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Two fuzzy covering rough set models and their generalizations over fuzzy lattices

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Abstract

By introducing the new concepts of fuzzy β-covering and fuzzy β-neighborhood, we define two new types of fuzzy covering rough set models which can be regarded as bridges linking covering rough set theory and fuzzy rough set theory. We show the properties of the two models, and reveal the relationships between the two models and some others. Moreover, we present the matrix representations of the newly defined lower and upper approximation operators so that the calculation of lower and upper approximations of subsets can be converted into operations on matrices. Finally, we generalize the models and their matrix representations to L-fuzzy covering rough sets which are defined over fuzzy lattices.

Keywords: Fuzzy rough set; Fuzzy β-neighborhood; Fuzzy β-covering; Fuzzy lattice

1. Introduction

Covering rough set theory [44] is an extension of classical rough set theory [26]. Starting in 2001, there is a recent surge of interest in covering based rough sets, and the existing studies show a great diversity of research on this topic [2,16,20,21,28,31,32,34,41,45–48]. Covering rough set as well as classical rough set is designed to process qualitative (discrete) data, and it faces great limitations when dealing with real-valued data sets since the values of attributes in databases could be both symbolic and real-valued [14]. Fuzzy set theory [43] however, is very useful to overcome these limitations, as it can deal effectively with vague concepts and graded indiscernibility. Nowadays, rough set theory and fuzzy set theory are two main tools being used to process uncertainty and incomplete information in the information systems. The two theories are related but distinct and complementary [27,40].

In the past two decades, the research on the hybridization between rough sets and fuzzy sets has attracted much attention. Intentions on combining rough set theory and fuzzy set theory can be found in different mathematical fields [17,19,25,33,39]. Dubois and Prade firstly proposed the concept of fuzzy rough sets [4,5] which combined these two theories and influenced numerous authors who used different fuzzy logical connectives and fuzzy relations to define fuzzy rough set models. Two essential works were done by Morsi et al. [24], and Radzikowska et al. [29]. The

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former study used both constructive and axiomatic approaches, and the latter defined fuzzy rough sets based on three general classes of fuzzy implications such as S-implicators, R-implicators and QL-implicators. Despite generalizing the fuzzy connectives, they still used fuzzy similarity relations. Greco et al. [8,9] were the first to use reflexive fuzzy relations and Wu et al. [37,38] were the first to consider general fuzzy relations. Mi et al. [22,23] considered conjunctions instead of t-norms and Hu et al. [11,12] studied fuzzy relations based on kernel functions. Furthermore, Yeung et al. [42] discussed two pairs of dual approximation operators from both constructive and axiomatic viewpoints. In reference [3], these different proposals were considered within a general Implicator-Conjunctor (IC)-based fuzzy rough set model that encapsulates all of them.

The most common fuzzy rough set is obtained by replacing the crisp binary relations and the crisp subsets with the fuzzy relations and the fuzzy subsets on the universe respectively. Recently, some fuzzy rough set models based on the concept of fuzzy covering were constructed [6,13,18,36], which can be viewed as bridges linking covering rough set theory and fuzzy rough set theory. In this paper, by introducing the concepts of fuzzy β-covering and fuzzy β-neighborhood, we define more general fuzzy covering rough set models. Properties of the models and their relationships with other rough sets are discussed.

A basic problem one faces in the investigation and application of rough sets is the calculation of lower and upper approximations for subsets of approximation spaces. For approximation spaces with large cardinals, such calculations would be tedious and prone to errors. An encouraging fact we find is that the lower and upper approximation operators we defined can be related to matrices, and further more, the calculation of lower and upper approximations can be converted into operations on matrices which greatly facilitate the calculation.

Besides the traditional fuzzy rough set, there are several other generalizations of the fuzzy rough set model appeared in the literature. For example, Radzikowska and Kerre have extended the fuzzy rough sets to L-fuzzy rough sets which defined over residuated lattices L [10,30]. In this paper, we further generalize the fuzzy covering rough set models by extending the range of fuzzy set from unit interval $[0, 1]$ to fuzzy lattice L. We call the generalized models L-fuzzy covering rough sets. In addition, the matrix representations of the fuzzy covering rough set models are also extend to the L-fuzzy covering rough sets models.

The remainder of this paper is structured as follows. In Section 2, some preliminary definitions are introduced. In Section 3, we introduce two new concepts, i.e., the fuzzy β-covering and the fuzzy β-neighborhood, and discuss their properties. Two fuzzy covering rough set models are defined in Section 4, and their matrix representations are introduced in Section 5. In Section 6, we generalize the models and their matrix representations to L-fuzzy covering rough sets. We conclude the paper in Section 7.

2. Preliminaries

Let us first review some notions in covering rough set theory and fuzzy set theory.

Definition 2.1. (See [46].) Let U be a universe and $\mathcal{C}$ be a family of subsets of U. If no element in $\mathcal{C}$ is empty and $U = \bigcup_{C\in\mathcal{C}} C$, then $\mathcal{C}$ is called a covering of U, and the ordered pair $(U, \mathcal{C})$ is called a covering approximation space.

For any $x \in U$, we can define the neighborhood of x as

$$N_x = \cap\{C \in \mathcal{C} : x \in C\}.$$

The following definition of lower and upper approximation operators are widely used in many references.

Definition 2.2. (See [32].) Let $(U, \mathcal{C})$ be a covering approximation space and $X \subseteq U$. The lower approximation $P^-(X)$ and the upper approximation $P^+(X)$ of X are defined as

$$P^-(X) = \{x \in U : N_x \subseteq X\}, \quad P^+(X) = \{x \in U : N_x \cap X \neq \emptyset\}.$$

Definition 2.3. (See [15].) Let U be a universal set. A fuzzy set $\widetilde{A}$, or rather a fuzzy subset $\widetilde{A}$ of U, is defined by a function assigning to each element x of U a value $\widetilde{A}(x) \in [0, 1]$. We denote by $F(U)$ the set of all fuzzy subsets of U, i.e., the set of all functions from U to $[0, 1]$, and call it the fuzzy power set of U.

For any $\widetilde{A}, \widetilde{B} \in F(U)$, we say that $\widetilde{A}$ is contained in $\widetilde{B}$, denoted by $\widetilde{A} \subset \widetilde{B}$, if $\widetilde{A}(x) \leq \widetilde{B}(x)$ for all $x \in U$, and we say that $\widetilde{A} = \widetilde{B}$ if and only if $\widetilde{A} \subset \widetilde{B}$ and $\widetilde{B} \subset \widetilde{A}$.

For any family $\alpha_i \in [0,1], i \in I$, we write $\vee_{i\in I}\alpha_i$ or $\vee\{\alpha_i : i \in I\}$ for the supremum of $\{\alpha_i : i \in I\}$, and $\wedge_{i\in I}\alpha_i$ or $\wedge\{\alpha_i : i \in I\}$ for the infimum of $\{\alpha_i : i \in I\}$. Given $\widetilde{A}, \widetilde{B} \in F(U)$, the union of $\widetilde{A}$ and $\widetilde{B}$, denoted as $\widetilde{A} \cup \widetilde{B}$, is defined by $(\widetilde{A} \cup \widetilde{B})(x) = \widetilde{A}(x) \vee \widetilde{B}(x)$ for all $x \in U$; The intersection of $\widetilde{A}$ and $\widetilde{B}$, denoted as $\widetilde{A} \cap \widetilde{B}$, is given by $(\widetilde{A} \cap \widetilde{B})(x) = \widetilde{A}(x) \wedge \widetilde{B}(x)$ for all $x \in U$.

Definition 2.4. (See [1].) A lattice is a partially ordered set in which any two elements a and b have a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$.

A lattice L is complete if any subset $A \subset L$ has a least upper bound (supremum) $\vee A$ and a greatest lower bound (infimum) $\wedge A$ in L.

A lattice L is completely distributive if the following conditions hold:

$$\bigwedge_{i\in I}(\bigvee_{j\in J_i} a_{ij}) = \bigvee_{f\in\prod_{i\in I}J_i}(\bigwedge_{i\in I} a_{if(i)}),$$

$$\bigvee_{i\in I}(\bigwedge_{j\in J_i} a_{ij}) = \bigwedge_{f\in\prod_{i\in I}J_i}(\bigvee_{i\in I} a_{if(i)}),$$

where $a_{ij} \in L$, I and J_i are nonempty index sets, and $f \in \prod_{i\in I} J_i$ means f is a mapping $f : I \to \cup_{i\in I} J_i$ such that $f(i) \in J_i$ for every $i \in I$.

For example, $([0,1], \vee, \wedge, 0, 1)$ is a complete completely distributive (CCD) lattice; If X is a nonempty set, $L = P(X)$, then $(L, \cup, \cap, \emptyset, X)$ is a CCD lattice; Let $F(X)$ be the collection of fuzzy sets in X, then $(F(X), \cup, \cap, \emptyset, X)$ is a CCD lattice.

Definition 2.5. (See [35].) A CCD lattice $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ with an order-reversing involution $' : L \to L$ is called a fuzzy lattice, and is represented as $L = (L, \wedge, \vee, ', \mathbf{0}, \mathbf{1})$, where $\mathbf{0}$ and $\mathbf{1}$ denote the least element and the greatest element of L respectively.

For example, $(\{0,1\}, \wedge, \vee, ', 0, 1)$ and $([0,1], \wedge, \vee, ', 0, 1)$ are both fuzzy lattices with $x' = 1 - x$, where $\wedge$ and $\vee$ denote the minimum and maximum respectively.

Definition 2.6. (See [7].) Let L be a fuzzy lattice. An L-fuzzy set $\widehat{X}$ of U is a mapping $\widehat{X} : U \to L$. We denote by $\widehat{F}(U)$ the set of all L-fuzzy sets in U. For $\widehat{X}, \widehat{Y} \in \widehat{F}(U)$, we define the relation $\widehat{X} \ll \widehat{Y}$, if $\widehat{X}(x) \le \widehat{Y}(x)$ for all $x \in U$. If $\widehat{X}, \widehat{Y} \in \widehat{F}(U)$, we define the operations $\sqcup$, $\sqcap$ and $'$ on $\widehat{F}(U)$ as follows:

$$(\widehat{X} \sqcup \widehat{Y})(x) = \widehat{X}(x) \vee \widehat{Y}(x), \quad (\widehat{X} \sqcap \widehat{Y})(x) = \widehat{X}(x) \wedge \widehat{Y}(x), \quad (\widehat{X})'(x) = (\widehat{X}(x))',$$

for all $x \in U$.

Obviously, the L-fuzzy set of U is a generalization of the fuzzy set of U by extending the range from the unit interval [0, 1] to the fuzzy lattice L.

3. Fuzzy β-covering and fuzzy β-neighborhoods

In this section, we introduce two new concepts, i.e., the fuzzy β-covering and the fuzzy β-neighborhood.

Definition 3.1. Let U be an arbitrary universal set, and $F(U)$ be the fuzzy power set of U. For each $\beta \in (0,1]$, we call $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$, with $\widetilde{C}_i \in F(U)$ $(i = 1, 2, \cdots, m)$, a fuzzy β-covering of U, if $(\cup_{i=1}^{m}\widetilde{C}_i)(x) \ge \beta$ for each $x \in U$. We also call $(U, \widetilde{\mathcal{C}})$ a fuzzy covering approximation space.

Note that the concept of fuzzy covering in papers [6,13,18,36] is a special case of fuzzy β-covering with $\beta = 1$.

Definition 3.2. Let $(U, \widetilde{\mathcal{C}})$ be a fuzzy covering approximation space, where $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ is a fuzzy β-covering of U. For each $x \in U$, we define the fuzzy β-neighborhood $\widetilde{N}_x^{\beta}$ of x as:

$$\widetilde{N}_x^{\beta} = \cap\{\widetilde{C}_i \in \widetilde{\mathcal{C}} : \widetilde{C}_i(x) \ge \beta\}.$$

Example 3.1. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. A set of fuzzy sets $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \widetilde{C}_4, \widetilde{C}_5\}$ of U is listed as follows. We can see that $\widetilde{\mathcal{C}}$ is a fuzzy β-covering of $U (0 < \beta \le 0.6)$.

	$\widetilde{C}_1$	$\widetilde{C}_2$	$\widetilde{C}_3$	$\widetilde{C}_4$	$\widetilde{C}_5$
x_1	0.7	0.6	0.4	0.3	0.1
x_2	0.5	0.3	0.2	0.4	0.6
x_3	0.4	0.2	0.5	0.6	0.7
x_4	0.6	0.4	0.3	0.3	0.5
x_5	0.3	0.7	0.6	0.8	0.3
x_6	0.2	0.7	0.1	0.7	0.2

Thus

$$\widetilde{N}_{x_1}^{0.5} = \widetilde{C}_1 \cap \widetilde{C}_2, \quad \widetilde{N}_{x_2}^{0.5} = \widetilde{C}_1 \cap \widetilde{C}_5, \quad \widetilde{N}_{x_3}^{0.5} = \widetilde{C}_3 \cap \widetilde{C}_4 \cap \widetilde{C}_5,$$
$$\widetilde{N}_{x_4}^{0.5} = \widetilde{C}_1 \cap \widetilde{C}_5, \quad \widetilde{N}_{x_5}^{0.5} = \widetilde{C}_2 \cap \widetilde{C}_3 \cap \widetilde{C}_4, \quad \widetilde{N}_{x_6}^{0.5} = \widetilde{C}_2 \cap \widetilde{C}_4,$$

and $\widetilde{N}_{x_i}^{0.5}$ of x_i $(i = 1, 2, 3, 4, 5, 6)$ are as follows.

	x_1	x_2	x_3	x_4	x_5	x_6
$\widetilde{N}_{x_1}^{0.5}$	0.6	0.3	0.2	0.4	0.3	0.2
$\widetilde{N}_{x_2}^{0.5}$	0.1	0.5	0.4	0.5	0.3	0.2
$\widetilde{N}_{x_3}^{0.5}$	0.1	0.2	0.5	0.3	0.3	0.1
$\widetilde{N}_{x_4}^{0.5}$	0.1	0.5	0.4	0.5	0.3	0.2
$\widetilde{N}_{x_5}^{0.5}$	0.3	0.2	0.2	0.3	0.6	0.1
$\widetilde{N}_{x_6}^{0.5}$	0.3	0.3	0.2	0.3	0.7	0.7

In a fuzzy covering approximation space $(U, \widetilde{\mathcal{C}})$, with $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ being a fuzzy β-covering of U for some $\beta \in (0, 1]$, we have the following properties of the fuzzy β-neighborhoods.

Proposition 3.1. *$\widetilde{N}_x^\beta(x) \ge \beta$ for each $x \in U$.*

Proof. For each $x \in U$, $\widetilde{N}_x^\beta(x) = (\cap_{\widetilde{C}_i(x) \ge \beta} \widetilde{C}_i)(x) = \wedge_{\widetilde{C}_i(x) \ge \beta} \widetilde{C}_i(x) \ge \beta$. □

Proposition 3.2. *$\forall x, y, z \in U$, if $\widetilde{N}_x^\beta(y) \ge \beta$ and $\widetilde{N}_y^\beta(z) \ge \beta$, then $\widetilde{N}_x^\beta(z) \ge \beta$.*

Proof. Let $I = \{1, 2, \cdots, m\}$. Since $\widetilde{N}_x^\beta(y) \ge \beta$, then for each $i \in I$, if $\widetilde{C}_i(x) \ge \beta$, we have $\widetilde{C}_i(y) \ge \beta$. And similarly, it follows from $\widetilde{N}_y^\beta(z) \ge \beta$ that $\widetilde{C}_i(y) \ge \beta$ which leads to $\widetilde{C}_i(z) \ge \beta$. So, for each $i \in I$, $\widetilde{C}_i(x) \ge \beta$ implies $\widetilde{C}_i(z) \ge \beta$, and hence $\widetilde{N}_x^\beta(z) = \wedge_{\widetilde{C}_i(x) \ge \beta} \widetilde{C}_i(z) \ge \beta$. □

Proposition 3.3. *For each $\beta \in (0, 1]$, there are*

$$\widetilde{C}_i \supset \cup\{\widetilde{N}_x^\beta : \widetilde{C}_i(x) \ge \beta, x \in U\}, \ i \in I = \{1, 2, \cdots, m\}.$$

Proof. For each $i \in I$, according to the definition of $\widetilde{N}_x^\beta$, $C_i(x) \ge \beta$ means $\widetilde{N}_x^\beta \subset \widetilde{C}_i$. So we have $\widetilde{C}_i \supset \cup\{\widetilde{N}_x^\beta : \widetilde{C}_i(x) \ge \beta, x \in U\}$. □

Proposition 3.4. *If $\beta_1 \le \beta_2$, then $\widetilde{N}_x^{\beta_1} \subset \widetilde{N}_x^{\beta_2}$ for all $x \in U$.*

Proof. For each $x \in U$, $\beta_1 \le \beta_2$ implies that $\{\widetilde{C}_i : \widetilde{C}_i(x) \ge \beta_1, \} \supset \{\widetilde{C}_i : \widetilde{C}_i(x) \ge \beta_2, \}$. Hence there is $\widetilde{N}_x^{\beta_1} = \cap\{\widetilde{C}_i : \widetilde{C}_i(x) \ge \beta_1\} \subset \cap\{\widetilde{C}_i : \widetilde{C}_i(x) \ge \beta_2\} = \widetilde{N}_x^{\beta_2}$ for all $x \in U$. □

Now we define the β-neighborhood in the fuzzy covering approximation space.

Definition 3.3. Let $(U, \widetilde{\mathcal{C}})$ be a fuzzy covering approximation space with $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ being a fuzzy β-covering of U for some $\beta \in (0, 1]$. For each $x \in U$, we define the β-neighborhood $\overline{N}_x^\beta$ of x as:

$$\overline{N}_x^\beta = \{y \in U : \widetilde{N}_x^\beta(y) \geq \beta\}.$$

Example 3.2. Let $(U, \widetilde{\mathcal{C}})$ be the fuzzy covering approximation space in Example 3.1 with $\beta = 0.5$, then we have

$$\overline{N}_{x_1}^\beta = \{x_1\}, \quad \overline{N}_{x_2}^\beta = \{x_2, x_4\}, \quad \overline{N}_{x_3}^\beta = \{x_3\},$$
$$\overline{N}_{x_4}^\beta = \{x_2, x_4\}, \quad \overline{N}_{x_5}^\beta = \{x_5\}, \quad \overline{N}_{x_6}^\beta = \{x_5, x_6\}.$$

We have the following properties of the β-neighborhoods in a fuzzy covering approximation space $(U, \widetilde{\mathcal{C}})$, where $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ is a fuzzy β-covering of U for some $\beta \in (0, 1]$.

Proposition 3.5. *$x \in \overline{N}_x^\beta(x)$ for each $x \in U$.*

Proof. It follows from Proposition 3.1 that $x \in \{y \in U : \widetilde{N}_x^\beta(y) \geq \beta\} = \overline{N}_x^\beta$ for each $x \in U$. □

Proposition 3.6. *$\forall x, y \in U$, if $x \in \overline{N}_y^\beta$, then $\overline{N}_x^\beta \subset \overline{N}_y^\beta$.*

Proof. $\forall z \in \overline{N}_x^\beta$, we have $\widetilde{N}_x^\beta(z) \geq \beta$. Furthermore, $x \in \overline{N}_y^\beta$ implies $\widetilde{N}_y^\beta(x) \geq \beta$. According to Proposition 3.2, there is $\widetilde{N}_y^\beta(z) \geq \beta$, and hence $z \in \overline{N}_y^\beta$. Thus $\overline{N}_x^\beta \subset \overline{N}_y^\beta$. □

4. Two fuzzy covering rough set models

In this section, we will define two rough set models on basis of the fuzzy β-neighborhoods and the β-neighborhoods. One model concerns the fuzzy lower and upper approximations of each fuzzy set in the fuzzy environment, and another concerns the crisp lower and upper approximations of each crisp set in the fuzzy environment. We will also discuss the properties of the defined lower and upper approximation operators and reveal their relationship with other rough set models.

4.1. A fuzzy covering rough set model for fuzzy subsets

Definition 4.1. Let $(U, \widetilde{\mathcal{C}})$ be a fuzzy covering approximation space with $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ being a fuzzy β-covering of U for some $\beta \in (0, 1]$. For each $X \in F(U)$, we define the lower approximation $\widetilde{P^-}X$ and the upper approximation $\widetilde{P^+}X$ of X as:

$$(\widetilde{P^-}X)(x) = \wedge_{y \in U}[(1 - \widetilde{N}_x^\beta(y)) \vee X(y)], \quad x \in U,$$
$$(\widetilde{P^+}X)(x) = \vee_{y \in U}[\widetilde{N}_x^\beta(y) \wedge X(y)], \quad x \in U.$$

If $\widetilde{P^-}(X) \neq \widetilde{P^+}(X)$, then X is called a fuzzy covering rough set.

Example 4.1. Let $(U, \widetilde{\mathcal{C}})$ be the fuzzy covering approximation space in Example 3.1. Then for

$$X = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.7}{x_5} + \frac{0.4}{x_6},$$
$$Y = \frac{0.55}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3} + \frac{0.45}{x_4} + \frac{0.45}{x_5} + \frac{0.35}{x_6}$$

and $\beta = 0.5$, we have

$$\widetilde{P^-}X = \frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.7}{x_5} + \frac{0.4}{x_6},$$
$$\widetilde{P^+}X = \frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.3}{x_3} + \frac{0.3}{x_4} + \frac{0.6}{x_5} + \frac{0.7}{x_6},$$

and

$$\widetilde{P^-}Y = \frac{0.55}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3} + \frac{0.45}{x_4} + \frac{0.45}{x_5} + \frac{0.35}{x_6},$$
$$\widetilde{P^+}Y = \frac{0.55}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.45}{x_5} + \frac{0.5}{x_6}.$$

Proposition 4.1. *Let $(U, \widetilde{\mathcal{C}}))$ be a fuzzy covering approximation space with $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ being a fuzzy β-covering of U for some $\beta \in (0, 1]$. For each $X, Y \in F(U)$, we have*

i). $\widetilde{P^-}(X') = (\widetilde{P^+}X)'$, $\widetilde{P^+}(X') = (\widetilde{P^-}X)'$, *where* $X'(x) = 1 - X(x)$, $x \in U$,
ii). $\widetilde{P^-}U = U$, $\widetilde{P^+}\emptyset = \emptyset$,
iii). $\widetilde{P^-}(X \cap Y) = \widetilde{P^-}X \cap \widetilde{P^-}Y$, $\widetilde{P^+}(X \cup Y) = \widetilde{P^+}X \cup \widetilde{P^+}Y$,
iv). *If* $X \subset Y$, *then* $\widetilde{P^-}X \subset \widetilde{P^-}Y$, $\widetilde{P^+}X \subset \widetilde{P^+}Y$,
v). $\widetilde{P^-}(X \cup Y) \supset \widetilde{P^-}X \cup \widetilde{P^-}Y$, $\widetilde{P^+}(X \cap Y) \subset \widetilde{P^+}X \cap \widetilde{P^+}Y$,
vi). *If* $1 - \widetilde{N}_x^\beta(x) \le X(x) \le \widetilde{N}_x^\beta(x)$ *for all* $x \in U$, *then* $\widetilde{P^-}X \subset X \subset \widetilde{P^+}X$.

Proof. *i*). Since

$$(\widetilde{P^+}(X'))(x) = \vee_{y \in U}[\widetilde{N}_x^\beta(y) \wedge X'(y)] = 1 - \wedge_{y \in U}[(1 - \widetilde{N}_x^\beta(y)) \vee X(y)] = 1 - (\widetilde{P^-}X)(x) = (\widetilde{P^-}X)'(x),$$

then $\widetilde{P^+}(X') = (\widetilde{P^-}X)'$. Replacing X by X' in this proof, we can obtain $\widetilde{P^-}(X') = (\widetilde{P^+}X)'$.

ii). It follows from $U(x) = 1$ and $\emptyset(x) = 0$ that for every $x \in U$,

$$(\widetilde{P^-}U)(x) = \wedge_{y \in U}[(1 - \widetilde{N}_x^\beta(y)) \vee U(y)] = 1 = U(x),$$
$$(\widetilde{P^+}\emptyset)(x) = \vee_{y \in U}[\widetilde{N}_x^\beta(y) \wedge \emptyset(y)] = 0 = \emptyset(x).$$

Thus we have $\widetilde{P^-}U = U$ and $\widetilde{P^+}\emptyset = \emptyset$.

iii). Since

$$\begin{aligned}(\widetilde{P^-}(X \cap Y))(x) &= \wedge_{y \in U}[(1 - \widetilde{N}_x^\beta(y)) \vee (X \cap Y)(y)] \\ &= \wedge_{y \in U}[((1 - \widetilde{N}_x^\beta(y)) \vee X(y)) \wedge ((1 - \widetilde{N}_x^\beta(y)) \vee Y(y))] \\ &= (\widetilde{P^-}X \cap \widetilde{P^-}Y)(x),\end{aligned}$$

then there is $\widetilde{P^-}(X \cap Y) = \widetilde{P^-}X \cap \widetilde{P^-}Y$.

Through a similar proof, we can obtain $\widetilde{P^+}(X \cup Y) = \widetilde{P^+}X \cup \widetilde{P^+}Y$.

iv). If $X \subset Y$, then $X(x) \le Y(x)$ for each $x \in U$. So we have

$$(\widetilde{P^-}X)(x) = \wedge_{y \in U}[(1 - \widetilde{N}_x^\beta(y)) \vee X(y)] \le \wedge_{y \in U}[(1 - \widetilde{N}_x^\beta(y)) \vee Y(y)] = (\widetilde{P^-}Y)(x).$$

Thus $\widetilde{P^-}X \subset \widetilde{P^-}Y$ holds, and similarly there is $\widetilde{P^+}X \subset \widetilde{P^+}Y$.

v). Since $X \subset X \cup Y$, $Y \subset X \cup Y$, $X \cap Y \subset X$ and $X \cap Y \subset Y$, using *iv*), we have

$$\widetilde{P^-}X \subset \widetilde{P^-}(X \cup Y), \quad \widetilde{P^-}Y \subset \widetilde{P^-}(X \cup Y),$$
$$\widetilde{P^+}(X \cap Y) \subset \widetilde{P^+}X, \quad \widetilde{P^+}(X \cap Y) \subset \widetilde{P^+}Y,$$

and thus

$$\widetilde{P^-}X \cup \widetilde{P^-}Y \subset \widetilde{P^-}(X \cup Y), \quad \widetilde{P^+}(X \cap Y) \subset \widetilde{P^+}X \cap \widetilde{P^+}Y.$$

vi). If for all $x \in U$, there is $1 - \widetilde{N}_x^\beta(x) \le X(x) \le \widetilde{N}_x^\beta(x)$, then

$$X(x) = \widetilde{N}_x^\beta(x) \wedge X(x) \le \vee_{y\in U}[\widetilde{N}_x^\beta(y) \wedge X(y)] = (\widetilde{P^+}X)(x),$$

$$(\widetilde{P^-}X)(x) = \wedge_{y\in U}[(1 - \widetilde{N}_x^\beta(y)) \vee X(y)] \le (1 - \widetilde{N}_x^\beta(x)) \vee X(x) = X(x),$$

and thus $\widetilde{P^-}X \subset X \subset \widetilde{P^+}X$. □

Now we give a practical example of the fuzzy covering rough set model defined in Definition 4.1.

Example 4.2. In medicine, we usually combine some kinds of medicines to treat a disease. Let $U = \{x_j : j = 1, 2, \cdots, n\}$ be the universe of n kinds of medicines, $V = \{y_i : i = 1, 2, \cdots, m\}$ be m main symptoms (for example, fever, cough, dizzy giddy, etc.) of a disease A (for example, H1N1, etc.), and $\widetilde{C}_i(x_j)$ denote the efficacy value of the medicine x_j for the symptom y_i $(i = 1, 2, \cdots, m, j = 1, 2, \cdots, n)$. Let β be the critical value. If we suppose that for each medicine $x_j \in U$, there is at least one symptom $y_i \in V$ such that the efficacy value of the medicine x_j for the symptom y_i is not less than β, then $\widetilde{\mathcal{C}} = \{\widetilde{C}_i : i = 1, 2, \cdots, m\}$ is a fuzzy β-covering of U. Then, for each medicine x_j, we consider the set of symptoms $\{y_i : \widetilde{C}_i(x_j) \ge \beta\}$. The fuzzy β-neighborhood $\widetilde{N}_{x_j}^\beta$ of x_j is a fuzzy set

$$\widetilde{N}_{x_j}^\beta(x_k) = [\cap\{\widetilde{C}_i : \widetilde{C}_i(x_j) \ge \beta\}](x_k) = \wedge_{\widetilde{C}_i(x_j)\ge\beta}\widetilde{C}_i(x_k), \quad k = 1, 2, \cdots, n,$$

which denotes the minimum value among all the efficacy values of each medicine x_k for treating the symptoms in $\{y_i : \widetilde{C}_i(x_j) \ge \beta\}$. If a fuzzy set X denotes the ability of all medicines in U to cure the disease A (according to a lot of experiments), since the inaccuracy of X, then we can take its approximate evaluation according to the lower and upper approximations of X.

We suppose that the fuzzy covering approximation space $(U, \widetilde{\mathcal{C}})$ is the space in Example 3.1 and X is fuzzy subset in Example 4.1.

$$X = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.7}{x_5} + \frac{0.4}{x_6},$$

$$\widetilde{P^-}X = \frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.7}{x_5} + \frac{0.4}{x_6},$$

$$\widetilde{P^+}X = \frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.3}{x_3} + \frac{0.3}{x_4} + \frac{0.6}{x_5} + \frac{0.7}{x_6}.$$

We can obtain the following results by analyzing $\widetilde{P^+}X$, $\widetilde{P^-}X$ and X under the critical value $\beta = 0.5$:

(1). Since $X(x_i) \ge 0.5$, $(\widetilde{P^+}X)(x_i) \ge 0.5$ and $(\widetilde{P^-}X)(x_i) \ge 0.5$ $(i = 1, 5)$, we conclude that the medicines x_1 and x_5 are most important for the treatment of disease A.
(2). As $X(x_4) \ge 0.5$ and $(\widetilde{P^-}X)(x_4) \ge 0.5$, the medicine x_4 is also important to the treatment of disease A.
(3). Since $X(x_i) < 0.5$ $(i = 2, 3, 6)$, $(\widetilde{P^+}X)(x_i) \ge 0.5$ $(i = 2, 6)$ and $(\widetilde{P^-}X)(x_i) \ge 0.5$ $(i = 2, 3)$, the medicines x_2, x_3, x_6 are less important than x_1, x_4, x_5 for the control of disease A.

4.2. A fuzzy covering rough set model for crisp subsets

Definition 4.2. Let $(U, \widetilde{\mathcal{C}})$ be a fuzzy covering approximation space with $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ being a fuzzy β-covering of U for some $\beta \in (0, 1]$. For each crisp subset $X \in P(U)$, we define the lower approximation $\overline{P^-}(X)$ and upper approximation $\overline{P^+}(X)$ of X as:

$$\overline{P^-}(X) = \{x : \overline{N}_x^\beta \subset X\},$$

$$\overline{P^+}(X) = \{x : \overline{N}_x^\beta \cap X \ne \emptyset\}.$$

If $\overline{P^-}(Y) \ne \overline{P^+}(Y)$, Y is called a fuzzy covering rough set.

The following properties of the operators $\overline{P^-}$ and $\overline{P^+}$ are the same as the properties of the operators in Definition 2.2 which were discussed in paper [32]. Here we omit their proofs.

Proposition 4.2. *Let $(U, \widetilde{\mathcal{C}})$ be a fuzzy covering approximation space with $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ being a fuzzy β-covering of U. Then for each $X, Y \in P(U)$, we have*

i). $\overline{P^-}(X') = (\overline{P^+}(X))'$, $\overline{P^+}(X') = (\overline{P^-}(X))'$, *where* $X' = U - X$,
ii). $\overline{P^-}(U) = U$, $\overline{P^+}(\emptyset) = \emptyset$,
iii). $\overline{P^-}(X \cap Y) = \overline{P^-}(X) \cap \overline{P^-}(Y)$, $\overline{P^+}(X \cup Y) = \overline{P^+}(X) \cup \overline{P^+}(Y)$,
iv). $\overline{P^-}(X) \subset \overline{P^-}(Y)$, $\overline{P^+}(X) \subset \overline{P^+}(Y)$, *if* $X \subset Y$,
v). $\overline{P^-}(X \cup Y) \supset \overline{P^-}(X) \cup \overline{P^-}(Y)$, $\overline{P^+}(X \cap Y) \subset \overline{P^+}(X) \cap \overline{P^+}(Y)$,
vi). $\overline{P^-}(X) \subset X \subset \overline{P^+}(X)$.

Now we give a practical example of the fuzzy covering rough set model defined in Definition 4.2.

Example 4.3. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a universe of six associated symptoms, $V = \{y_1, y_2, y_3, y_4, y_5\}$ be a set of sufferers and $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \widetilde{C}_4, \widetilde{C}_5\}$ be a set of fuzzy subsets of U, where $\widetilde{C}_i(x_j)$ denotes the critical value of the corresponding sufferer $y_i \in V$ showing the symptom $x_j \in U$.

	$\widetilde{C}_1$	$\widetilde{C}_2$	$\widetilde{C}_3$	$\widetilde{C}_4$	$\widetilde{C}_5$
x_1	0.7	0.6	0.4	0.3	0.1
x_2	0.5	0.3	0.2	0.4	0.6
x_3	0.4	0.2	0.5	0.6	0.7
x_4	0.6	0.4	0.3	0.3	0.5
x_5	0.3	0.7	0.6	0.8	0.3
x_6	0.2	0.7	0.1	0.7	0.2

Then $(U, \widetilde{\mathcal{C}})$ is same to the fuzzy covering approximation space introduced in Example 3.1. Suppose that the symptoms in $X = \{x_3, x_5\}$ are all involved in a certain disease A, taking $\beta = 0.5$ as the critical value, then we can calculate the lower approximation and upper approximation of X as follows.

$$\overline{P^-}(X) = \{x_3, x_5\}, \qquad \overline{P^+}(X) = \{x_3, x_5, x_6\}.$$

Since $\{x : \widetilde{C}_3(x) \geq 0.5\} = \{x_3, x_5\} \subset \overline{P^-}(X)$, $\{x : \widetilde{C}_i(x) \geq 0.5\} \cap \overline{P^+}(X) \neq \emptyset$, $(i = 2, 4, 5)$ and $\{x : \widetilde{C}_1(x) \geq 0.5\} \cap \overline{P^+}(X) = \emptyset$, then y_3 must be suffered from the disease A, and should be given treatment immediately. The sufferers y_2, y_4, y_5 are not necessarily ill with the disease and should be further considered by the doctor, and y_1 has nothing to do with the disease A.

Let $(U, \widetilde{\mathcal{C}})$ be a fuzzy covering approximation space, we call $\widetilde{P^-}$, $\widetilde{P^+}$ and $\overline{P^-}$, $\overline{P^+}$ the fuzzy covering approximation operators. $\widetilde{P^-}$, $\widetilde{P^+}$ are fuzzy approximation operators in the fuzzy environment, and $\overline{P^-}$, $\overline{P^+}$ are crisp approximation operators in the fuzzy environment.

4.3. Relationships between the two models and some other rough set models

The two types of fuzzy covering rough set models we defined above can be viewed as bridges linking covering rough set and fuzzy rough set theory.

If $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ is a fuzzy β-covering of U, we can define a fuzzy relation $\widetilde{R}$ on the universe U as:

$$\widetilde{R}(x, y) = \widetilde{N}_x^\beta(y), \quad x, y \in U,$$

then the fuzzy relation $\widetilde{R}$ induced by the fuzzy β-covering $\widetilde{\mathcal{C}}$ is related to all $\widetilde{C}_i \in \widetilde{\mathcal{C}}$. Thus the fuzzy covering rough set model defined in Section 4.1 can be view as a fuzzy rough set model proposed by Dubois and Prade in [5]:

$$(\widetilde{P^-}X)(x) = \wedge_{y \in U}[(1 - \widetilde{R}(x, y)) \vee X(y)], \quad x \in U,$$
$$(\widetilde{P^+}X)(x) = \vee_{y \in U}[\widetilde{R}(x, y) \wedge X(y)], \quad x \in U,$$

for each $X \in F(U)$.

Obviously, for each $\beta' < \beta$, $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ is also a fuzzy β'-covering of U. We can also define the fuzzy covering rough set model as follows:

$$(\widetilde{P'^-}X)(x) = \wedge_{y\in U}[(1-\widetilde{N}_x^{\beta'}(y)) \vee X(y)], \quad x \in U,$$
$$(\widetilde{P'^+}X)(x) = \vee_{y\in U}[\widetilde{N}_x^{\beta'}(y) \wedge X(y)], \quad x \in U,$$

for each $X \in F(U)$.

It follows from Proposition 3.4 that $\widetilde{N}_x^{\beta'} \subset \widetilde{N}_x^{\beta}$. Then we can easily obtain

$$\widetilde{P'^-}X \supset \widetilde{P^-}X, \quad \widetilde{P'^+}X \subset \widetilde{P^+}X$$

for each $X \in F(U)$. Thus, for each $X \in F(U)$, we define an order relation $\prec$ for every two lower and upper approximation operator pairs of X as:

$$(\widetilde{P'^-}X, \widetilde{P'^+}X) \prec (\widetilde{P^-}X, \widetilde{P^+}X) \text{ if and only if } \widetilde{P'^-}X \supset \widetilde{P^-}X \text{ and } \widetilde{P'^+}X \subset \widetilde{P^+}X.$$

We can obtain a series of fuzzy covering rough approximation pairs of $X \in F(U)$ varying with each $\beta' \in (0, \beta]$, each pair of which can be regarded as the model proposed by Dubois and Prade, and they can be arranged linearly according to order relation $\prec$. Thus, we can choose any pair through selecting the degree β' in practice.

The following proposition implies that, for each fixed $\beta \in (0, 1]$, the fuzzy covering rough set model defined in Section 4.2 can be viewed as a covering rough set model in Definition 2.2 which was proposed by Samanta and Chakraborty in [32].

Proposition 4.3. *Let $(U, \widetilde{\mathcal{C}})$ be a fuzzy covering approximation space with $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ being a fuzzy β-covering of U.*

i). Let $C_i = \{x : \widetilde{C}_i(x) \geq \beta\}$ $(i = 1, 2, \cdots, m)$, then $\mathcal{C} = \{C_1, C_2, \cdots, C_m\}$ is a covering of U;
ii). Let N_x denote the neighborhood of x in the covering approximation space $(U, \mathcal{C})$, then $\overline{N}_x^{\beta} = N_x$.

Proof. *i*). Since $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ is a fuzzy β-covering of U, then for each $x \in U$, there exists $\widetilde{C}_i$ such that $\widetilde{C}_i(x) \geq \beta$. Thus $x \in C_i$, and hence $\mathcal{C} = \{C_1, C_2, \cdots, C_m\}$ is a covering of U.

ii).

$$\begin{aligned}\overline{N}_x^{\beta} &= \{y \in U : \widetilde{N}_x^{\beta}(y) \geq \beta\} = \{y \in U : (\cap_{\widetilde{C}_i(x)\geq\beta}\widetilde{C}_i)(y) \geq \beta\} \\ &= \{y \in U : (\cap_{x\in\overline{C}_i}\widetilde{C}_i)(y) \geq \beta\} = \{y : x \in \overline{C}_i \to y \in \overline{C}_i,\ i = 1, 2, \cdots, m\} \\ &= \cap\{\overline{C_i} : x \in \overline{C_i}\} = N_x.\end{aligned}$$

Thus, for each fixed $\beta \in (0, 1]$, the fuzzy covering approximation space $(U, \widetilde{\mathcal{C}})$ induces a covering approximation space $(U, \mathcal{C})$, and the fuzzy covering rough set model defined in Subsection 4.2 can be viewed as a covering rough set model in the covering approximation space

$$\overline{P^-}(X) = \{x \in U : N_x \subset X\}, \quad \overline{P^+}(X) = \{x \in U : N_x \cap X \neq \emptyset\}$$

for each $X \in P(U)$. Therefore, the properties of the operators in Proposition 4.2 can be deduced directly from the properties of the operators in Definition 2.2, which have been discussed in Ref. [32].

Furthermore, both fuzzy covering rough sets defined in Section 4 can be viewed as generalizations of the covering rough set.

Suppose that $(U, \mathcal{C})$ is a covering approximation space, where $\mathcal{C} = \{C_1, C_2, \cdots, C_m\}$. For each i $(1 \leq i \leq m)$, $x \in U$ and $X \in P(U)$, if we set

$$\widetilde{C}_i(y) = \begin{cases} 1, & y \in C_i \\ 0, & y \notin C_i \end{cases}, \quad \widetilde{N}_x(y) = \begin{cases} 1, & y \in N_x \\ 0, & y \notin N_x \end{cases}, \quad \widetilde{X}(y) = \begin{cases} 1, & y \in X \\ 0, & y \notin X \end{cases}$$

for every $y \in U$, then every crisp subset of U can be consider as a fuzzy subset of U, and thus $N_x = \widetilde{N}_x^{\beta} = \overline{N}_x^{\beta}$ for every $x \in U$ and $\beta \in (0, 1]$. So we have

$$\widetilde{P^-}X = \overline{P^-}(X) = P^-(X), \quad \widetilde{P^+}X = \overline{P^+}(X) = P^+(X)$$

for each $X \in P(U)$. Therefore, the two fuzzy covering rough set models in Section 4 are both generalizations of the covering rough set models in Definition 2.2. □

5. Matrix representations of lower and upper approximation operators

In this section, we will present matrix representations of the lower and upper approximation operators defined in Definitions 4.1 and 4.2. The matrix representations of the approximation operators makes it possible to calculate the lower and upper approximations of subsets through the operations on matrices, which is algorithmic, and can easily be implemented through the computer.

Definition 5.1. Let $A=(a_{ik})_{n\times m}$ and $B=(b_{kj})_{m\times l}$ be two matrices. We define $C=A\cdot B=(c_{ij})_{n\times l}$ and $D=A*B=(d_{ij})_{n\times l}$ as follows:

$$c_{ij}=\vee_{k=1}^{m}(a_{ik}\wedge b_{kj}), i=1,2,\cdots,n,\ j=1,2,\cdots,l,$$
$$d_{ij}=\wedge_{k=1}^{m}[(1-a_{ik})\vee b_{kj}], i=1,2,\cdots,n,\ j=1,2,\cdots,l.$$

Definition 5.2. Let $U=\{x_1,x_2,\cdots,x_n\}$ be a finite universe and $\widetilde{\mathcal{C}}=\{\widetilde{C}_1,\widetilde{C}_2,\cdots,\widetilde{C}_m\}$ be a fuzzy β-covering of U. We call $M=(\widetilde{C}_j(x_i))_{n\times m}$ a matrix representation of $\widetilde{\mathcal{C}}$, and call the Boolean matrix $M_\beta=(t_{ij})_{n\times m}$ a β-matrix representation of $\widetilde{\mathcal{C}}$, where

$$t_{ij}=\begin{cases}1, & \widetilde{C}_j(x_i)\geq\beta\\ 0, & \widetilde{C}_j(x_i)<\beta.\end{cases}$$

Example 5.1. Let $(U,\widetilde{\mathcal{C}})$ be the fuzzy covering approximation space in Example 3.1. For two orders $\{\widetilde{C}_1,\widetilde{C}_2,\widetilde{C}_3,\widetilde{C}_4,\widetilde{C}_5\}$ and $\{\widetilde{C}_1,\widetilde{C}_5,\widetilde{C}_3,\widetilde{C}_4,\widetilde{C}_2\}$ of $\widetilde{\mathcal{C}}$, M, N are both matrix representations of $\widetilde{\mathcal{C}}$, while $M_{0.5}$, $N_{0.5}$ are both 0.5-matrix representations of $\widetilde{\mathcal{C}}$, where

$$M=\begin{pmatrix}0.6&0.1&0.1&0.3&0.3\\0.3&0.5&0.2&0.2&0.3\\0.2&0.4&0.5&0.2&0.2\\0.4&0.5&0.3&0.3&0.3\\0.3&0.3&0.3&0.6&0.7\\0.2&0.2&0.1&0.1&0.7\end{pmatrix},\quad N=\begin{pmatrix}0.6&0.3&0.1&0.3&0.1\\0.3&0.3&0.2&0.2&0.5\\0.2&0.2&0.5&0.2&0.4\\0.4&0.3&0.3&0.3&0.5\\0.3&0.7&0.3&0.6&0.3\\0.2&0.7&0.1&0.1&0.2\end{pmatrix},$$

and

$$M_{0.5}=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix},\quad N_{0.5}=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&1\\0&0&1&0&0\\0&0&0&0&1\\0&1&0&1&0\\0&1&0&0&0\end{pmatrix}.$$

From this example we can see that for a fixed order of all elements of U, different orders of $\widetilde{\mathcal{C}}$ lead to different matrix representations of $\widetilde{\mathcal{C}}$, and the matrices can be transformed into each other by list exchanges.

Proposition 5.1. *Let $U=\{x_1,x_2,\cdots,x_n\}$ be a finite universe of which the order of elements is given, and $\widetilde{\mathcal{C}}=\{\widetilde{C}_1,\widetilde{C}_2,\cdots,\widetilde{C}_m\}$ be a fuzzy β-covering of U. If, under two different orders of $\widetilde{\mathcal{C}}$, M, N are two matrix representations of $\widetilde{\mathcal{C}}$, and M_β, N_β are two β-matrix representations of $\widetilde{\mathcal{C}}$ respectively. Then $M_\beta*M^T=N_\beta*N^T$ and $M_\beta*M_\beta^T=N_\beta*N_\beta^T$.*

Proof. M and N can be transformed into each other through list exchanges, so do M_β and N_β. Without loss of generality, we assume that they can be denoted by block matrix columns as

$$M=\{\alpha_1,\cdots,\alpha_p,\cdots,\alpha_q,\cdots,\alpha_m\},\ N=\{\alpha_1,\cdots,\alpha_q,\cdots,\alpha_p,\cdots,\alpha_m\},$$
$$M_\beta=\{t_1,\cdots,t_p,\cdots,t_q,\cdots,t_m\},\ N_\beta=\{t_1,\cdots,t_q,\cdots,t_p,\cdots,t_m\}.$$

Suppose that $M_\beta*M^T=(a_{ij})_{n\times n}$ and $N_\beta*N^T=(b_{ij})_{n\times n}$. Then

$$
\begin{aligned}
a_{ij} &= (t_{j1}, \cdots, t_{jp}, \cdots, t_{jq}, \cdots, t_{jn}) * (\widetilde{C}_1(x_i), \cdots, \widetilde{C}_p(x_i), \cdots, \widetilde{C}_q(x_i), \cdots, \widetilde{C}_n(x_i))^T \\
&= \wedge_{k=1}^{m}[(1 - t_{jk}) \vee \widetilde{C}_k(x_i)] \\
&= (t_{j1}, \cdots, t_{jq}, \cdots, t_{jp}, \cdots, t_{jn}) * (\widetilde{C}_1(x_i), \cdots, \widetilde{C}_q(x_i), \cdots, \widetilde{C}_p(x_i), \cdots, \widetilde{C}_n(x_i))^T \\
&= b_{ij},
\end{aligned}
$$

$i, j = 1, 2, \cdots, n$. Thus, we have $M_\beta * M^T = N_\beta * N^T$.

In a similar way, we can obtain the proof of $M_\beta * M_\beta^T = N_\beta * N_\beta^T$. □

Proposition 5.2. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universe of which the order of elements is given, and $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ be a fuzzy β-covering of U. If M is a matrix representation of $\widetilde{\mathcal{C}}$, and M_β is a β-matrix representation of $\widetilde{\mathcal{C}}$, then*

$$M_\beta * M^T = (\widetilde{N}_{x_i}^{\beta}(x_j))_{n\times n}.$$

Proof. As $\widetilde{\mathcal{C}}$ is a fuzzy β-covering of U, and $M_\beta = (t_{ik})_{n\times m}$ is a β-matrix representation of $\widetilde{\mathcal{C}}$, for each i $(1 \le i \le n)$, there is k $(1 \le k \le m)$ such that $t_{ik} = 1$. Suppose that $M_\beta * M^T = (c_{ij})_{n\times n}$, then

$$
\begin{aligned}
c_{ij} &= \wedge_{k=1}^{m}[(1 - t_{ik}) \vee \widetilde{C}_k(x_j)] = \wedge_{t_{ik}=1}[(1 - t_{ik}) \vee \widetilde{C}_k(x_j)] \\
&= \wedge_{t_{ik}=1}\widetilde{C}_k(x_j) = \wedge_{\widetilde{C}_k(x_i)\ge\beta}\widetilde{C}_k(x_j) \\
&= (\cap_{\widetilde{C}_k(x_i)\ge\beta}\widetilde{C}_k)(x_j) = \widetilde{N}_{x_i}^{\beta}(x_j), \quad i, j = 1, 2, \cdots, n.
\end{aligned}
$$

Thus,

$$M_\beta * M^T = (\widetilde{N}_{x_i}^{\beta}(x_j))_{n\times n}. \qquad \square$$

Example 5.2. We can calculate $M_\beta * M^T$ and $N_\beta * N^T$ in Example 5.1 as follows.

$$
M_\beta * M^T = N_\beta * N^T = \begin{pmatrix}
0.6 & 0.3 & 0.2 & 0.4 & 0.3 & 0.2 \\
0.1 & 0.5 & 0.4 & 0.5 & 0.3 & 0.2 \\
0.1 & 0.2 & 0.5 & 0.3 & 0.3 & 0.1 \\
0.1 & 0.5 & 0.4 & 0.5 & 0.3 & 0.2 \\
0.3 & 0.2 & 0.2 & 0.3 & 0.6 & 0.1 \\
0.3 & 0.3 & 0.2 & 0.3 & 0.7 & 0.7
\end{pmatrix} = (\widetilde{N}_{x_i}^{\beta}(x_j))_{n\times n}.
$$

Next we show that the calculations of the lower and upper approximations $\widetilde{P^-}X$ and $\widetilde{P^+}X$ of fuzzy set X can be converted to operations on matrices.

Proposition 5.3. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universe of which the order of elements is given, and $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ be a fuzzy β-covering of U. If M is a matrix representation of $\widetilde{\mathcal{C}}$, and M_β is β-matrix representation of $\widetilde{\mathcal{C}}$, then for every $X \in F(U)$, we have*

$$\widetilde{P^-}X = (M_\beta * M^T) * X, \quad \widetilde{P^+}X = (M_\beta * M^T) \cdot X.$$

Proof. For each i $(1 \le i \le n)$, we have

$$
\begin{aligned}
&((M_\beta * M^T) * X)(x_i) = \wedge_{k=1}^{m}[(1 - \widetilde{N}_{x_i}^{\beta}(x_k)) \vee X(x_k)] = (\widetilde{P^-}X)(x_i), \\
&((M_\beta * M^T) \cdot X)(x_i) = \vee_{k=1}^{m}(\widetilde{N}_{x_i}^{\beta}(x_k) \wedge X(x_k)) = (\widetilde{P^+}X)(x_i).
\end{aligned}
$$

Hence, $\widetilde{P^-}X = (M_\beta * M^T) * X$ and $\widetilde{P^+}X = (M_\beta * M^T) \cdot X$ can be followed. □

Example 5.3. The $\widetilde{P^-}X$ and $\widetilde{P^+}X$ in Example 4.1 can be calculated as follows.

$$(M_\beta * M^T) * X = \begin{pmatrix} 0.6 & 0.3 & 0.2 & 0.4 & 0.3 & 0.2 \\ 0.1 & 0.5 & 0.4 & 0.5 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.5 & 0.3 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.4 & 0.5 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.2 & 0.3 & 0.6 & 0.1 \\ 0.3 & 0.3 & 0.2 & 0.3 & 0.7 & 0.7 \end{pmatrix} * \begin{pmatrix} 0.6 \\ 0.4 \\ 0.3 \\ 0.5 \\ 0.7 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.7 \\ 0.4 \end{pmatrix} = \widetilde{P^-} X,$$

$$M_\beta * M^T) \cdot X = \begin{pmatrix} 0.6 & 0.3 & 0.2 & 0.4 & 0.3 & 0.2 \\ 0.1 & 0.5 & 0.4 & 0.5 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.5 & 0.3 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.4 & 0.5 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.2 & 0.3 & 0.6 & 0.1 \\ 0.3 & 0.3 & 0.2 & 0.3 & 0.7 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.4 \\ 0.3 \\ 0.5 \\ 0.7 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.3 \\ 0.3 \\ 0.6 \\ 0.7 \end{pmatrix} = \widetilde{P^+} X.$$

We use χ_X to denote the characteristic function of the crisp subset $X \subset U$. In the following, we discuss the matrix representations of the lower and upper approximations $\overline{P^-}(X)$ and $\overline{P^+}(X)$ of every crisp subset X.

Proposition 5.4. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universe of which the order of elements is given, and $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ be a fuzzy β-covering of U. If M_β is a β-matrix representation of $\widetilde{\mathcal{C}}$, then*

$$M_\beta * M_\beta^T = (\chi_{\overline{N}^\beta_{x_i}}(x_j))_{n\times n}.$$

Proof. Denote $M_\beta * M_\beta^T = (d_{ij})_{n\times n}$. If $d_{ij} = 1$, then $\wedge_{k=1}^m [(1 - t_{ik}) \vee t_{jk}] = 1$. This implies that if $t_{ik} = 1$ then $t_{jk} = 1$. Therefore, $\widetilde{C}_k(x_i) \geq \beta$ leads to $\widetilde{C}_k(x_j) \geq \beta$, and hence $x_j \in \overline{N}^\beta_{x_i}$, i.e., $\chi_{\overline{N}^\beta_{x_i}}(x_j) = 1 = d_{ij}$.

If $d_{ij} = 0$, then $\wedge_{k=1}^m [(1 - t_{ik}) \vee t_{jk}] = 0$, which implies that there is k such that $t_{jk} = 0$ and $t_{ik} = 1$. This indicates that $\widetilde{C}_k(x_j) < \beta$ and $\widetilde{C}_k(x_i) \geq \beta$. So we have $x_j \notin \overline{N}^\beta_{x_i}$, namely, $\chi_{\overline{N}^\beta_{x_i}}(x_j) = 0 = d_{ij}$.

Thus, we have proved $M_\beta * M_\beta^T = (\chi_{\overline{N}^\beta_{x_i}}(x_j))_{n\times n}$. □

Example 5.4. For M_β and N_β in Example 5.1, we have

$$M_\beta * M_\beta^T = N_\beta * N_\beta^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} = (\chi_{\overline{N}^\beta_{x_i}}(x_j))_{n\times n}.$$

Proposition 5.5. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universe of which the order of elements is given, and $\widetilde{\mathcal{C}} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ be a fuzzy β-covering of U. If M_β is a β-matrix representation of $\widetilde{\mathcal{C}}$, then for every $X \in P(U)$, we have*

$$\chi_{\overline{P^-}(X)} = (M_\beta * M_\beta^T) * \chi_X, \quad \chi_{\overline{P^+}(X)} = (M_\beta * M_\beta^T) \cdot \chi_X.$$

Proof.

$$\begin{aligned} ((M_\beta * M_\beta^T) * \chi_X)(x_i) = 1 \;&\Leftrightarrow\; \wedge_{k=1}^m [(1 - \chi_{\overline{N}^\beta_{x_i}}(x_k) \vee \chi_X(x_k)] = 1 \\ \Leftrightarrow\;& \chi_{\overline{N}^\beta_{x_i}}(x_k) = 1 \rightarrow \chi_X(x_k) = 1,\ k = 1, 2, \cdots, m \\ \Leftrightarrow\;& x_k \in \overline{N}^\beta_{x_i} \rightarrow x_k \in X,\ k = 1, 2, \cdots, m \\ \Leftrightarrow\;& \overline{N}^\beta_{x_i} \subset X \;\Leftrightarrow\; x_i \in \overline{P^-}(X) \;\Leftrightarrow\; \chi_{\overline{P^-}(X)}(x_i) = 1, \end{aligned}$$

and

$$((M_\beta * M_\beta^T)\cdot \chi_X)(x_i)=1 \;\Leftrightarrow\; \vee_{k=1}^m(\chi_{\overline{N}_{x_i}^\beta}(x_k)\wedge \chi_X(x_k))=1$$
$$\Leftrightarrow\quad \exists k,\;\; \chi_{\overline{N}_{x_i}^\beta}(x_k)=\chi_X(x_k)=1 \;\Leftrightarrow\; \exists k,\;\; x_k\in \overline{N}_{x_i}^\beta\cap X$$
$$\Leftrightarrow\quad \overline{N}_{x_i}^\beta\cap X\neq\emptyset \;\Leftrightarrow\; x_i\in\overline{P^+}(X) \;\Leftrightarrow\; \chi_{\overline{P^+}(X)}(x_i)=1,$$

then we have

$$\chi_{\overline{P^-}(X)}=(M_\beta * M_\beta^T)*\chi_X,\quad \chi_{\overline{P^+}(X)}=(M_\beta * M_\beta^T)\cdot\chi_X. \qquad \square$$

Example 5.5. For $\overline{P^-}(X)$ and $\overline{P^+}(X)$ in Example 4.2, there are

$$(M_\beta * M^T)*\chi_X=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&1&0&0\\0&0&1&0&0&0\\0&1&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&1&1\end{pmatrix}*\begin{pmatrix}1\\0\\0\\1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\\1\\0\end{pmatrix}=\chi_{\overline{P^-}(X)},$$

$$(M_\beta * M_\beta^T)\cdot\chi_X=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&1&0&0\\0&0&1&0&0&0\\0&1&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&1&1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\\1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\0\\1\\1\\1\end{pmatrix}=\chi_{\overline{P^+}(X)}.$$

6. Generalizations of the models over fuzzy lattice

As we discussed in Section 4.3, the two fuzzy covering rough set models in Section 4 are both generalizations of the covering rough set models in Definition 2.2. In the following, we further generalize the models and their matrix representations to L-fuzzy covering rough sets which are defined over fuzzy lattices.

6.1. L-fuzzy β-covering and L-fuzzy β-neighborhoods

Definition 6.1. Let $L=(L,\wedge,\vee,',\mathbf{0},\mathbf{1})$ be a fuzzy lattice. Suppose that U is an arbitrary universal set and $\widehat{F}(U)$ is the collection of all the L-fuzzy sets from U to L, i.e., the set of all functions from U to L. For some $\beta>\mathbf{0}$ ($\beta\in L$), we call $\widehat{\mathcal{C}}=\{\widehat{C}_1,\widehat{C}_2,\cdots,\widehat{C}_m\}$ a L-fuzzy β-covering of U, if $(\sqcup_{i=1}^m\widehat{C}_i)(x)\geq\beta$ for each $x\in U$, where $\widehat{C}_i\in\widehat{F}(U)$ ($i=1,2,\cdots,m$). We also call $(U,\widehat{\mathcal{C}})$ a L-fuzzy covering approximation space.

Definition 6.2. Let $(U,\widehat{\mathcal{C}})$ be a L-fuzzy covering approximation space, where $\widehat{\mathcal{C}}=\{\widehat{C}_1,\widehat{C}_2,\cdots,\widehat{C}_m\}$ is a L-fuzzy β-covering of U. For each $x\in U$, we define the L-fuzzy β-neighborhood $\widehat{N}_x^\beta$ and β-neighborhood $\overline{N}_x^\beta$ of x as

$$\widehat{N}_x^\beta=\sqcap\{\widehat{C}_i:\widehat{C}_i(x)\geq\beta\},$$
$$\overline{N}_x^\beta=\{y\in U:\widehat{N}_x^\beta(y)\geq\beta\}.$$

The properties of L-fuzzy β-neighborhoods and β-neighborhoods and their proofs are similar to the properties of fuzzy β-neighborhoods and β-neighborhoods discussed in Section 3. We next list some properties and omit their proofs.

Proposition 6.1. *Suppose $(U,\widehat{\mathcal{C}})$ is a L-fuzzy covering approximation space, where $\widehat{\mathcal{C}}=\{\widehat{C}_1,\widehat{C}_2,\cdots,\widehat{C}_m\}$ is a L-fuzzy β-covering of U for some $\beta\in L-\{\mathbf{0}\}$. Then the following are correct*

i). $\widehat{N}_x^\beta(x)\geq\beta$ *for each* $x\in U$,
ii). $\forall x,y,z\in U$, *if* $\widehat{N}_x^\beta(y)\geq\beta$ *and* $\widehat{N}_y^\beta(z)\geq\beta$, *then* $\widehat{N}_x^\beta(z)\geq\beta$,

iii). *for each* $i \in \{1, 2, \cdots, m\}$, $\sqcup\{\widehat{N}_x^\beta : \widehat{C}_i(x) \geq \beta, x \in U\} \ll \widehat{C}_i$,
iv). *if* $\beta_1 < \beta_2$, *then* $\widehat{N}_x^{\beta_1} \ll \widehat{N}_x^{\beta_2}$ *for all* $x \in U$,
v). $x \in \overline{N}_x^\beta(x)$ *for each* $x \in U$,
vi). $\forall x, y \in U$, *if* $x \in \overline{N}_y^\beta$ *then* $\overline{N}_x^\beta \subset \overline{N}_y^\beta$.

Example 6.1. Let $U = \{x_1, x_2, x_3\}$ be the universe, $L = \{1, 2, 3, 6\}$ be the set of all positive divisors of 6, and $(L, \wedge, \vee, ', \mathbf{0}, \mathbf{1})$ be a fuzzy lattice in which $\mathbf{0} = 1$, $\mathbf{1} = 6$, the join $a \vee b$ and the meet $a \wedge b$ be the least common multiple and greatest common divisor of a, b respectively. For the order-reversing involution $'$

$$1' = 6, \quad 2' = 3, \quad 3' = 2, \quad 6' = 1$$

and the following L-fuzzy 2-covering $\widehat{\mathcal{C}} = \{\widehat{C}_1, \widehat{C}_2, \widehat{C}_3\}$ of U

	$\widetilde{C}_1$	$\widetilde{C}_2$	$\widetilde{C}_3$
x_1	3	2	6
x_2	1	3	2
x_3	3	6	1

we have $\widehat{N}_{x_1}^2 = \widehat{C}_2 \sqcap \widehat{C}_3$, $\widehat{N}_{x_2}^2 = \widehat{C}_3$ and $\widehat{N}_{x_3}^2 = \widehat{C}_2$. Furthermore, $\widehat{N}_{x_i}^2$ and $\overline{N}_{x_i}^2$ of x_i $(i = 1, 2, 3)$ can be calculated as follows.

	x_1	x_2	x_3
$\widehat{N}_{x_1}^2$	2	1	1
$\widehat{N}_{x_2}^2$	6	2	1
$\widehat{N}_{x_3}^2$	2	3	6

$$\overline{N}_{x_1}^2 = \{x_1\}, \quad \overline{N}_{x_2}^2 = \{x_1, x_2\}, \quad \overline{N}_{x_3}^2 = \{x_1, x_3\}.$$

6.2. Two types of L-fuzzy covering rough set models

Definition 6.3. Let $(U, \widehat{\mathcal{C}})$ be a L-fuzzy covering approximation space with $\widehat{\mathcal{C}} = \{\widehat{C}_1, \widehat{C}_2, \cdots, \widehat{C}_m\}$ being a L-fuzzy β-covering of U. For each L-fuzzy subset $X \in \widehat{F}(U)$, we define the lower approximation $\widehat{P^-}X$ and the upper approximation $\widehat{P^+}X$ of X as:

$$(\widehat{P^-}X)(x) = \wedge_{y \in U}[(\mathbf{1} - \widehat{N}_x^\beta(y)) \vee X(y)], \quad (\widehat{P^+}X)(x) = \vee_{y \in U}[\widehat{N}_x^\beta(y) \wedge X(y)], \quad x \in U.$$

For each crisp subset $X \in P(U)$, we define the lower approximation $\overline{P^-}(X)$ and the upper approximation $\overline{P^+}(X)$ of X as:

$$\overline{P^-}(X) = \{x : \overline{N}_x^\beta \subset X\}, \quad \overline{P^+}(X) = \{x : \overline{N}_x^\beta \cap X \neq \emptyset\}.$$

For an L-fuzzy subset $X \in \widehat{F}(U)$ and the crisp subset $Y \in P(U)$, if $\widehat{P^-}X \neq \widehat{P^+}X$, as well as $\overline{P^-}(Y) \neq \overline{P^+}(Y)$, we call both X and Y the L-fuzzy covering rough sets. The lower and upper approximation operators $\widehat{P^-}$, $\widehat{P^+}$ and $\overline{P^-}$, $\overline{P^+}$ have all the properties in Propositions 4.1 and 4.2 respectively, so we omit them here.

Example 6.2. Let $(U, \widehat{\mathcal{C}})$ be the L-fuzzy 2-covering approximation space in Example 6.1. For the L-fuzzy subset $X = \frac{2}{x_1} + \frac{3}{x_2} + \frac{3}{x_3}$ and the crisp subset $Y = \{x_1, x_3\}$, we have:

$$\widehat{P^-}X = \frac{6}{x_1} + \frac{1}{x_2} + \frac{6}{x_3}, \quad \widehat{P^+}X = \frac{2}{x_1} + \frac{2}{x_2} + \frac{6}{x_3};$$
$$\overline{P^-}(Y) = \{x_1, x_3\}, \quad \overline{P^+}(Y) = \{x_1, x_2, x_3\}.$$

6.3. Matrix representations of lower and upper approximations of L-fuzzy covering rough sets

Definition 6.4. Let $L = (L, \wedge, \vee, ', \mathbf{0}, \mathbf{1})$ be a fuzzy lattice, $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universe and $\widehat{\mathcal{C}} = \{\widehat{C}_1, \widehat{C}_2, \cdots, \widehat{C}_m\}$ be a L-fuzzy β-covering of U. We call $M = (\widehat{C}_j(x_i))_{n\times m}$ a matrix representation of $\widehat{\mathcal{C}}$, and call the matrix $M_\beta = (t_{ij})_{n\times m}$ a β-matrix representation of $\widehat{\mathcal{C}}$, where

$$t_{ij} = \begin{cases} \mathbf{1}, & \widehat{C}_j(x_i) \geq \beta \\ \mathbf{0}, & \widehat{C}_j(x_i) < \beta \end{cases}.$$

For each $X \subset U$, we set $\widehat{\chi}_X = (t_1, t_2, \cdots, t_n)^T$, where $t_i = \begin{cases} \mathbf{1}, & x_i \in X \\ \mathbf{0}, & x_i \notin X \end{cases}, i = 1, 2, \cdots, n$.

Definition 6.5. Let $L = (L, \wedge, \vee, ', \mathbf{0}, \mathbf{1})$ be a fuzzy lattice, $A = (a_{ik})_{n\times m}$ and $B = (b_{kj})_{m\times l}$ be two lattice valued matrices, i.e., $a_{ik}, b_{kj} \in L$. We define $C = A \cdot B = (c_{ij})_{n\times l}$ and $D = A * B = (d_{ij})_{n\times l}$ as follows:

$$c_{ij} = \vee_{k=1}^{m}(a_{ik} \wedge b_{kj}), i = 1, 2, \cdots, n, j = 1, 2, \cdots, l,$$
$$d_{ij} = \wedge_{k=1}^{m}[a'_{ik} \vee b_{kj}], i = 1, 2, \cdots, n, j = 1, 2, \cdots, l.$$

The proofs of the following two propositions are similar to that of Propositions 5.1, 5.2, 5.3, 3.4 and 5.5 respectively.

Proposition 6.2. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be an ordered finite universe, and $\widehat{\mathcal{C}} = \{\widehat{C}_1, \widehat{C}_2, \cdots, \widehat{C}_m\}$ be a L-fuzzy β-covering of U. If, under two different orders of $\widehat{\mathcal{C}}$, M and N are two matrix representations of $\widehat{\mathcal{C}}$, M_β and N_β are two β-matrix representations of $\widehat{\mathcal{C}}$, then*

$$M_\beta * M^T = N_\beta * N^T = (\widehat{N}_{x_i}^{\beta}(x_j))_{n\times n},$$
$$M_\beta * M_\beta^T = N_\beta * N_\beta^T = (\chi_{\overline{N}_{x_i}^{\beta}}(x_j))_{n\times n}.$$

Proposition 6.3. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be an ordered finite universe of which the order of elements is given, and $\widehat{\mathcal{C}} = \{\widehat{C}_1, \widehat{C}_2, \cdots, \widehat{C}_m\}$ be a L-fuzzy β-covering of U. If M is a matrix representation of $\widehat{\mathcal{C}}$ and M_β is β-matrix representation of $\widehat{\mathcal{C}}$, then for every $X \in \widehat{F}(U)$ and $Y \in P(U)$, we have*

$$\widehat{P^-}X = (M_\beta * M^T) * X, \quad \widehat{P^+}X = (M_\beta * M^T) \cdot X.$$
$$\widehat{\chi}_{\overline{P^-}(Y)} = (M_\beta * M_\beta^T) * \widehat{\chi}_Y, \quad \widehat{\chi}_{\overline{P^+}(Y)} = (M_\beta * M_\beta^T) \cdot \widehat{\chi}_Y.$$

Example 6.3. In Example 6.2, since $\mathbf{1} = 6$ and $\mathbf{0} = 1$, we can calculate $\widehat{P^-}X$, $\widehat{P^+}X$ and $\overline{P^-}(Y)$, $\overline{P^+}(Y)$ as follows.

$$(M_2 * M^T) * X = \left(\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix} * \begin{pmatrix} 3 & 1 & 3 \\ 2 & 3 & 6 \\ 6 & 2 & 1 \end{pmatrix}\right) * \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$
$$= \left(\begin{pmatrix} 1 & 6 & 6 \\ 1 & 1 & 6 \\ 1 & 6 & 1 \end{pmatrix} * \begin{pmatrix} 3 & 1 & 3 \\ 2 & 3 & 6 \\ 6 & 2 & 1 \end{pmatrix}\right) * \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ 2 & 3 & 6 \end{pmatrix} * \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix} = \widehat{P^-}X,$$

$$(M_2 * M^T) \cdot X = \left(\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix} * \begin{pmatrix} 3 & 1 & 3 \\ 2 & 3 & 6 \\ 6 & 2 & 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$
$$= \left(\begin{pmatrix} 1 & 6 & 6 \\ 1 & 1 & 6 \\ 1 & 6 & 1 \end{pmatrix} * \begin{pmatrix} 3 & 1 & 3 \\ 2 & 3 & 6 \\ 6 & 2 & 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ 6 & 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} = \widehat{P^+}X.$$

$$\begin{aligned} (M_2 * M_2^T) * \widehat{\chi}_Y &= (\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix} * \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}^T) * \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \\ &= (\begin{pmatrix} 1 & 6 & 6 \\ 1 & 1 & 6 \\ 1 & 6 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 6 & 6 \\ 1 & 1 & 6 \\ 1 & 6 & 1 \end{pmatrix}^T) * \begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 1 & 1 \\ 6 & 6 & 1 \\ 6 & 1 & 6 \end{pmatrix} * \begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \\ &= \widehat{\chi}_{\overline{P^-}(Y)}, \end{aligned}$$

$$\begin{aligned} (M_2 * M_2^T) \cdot \widehat{\chi}_Y &= (\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix} * \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}^T) \cdot \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix} = \widehat{\chi}_{\overline{P^+}(Y)}. \end{aligned}$$

Thus, we have:

$$\begin{aligned} &\widehat{P^-}X = \frac{6}{x_1} + \frac{1}{x_2} + \frac{6}{x_3}, \quad \widehat{P^+}X = \frac{2}{x_1} + \frac{2}{x_2} + \frac{6}{x_3}; \\ &\overline{P^-}(Y) = \{x_1, x_3\}, \quad \overline{P^+}(Y) = \{x_1, x_2, x_3\}. \end{aligned}$$

7. Conclusion

We defined two fuzzy covering rough set models which are more general than the existing ones and can be regarded as bridges linking covering rough set theory and fuzzy rough set theory. The encouraging thing is that we also found the matrix representations of the defined lower and upper approximation operators, which made it possible to calculate the lower and upper approximations of subsets through computer. Moreover, the models and their matrix representations were further generalized to L-fuzzy covering rough sets which are defined over fuzzy lattices. The new concepts and the matrix tool introduced in this work may be helpful to the investigation of fuzzy rough set theory and its applications.

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