_2^{\beta_j}}(u)(v)$. From $\mathbb{B}_1^{\beta_j} \subseteq \mathbb{B}_2^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$ we know that $N^{\beta_j}_{\mathbb{B}_1^{\beta_j}}(u) \supseteq N^{\beta_j}_{\mathbb{B}_2^{\beta_j}}(u)$, i.e. $\widehat{N}^{\beta_j}_{\mathbb{B}_1^{\beta_j}}(u)(v) \ge \widehat{N}^{\beta_j}_{\mathbb{B}_2^{\beta_j}}(u)(v)$. Thus, according to Definition 3.13, if $\widehat{N}^{\beta_j}_{\mathbb{B}_1^{\beta_j}}(u)(v) \vee \widehat{N}^{\beta_j}_{\mathbb{B}_1^{\beta_j}}(v)(u) < \lambda$, then $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j}}(u)(v) = 0$; moreover, $\widehat{N}^{\beta_j}_{\mathbb{B}_2^{\beta_j}}(u)(v) \vee \widehat{N}^{\beta_j}_{\mathbb{B}_2^{\beta_j}}(v)(u) < \lambda$ must hold, that is $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j}}(u)(v) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j}}(u)(v) = 0$. Else

$$\widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j}}(u)(v) = O\left(\widehat{N}^{\beta_j}_{\mathbb{B}_1^{\beta_j}}(u)(v), \widehat{N}^{\beta_j}_{\mathbb{B}_1^{\beta_j}}(v)(u)\right) = O\left(N^{\beta_j}_{\mathbb{B}_1^{\beta_j}}(u)(v), N^{\beta_j}_{\mathbb{B}_1^{\beta_j}}(v)(u)\right)$$

$$\ge O\left(N^{\beta_j}_{\mathbb{B}_2^{\beta_j}}(u)(v), N^{\beta_j}_{\mathbb{B}_2^{\beta_j}}(v)(u)\right) = O\left(\widehat{N}^{\beta_j}_{\mathbb{B}_2^{\beta_j}}(u)(v), \widehat{N}^{\beta_j}_{\mathbb{B}_2^{\beta_j}}(v)(u)\right) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j}}(u)(v) \ge 0.$$

That is $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j}}(u) \supseteq \widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j}}(u)$.

Combining Cases 1 and 2, we can conclude that $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j}}(u) \supseteq \widehat{ON}^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j}}(u)$ if $\mathbb{B}_1^{\beta_j} \subseteq \mathbb{B}_2^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$. Similarly, we can obtain that $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j}}(u) \supseteq \widehat{GN}^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j}}(u)$. □

Proposition 3.26. *Suppose $(U,\mathbb{C}')$ is a VSFβ-CAS. For each $u \in U$, the following statements hold.*
(1) If $\lambda_1 \le \lambda_2$, then $\widehat{OW}^{\beta_j,\lambda_1}_{\mathbb{C}^{\beta_j}}(u) \supseteq \widehat{OW}^{\beta_j,\lambda_2}_{\mathbb{C}^{\beta_j}}(u)$, $\widehat{GW}^{\beta_j,\lambda_1}_{\mathbb{C}^{\beta_j}}(u) \supseteq \widehat{GW}^{\beta_j,\lambda_2}_{\mathbb{C}^{\beta_j}}(u)$;
(2) If $\beta_1 \le \beta_2$, then $\widehat{OW}^{\beta_1,\lambda}_{\mathbb{C}^{\beta_1}}(u) \subseteq \widehat{OW}^{\beta_2,\lambda}_{\mathbb{C}^{\beta_2}}(u)$, $\widehat{GW}^{\beta_1,\lambda}_{\mathbb{C}^{\beta_1}}(u) \subseteq \widehat{GW}^{\beta_2,\lambda}_{\mathbb{C}^{\beta_2}}(u)$;
(3) If $\mathbb{B}_1^{\beta_j} \subseteq \mathbb{B}_2^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$, then $\widehat{OW}^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j}}(u) \supseteq \widehat{OW}^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j}}(u)$, $\widehat{GW}^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j}}(u) \supseteq \widehat{GW}^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j}}(u)$.

Proof. According to Proposition 3.21, the proposition is obvious. □

We next discuss the relationship among two pairs of parameterized Fβ-NOs.

Proposition 3.27. *Suppose $(U,\mathbb{C}')$ is a VSFβ-CAS, G is dual to O. For each $u \in U$, the following statements hold.*
(1) $\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u) \supseteq \widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) = \widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$;
(2) $\widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u) \supseteq \widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$;
(3) $\widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) \subseteq \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) = \widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$.

Proof. (1) For any given $u \in U$, two cases arise.

Case 1: According to Definitions 2.4 and 3.13, for each $v \in U$, if $C_i(u) = C_i(v)(i = 1,2,\cdots,n)$, then $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) = 1$.

Case 2: According to Definition 2.4, for each $v \in U$, $C_{i_0}(u) \neq C_{i_0}(v)$ if there exists a fuzzy set $C_{i_0} \in \mathbb{C}'$. If $\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \vee \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u) < \lambda$, then $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = 0 \le \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v)$; else

$$\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = O\left(\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v), \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u)\right) \le \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v).$$

Combining Cases 1 and 2, we can conclude that $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) \subseteq \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)$. According to Proposition 3.21, $\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u) \supseteq \widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) = \widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ is obvious.

(2) It can be proven using a method analogous to (1).

(3) According to Definition 3.13, for each $v \in U$, if $\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \vee \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u) < \lambda$, $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = 0$; else, due to G is dual to O, we have

$$\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = O\left(\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v), \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u)\right) \le G\left(\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v), \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u)\right) = \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v).$$

Thus, we can obtain that $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) \subseteq \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$. According to Proposition 3.21, $\widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) \subseteq \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) = \widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ is obvious. □

Proposition 3.28. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. For each $E \in F(U)$, the following statements are equivalent.*
(1) $\widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) \subseteq \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u) \subseteq \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) = \widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$;
(2) $\lambda = 0$.

Proof. It follows from Proposition 3.27 that it is only necessary to show that $\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u) \subseteq \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ and $\lambda = 0$ are equivalent.

(1)⇒(2) For each $u, v \in U$, from $\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u) \subseteq \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ we know that $\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \le \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v)$. Assuming $\lambda \neq 0$, i.e. $\lambda \in (0,1]$. $\exists u, v \in U$, s.t. $0 < \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \vee \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u) < \lambda$, then $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = 0$, that is $\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) > \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = 0$, it contradicts $\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \le \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v)$. Thus, $\lambda = 0$.

(2)⇒(1) For each $u, v \in U$, $\lambda = 0$ and $\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v) \in [0,1]$ means that $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) = G\left(\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v), \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(v)(u)\right) \ge \widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u)(v)$. Thus, $\widehat{N}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u) \subseteq \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$. □

Proposition 3.29. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. For each $E \in F(U)$, the following statements are equivalent.*
(1) $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) = \widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) \subseteq \widehat{W}^{\beta_j}_{\mathbb{C}^{\beta_j}}(u) \subseteq \widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u) = \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$;
(2) $\lambda = 0$.

Proof. This can be proven using a method analogous to the one used in Proposition 3.28. □

4. Four new types of FCRS models based on fuzzy β-covering relation in VSFβ-CAS

In following section, we prove these Fβ-NOs conform to the structural requirements of fuzzy β-covering relations, and construct the new Fβ-CRS models, their basic properties and interrelationships relationship among different models are discussed at the same time.

4.1. Fβ-CRS models based on fuzzy β-covering relation

Proposition 4.1. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS, our presented parameterized Fβ-NOs $\widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$, $\widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$, $\widehat{OM}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ and $\widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)$ are fuzzy β-covering relations.*

Proof. According to Definition 2.7 and Proposition 3.19, the proposition is obvious. □

Definition 4.2. Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS, $\mathbb{B}^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$. For each $u, v \in U$, we define the l-th type of parameterized fuzzy β-covering relations $R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u, v)$ $(l = 1, 2, 3, 4)$ as:

$$R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},1}(u, v) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v), \tag{15}$$

$$R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},2}(u, v) = \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v), \tag{16}$$

$$R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},3}(u, v) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) \wedge \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v), \tag{17}$$

$$R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},4}(u, v) = \widehat{ON}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) \vee \widehat{GN}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v). \tag{18}$$

The fuzzy similarity class $[u]^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}$ of u with respect to $\mathbb{B}^{\beta_j}$ is defined as a fuzzy set on U. $[u]^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}$ is interpreted as a fuzzy neighborhood associated with $\mathbb{B}^{\beta_j}$ when $R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}$ constitutes a fuzzy β-covering relation. Specifically, for any $v \in U$, it is given by $[u]^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(v) = R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)$.

Remark 4.3. According to Proposition 3.21, $\forall u, v \in U$, we can easily obtain that,

$$R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},1}(u,v) = \widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v),\ R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},2}(u,v) = \widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v),$$

$$R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},3}(u,v) = \widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) \wedge \widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v),\ R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},4}(u,v) = \widehat{OW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v) \vee \widehat{GW}^{\beta_j,\lambda}_{\mathbb{C}^{\beta_j}}(u)(v).$$

Proposition 4.4. *Suppose* $(U, \mathbb{C}')$ *is a VSF*β*-CAS. The following statements hold* ($l = 1, 2, 3, 4$).
(1) If $\lambda_1 \le \lambda_2$*, then* $R^{\beta_j,\lambda_1}_{\mathbb{B}^{\beta_j},l} \supseteq R^{\beta_j,\lambda_2}_{\mathbb{B}^{\beta_j},l}$;
(2) If $\beta_1 \le \beta_2$*, then* $R^{\beta_1,\lambda}_{\mathbb{B}^{\beta_1},l} \subseteq R^{\beta_2,\lambda}_{\mathbb{B}^{\beta_2},l}$;
(3) If $\mathbb{B}^{\beta_j}_1 \subseteq \mathbb{B}^{\beta_j}_2 \subseteq \mathbb{C}^{\beta_j}$*, then* $R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j}_1,l} \supseteq R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j}_2,l}$.

Proof. According to Definition 4.2, the proposition is obvious. □

Based on our proposed parameterized fuzzy β-covering relations, four new types of FCRS models will be constructed in VSFβ-CAS.

Definition 4.5. Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS, $\beta_j \in (0,1]$, $\lambda \in [0,1]$, $\lambda \le \beta_j (\forall j \in \Lambda)$ and $\mathbb{B}^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$. For each $u \in U$ and $E \in F(U)$, on the basis of four kinds of parameterized fuzzy β-covering relations, we define the fuzzy lower and upper approximations of E as:

$$\underline{R_{\mathbb{B}^{\beta_j},l}}(E)(u) = \bigwedge_{v\in U} \left\{ \left(1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\right) \vee E(v) \right\}, \tag{19}$$

$$\overline{R_{\mathbb{B}^{\beta_j},l}}(E)(u) = \bigvee_{v\in U} \left\{ R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v) \wedge E(v) \right\}. \tag{20}$$

If $\underline{R_{\mathbb{B}^{\beta_j},l}}(E) \neq \overline{R_{\mathbb{B}^{\beta_j},l}}(E)$, then E is the FCRS of the l-th type, where $l = 1, 2, 3, 4$; otherwise it is definable. The ordered pair $\left(\overline{R_{\mathbb{B}^{\beta_j},l}}(E), \underline{R_{\mathbb{B}^{\beta_j},l}}(E)\right)$ is called the l-FCRS model of E.

Example 4.6. (Following Example 3.22) Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. Let $\beta = 0.6$, $\lambda = 0.37$, $O = O_{mp}$ and $G = G_{M_p}(p = 1)$. Then for $A = \frac{0.64}{u_1} + \frac{0.46}{u_2} + \frac{0.75}{u_3} + \frac{0.35}{u_4} + \frac{0.54}{u_5} + \frac{0.72}{u_6}$, based on Definition 4.5, we have

$$\overline{R_{\mathbb{B}^{0.6},1}}(E)(u) = \frac{0.45}{u_1} + \frac{0.46}{u_2} + \frac{0.54}{u_3} + \frac{0.35}{u_4} + \frac{0.54}{u_5} + \frac{0.47}{u_6}, \underline{R_{\mathbb{B}^{0.6},1}}(E)(u) = \frac{0.64}{u_1} + \frac{0.53}{u_2} + \frac{0.75}{u_3} + \frac{0.55}{u_4} + \frac{0.75}{u_5} + \frac{0.72}{u_6};$$

$$\overline{R_{\mathbb{B}^{0.6},2}}(E)(u) = \frac{0.37}{u_1} + \frac{0.46}{u_2} + \frac{0.54}{u_3} + \frac{0.35}{u_4} + \frac{0.54}{u_5} + \frac{0.46}{u_6}, \underline{R_{\mathbb{B}^{0.6},2}}(E)(u) = \frac{0.64}{u_1} + \frac{0.62}{u_2} + \frac{0.75}{u_3} + \frac{0.63}{u_4} + \frac{0.75}{u_5} + \frac{0.72}{u_6};$$

$$\overline{R_{\mathbb{B}^{0.6},3}}(E)(u) = \frac{0.45}{u_1} + \frac{0.46}{u_2} + \frac{0.54}{u_3} + \frac{0.35}{u_4} + \frac{0.54}{u_5} + \frac{0.47}{u_6}, \underline{R_{\mathbb{B}^{0.6},3}}(E)(u) = \frac{0.64}{u_1} + \frac{0.53}{u_2} + \frac{0.75}{u_3} + \frac{0.55}{u_4} + \frac{0.75}{u_5} + \frac{0.72}{u_6};$$

$$\overline{R_{\mathbb{B}^{0.6},4}}(E)(u) = \frac{0.37}{u_1} + \frac{0.46}{u_2} + \frac{0.54}{u_3} + \frac{0.35}{u_4} + \frac{0.54}{u_5} + \frac{0.46}{u_6}, \underline{R_{\mathbb{B}^{0.6},4}}(E)(u) = \frac{0.64}{u_1} + \frac{0.62}{u_2} + \frac{0.75}{u_3} + \frac{0.63}{u_4} + \frac{0.75}{u_5} + \frac{0.72}{u_6}.$$

Remark 4.7. We observe that in Example 4.6, the upper and lower approximations of 1-FCRS model are equal to those of 3-FCRS model, and the upper and lower approximations of 2-FCRS model are equal to those of 4-FCRS model. This is due to the small amount of data and simple structure in this example. In the following content, we will rigorously prove the inclusion relationships among these four models, and further verify their differences through experiments.

In the following, we present some properties of the four FCRS models. We can directly get the relation that upper approximations contain lower approximations in the four models based on fuzzy β-covering relations.

Proposition 4.8. *Suppose* $(U, \mathbb{C}')$ *is a VSF*β*-CAS. For each* $E \in F(U)$, $\underline{R_{\mathbb{B}^{\beta_j},l}}(E) \subseteq E \subseteq \overline{R_{\mathbb{B}^{\beta_j},l}}(E)$, *l=1,2,3,4.*

Proof. According to Definition 4.5, for each $u \in U$ and $E \in F(U)$,

$$\underline{R_{\mathbb{B}^{\beta_j},l}}(E)(u) = \bigwedge_{v\in U}\left\{\left(1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\right) \vee E(v)\right\} \le \left(1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,u)\right) \vee E(u) = (1-1)\vee E(u)$$
$$= E(u)$$
$$= 1 \wedge E(u) = R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,u) \wedge E(u) \le \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\wedge E(v)\right\} = \overline{R_{\mathbb{B}^{\beta_j},l}}(E)(u).$$

Thus, we have $\underline{R_{\mathbb{B}^{\beta_j},l}}(E) \subseteq E \subseteq \overline{R_{\mathbb{B}^{\beta_j},l}}(E)$, l=1,2,3,4. □

Proposition 4.9. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS, l=1,2,3,4. For each $E, F \in F(U)$, the following statements hold.*
(1) $\overline{R_{\mathbb{B}^{\beta_j},l}}(\varnothing) = \varnothing$ and $\underline{R_{\mathbb{B}^{\beta_j},l}}(U) = U$;
(2) $\overline{R_{\mathbb{B}^{\beta_j},l}}(\neg E) = \neg\underline{R_{\mathbb{B}^{\beta_j},l}}(E)$ and $\underline{R_{\mathbb{B}^{\beta_j},l}}(\neg E) = \neg\overline{R_{\mathbb{B}^{\beta_j},l}}(E)$, where $\neg E(u) = 1 - E(u)$ for each $u \in U$;
(3) $\overline{R_{\mathbb{B}^{\beta_j},l}}(E\cup F) = \overline{R_{\mathbb{B}^{\beta_j},l}}(E) \cup \overline{R_{\mathbb{B}^{\beta_j},l}}(F)$ and $\underline{R_{\mathbb{B}^{\beta_j},l}}(E\cap F) = \underline{R_{\mathbb{B}^{\beta_j},l}}(E) \cap \underline{R_{\mathbb{B}^{\beta_j},l}}(F)$;
(4) $\overline{R_{\mathbb{B}^{\beta_j},l}}(E\cap F) \subseteq \overline{R_{\mathbb{B}^{\beta_j},l}}(E) \cap \overline{R_{\mathbb{B}^{\beta_j},l}}(F)$ and $\underline{R_{\mathbb{B}^{\beta_j},l}}(E\cup F) \supseteq \underline{R_{\mathbb{B}^{\beta_j},l}}(E) \cup \underline{R_{\mathbb{B}^{\beta_j},l}}(F)$;
(5) $E \subseteq F \Rightarrow \overline{R_{\mathbb{B}^{\beta_j},l}}(E) \subseteq \overline{R_{\mathbb{B}^{\beta_j},l}}(F)$ and $\underline{R_{\mathbb{B}^{\beta_j},l}}(E) \subseteq \underline{R_{\mathbb{B}^{\beta_j},l}}(F)$.

Proof. (1) For each $u, v \in U$, it follows that,

$$\overline{R_{\mathbb{B}^{\beta_j},l}}(\varnothing)(u) = \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\wedge \varnothing(v)\right\} = 0 = \varnothing,$$
$$\underline{R_{\mathbb{B}^{\beta_j},l}}(U)(u) = \bigwedge_{v\in U}\left\{\left(1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\right)\vee U(v)\right\} = 1 = U.$$

Thus, we have $\overline{R_{\mathbb{B}^{\beta_j},l}}(\varnothing) = \varnothing$, $\underline{R_{\mathbb{B}^{\beta_j},l}}(U) = U$.

(2) For each $u, v \in U$, it follows that,

$$\underline{R_{\mathbb{B}^{\beta_j},l}}(\neg E)(u) = \bigwedge_{v\in U}\left\{\left(1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\right)\vee (\neg E)(v)\right\} = \bigwedge_{v\in U}\left\{\left(1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\right)\vee (1-E(v))\right\}$$
$$= \bigwedge_{v\in U}\left\{1 - \left(R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\wedge E(v)\right)\right\} = 1 - \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\wedge E(v)\right\}$$
$$= \neg\overline{R_{\mathbb{B}^{\beta_j},l}}(E)(u),$$

and

$$\overline{R_{\mathbb{B}^{\beta_j},l}}(\neg E)(u) = 1 - \underline{R_{\mathbb{B}^{\beta_j},l}}(\neg(\neg E))(u) = 1 - \underline{R_{\mathbb{B}^{\beta_j},l}}(E)(u) = \neg\underline{R_{\mathbb{B}^{\beta_j},l}}(E).$$

Thus, we have $\overline{R_{\mathbb{B}^{\beta_j},l}}(\neg E) = \neg\underline{R_{\mathbb{B}^{\beta_j},l}}(E)$, $\underline{R_{\mathbb{B}^{\beta_j},l}}(\neg E) = \neg\overline{R_{\mathbb{B}^{\beta_j},l}}(E)$.

(3) For each $u, v \in U$, it follows that,

$$\overline{R_{\mathbb{B}^{\beta_j},l}}(E\cup F)(u) = \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\wedge (E\cup F)(v)\right\} = \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\wedge \left(E(v)\vee F(v)\right)\right\}$$
$$= \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\wedge E(v)\right\} \vee \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\wedge F(v)\right\}$$
$$= \overline{R_{\mathbb{B}^{\beta_j},l}}(E)(u) \vee \overline{R_{\mathbb{B}^{\beta_j},l}}(F)(u) = \left(\overline{R_{\mathbb{B}^{\beta_j},l}}(E) \cup \overline{R_{\mathbb{B}^{\beta_j},l}}(F)\right)(u),$$

$$\underline{R_{\mathbb{B}^{\beta_j},l}}(E\cap F)(u) = \bigwedge_{v\in U}\left\{\left(1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\right)\vee (E\cap F)(v)\right\}$$
$$= \bigwedge_{v\in U}\left\{\left(1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\right)\vee \left(E(v)\wedge F(v)\right)\right\}$$
$$= \bigwedge_{v\in U}\left\{\left(1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\right)\vee E(v)\right\} \wedge \bigwedge_{v\in U}\left\{\left(1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\right)\vee F(v)\right\}$$
$$= \underline{R_{\mathbb{B}^{\beta_j},l}}(E)(u) \wedge \underline{R_{\mathbb{B}^{\beta_j},l}}(F)(u) = \left(\underline{R_{\mathbb{B}^{\beta_j},l}}(E) \cap \underline{R_{\mathbb{B}^{\beta_j},l}}(F)\right)(u).$$

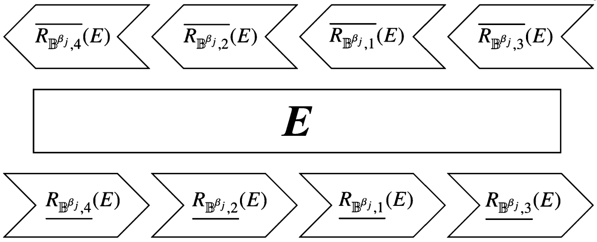

Fig. 2. Interrelationships among four types of Fβ-CRS models.

Thus, we have $\underline{R_{\mathbb{B}^{\beta_j},l}}(E \cap F) = \underline{R_{\mathbb{B}^{\beta_j},l}}(E) \cap \underline{R_{\mathbb{B}^{\beta_j},l}}(F)$ and $\overline{R_{\mathbb{B}^{\beta_j},l}}(E \cup F) = \overline{R_{\mathbb{B}^{\beta_j},l}}(E) \cup \overline{R_{\mathbb{B}^{\beta_j},l}}(F)$.

(4) For each $u, v \in U$, it follows that,

$$\begin{aligned}
\overline{R_{\mathbb{B}^{\beta_j},l}}(E \cap F)(u) &= \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v) \wedge (E\cap F)(v)\right\} = \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v) \wedge \left(E(v)\wedge F(v)\right)\right\}\\
&\le \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v) \wedge E(v)\right\} \wedge \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v) \wedge F(v)\right\}\\
&= \overline{R_{\mathbb{B}^{\beta_j},l}}(E)(u) \wedge \overline{R_{\mathbb{B}^{\beta_j},l}}(F)(u) = \left(\overline{R_{\mathbb{B}^{\beta_j},l}}(E) \cap \overline{R_{\mathbb{B}^{\beta_j},l}}(F)\right)(u);
\end{aligned}$$

Similarly, we have $\underline{R_{\mathbb{B}^{\beta_j},l}}(E\cup F)(u) \ge \left(\underline{R_{\mathbb{B}^{\beta_j},l}}(E) \cup \underline{R_{\mathbb{B}^{\beta_j},l}}(F)\right)(u)$. Thus, we have $\underline{R_{\mathbb{B}^{\beta_j},l}}(E\cup F) \supseteq \underline{R_{\mathbb{B}^{\beta_j},l}}(E) \cup \underline{R_{\mathbb{B}^{\beta_j},l}}(F)$, $\overline{R_{\mathbb{B}^{\beta_j},l}}(E\cap F) \subseteq \overline{R_{\mathbb{B}^{\beta_j},l}}(E) \cap \overline{R_{\mathbb{B}^{\beta_j},l}}(F)$.

(5) For each $u \in U$, one can deduce that $E(u) \le F(u)$ when $E \subseteq F$ is given. It follows that: $\forall u, v \in U$

$$\overline{R_{\mathbb{B}^{\beta_j},l}}(E)(u) = \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v) \wedge E(v)\right\} \le \bigvee_{v\in U}\left\{R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v) \wedge F(v)\right\} = \overline{R_{\mathbb{B}^{\beta_j},l}}(F)(u),$$

$$\underline{R_{\mathbb{B}^{\beta_j},l}}(E)(u) = \bigwedge_{v\in U}\left\{\left(1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\right) \vee E(v)\right\} \le \bigwedge_{v\in U}\left\{\left(1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v)\right) \vee F(v)\right\} = \underline{R_{\mathbb{B}^{\beta_j},l}}(F)(u).$$

Thus, we have $\overline{R_{\mathbb{B}^{\beta_j},l}}(E) \subseteq \overline{R_{\mathbb{B}^{\beta_j},l}}(F)$, $\underline{R_{\mathbb{B}^{\beta_j},l}}(E) \subseteq \underline{R_{\mathbb{B}^{\beta_j},l}}(F)$. ☐

4.2. Interrelationships among different Fβ-CRS models

In this subsection, we will focus on the interrelationships among the different approximation operators discussed in the previous subsections.

Proposition 4.10. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. For each $E \in F(U)$, the following statements hold.*
(1) $\overline{R_{\mathbb{B}^{\beta_j},3}}(E) \subseteq \overline{R_{\mathbb{B}^{\beta_j},1}}(E) \subseteq \overline{R_{\mathbb{B}^{\beta_j},2}}(E) \subseteq \overline{R_{\mathbb{B}^{\beta_j},4}}(E)$;
(2) $\underline{R_{\mathbb{B}^{\beta_j},4}}(E) \subseteq \underline{R_{\mathbb{B}^{\beta_j},2}}(E) \subseteq \underline{R_{\mathbb{B}^{\beta_j},1}}(E) \subseteq \underline{R_{\mathbb{B}^{\beta_j},3}}(E)$.

Proof. From Definition 4.5 and Proposition 3.27(3), it can be simply proved that the conclusion holds. ☐

Combining the Proposition 4.8 one obtains the Fig. 2, we can get that the maximum element is $\overline{R_{\mathbb{B}^{\beta_j},4}}(E)$, and the minimum element is $\underline{R_{\mathbb{B}^{\beta_j},4}}(E)$.

In the following we discuss the interrelationships among the approximation operators in [20] and the proposed in this paper.

Proposition 4.11. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. For each $E \in F(U)$, the following statements are equivalent.*
(1) $\underline{R_{\mathbb{B}^{\beta_j},4}}(E) \subseteq \underline{R_{\mathbb{B}^{\beta_j},2}}(E) \subseteq \underline{Appr}(E) \subseteq \underline{R_{\mathbb{B}^{\beta_j},1}}(E) \subseteq \underline{R_{\mathbb{B}^{\beta_j},3}}(E) \subseteq E \subseteq \overline{R_{\mathbb{B}^{\beta_j},3}}(E) \subseteq \overline{R_{\mathbb{B}^{\beta_j},1}}(E) \subseteq \overline{Appr}(E) \subseteq \overline{R_{\mathbb{B}^{\beta_j},2}}(E) \subseteq \overline{R_{\mathbb{B}^{\beta_j},4}}(E)$;
(2) $\lambda = 0$.

Proof. It can be simply proved that the conclusion holds by Definition 2.6, Propositions 3.28 and 4.10. ☐

It can be found that among these approximation operators, $\underline{R_{\mathbb{B}^{\beta_j},4}}(E)$ is the minimum operator, and $\overline{R_{\mathbb{B}^{\beta_j},4}}(E)$ is the maximum operator when $\lambda = 0$.

5. Fβ-CRS models application to TWD

In this section, we will define the fuzzy and crisp three-way regions according to the proposed models, respectively. Furthermore, a new TWD methods are designed using them respectively.

5.1. TWD methods based on Fβ-CRS models

Firstly, the fuzzy positive region, fuzzy negative region and fuzzy boundary region are defined using the proposed models, then their properties are discussed.

Definition 5.1. Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS, $\beta_j \in (0,1]$, and $\mathbb{B}^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$. For each $u \in U$ and $E \in F(U)$, the fuzzy positive region $POS_{\mathbb{B}^{\beta_j},l}(E)$, fuzzy negative region $NEG_{\mathbb{B}^{\beta_j},l}(E)$, and fuzzy boundary region $BND_{\mathbb{B}^{\beta_j},l}(E)$ of E are defined as

$$POS_{\mathbb{B}^{\beta_j},l}(E)(u) = \underline{R_{\mathbb{B}^{\beta_j},l}}(E)(u), \tag{21}$$

$$NEG_{\mathbb{B}^{\beta_j},l}(E)(u) = 1 - \overline{R_{\mathbb{B}^{\beta_j},l}}(E)(u), \tag{22}$$

$$BND_{\mathbb{B}^{\beta_j},l}(E)(u) = \overline{R_{\mathbb{B}^{\beta_j},l}}(E)(u) - \underline{R_{\mathbb{B}^{\beta_j},l}}(E)(u). \tag{23}$$

Proposition 5.2. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. For each $E \in F(U)$, the following statements hold.*
(1) $POS_{\mathbb{B}^{\beta_j},l}(E) \subseteq E$ and $NEG_{\mathbb{B}^{\beta_j},l}(E)(u) \subseteq \neg E$;
(2) $\sum_{u\in U} POS_{\mathbb{B}^{\beta_j},l}(E)(u) + \sum_{u\in U} NEG_{\mathbb{B}^{\beta_j},l}(E)(u) + \sum_{u\in U} BND_{\mathbb{B}^{\beta_j},l}(E)(u) = |U|$.

Proof. According to Definition 5.1, the proposition is obvious. □

Proposition 5.3. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. For each $E \in F(U)$, the following statements hold.*
(1) $POS_{\mathbb{B}^{\beta_j},4}(E) \subseteq POS_{\mathbb{B}^{\beta_j},2}(E) \subseteq POS_{\mathbb{B}^{\beta_j},1}(E) \subseteq POS_{\mathbb{B}^{\beta_j},3}(E)$;
(2) $NEG_{\mathbb{B}^{\beta_j},4}(E) \subseteq NEG_{\mathbb{B}^{\beta_j},2}(E) \subseteq NEG_{\mathbb{B}^{\beta_j},1}(E) \subseteq NEG_{\mathbb{B}^{\beta_j},3}(E)$;
(3) $BND_{\mathbb{B}^{\beta_j},3}(E) \subseteq BND_{\mathbb{B}^{\beta_j},1}(E) \subseteq BND_{\mathbb{B}^{\beta_j},2}(E) \subseteq BND_{\mathbb{B}^{\beta_j},4}(E)$.

Proof. According to Definition 5.1 and Proposition 4.10, the proposition is obvious. □

The fuzzy positive region and fuzzy negative region can be respectively regarded as the acceptance decision and rejection decision for any object $u \in U$ with respect to the concept E. Based on this characteristic, we can convert the fuzzy three-way regions into crisp three-way regions by setting a pair of thresholds.

Definition 5.4. Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS, $\beta_j \in (0,1]$, (θ, δ) is a pair of thresholds. For each $u \in U$ and $E \in F(U)$, the positive region $\overline{POS}_{\mathbb{B}^{\beta_j},l}(E)$, negative region $\overline{NEG}_{\mathbb{B}^{\beta_j},l}(E)$, and boundary region $\overline{BND}_{\mathbb{B}^{\beta_j},l}(E)$ of E are defined as

$$\overline{POS}_{\mathbb{B}^{\beta_j},l}(E) = \{u \in U \mid POS_{\mathbb{B}^{\beta_j},l}(E)(u) - NEG_{\mathbb{B}^{\beta_j},l}(E)(u) \geq \theta\}, \tag{24}$$

$$\overline{NEG}_{\mathbb{B}^{\beta_j},l}(E) = \{u \in U \mid POS_{\mathbb{B}^{\beta_j},l}(E)(u) - NEG_{\mathbb{B}^{\beta_j},l}(E)(u) \leq \delta\}, \tag{25}$$

$$\overline{BND}_{\mathbb{B}^{\beta_j},l}(E) = \{u \in U \mid \delta < POS_{\mathbb{B}^{\beta_j},l}(E)(u) - NEG_{\mathbb{B}^{\beta_j},l}(E)(u) < \theta\}. \tag{26}$$

On the basis of these, the TWD algorithm is designed as outlined in Algorithm 1.

In terms of time complexity, the main loop of steps 2-6 has a total of m iterations, each iteration traverses all the objects to calculate the fuzzy relationship, resulting in the time complexity is $O(m^2)$. In terms of space complexity, the set storage has a space complexity of $O(m)$, and the fuzzy relationship matrix is stored as a two-dimensional array of $m \times m$, so the space complexity is $O(m^2)$.

5.2. Experimental analysis

5.2.1. Experimental setup

In the experiments, the four models proposed in this paper (l-FCRSM, $l = 1, 2, 3, 4$) will be used for TWD. Two existing Fβ-CRS models based on Fβ-NO (Jiang et al. [20] and Ma et al. [13]) are selected for comparative experiments to verify the superiority and rationality of the algorithm proposed in this study. Both of the above algorithms use the Fβ-NOs based on fuzzy β-neighborhoods.

Four public datasets were selected for the experiment, and their characteristic descriptions are detailed in Table 7. The data preprocessing process is as follows: all datasets were standardized to [0,1]; before applying the algorithm in this study for TWD, the standardized data were further processed to calculate the fuzzy β-covering of conditional attributes using the formula $C_j(u_i) = a_j(u_i)/\max_{i=1}^{n}\{a_j(u_i)\}$. In addition, for the binary decision classes $D = \{D_1, D_2\}$ in the real dataset, a random strategy [20] for fuzzy

Algorithm 1 TWD algorithm based on l-FCRS model.

Input: A VSFβ-CAS $(U, \mathbb{C}')$, θ, δ and β.
Output: Crisp three-way regions.
Initialize $\overline{POS}_{\mathbb{B}^{\beta_j},l}(E) = \varnothing, \overline{NEG}_{\mathbb{B}^{\beta_j},l}(E) = \varnothing, \overline{BND}_{\mathbb{B}^{\beta_j},l}(E) = \varnothing$;
for any $u \in U$ **do**
  Calculate $\beta_j = \bigwedge \left(\bigvee_{C \in \mathbb{C}'} C(u)\right)$;
  if $\beta_j \geq \beta$ **then**
    $\beta_j \leftarrow \beta$;
  end if
  Calculate $R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u, v)$, $\forall v \in U$;
  Calculate $\underline{R_{\mathbb{B}^{\beta_j},l}}(E)(u)$ and $\overline{R_{\mathbb{B}^{\beta_j},l}}(E)(u)$;
  Calculate $\overline{POS}_{\mathbb{B}^{\beta_j},l}(E)(u)$, $NEG_{\mathbb{B}^{\beta_j},l}(E)(u)$ and $BND_{\mathbb{B}^{\beta_j},l}(E)(u)$;
end for
if $POS_{\mathbb{B}^{\beta_j},l}(E)(u) - NEG_{\mathbb{B}^{\beta_j},l}(E)(u) \geq \theta$ **then**
  Make $u \in \overline{POS}_{\mathbb{B}^{\beta_j},l}(E)$;
else if $POS_{\mathbb{B}^{\beta_j},l}(E)(u) - NEG_{\mathbb{B}^{\beta_j},l}(E)(u) \leq \delta$ **then**
  Make $u \in \overline{NEG}_{\mathbb{B}^{\beta_j},l}(E)$;
else
  Make $u \in \overline{BND}_{\mathbb{B}^{\beta_j},l}(E)$;
end if
return Crisp three-way regions $\overline{POS}_{\mathbb{B}^{\beta_j},l}(E)$, $\overline{NEG}_{\mathbb{B}^{\beta_j},l}(E)$ and $\overline{BND}_{\mathbb{B}^{\beta_j},l}(E)$.

Table 7
The characteristic descriptions of datasets.

No.	Datasets	Samples	Conditional attributes	$[D_1;D_2]$
1	SPECTF	267	44	[212;55]
2	ThoracicSurgery	470	16	[70;400]
3	PopFailures	540	18	[494;46]
4	WDBC	569	30	[357;212]

concept E is used. That is, when the sample $u \in D_1$, its fuzzy membership degree is a random value within $[0.5, 1]$, and conversely, it is a random value within $[0, 0.5]$.

In the TWD experiments, β was set to 0.5 for all six methods, and the pair of threshold (θ, δ) was (0.1,-0.1). The overlap function and grouping function used in the l-FCRSM are O_{mM} and G_{mM} in Examples 2.11 and 2.12, respectively. The experiment was run in the Python 3.13, with a computer configuration of an Intel (R) Core (TM) i7-14700KF 3.40 GHz CPU and 32.0 GB RAM.

5.2.2. Experimental results

The following three measurements (acceptance-decision rate ADR, rejection-decision rate RDR and uncertainty-decision rate UDR) are used to show the decision results for samples divided into three-way regions by different models, as defined below.

$$ADR = \frac{|\overline{POS}_{\mathbb{B}^{\beta_j},l}(E)|}{|U|} \times 100\%, RDR = \frac{|\overline{NEG}_{\mathbb{B}^{\beta_j},l}(E)|}{|U|} \times 100\%, UDR = \frac{|\overline{BND}_{\mathbb{B}^{\beta_j},l}(E)|}{|U|} \times 100\%. \tag{27}$$

Fig. 3 shows that the comparison of ADR, UDR and RDR for TWD methods with four datasets. As shown in the figure, in the four datasets, the ADR and RDR of 1-FCRSM and 3-FCRSM (except for the WDBC dataset) are higher than those of other comparative models, reflecting that their decision-making results have higher certainty. On the other hand, the UDR of 1-FCRSM and 3-FCRSM are lower than those of other models, demonstrating their advantages in reducing decision-making uncertainty. In contrast, 2-FCRSM and 4-FCRSM, similar to the model proposed by Ma et al., tend to classify more objects into the boundary region. This may be due to the fact that they are looser than others, leading to a poor calculation of granularity.

After calculating the decision results for all samples in the four datasets, we further evaluate the performance of the proposed TWD methods with the Fβ-CRS models using three other measurements (error rate ER, precision rate PR, and recall rate RR), as defined below.

$$ER = \frac{|\overline{POS}_{\mathbb{B}^{\beta_j},l}(E) \cap D_2| + |\overline{NEG}_{\mathbb{B}^{\beta_j},l}(E) \cap D_1|}{|U|} \times 100\%, \tag{28}$$

$$PR = \frac{|\overline{POS}_{\mathbb{B}^{\beta_j},l}(E) \cap D_1| + |\overline{NEG}_{\mathbb{B}^{\beta_j},l}(E) \cap D_2|}{|\overline{POS}_{\mathbb{B}^{\beta_j},l}(E)| + |\overline{NEG}_{\mathbb{B}^{\beta_j},l}(E)|} \times 100\%, \tag{29}$$

$$RR = \frac{|\overline{POS}_{\mathbb{B}^{\beta_j},l}(E)| + |\overline{NEG}_{\mathbb{B}^{\beta_j},l}(E)|}{|U|} \times 100\%. \tag{30}$$

Fig. 4 shows that the comparison of ER, PR and RR for TWD methods with four datasets. As shown in the figure, in the four datasets, the error rates of the four new models are almost 0%, indicating that they can significantly reduce the decision-making

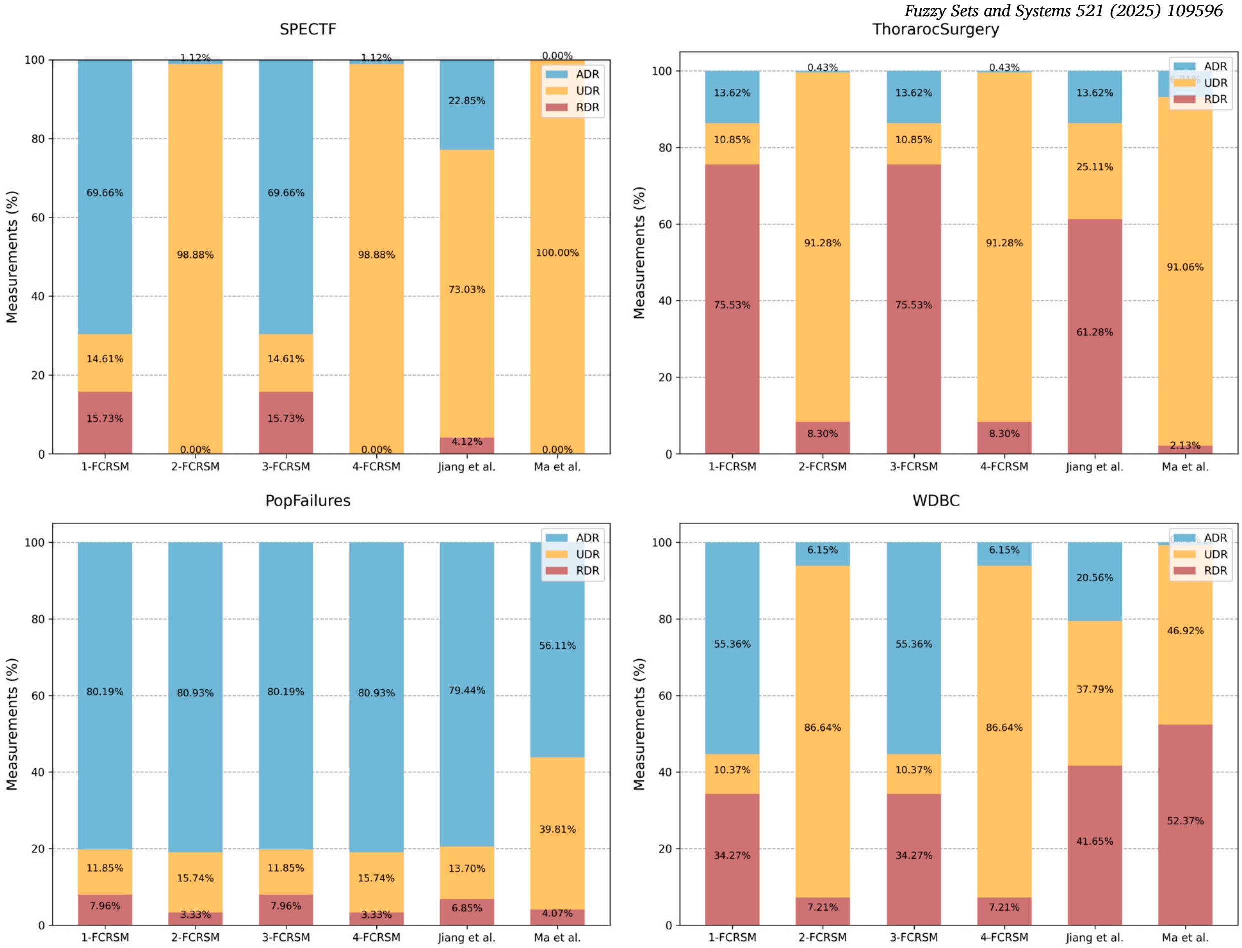

Fig. 3. Comparison of ADR, UDR and RDR for TWD methods with four datasets.

errors. At the same time, the precision rates of these models are higher than those of other comparative models, reflecting the effective improvement of the decision-making precision. In addition, the recall rate of 1-FCRSM and 3-FCRSM are higher, which shows that the two models significantly enhance the recall ability of the target categories in the TWD task. It is worth noting that the three measurement indicators of the model proposed by Ma et al. in the SPECTF dataset are all 0, which is caused by the fact that this model divides all samples into the boundary region.

In summary, the four models proposed in this paper are reasonable and superior in TWD.

6. Fβ-CRS models application to ATR

6.1. Variable-scale ATR methods based on FCRS models in VSFβ-CAS

In existing ATR research, improper selection of the parameter β often leads to abnormal interruption of the algorithm. We use the following Example to illustrate this drawback of ATR in Fβ-CAS.

Example 6.1. Let $U = \{u_1, u_2, u_3, u_4\}$ and $\mathbb{C} = \{C_1, C_2, C_3\}$ be a family of fuzzy sets of U, where

$$C_1 = \frac{0.37}{u_1} + \frac{0.47}{u_2} + \frac{0.28}{u_3} + \frac{0.75}{u_4}, C_2 = \frac{0.65}{u_1} + \frac{0.44}{u_2} + \frac{0.44}{u_3} + \frac{0.44}{u_4}, C_3 = \frac{0.4}{u_1} + \frac{0.5}{u_2} + \frac{0.5}{u_3} + \frac{0.59}{u_4}.$$

According to the existing ATR algorithm in the Fβ-CAS, we first need to calculate the fuzzy β-coverings $\mathbb{C}_1 = \{C_1\}$, $\mathbb{C}_2 = \{C_2\}$ and $\mathbb{C}_3 = \{C_3\}$ respectively. Assume that $\beta = 0.45$. Taking $\mathbb{C}_1$ as an example, according to Definition 2.1, $C_1(u_1)$ and $C_1(u_3)$ do not exist under this condition, which means $\mathbb{C}_1$ does not exist. This indicates that in the Fβ-CAS, an incorrect selection of the parameter β will cause the algorithm to abort abnormally.

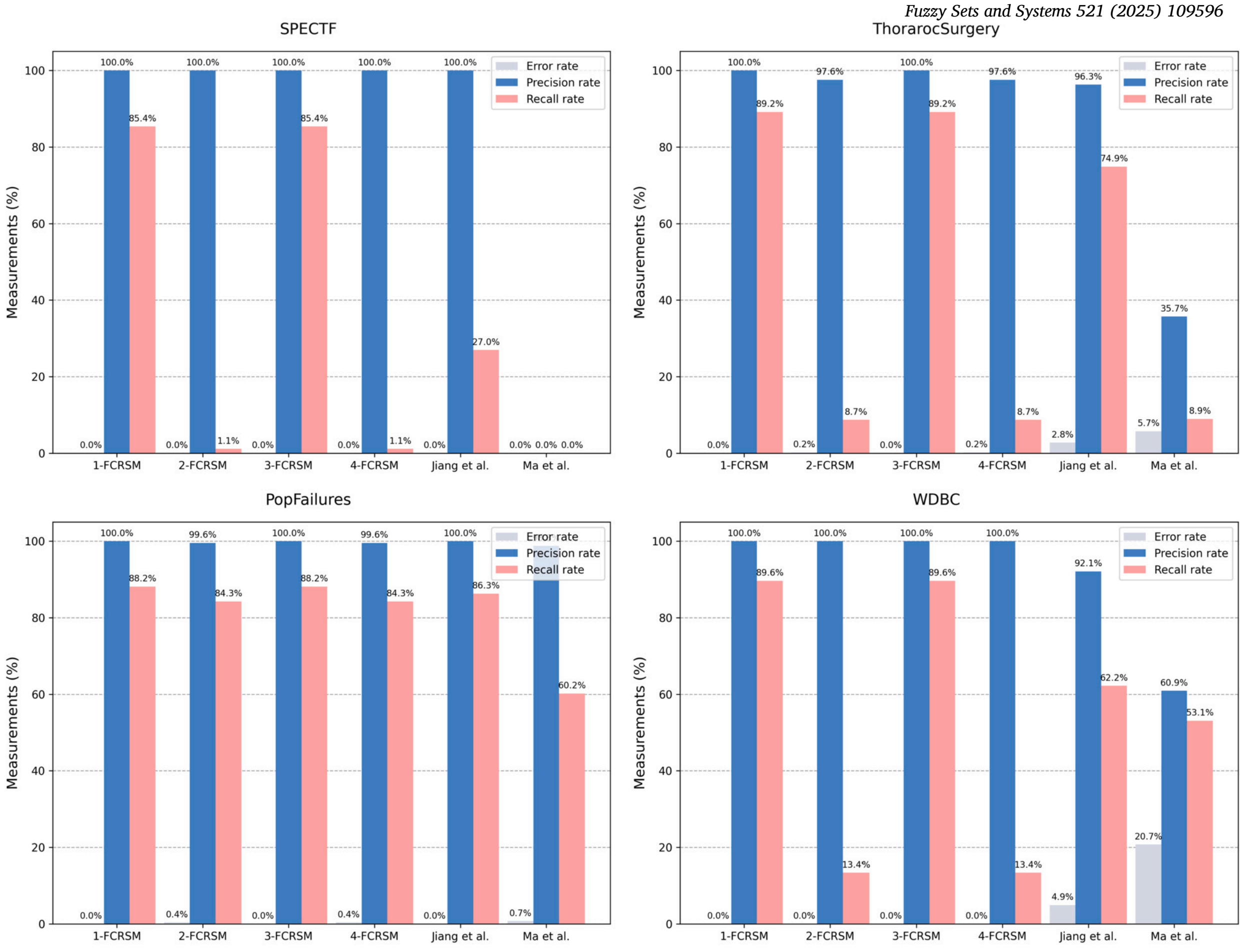

Fig. 4. Comparison of ER, PR and RR for TWD methods with four datasets.

Based on this problem, we designed the variable-scale ATR methods. We define two monotonic uncertainty measures on the basis of the proposed models, named fuzzy dependency functions and fuzzy composite functions. Furthermore, some new variable-scale ATR methods are designed using them respectively.

Definition 6.2. Consider $(U, \mathbb{C}')$ as a VSFβ-CAS, $\mathbf{D}$ as a decision target, and U can be divided into k equivalence classes by $\mathbf{D}$, formulated as $U/\mathbf{D} = \{D_1, D_2, \cdots, D_k\}$, then $(U, \mathbb{C}', \mathbf{D})$ is named a variable-scale fuzzy β-covering decision table (VSFβ-CDT).

Definition 6.3. Suppose $(U, \mathbb{C}', \mathbf{D})$ is a VSFβ-CDT, $\beta_j \in (0,1]$, $\lambda \in [0,1]$, $\lambda \le \beta_j (\forall j \in \Lambda)$ and $\mathbb{B}^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$. For each $u \in U$, on the basis of four kinds of parameterized fuzzy β-covering relations, we define the fuzzy lower and upper approximations of $D_k \in \mathbf{D}$ in relation to $\mathbb{B}^{\beta_j}$ as:

$$\underline{R_{\mathbb{B}^{\beta_j},l}}(D_k)(u) = \bigwedge_{v \notin D_k} \left\{ 1 - R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v) \right\}, \tag{31}$$

$$\overline{R_{\mathbb{B}^{\beta_j},l}}(D_k)(u) = \bigvee_{v \in D_k} \left\{ R^{\beta_j,\lambda}_{\mathbb{B}^{\beta_j},l}(u,v) \right\}. \tag{32}$$

Definition 6.4. Suppose $(U, \mathbb{C}', \mathbf{D})$ is a VSFβ-CDT, $\beta_j \in (0,1]$, $\mathbb{B}^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$. For each $u \in U$, we define the fuzzy positive region $POS_{\mathbb{B}^{\beta_j},l}(\mathbf{D})$ of $\mathbf{D}$ in relation to $\mathbb{B}^{\beta_j}$ as

$$POS_{\mathbb{B}^{\beta_j},l}(\mathbf{D})(u) = \bigcup_{k=1}^{r} \underline{R_{\mathbb{B}^{\beta_j},l}}(D_k)(u). \tag{33}$$

Proposition 6.5. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. The following statements hold.*
(1) If $\beta_1 \leq \beta_2$, then $POS_{\mathbb{B}^{\beta_1},l}(\boldsymbol{D}) \supseteq POS_{\mathbb{B}^{\beta_2},l}(\boldsymbol{D})$;
(2) If $\mathbb{B}_1^{\beta_j} \subseteq \mathbb{B}_2^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$, then $POS_{\mathbb{B}_1^{\beta_j},l}(\boldsymbol{D}) \subseteq POS_{\mathbb{B}_2^{\beta_j},l}(\boldsymbol{D})$.

Proof. (1) Based on Proposition 4.4(2), we have $R^{\beta_1,\lambda}_{\mathbb{B}^{\beta_1},l} \subseteq R^{\beta_2,\lambda}_{\mathbb{B}^{\beta_2},l}$. $\forall u \in U$, we can derive that $\underline{R_{\mathbb{B}^{\beta_1},l}}(D_k)(u) \geq \underline{R_{\mathbb{B}^{\beta_2},l}}(D_k)(u)$ by Definition 4.5. Thus, we have $POS_{\mathbb{B}^{\beta_1},l}(\mathbf{D})(u) \geq POS_{\mathbb{B}^{\beta_2},l}(\mathbf{D})(u)$, that is, $POS_{\mathbb{B}^{\beta_1},l}(\mathbf{D}) \supseteq POS_{\mathbb{B}^{\beta_2},l}(\mathbf{D})$.

(2) By Proposition 4.4(3), we can determine that $R^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j},l} \supseteq R^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j},l}$. $\forall u \in U$, we can derive that $\underline{R_{\mathbb{B}_1^{\beta_j},l}}(D_k)(u) \leq \underline{R_{\mathbb{B}_2^{\beta_j},l}}(D_k)(u)$. Thus, we have $POS_{\mathbb{B}_1^{\beta_j},l}(\mathbf{D})(u) \leq POS_{\mathbb{B}_2^{\beta_j},l}(\mathbf{D})(u)$, that is $POS_{\mathbb{B}_1^{\beta_j},l}(\mathbf{D}) \subseteq POS_{\mathbb{B}_2^{\beta_j},l}(\mathbf{D})$. □

Definition 6.6. Suppose $(U, \mathbb{C}', \mathbf{D})$ is a VSFβ-CDT, $\beta_j \in (0,1]$, $\mathbb{B}^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$. For each $u \in U$, we define the fuzzy dependency function of $\mathbf{D}$ in relation to $\mathbb{B}^{\beta_j}$ as

$$DF_{\mathbb{B}^{\beta_j},l}(\mathbf{D}) = \frac{\sum_{u \in U} POS_{\mathbb{B}^{\beta_j},l}(\mathbf{D})(u)}{|U|}. \tag{34}$$

Remark 6.7. The fuzzy dependence function, also known as dependency degree, represents the ratio of the positive region to the entire set of all samples in the feature space. The positive region is the set of samples that can be correctly classified into the decision class by the attribute subset. When no samples can be correctly classified, the positive region is empty, and at this time, $DF_{\mathbb{B}^{\beta_j},l}(\mathbf{D}) = 0$; when all samples can be correctly classified, the positive region is equal to the universal set, and at this time, $DF_{\mathbb{B}^{\beta_j},l}(\mathbf{D}) = 1$. Therefore, the larger value of this function, the stronger classification ability of the attribute subset for the samples, and vice versa.

Proposition 6.8. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. The following statements hold.*
(1) If $\beta_1 \leq \beta_2$, then $DF_{\mathbb{B}^{\beta_1},l}(\boldsymbol{D}) \geq DF_{\mathbb{B}^{\beta_2},l}(\boldsymbol{D})$;
(2) If $\mathbb{B}_1^{\beta_j} \subseteq \mathbb{B}_2^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$, then $DF_{\mathbb{B}_1^{\beta_j},l}(\boldsymbol{D}) \leq DF_{\mathbb{B}_2^{\beta_j},l}(\boldsymbol{D})$.

Proof. Similarly to Proposition 6.5, the proposition is obvious. □

Definition 6.9. Suppose $(U, \mathbb{C}', \mathbf{D})$ is a VSFβ-CDT, $\beta_j \in (0,1]$, $\mathbb{B}^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$. We define the significance $SIG_{DF,l}\left(\mathbf{C}, \mathbb{B}^{\beta_j}, \mathbf{D}\right)$ of $\mathbf{C} \in \mathbb{C}' - \mathbb{B}^{\beta_j}$ in relation to $\mathbb{B}^{\beta_j}$ for $\mathbf{D}$ as

$$SIG_{DF,l}\left(\mathbf{C}, \mathbb{B}^{\beta_j}, \mathbf{D}\right) = DF_{\mathbb{B}^{\beta_j} \cup \{\mathbf{C}\},l}(\mathbf{D}) - DF_{\mathbb{B}^{\beta_j},l}(\mathbf{D}). \tag{35}$$

In addition, to measure the uncertainty of decision targets under the fuzzy β-covering, inspired by Zou et al. [23], we propose a fuzzy composite function by combining data distribution.

Definition 6.10. Suppose $(U, \mathbb{C}', \mathbf{D})$ is a VSFβ-CDT, $\beta_j \in (0,1]$, $\mathbb{B}^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$. For each $u \in U$, we define the fuzzy composite function of $\mathbf{D}$ in relation to $\mathbb{B}^{\beta_j}$ as

$$CF_{\mathbb{B}^{\beta_j},l}(\mathbf{D}) = -\sum_{k=1}^{r} \frac{|D_k|}{U} \log \frac{\sum_{u \in U} \underline{R_{\mathbb{B}^{\beta_j},l}}(D_k)(u)}{\sum_{u \in U} \overline{R_{\mathbb{B}^{\beta_j},l}}(D_k)(u)}. \tag{36}$$

Remark 6.11. The fuzzy composite function is constructed by considering the boundary region of the fuzzy rough set and the data distribution. Here, $|C_l|/|U|$ represents the weight of the lth decision class in the universe of discourse. The larger ratio of the lower approximation to the upper approximation, the smaller boundary region and the lower uncertainty. Thus, a larger value of the fuzzy composite function indicates greater uncertainty, and vice versa.

Proposition 6.12. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. The following statements hold.*
(1) $CF_{\mathbb{B}^{\beta_j},l}(\boldsymbol{D}) \geq 0$;
(2) If $\sum_{u \in U} \underline{R_{\mathbb{B}^{\beta_j},l}}(D_k)(u) / \sum_{u \in U} \overline{R_{\mathbb{B}^{\beta_j},l}}(D_k)(u) = 1$, then $CF_{\mathbb{B}^{\beta_j},l}(\boldsymbol{D}) = 0$.

Proof. According to Definition 6.10, the proposition is obvious. □

Proposition 6.13. *Suppose $(U, \mathbb{C}')$ is a VSFβ-CAS. If $\mathbb{B}_1^{\beta_j} \subseteq \mathbb{B}_2^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$, then $CF_{\mathbb{B}_1^{\beta_j},l}(\boldsymbol{D}) \geq CF_{\mathbb{B}_2^{\beta_j},l}(\boldsymbol{D})$.*

Proof. According to Proposition 4.4(3), we can determine that $R^{\beta_j,\lambda}_{\mathbb{B}_1^{\beta_j},l} \supseteq R^{\beta_j,\lambda}_{\mathbb{B}_2^{\beta_j},l}$. For each $u \in U$, we can derive that $\underline{R}_{\mathbb{B}_1^{\beta_j},l}(D_k)(u) \le \underline{R}_{\mathbb{B}_2^{\beta_j},l}(D_k)(u)$ and $\overline{R}_{\mathbb{B}_1^{\beta_j},l}(D_k)(u) \ge \overline{R}_{\mathbb{B}_2^{\beta_j},l}(D_k)(u)$, then $\sum_{u\in U} \underline{R}_{\mathbb{B}_1^{\beta_j},l}(D_k)(u) / \sum_{u\in U} \overline{R}_{\mathbb{B}_1^{\beta_j},l}(D_k)(u) \le \sum_{u\in U} \underline{R}_{\mathbb{B}_2^{\beta_j},l}(D_k)(u) / \sum_{u\in U} \overline{R}_{\mathbb{B}_2^{\beta_j},l}(D_k)(u)$. Thus, we have $CF_{\mathbb{B}_1^{\beta_j},l}(\mathbf{D}) \ge CF_{\mathbb{B}_2^{\beta_j},l}(\mathbf{D})$. □

Definition 6.14. Suppose $(U, \mathbb{C}', \mathbf{D})$ is a VSFβ-CDT, $\beta_j \in (0,1]$, $\mathbb{B}^{\beta_j} \subseteq \mathbb{C}^{\beta_j}$. The significance $SIG_{CF,l}\left(\mathbf{C}, \mathbb{B}^{\beta_j}, \mathbf{D}\right)$ of $\mathbf{C} \in \mathbb{C}' - \mathbb{B}^{\beta_j}$ in relation to $\mathbb{B}^{\beta_j}$ for $\mathbf{D}$ is defined as

$$SIG_{CF,l}\left(\mathbf{C}, \mathbb{B}^{\beta_j}, \mathbf{D}\right) = CF_{\mathbb{B}^{\beta_j},l}(\mathbf{D}) - CF_{\mathbb{B}^{\beta_j}\cup\{\mathbf{C}\},l}(\mathbf{D}). \tag{37}$$

Especially, $SIG_{CF,l}\left(\mathbf{C}, \mathbb{B}^{\beta_j}, \mathbf{D}\right) = +\infty$ when $\mathbb{B}^{\beta_j} = \varnothing$.

On the basis of the four constructed models, fuzzy dependency functions and fuzzy composite functions, the variable-scale ATR algorithm is designed as outlined in Algorithm 2.

Algorithm 2 Variable-scale ATR algorithm based on fuzzy dependency function (or fuzzy composite function).

Input: A original decision table $(U, A, \mathbf{D})$, β and λ.
Output: Subset of condition attributes $Result$.

1: Normalize a for any $a \in A$;
2: Initialize $Result = \varnothing, \mathbb{B} = \mathbb{C}$;
3: **while** $\mathbb{B} \ne \varnothing$ **do**
4:     **for** any $\mathbf{C} \in \mathbb{B}$ **do**
5:         **for** any $u \in U$ **do**
6:             Calculate $\beta_j = \bigwedge\left(\bigvee_{C\in Result\cup\{\mathbf{C}\}} C(u)\right)$;
7:             **if** $\beta_j \ge \beta$ **then**
8:                 $\beta_j \leftarrow \beta$;
9:             **end if**
10:             Calculate $R^{\beta_j,\lambda}_{\{Result\cup\{\mathbf{C}\}\}^{\beta_j},l}(u,v)$, $\forall v \in U$;
11:             Calculate $\underline{R}_{\{Result\cup\{\mathbf{C}\}\}^{\beta_j},l}(D_k)(u)$ and $\overline{R}_{\{Result\cup\{\mathbf{C}\}\}^{\beta_j},l}(D_k)(u)$ for any $D_k \in \mathbf{D}$;
12:         **end for**
13:         Calculate $DF_{\{Result\cup\{\mathbf{C}\}\}^{\beta_j},l}(\mathbf{D})$ $\left(or\ CF_{\{Result\cup\{\mathbf{C}\}\}^{\beta_j},l}(\mathbf{D})\right)$;
14:     **end for**
15:     Find $\mathbf{C}' = \arg\max_{\mathbf{C}\in\mathbb{B}} DF_{\{Result\cup\{\mathbf{C}\}\}^{\beta_j},l}(\mathbf{D})$ $\left(or\ \mathbf{C}' = \arg\min_{\mathbf{C}\in\mathbb{B}} CF_{\{Result\cup\{\mathbf{C}\}\}^{\beta_j},l}(\mathbf{D})\right)$;
16:     Calculate $SIG_{DF,l}\left(\mathbf{C}', Result, \mathbf{D}\right)$ $\left(or\ SIG_{CF,l}\left(\mathbf{C}', Result, \mathbf{D}\right)\right)$;
17:     **if** $SIG_{DF,l}\left(\mathbf{C}', Result, \mathbf{D}\right) > 0$ $\left(or\ SIG_{CF,l}\left(\mathbf{C}', Result, \mathbf{D}\right) > 0\right)$ **then**
18:         $Result \leftarrow Result \cup \{\mathbf{C}\}$;
19:         $\mathbb{B} \leftarrow \mathbb{B} - Result$;
20:     **else**
21:         **return** $Result$
22:     **end if**
23: **end while**
24: **return** $Result$

A detailed analysis of the time and space complexity of Algorithm 2 are presented as follows. For the time complexity: the normalization step involves processing m objects for each of the n attributes, resulting in a time complexity of $O(mn)$. Initializing the sets $Result$ and $\mathbb{B}$ is a constant-time operation with a complexity of $O(1)$. In the main loop (Steps 3–24), the outer while loop runs at most n times. The inner for loop (Steps 4-13) processes each candidate attribute $\mathbf{C} \in \mathbb{B}$, and computing the fuzzy relation requires iterating over all pairs of objects, leading to a time complexity of $O(m^2)$ per attribute. Since the size of $\mathbb{B}$ decreases from n to 1, the total number of inner loop iterations is approximately $O(n^2)$, making the total time complexity of the inner loop $O(m^2n^2)$. The time complexity of calculating the degree of dependency and selecting the optimal attribute are $O(m)$ and $O(n^2)$, respectively, so the dominant time complexity is $O(m^2n^2)$.

For the space complexity: storing the fuzzy relation matrix requires an $m \times m$ matrix, resulting in a space complexity of $O(m^2)$. The storage space for the sets $Result$ and $\mathbb{B}$ is $O(n)$, and the constant space required for intermediate variables is $O(1)$. Thus, the space complexity is $O(m^2)$. In conclusion, the algorithm has a time complexity of $O(m^2n^2)$ and a space complexity of $O(m^2)$.

Example 6.15. As shown in Table 8, $(U, A, \mathbf{D})$ is a original decision table. According to Algorithm 2, we can obtain an attribute subset $Result$ from the decision table in the case of 2-FCRS with fuzzy composite function.

Firstly, the decision table $(U, A, \mathbf{D})$ is normalized to obtain the new decision table in Table 9 by Formula (38). Let fuzzy β-covering, denoted as $\mathbb{C} = \{C_1, C_2, C_3, C_4\} = \{a_1, a_2, a_3, a_4\}$, be total attributes from Table 9.

$$r_{ij} = u_{ij} \Bigg/ \sqrt{\sum_{i=1}^{m} u_{ij}^2}. \tag{38}$$

Table 8
The original decision table.

U	a_1	a_2	a_3	a_4	**D**
u_1	4	6	4	7	1
u_2	5	4	5	8	1
u_3	3	4	5	3	2
u_4	8	4	6	4	3

Table 9
Normalized decision table from Table 8.

U	C_1	C_2	C_3	C_4	**D**
u_1	0.3746	0.6547	0.3961	0.5959	1
u_2	0.4683	0.4364	0.4951	0.6810	1
u_3	0.2810	0.4364	0.4951	0.2554	2
u_4	0.7493	0.4364	0.5941	0.3405	3

Let $\beta = 0.45$, $\lambda = 0.26$, $G = G_{M_p}(p = 2)$, an attribute subset $Result$ can be calculated. According to Algorithm 2, we need to compute the fuzzy β-covering relations related to $\{C_1\}$, $\{C_2\}$, $\{C_3\}$ and $\{C_4\}$ respectively. Firstly, it is calculated from $\beta_j = \bigwedge\left(\bigvee_{C\in\{C_1\}} C(u)\right)$ that $\beta_j = 0.2810$ for $\{C_1\}$, and the algorithm will then be able to run normally. This indicates that in the VSFβ-CAS, the abnormal termination caused by the initial selection of an incorrect β can be effectively avoided, which greatly improves the flexibility of the algorithm. Next, it can be calculated according to Definition 4.2 that:

$$R^{0.2810,\lambda}_{\{C_1\}^{0.2810},2} = \begin{pmatrix} 1 & 0.2193 & 0.1404 & 0.5614 \\ 0.2193 & 1 & 0.2193 & 0.5614 \\ 0.1404 & 0.2193 & 1 & 0.5614 \\ 0.5614 & 0.5614 & 0.5614 & 1 \end{pmatrix}, R^{0.4364,\lambda}_{\{C_2\}^{0.4364},2} = \begin{pmatrix} 1 & 0.4286 & 0.4286 & 0.4286 \\ 0.4286 & 1 & 1 & 1 \\ 0.4286 & 1 & 1 & 1 \\ 0.4286 & 1 & 1 & 1 \end{pmatrix},$$

$$R^{0.3961,\lambda}_{\{C_3\}^{0.3961},2} = \begin{pmatrix} 1 & 0.2451 & 0.2451 & 0.3529 \\ 0.2451 & 1 & 1 & 0.3529 \\ 0.2451 & 1 & 1 & 0.3529 \\ 0.3529 & 0.3529 & 0.3529 & 1 \end{pmatrix}, R^{0.2554,\lambda}_{\{C_4\}^{0.2554},2} = \begin{pmatrix} 1 & 0.4638 & 0.3551 & 0.3551 \\ 0.4638 & 1 & 0.4638 & 0.4638 \\ 0.3551 & 0.4638 & 1 & 0.1159 \\ 0.3551 & 0.4638 & 0.1159 & 1 \end{pmatrix}.$$

By Definition 6.3, we can determine that

$$\underline{R_{\{C_1\}^{0.2810},2}} = \begin{pmatrix} 0.4386 & 0.4386 & 0 & 0 \\ 0 & 0 & 0.4386 & 0 \\ 0 & 0 & 0 & 0.4386 \end{pmatrix}, \underline{R_{\{C_2\}^{0.4364},2}} = \begin{pmatrix} 0.5714 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\underline{R_{\{C_3\}^{0.3961},2}} = \begin{pmatrix} 0.6471 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6471 \end{pmatrix}, \underline{R_{\{C_4\}^{0.2554},2}} = \begin{pmatrix} 0.6449 & 0.5362 & 0 & 0 \\ 0 & 0 & 0.5362 & 0 \\ 0 & 0 & 0 & 0.5362 \end{pmatrix};$$

$$\overline{R_{\{C_1\}^{0.2810},2}} = \begin{pmatrix} 1 & 1 & 0.2193 & 0.5614 \\ 0.1404 & 0.2193 & 1 & 0.5614 \\ 0.5614 & 0.5614 & 0.5614 & 1 \end{pmatrix}, \overline{R_{\{C_2\}^{0.4364},2}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0.4286 & 1 & 1 & 1 \\ 0.4286 & 1 & 1 & 1 \end{pmatrix},$$

$$\overline{R_{\{C_3\}^{0.3961},2}} = \begin{pmatrix} 1 & 1 & 1 & 0.3529 \\ 0.2451 & 1 & 1 & 0.3529 \\ 0.3529 & 0.3529 & 0.3529 & 1 \end{pmatrix}, \overline{R_{\{C_4\}^{0.2554},2}} = \begin{pmatrix} 1 & 1 & 0.4638 & 0.4638 \\ 0.3551 & 0.4638 & 1 & 0.1159 \\ 0.3551 & 0.4638 & 0.1159 & 1 \end{pmatrix}.$$

By Definition 6.10 and 6.14, there are $CF_{\{C_1\}^{0.2810},2}(\mathbf{D}) = 2.0184$, $CF_{\{C_2\}^{0.4364},2}(\mathbf{D}) = 2.4037$, $CF_{\{C_3\}^{0.3961},2}(\mathbf{D}) = 2.1042$, and $CF_{\{C_4\}^{0.2554},2}(\mathbf{D}) = 1.5804$. From Algorithm 2, we get $SIG_{CF,2}(C_4, \emptyset, \mathbf{D}) = +\infty$, so $Result = \{C_4\}$ and $\mathbb{B} = \{C_1, C_2, C_3\}$, we need to continue calculating.

Taking $\{C_4, C_1\}$ as an example, calculated from $\beta_j = \bigwedge\left(\bigvee_{C\in\{C_4,C_1\}} C(u)\right)$ to get $\beta_j = 0.2810$, we can determine that

$$R^{0.2810,\lambda}_{\{C_4,C_1\}^{0.2810},2} = \begin{pmatrix} 1 & 0.2193 & 0.1404 & 0.1404 \\ 0.2193 & 1 & 0.2193 & 0.2193 \\ 0.1404 & 0.2193 & 1 & 0.5614 \\ 0.1404 & 0.2193 & 0.5614 & 1 \end{pmatrix}, \underline{R_{\{C_4,C_1\}^{0.2810},2}} = \begin{pmatrix} 0.8597 & 0.7807 & 0 & 0 \\ 0 & 0 & 0.4386 & 0 \\ 0 & 0 & 0 & 0.4386 \end{pmatrix},$$

$$\overline{R_{\{C_4,C_1\}^{0.2810},2}} = \begin{pmatrix} 1 & 1 & 0.2193 & 0.2193 \\ 0.1404 & 0.2193 & 1 & 0.5614 \\ 0.1404 & 0.2193 & 0.5614 & 1 \end{pmatrix}, CF_{\{C_4,C_1\}^{0.2810},2}(\mathbf{D}) = 1.3515.$$

Table 10
The characteristic descriptions of datasets.

No.	Datasets	Conditional attributes	Samples	Classes
1	BreastCancer	9	116	2
2	Wine	13	178	3
3	ThoraSurgery	16	470	2
4	Lymphography	18	148	4
5	PopFailures	18	540	2
6	WDBC	30	569	2
7	SPECTF	44	267	2
8	ColonTumor	2000	62	2

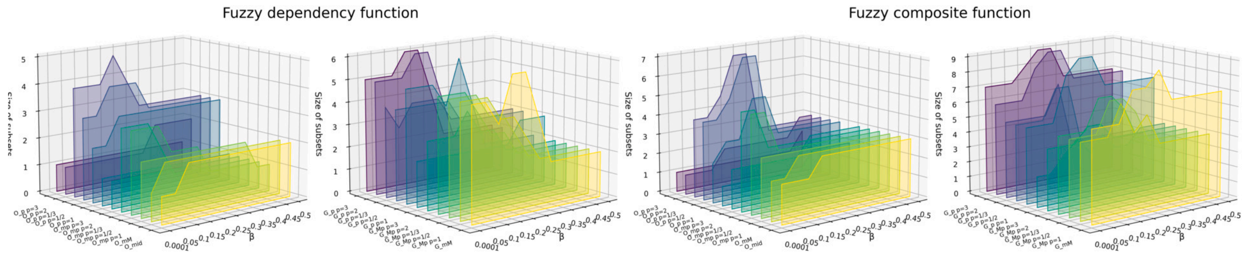

Fig. 5. Size of subsets (BreastCancer).

Similarly, we can calculate $CF_{\{C_4,C_2\}^{0.4364},2}(\mathbf{D}) = 1.3559$ and $CF_{\{C_4,C_3\}^{0.45},2}(\mathbf{D}) = 1.0397$ (Since $\beta_j = 0.4951 \geq \beta$, $\beta_j = \beta = 0.45$), we obtain $SIG_{CF,2}(C_3, \{C_4\}, \mathbf{D}) = 0.5407 > 0$, $Result = \{C_4, C_3\}$ and $\mathbb{B} = \{C_1, C_2\}$. We still need to calculate $CF_{\{C_4,C_3,C_1\}^{0.45},2}(\mathbf{D}) = 1.0234$ and $CF_{\{C_4,C_3,C_2\}^{0.45},2}(\mathbf{D}) = 1.0397$. Since $SIG_{CF,2}(C_1, \{C_4, C_3\}, \mathbf{D}) = 0.0163 > 0$, then $Result = \{C_4, C_3, C_1\}$ and $\mathbb{B} = \{C_2\}$. Next, we can calculate $CF_{\{C_4,C_3,C_1,C_2\}^{0.45},2}(\mathbf{D}) = 1.0234$ and $SIG_{CF,2}(C_2, \{C_4, C_3, C_1\}, \mathbf{D}) = 0$, so the final reduction $Result = \{C_4, C_3, C_1\} = \{a_4, a_3, a_1\}$.

6.2. Experimental analysis

6.2.1. Experimental setup

In the experiments, the four models proposed in this paper will be used for ATR using fuzzy dependency functions (l-FCRSM(DF), $l = 1, 2, 3, 4$) and fuzzy composite functions (l-FCRSM(CF), $l = 1, 2, 3, 4$), respectively. Two existing FCRS models based on fuzzy β-covering relations (FβCRSM [27] and OFβCRSM [28]) are selected for comparative experiments to verify the superiority and rationality of the algorithm proposed in this study. Both of these methods use the fuzzy dependency function to measure attribute significance and select the optimal attribute subsets through a forward greedy strategy.

Eight public datasets were selected for the experiment, and their characteristic descriptions are detailed in Table 10. The data preprocessing process is as follows: all datasets were standardized to [0,1]; before applying the algorithm in this study for ATR, the β-normalization method ($data = (data + \beta)/(1 + \beta)$) was used to further map the standardized data to [β,1] to minimizing the loss of information.

During the experiment, the method in this study and the comparative algorithms were quantitatively compared from two indicators: the size of the attribute subsets and the classification performance. For classification performance evaluation, two classic classifiers, namely Decision Tree (CART) and K-Nearest Neighbors (KNN, K = 3), were adopted. The classification performance of all datasets was obtained through the ten-fold cross-validation method.

6.2.2. Parameter analysis

Based on the theoretical framework of Algorithm 2, the selection of the parameter β and the function configuration may affect the results of ATR, and thus have an impact on the classification performance of the dataset. Therefore, the aim of this section is to determine, through experiments, the optimal parameter values and function combinations that can achieve the highest classification performance. In the experiment, the value range of the parameter β is set to (0, 0.5]; and it is incremented by a step size of 0.05, the functions are chosen in Examples 2.11 and 2.12 ($p = 1/3, 1/2, 1, 2, 3$).

By configuring different parameter values and function types for the l-FCRSM ($l = 1, 2$) methods, the ATR results and classification performance data of eight datasets are obtained. Figs. 5-12 show the sizes of the attribute subsets selected by the l-FCRSM methods on eight datasets under different β and function configurations. The Figures show that the size of the attribute subsets of these methods do not significantly decrease as the β value increases, which is particularly different from the traditional ATR methods based on the Fβ-CRS model - the size of the attribute subset of the latter often sharply decreases to a single attribute as the β value increases.

Figs. 13-20 present the average classification performance of these methods on eight datasets under different β and function configurations. The dashed line shows the average of classification performance for different β given a function; the solid line shows the average of classification performance for different functions given β. The Figures show that as the β value increases, the overall

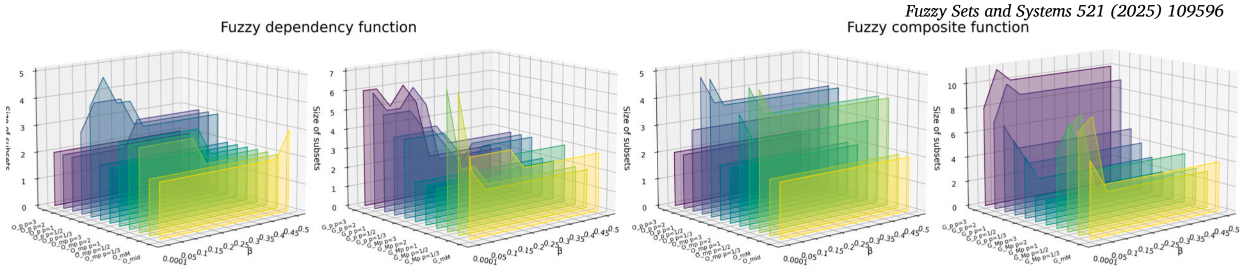

Fig. 6. Size of subsets (Wine).

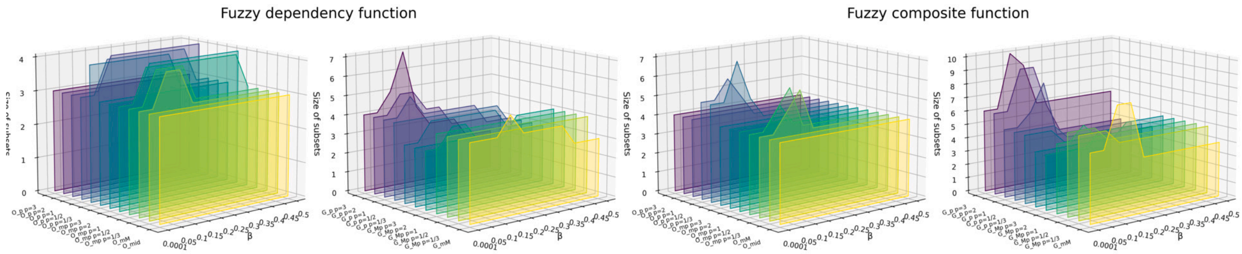

Fig. 7. Size of subsets (ThoraSurgery).

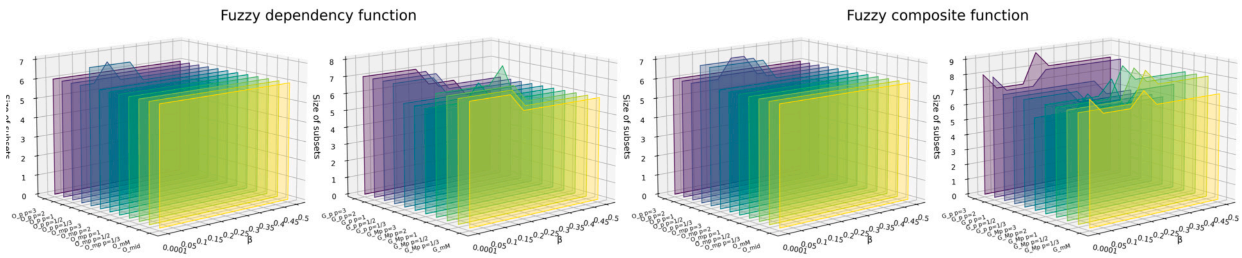

Fig. 8. Size of subsets (Lymphography).

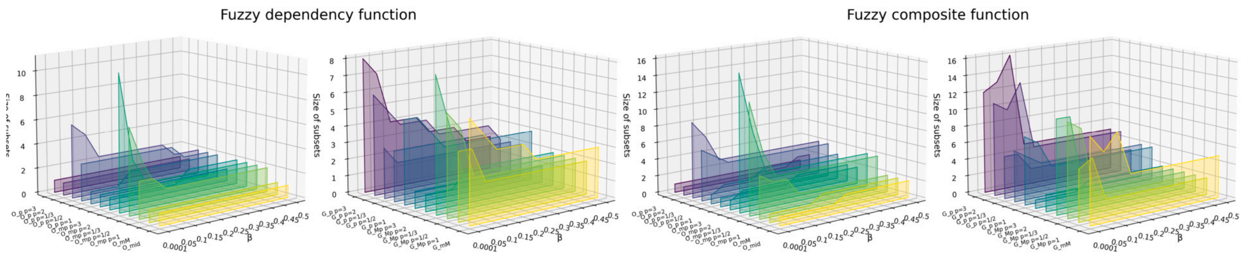

Fig. 9. Size of subsets (PopFailures).

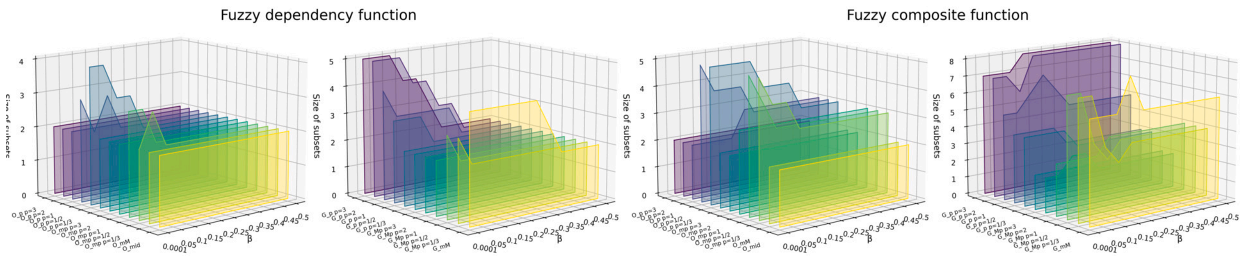

Fig. 10. Size of subsets (WDBC).

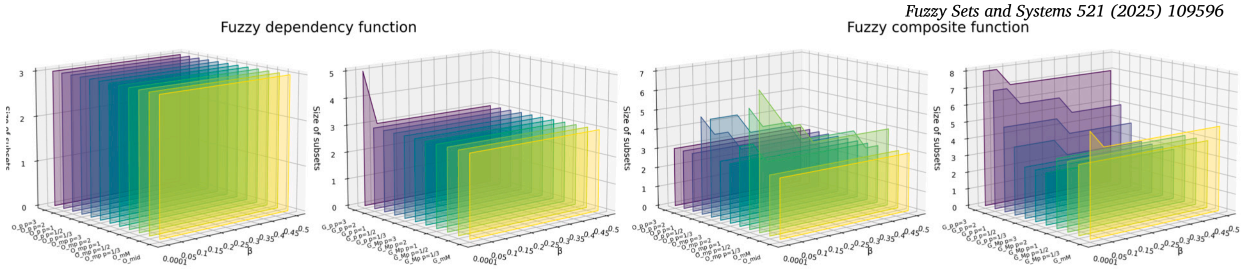

Fig. 11. Size of subsets (SPECTF).

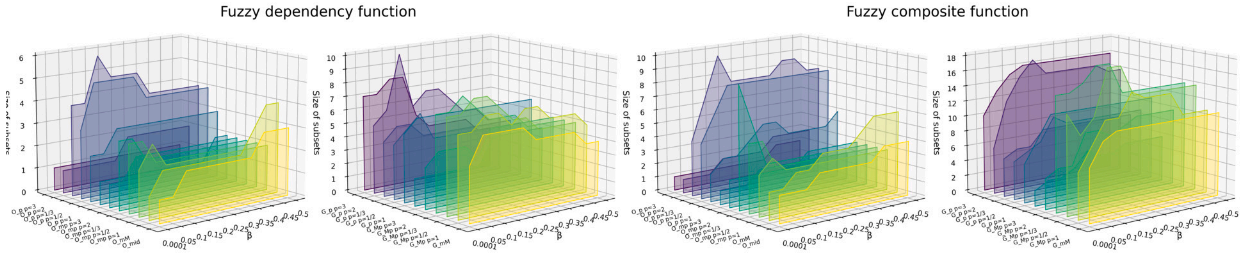

Fig. 12. Size of subsets (ColonTumor).

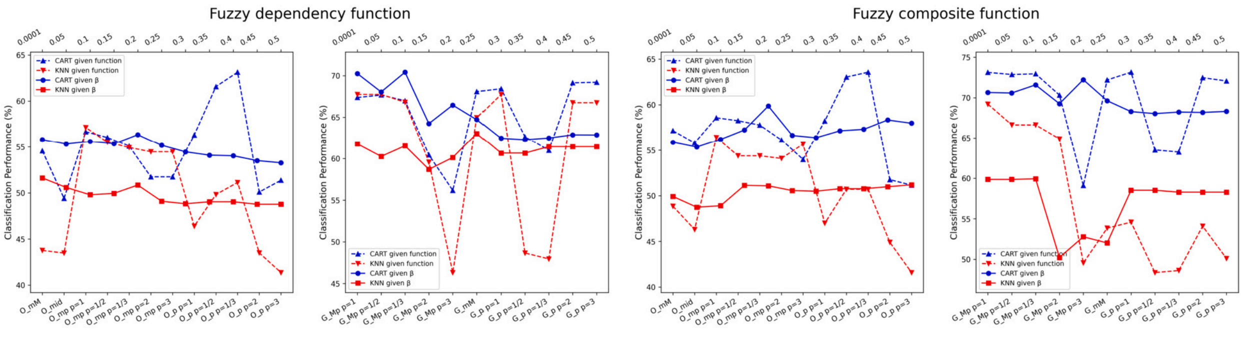

Fig. 13. Variation of average classification performance with β and functions (BreastCancer).

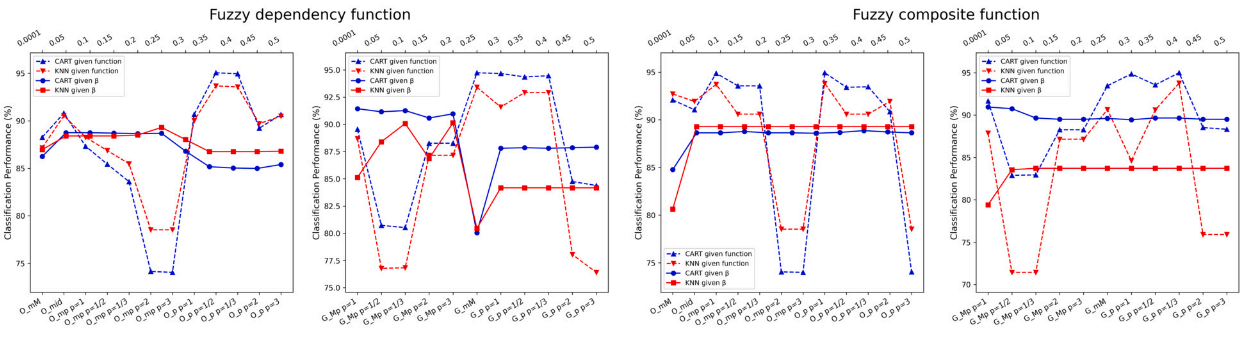

Fig. 14. Variation of average classification performance with β and functions (Wine).

classification performance of the datasets does not show a significant downward trend, and some datasets even maintain stable performance. This experimental phenomenon indicates that the variable-scale ATR methods effectively alleviate the problems of attribute subset sizes plummet and classification performance degradation caused by improper selection of the parameter β, and significantly improve the flexibility of selecting the parameter β in practical applications.

6.2.3. Experimental results

From Figs. 13-20, we can observe that the average classification performance achieved by our methods are generally higher when the functions are used O_p and G_p ($p = 1/3, 1/2, 1, 2, 3$). Therefore, in this subsection, we select these functions for our experiments. Fig. 21 shows the average time taken by different models we proposed to select an attribute subset under different datasets.

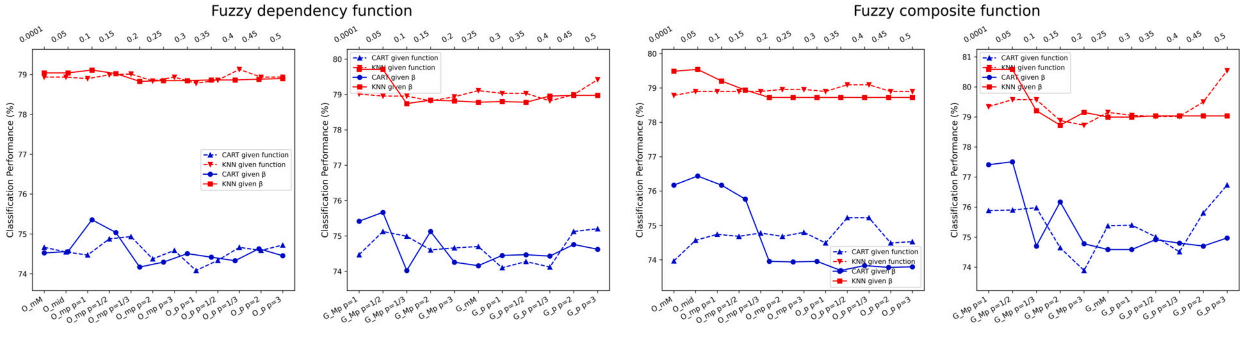

Fig. 15. Variation of average classification performance with β and functions (ThoraSurgery).

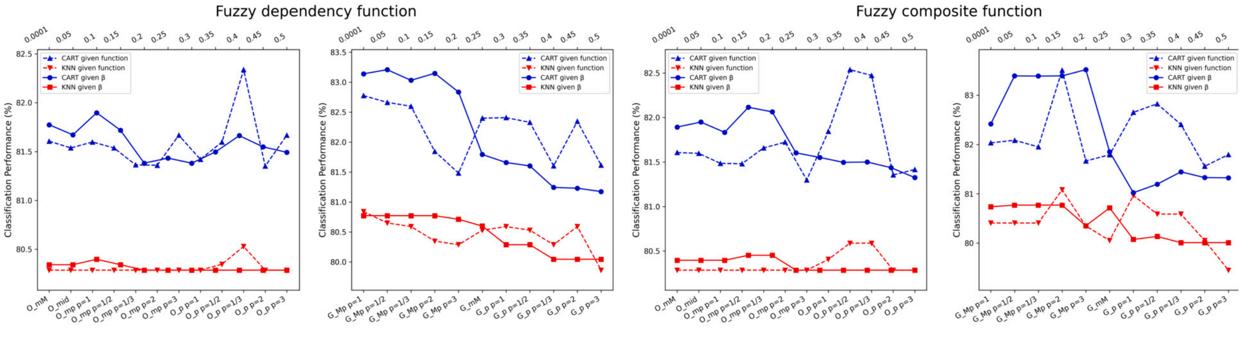

Fig. 16. Variation of average classification performance with β and functions (Lymphography).

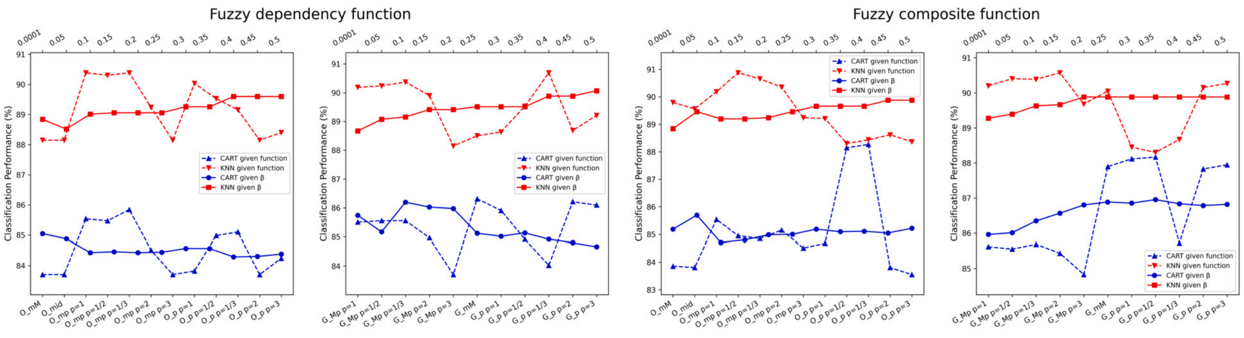

Fig. 17. Variation of average classification performance with β and functions (PopFailures).

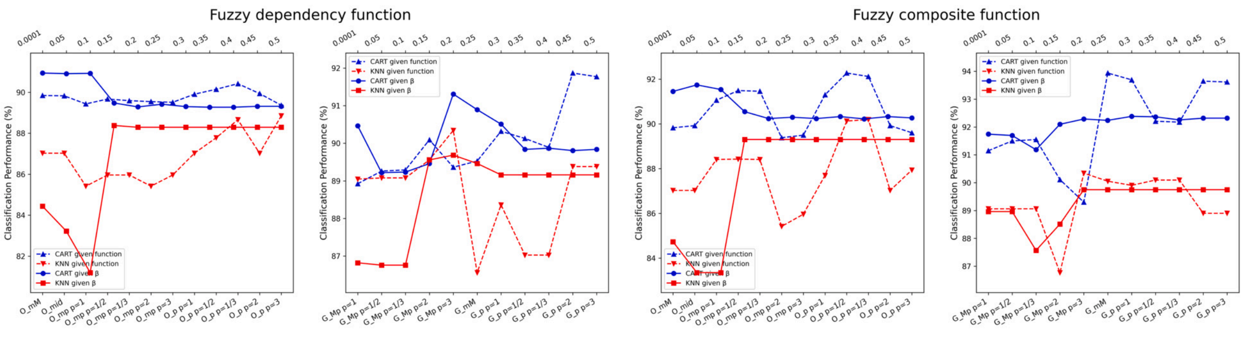

Fig. 18. Variation of average classification performance with β and functions (WDBC).

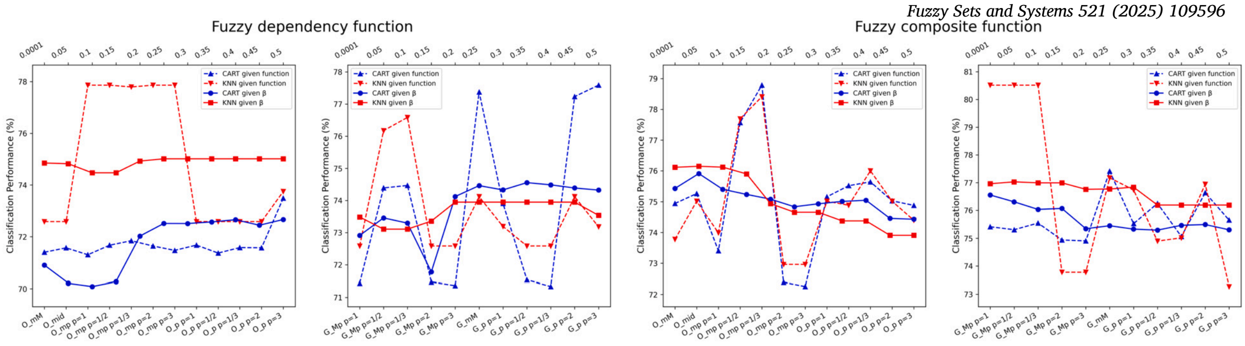

Fig. 19. Variation of average classification performance with β and functions (SPECTF).

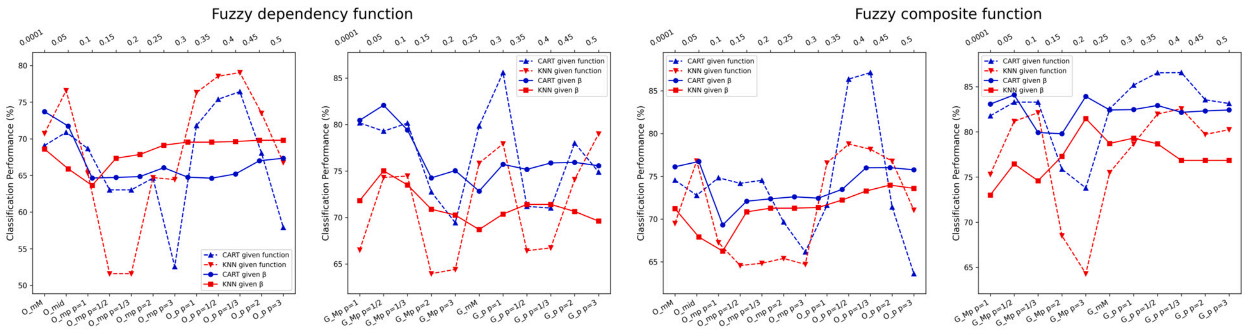

Fig. 20. Variation of average classification performance with β and functions (ColonTumor).

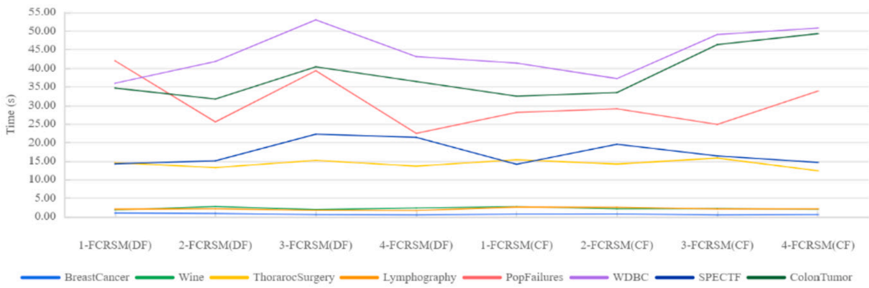

Fig. 21. The average time taken by different models we proposed to select an attribute subset.

Table 11
The size of subsets selected by methods.

Dataset	All	FβCRSM	OFβCRSM	1-FCRSM(DF)		2-FCRSM(DF)		3-FCRSM(DF)		4-FCRSM(DF)		1-FCRSM(CF)		2-FCRSM(CF)		3-FCRSM(CF)		4-FCRSM(CF)	
				CART	KNN	CART	KNN	CART	KNN	CART	KNN	CART	KNN	CART	KNN	CART	KNN	CART	KNN
BreastCancer	9	6	6	4	4	5	4	4	3	5	4	7	4	5	5	7	4	5	5
Wine	13	4	3	4	2	4	3	4	2	4	3	3	2	5	3	4	2	5	3
ThoraSurgery	16	10	1	4	4	5	2	4	4	5	2	5	5	9	6	5	5	9	6
Lymphography	18	11	9	7	7	7	7	7	7	7	7	7	7	7	8	7	7	7	8
PopFailures	18	4	5	1	2	4	2	1	2	4	2	4	2	4	10	4	2	4	10
WDBC	30	3	3	4	3	4	2	3	2	4	3	4	4	8	4	4	3	5	4
SPECTF	44	4	2	3	3	3	5	3	3	3	5	4	3	7	5	4	3	7	4
ColonTumor	2000	12	15	3	5	5	5	3	4	5	5	8	3	10	8	8	8	10	15
Average	268.50	6.75	5.50	3.75	3.75	4.63	3.75	3.63	3.38	4.63	3.88	5.25	3.75	6.88	6.13	5.38	4.25	6.50	6.88

Table 11 presents the sizes of attribute subsets using the ten methods. Among these, the values in the CART and KNN columns represent the number of attributes obtained by performing attribute reduction using the corresponding FCRSM before applying the classifier. When using the respective FCRSMs, aggregation functions and the parameter β were appropriately selected to achieve high classification performance. The data in the tables clearly show that all algorithms are effective in the attribute selection task and can significantly reduce the number of attributes in the dataset. Specifically, for the CART classifier, the average reduction rate of all l-FCRSMs ($l = 1, 2, 3, 4$) is 98.45% when fuzzy dependency function is used, and 97.77% when fuzzy composite function is used.

Table 12
Classification performance (mean±std%) with CART.

Dataset	All	FβCRSM	OFβCRSM	1-FCRSM(DF)	2-FCRSM(DF)	3-FCRSM(DF)	4-FCRSM(DF)	1-FCRSM(CF)	2-FCRSM(CF)	3-FCRSM(CF)	4-FCRSM(CF)
BreastCancer	66.29±9.09	65.53±9.03	73.94±8.37	69.02±7.59	77.50±7.16	68.94±9.49	77.42±7.37	74.17±7.36	76.59±8.20	73.33±15.09	76.59±8.20
Wine	92.65±7.74	95.00±5.80	88.17±8.19	95.56±5.98	95.56±5.98	95.56±5.98	95.56±5.98	94.93±3.01	95.56±5.98	95.56±5.98	95.52±4.85
ThoraSurgery	79.79±4.78	82.55±3.13	79.57±3.04	76.60±5.87	78.30±5.19	76.81±7.15	78.51±5.74	78.09±6.17	80.21±3.57	78.30±4.44	80.21±3.81
Lymphography	81.05±11.12	79.67±8.08	80.43±10.61	84.38±7.62	84.38±7.62	84.38±7.62	84.38±7.62	83.71±7.08	84.38±7.62	83.71±8.24	84.38±7.62
PopFailures	92.59±4.54	91.11±3.78	92.41±2.26	86.85±2.92	89.26±3.19	86.85±2.92	89.44±2.75	89.07±3.65	89.26±3.69	89.26±3.87	89.26±3.29
WDBC	93.85±2.63	90.34±3.85	92.63±3.57	93.31±2.83	94.91±2.87	92.97±3.15	94.91±2.87	94.03±3.61	95.08±3.02	94.21±4.08	94.91±2.65
SPECTF	79.37±6.39	79.07±7.93	74.57±10.25	76.01±7.68	78.65±8.88	76.04±6.43	79.00±8.92	78.93±7.62	79.06±4.66	78.92±7.68	79.80±4.98
ColonTumor	80.48±21.87	77.14±16.78	80.48±20.56	86.67±14.53	88.33±15.00	86.67±14.53	88.33±15.00	88.33±13.02	88.57±10.58	86.90±9.95	88.57±10.58
Average	83.26±8.52	82.55±7.30	82.78±8.36	83.55±6.88	85.86±6.99	83.53±7.16	85.94±7.03	85.16±6.44	86.09±5.92	85.02±7.42	86.16±5.75

Table 13
Classification performance (mean±std%) with KNN.

Dataset	All	FβCRSM	OFβCRSM	1-FCRSM(DF)	2-FCRSM(DF)	3-FCRSM(DF)	4-FCRSM(DF)	1-FCRSM(CF)	2-FCRSM(CF)	3-FCRSM(CF)	4-FCRSM(CF)
BreastCancer	50.76±17.58	50.76±17.58	49.85±14.31	63.18±15.53	74.70±16.04	55.08±8.84	74.70±16.04	63.18±15.53	68.11±13.52	63.18±15.53	68.11±13.52
Wine	75.33±6.15	74.77±5.64	72.09±9.06	93.27±5.97	93.82±6.78	93.27±5.97	93.82±6.78	93.27±5.97	93.82±6.78	93.27±5.97	93.82±6.78
ThoraSurgery	81.06±3.49	78.94±5.16	81.06±2.77	80.21±3.69	80.21±4.95	79.36±4.15	80.21±4.95	80.43±3.13	81.70±3.83	80.43±3.13	81.70±3.83
Lymphography	81.10±4.95	81.81±5.86	76.38±10.42	80.95±10.13	80.95±10.13	80.95±10.13	80.95±10.13	80.95±10.13	81.71±8.09	80.95±10.13	81.71±8.09
PopFailures	92.96±2.16	91.48±3.33	92.78±1.00	90.74±2.34	90.74±2.34	90.74±2.34	90.74±2.34	90.74±2.34	90.74±2.19	90.74±2.34	90.74±2.19
WDBC	91.39±3.45	90.69±2.82	92.62±3.02	91.39±3.45	90.34±4.23	90.34±4.23	91.39±3.45	91.39±3.45	91.92±2.84	91.39±3.45	91.92±2.84
SPECTF	77.54±7.44	76.41±7.81	74.15±8.58	77.19±6.40	78.26±5.99	77.19±6.40	78.26±5.99	77.18±6.87	77.88±6.60	77.18±6.87	80.51±3.29
ColonTumor	80.24±18.01	88.33±13.02	81.90±15.76	81.90±15.76	85.24±15.71	83.81±12.36	83.81±10.58	82.14±15.33	83.57±12.93	83.57±12.93	83.57±14.92
Average	78.80±7.90	79.15±7.65	77.60±8.12	82.35±7.91	84.28±8.27	81.34±6.80	84.24±7.53	82.41±7.84	83.68±7.10	82.59±7.54	84.01±6.93

For the KNN classifier, the average reduction rate of all l-FCRSMs ($l = 1,2,3,4$) is 98.63% when fuzzy dependency function is used, and 98.04% when fuzzy composite function is used. It can be preliminarily understood that l-FCRSM ($l = 1,2,3,4$) has the highest reduction rate when the fuzzy dependency function is used, which suggests that it can sufficiently reduce the dimensionality of the dataset; however, more features are retained when the fuzzy composite function is used.

Tables 12 and 13 display the classification performances (mean±std) of attribute subsets derived from ten ATR methods under the CART and KNN classifiers, respectively. Underlined values denote the highest classification performance within the same dataset. Attribute subsets generated by the 4-FCRSM(CF) and 2-FCRSM(DF) methods demonstrate the highest average classification performance under two classifiers, respectively. Notably, among the eight methods introduced in this study, the average classification performance of the CART and KNN classifiers improved by at best 2.9% and 5.48% when compared to the original dataset. Additionally, a further analysis of the 16 cases with the highest classification performance (excluding duplicates) reveals that our methods account for 10 cases, and the remaining two comparative methods account for 3 cases. These results indicate that the method proposed here outperforms other ATR methods based on fuzzy β-covering similarity relations in terms of classification performance.

In addition to this, a comprehensive comparison of the fuzzy dependency function and the fuzzy composite function reveals that the average of the classification performance of all our methods using the fuzzy dependency function is 83.89%, while the average of the classification performance of all our methods using the fuzzy composite function is 84.39%. This suggests that the methods using fuzzy composite function are more inclined to obtain a higher classification accuracy.

In summary, all ten ATR methods can effectively select important attributes and improve the classification performance of the model. Among them, the method using fuzzy dependency function obtains a shorter attribute subset length, while the method using fuzzy composite function tends to obtain a higher attribute subset classification performance. Overall, our proposed method outperforms other methods because of the fuzzy β-covering similarity relations at different levels.

6.2.4. Statistical test

To explore the statistical correlation between the proposed method and the comparative methods, this paper conducts the Friedman test [34] and the Nemenyi test [35] based on the classification performances of eight datasets. The Friedman test is used to determine whether there are significant differences among different methods, and its expression is shown as

$$F_F = \frac{(d-1)\chi_F^2}{d(n-1)-\chi_F^2} \sim F(n-1,(n-1)(d-1)), \tag{39}$$

$$\chi_F^2 = \frac{12d}{n(n+1)}\left(\sum_{i=1}^{n} r_i^2 - \frac{n(n+1)^2}{4}\right), \tag{40}$$

where n is the number of methods, d is the number of datasets, and r_i represents the average ranking of the i-th method on d datasets. When $\alpha = 0.1$, $n = 10$, and $d = 8$, the critical value is $F(7,63) = 1.733$.

Based on the classification performance data in Tables 12 and 13, we calculate the rankings of ten ATR methods on eight datasets based on the CART and KNN classifiers, and the results are listed in Table 14 and 15 respectively. According to the average rankings of each method under the two types of classifiers, $F_F = 3.577$, $\chi_F^2 = 24.348$ for the CART classifier, and the $F_F = 1.822$, $\chi_F^2 = 14.872$

Table 14
Ranking of classification performance with CART.

Dataset	FβCRSM	OFβCRSM	1-FCRSM(DF)	2-FCRSM(DF)	3-FCRSM(DF)	4-FCRSM(DF)	1-FCRSM(CF)	2-FCRSM(CF)	3-FCRSM(CF)	4-FCRSM(CF)
BreastCancer	10	6	8	1	9	2	5	3.5	7	3.5
Wine	8	10	3.5	3.5	3.5	3.5	9	3.5	3.5	7
ThoraSurgery	1	4	10	6.5	9	5	8	2.5	6.5	2.5
Lymphography	10	9	3.5	3.5	3.5	3.5	7.5	3.5	7.5	3.5
PopFailures	2	1	9.5	5.5	9.5	3	8	5.5	5.5	5.5
WDBC	10	9	7	3	8	3	6	1	5	3
SPECTF	2	10	9	7	8	4	5	3	6	1
ColonTumor	10	9	7.5	4	7.5	4	4	1.5	6	1.5
Average	6.63	7.25	7.25	4.25	7.25	3.50	6.56	3.00	5.88	3.44

Table 15
Ranking of classification performance with KNN.

Dataset	FβCRSM	OFβCRSM	1-FCRSM(DF)	2-FCRSM(DF)	3-FCRSM(DF)	4-FCRSM(DF)	1-FCRSM(CF)	2-FCRSM(CF)	3-FCRSM(CF)	4-FCRSM(CF)
BreastCancer	9	10	6	1.5	8	1.5	6	3.5	6	3.5
Wine	9	10	6.5	2.5	6.5	2.5	6.5	2.5	6.5	2.5
ThoraSurgery	10	3	7	7	9	7	4.5	1.5	4.5	1.5
Lymphography	1	10	6.5	6.5	6.5	6.5	6.5	2.5	6.5	2.5
PopFailures	2	1	6.5	6.5	6.5	6.5	6.5	6.5	6.5	6.5
WDBC	8	1	5.5	9.5	9.5	5.5	5.5	2.5	5.5	2.5
SPECTF	9	10	5.5	2.5	5.5	2.5	7.5	4	7.5	1
ColonTumor	1	9.5	9.5	2	3.5	3.5	8	6	6	6
Average	6.13	6.81	6.63	4.75	6.88	4.44	6.38	3.63	6.13	3.25

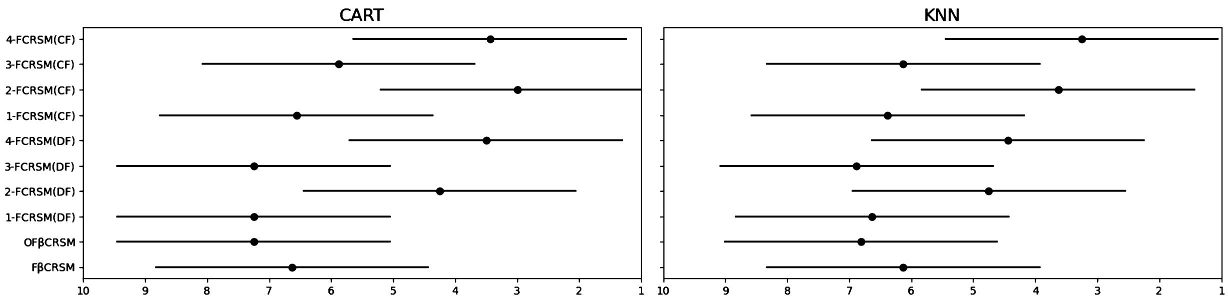

Fig. 22. CD graphs of Nemenyi test.

for the KNN classifier. Since the Friedman test values of both classifiers exceed the critical threshold, it indicates that there are statistically significant differences in the classification performances of the ten ATR methods.

To further analyze the differences in classification performance among the ten ATR methods, this paper uses the Nemenyi test for statistical inference. Let n and d represent the number of methods and the number of datasets, respectively. When the difference in the average rankings of two methods exceeds the Critical Distance (CD), it is determined that there are significant differences in their performance. Under the conditions of a significance level of $\alpha = 0.1$ and the number of methods $n = 10$, the Studentized range statistic $q_\alpha = 2.920$ is obtained by looking up, and the critical distance $\mathrm{CD}_{0.1} = 4.4204$ is calculated by

$$CD_\alpha = q_\alpha \sqrt{\frac{n(n+1)}{6d}}. \tag{41}$$

Based on this critical distance and the average rankings of each method under the two types of classifiers, the results of the Nemenyi test are shown in Fig. 22. The analysis shows that: in the environments of both classifiers, the average ranking of our method ranks first among the ten methods. From this, it can be inferred that our method is significantly superior to the other ten comparative methods in a statistical sense.

7. Conclusion

VSFβ-CAS is an extension of the traditional space that takes into account the information gap characterized by different combinations of attributes or features. In this paper, some Fβ-CRS models and their applications to VSFβ-CAS are considered. Overall, the paper makes the following contributions:

- We propose a VSFβ-CAS that allows different β parameters to be chosen for different covering within the same framework. It is a completely different concept from Fβ-CAS or VSFβ-CGA.
- Four types of Fβ-NOs satisfying reflexivity and symmetry are proposed in VSFβ-CAS, which include parameterized fuzzy β-neighborhoods and parameterized fuzzy complementary β-neighborhoods based on overlap functions or grouping functions, and they can also filter out the noise effectively.
- We construct four types of Fβ-CRS models with inclusion relationship on the basis of the fuzzy β-covering relations, and discusses the basic properties of the models and interrelationships relationship among different models.
- Fuzzy and crisp three-way regions are defined using the four proposed models, and based on this, a TWD method is designed, which has more flexibility and the ability to describe uncertain information.
- Two monotonic uncertainty measures are defined based on the proposed models, named fuzzy dependency functions and fuzzy composite functions, and new variable-scale ATR methods are designed using them respectively.

Regarding the upcoming research, we believe that there are several directions for expansion: (1) multi-granularity [36] or variable precision structure in VSFβ-CAS; (2) construction of fuzzy rough sets by using other novel aggregation functions or fuzzy β-co-neighborhood; (3) fabrication of granular-ball rough sets [37,38] by using fuzzy relations or neighborhood; and (4) application of the above ideas in group decision-making, fuzzy clustering or other researches [39,40].

CRediT authorship contribution statement

Yaoyao Fan: Writing – original draft, Validation, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Xiaohong Zhang:** Writing – review & editing, Supervision, Methodology, Funding acquisition, Conceptualization. **Jun Liu:** Writing – review & editing. **Jingqian Wang:** Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Xiaohong Zhang reports financial support was provided by National Natural Science Foundation of China. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

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