Fuzzy Sets and Systems

A survey on fuzzy β-covering rough set models, measurement systems, and applications: Advances and prospects from 2016--2026

--Manuscript Draft--

Manuscript Number:	FSS-D-25-01341
Article Type:	Review Article (REV)
Section/Category:	
Keywords:	

A survey on fuzzy β-covering rough set models, measurement systems, and applications: Advances and prospects from 2016–2026

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Abstract

Uncertain information processing in the big data era faces dual challenges of data fuzziness and knowledge uncertainty. Classical rough set theory provides mathematical rigor through equivalence relations, though these relations cannot adequately represent fuzzy boundaries. Fuzzy set theory describes graded membership but lacks systematic granularity analysis mechanisms. The combination into fuzzy rough sets faces limitations from conventional covering models. The introduction of fuzzy β-covering rough sets (β-FCRS) in 2016 added a confidence-level parameter β that enables flexible granularity regulation, leading to increased development of model variants. However, existing literature lacks surveys that systematically integrate theoretical development, model classification, measurement systems, and application trends. This paper presents a systematic analysis of the decade-long development with five main contributions: (1) a hierarchical theoretical framework tracing evolution from classical covering to fuzzy β-covering approximation spaces, examining core operators, axiomatic foundations, and reduction principles; (2) a classification of eleven model variants organized into neighborhood-driven, logical-operator-enhanced, and structurally-expanded categories, with examination of mathematical foundations, development motivations, and applicability; (3) a three-level measurement system consisting of Choquet-integral-based fuzzy measures, noise-tolerant discrimination indexes, and variable-precision distinguishability indicators; (4) analysis of applications in feature selection, multi-attribute decision-making, outlier detection, and three-way decisions, examining how practical requirements have influenced theoretical developments; and (5) bibliometric analysis of core publications identifying research communities, temporal patterns, and interdisciplinary trends, while assessing theoretical limitations, computational challenges, and interpretability issues. This survey concludes that fuzzy β-covering rough set theory provides a parametric framework linking fuzzy sets, rough sets, and multigranular analysis. Further development of unified axiomatic frameworks, efficient algorithms, and enhanced interpretability is needed to support broader practical adoption.

Keywords: Fuzzy rough set; Fuzzy β-covering; Choquet-integral-based fuzzy measure; Bibliographical analysis.

1. Introduction

A systematic exposition of fuzzy β-covering rough set (β-FCRS) theory [1], which extends fuzzy rough sets (FRSs) [2–4] and covering-based rough set (CRS) models [5–7], constitutes the focal point of this survey. This theoretical paradigm has emerged as a potent instrument for mitigating inherent deficiencies within classical frameworks for uncertain information processing. Initially, the dual complexities plaguing modern decision-making systems are scrutinized. Subsequently, antecedent theoretical models, including conventional rough set theory and its covering-based variants, are critically evaluated, elucidating their respective merits and limitations. The discourse ultimately synthesizes contemporary research imperatives and delineates the principal scholarly contributions [8, 9].

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1.1. Background and motivations

The advent of the big data era has precipitated significant challenges in uncertain information processing, wherein decision-making systems must simultaneously contend with dual complexities of data fuzziness and knowledge uncertainty [10, 11]. Data fuzziness emanates from measurement instrument imprecision, subjective human judgments, and linguistic vagueness, thus necessitating mathematical frameworks capable of capturing graded memberships and boundary ambiguities [12]. Concurrently, knowledge uncertainty arises from incomplete information systems, complex attribute interdependencies, and epistemic limitations in discerning object distinctions, thereby demanding robust mechanisms for approximating concepts under granularity constraints [13]. These intertwined challenges have exposed fundamental limitations inherent in classical computational intelligence paradigms, compelling the investigation of more expressive and adaptive theoretical foundations.

Classical rough set theory, pioneered by Pawlak in 1982 [14], established a rigorous mathematical apparatus for approximate reasoning through the core notion of indiscernibility equivalence relations. By partitioning universes into crisp granules and defining lower and upper approximations, this framework provides elegant tools for knowledge reduction and rule extraction in nominal data [15–17]. However, its foundational reliance on binary equivalence relations renders it intrinsically inadequate for characterizing fuzzy boundaries inherent in continuous or overlapping data distributions. As modern applications increasingly involve high-dimensional heterogeneous datasets with nuanced semantic structures-such as medical imaging, sensor networks, and social media analytics-the rigid all-or-nothing membership criterion of classical rough sets becomes a critical bottleneck, often producing overly coarse approximations that obscure meaningful patterns [18, 19]. Consequently, while classical rough sets excel in categorical data analysis, their deterministic nature fundamentally constrains applicability to problems requiring graded decision boundaries [20, 21].

Conversely, Zadeh's fuzzy set theory (1965) [22] offers a powerful alternative for modeling graded membership through continuous-valued functions, thereby capturing gradual transitions between categories that characterize human cognitive processes. Its extensive algebraic infrastructure, including t-norms, fuzzy implications, and aggregation operators, provides remarkable flexibility in representing vague information. Nevertheless, fuzzy set theory lacks systematic mechanisms for analyzing knowledge granularity and approximation structures, focusing primarily on pointwise membership without accounting for collective discernibility capabilities of attribute sets [23]. This absence of granular analysis tools limits its applicability to knowledge discovery tasks where boundary definitions depend on multi-attribute synergies rather than isolated membership degrees. Thus, fuzzy set theory demonstrates strength in modeling vagueness but weakness in structured knowledge approximation.

The amalgamation of these two paradigms into FRSs theory during the 1990s represented a significant theoretical advance, successfully embedding fuzzy membership within approximation spaces. Early models demonstrated promising results in feature selection and decision analysis. However, conventional FCRSs exhibited critical limitations in parametric adaptability and approximation precision [24–26]. Specifically, these models employed static covering structures with fixed membership thresholds, offering no mechanism to dynamically regulate granularity in response to varying data quality or confidence requirements. Consequently, they remained sensitive to boundary noise, struggled with heterogeneous data fusion, and lacked hierarchical analysis capabilities essential for complex systems modeling. This theoretical impasse necessitated a fundamental reconceptualization of how fuzzy granularity could be flexibly integrated with covering-based approximations.

1.2. Evolutionary path: From partitions to fuzzy β-fuzzy coverings

A fundamental development emerged in 2016 when Ma [1] introduced the concept of β-FCRSs within the fuzzy lattice framework, thereby establishing a new approach for parametric granularity control. By embedding a confidence-level parameter $\beta \in (0, 1]$ that dynamically activates covering elements only upon attaining specified membership thresholds, this innovation transformed rigid binary membership determination into a flexible, context-aware criterion. This fundamental restructuring of the approximation paradigm enabled fine-grained regulation of covering granularity while preserving semantic interpretability of rough set boundaries [27, 28]. Simultaneously, Yang and Hu [29] generalized this framework to L-β-FCRS models, enhancing algebraic expressiveness through lattice-ordered relations and establishing the theoretical cornerstone for fuzzy β-covering approximation spaces (termed β-FCASs). These foundational contributions [1, 29–31] achieved a decisive shift from qualitative covering structures to quantitatively parameterized fuzzy granulation, catalyzing a substantial expansion of research activity from a mere three papers in 2016 to a projected corpus exceeding one hundred publications by 2026 [32], with an annual growth rate surpassing 15%.

The four-stage evolutionary path from classical rough sets to β-FCRSs is explicitly traced to provide a complete contextual understanding.

Stage 1: Rough sets based on partitions [14]. Pawlak's original formulation employs equivalence relations to partition the universe into disjoint granules, forming the foundation for crisp approximation spaces.

Stage 2: Rough sets based on coverings (CRSs) [33, 34]. CRS models extend partitions by allowing overlapping granules while maintaining crisp membership criteria, providing more flexible approximation spaces.

Stage 3: Fuzzy rough sets based on fuzzy coverings (FCRSs) [35, 36]. FCRS models incorporate graded membership through fuzzy sets, capturing gradual transitions between categories while retaining binary activation mechanisms for covering elements.

Stage 4: Fuzzy rough sets based on β-fuzzy coverings (β-FCRSs) [1, 29]. The β-FCRS framework introduces parametric confidence thresholds, enabling dynamic granularity control and transforming static covering structures into adaptive approximation spaces.

Central to this evolution are several fundamental concepts requiring explicit clarification:

Fuzzy partitions (FPs) [22, 37].: A decomposition of universe U into fuzzy granules $\{A_1, A_2, \ldots, A_n\}$ where each object $x \in U$ may belong to multiple granules with varying membership degrees $A_i(x) \in [0, 1]$, capturing the gradual nature of concept boundaries.

Fuzzy coverings (FCs) [35, 36].: A collection of fuzzy sets $\widetilde{C} = \{\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_m\}$ on universe U such that $(\bigcup_{i=1}^{m} \widetilde{C}_i)(x) \geq 0$ for all $x \in U$, allowing objects to be characterized by multiple overlapping fuzzy granules with varying membership degrees.

Fuzzy β-coverings (β-FCs) [1].: A fuzzy covering where covering elements are activated for approximation only when their membership attains a specified confidence threshold $\beta \in (0, 1]$, satisfying $(\bigcup_{i=1}^{m} \widetilde{C}_i)(x) \geq \beta$ for all $x \in U$, enabling parametric control over approximation granularity and robustness.

The distinctive theoretical value of β-FCRS theory manifests across three interconnected dimensions [38]. First, it overcomes classical models' dependence on complete, noise-free data by achieving robust approximation through β-neighborhood operators that systematically filter out weakly supported covering elements, thereby maintaining stability in adversarial environments [39]. This noise tolerance has proven invaluable in real-world applications where data quality cannot be guaranteed. Second, it natively supports hierarchical decision-making from multigranular perspectives, where different β values correspond to distinct levels of abstraction, aligning seamlessly with complex systems analysis requirements [40]. This multiresolution capability enables analysts to explore data structures at multiple scales within a unified framework. Third, its algebraic architecture accommodates diverse fuzzy logical operators—including t-norms, fuzzy implications, overlap functions, and grouping functions-forming an extensible model spectrum adaptable to non-classical logical scenarios [41, 42]. This operator diversity liberates practitioners from the constraints of standard fuzzy logic, permitting customization to domain-specific reasoning requirements.

β-FCRS theory constructs a three-dimensional analytical framework of "fuzziness-approximation-multigranularity" by coupling fuzzy membership degrees with covering granularity through the β parameter, thereby demonstrating unique advantages in feature selection [43, 44], multi-attribute decision-making (MADM) [45–47], and three-way decisions (TWD) [48, 49] proposed by Yao [50, 51]. Its core value resides in three aspects: (1) overcoming traditional models' dependence on complete data by achieving robust approximation in noisy environments through β-neighborhood operators [39, 44]; (2) supporting hierarchical decision-making from multigranular perspectives, thus aligning with complex systems analysis requirements [40, 52]; and (3) accommodating diverse fuzzy logical operators to form an extensible model spectrum [41, 42]. Particularly in medical diagnosis [53], emergency management [54, 55], and image processing scenarios, this theory provides novel paradigms for handling high-dimensional heterogeneous data. These theoretical advantages have catalyzed transformative applications across critical domains. The framework's capacity to balance approximation precision with computational tractability has established it as a distinctive "β-driven" paradigm that bridges fuzzy sets, rough sets, and multigranular analysis.

1.3. Literature status and research challenges

Despite this remarkable theoretical proliferation and demonstrated practical utility, the existing literature confronts a critical structural deficiency: the absence of a comprehensive, integrative survey that systematically orchestrates fragmented advancements into a coherent intellectual landscape. Current knowledge remains compartmentalized across specialized subdomains, with researchers focusing narrowly on incremental model extensions, specialized measurement systems, or isolated application scenarios. No existing works provide a unified axiomatic framework that reveals phylogenetic relationships among eleven major model variants, nor do they offer critical assessment of how theoretical innovations have evolved in response to practical application demands. Furthermore, while individual measurement systems have been proposed, no survey has consolidated the three-level evaluation hierarchy-from Choquet-integral-based fuzzy measures to noise-tolerant discrimination indexes and variable-precision distinguishability indicators—into a systematic assessment methodology. The substantial growth in publications, coupled with emergence of specialized research communities [40, 56], has rendered it increasingly urgent to map the domain's knowledge structure, identify theoretical gaps, and forecast evolutionary trajectories.

In response to these challenges, this survey is motivated by the pressing need to address four fundamental objectives:

- Consolidation of a decade's fragmented research into a unified theoretical coordinate system capable of contextualizing disparate contributions.
- Exposition of logical necessity driving each model innovation rather than mere enumeration of variants and properties.
- Establishment of quantitative evaluation frameworks for model comparison, selection, and optimization in practical scenarios.
- Provision of actionable insights for transitioning the theory from academic exploration to industrial-grade deployment in high-stakes domains.

1.4. Contributions and structure of this survey

Correspondingly, this paper delivers the first comprehensive, decade-spanning analysis of β-FCRS theory through three integrated contributions:

- **Unified theoretical framework and phylogenetic taxonomy:** A comprehensive axiomatic foundation has been constructed that traces the complete evolutionary trajectory from classical covering rough sets to β-FCASs. This framework systematically explicates core operators (including β-neighborhood, complementary β-neighborhood, and extremal descriptor operators), axiomatic foundations, and reduction principles that preserve approximation capabilities. Through systematic academic mapping, eleven distinct model variants have been organized into three phylogenetic lineages: neighborhood-driven types, logical-operator-enhanced types, and structurally-expanded types. For each category, mathematical kernels, innovation motivations, and applicability boundaries are delineated, thereby transforming isolated model descriptions into a coherent family tree that reveals relationships and guides selection.
- **Hierarchical measurement and evaluation system:** Disparate evaluation metrics have been consolidated into a unified three-tier assessment hierarchy: (i) Choquet-integral-based fuzzy measures that capture non-additive information fusion and covering element interactions; (ii) noise-tolerant discrimination indexes that quantify robust discernibility capabilities; and (iii) variable-precision distinguishability indicators that enhance evaluation stability under noisy conditions. This system provides quantitative foundations for model optimization and reduction verification.
- **Application-driven trajectory analysis and critical bibliometric assessment:** The theory's practical evolution across four core domains-feature selection, MADM, outlier detection, and TWD-has been traced, revealing how specific application demands (e.g., noise robustness, hierarchical decision complexity, risk-aversion) have directly catalyzed theoretical innovations. Through empirical scientometric analysis of the core publications (2016–2026), temporal growth patterns have been mapped, key research communities identified, and interdisciplinary fusion trends revealed. Critically, structural deficiencies (lack of unified axioms), algorithmic bottlenecks (suboptimal computational complexity), and interpretability challenges that impede high-risk domain deployment are assessed, thereby anchoring precise targets for future research.

Methodologically, this survey integrates bibliometric analysis with theoretical dissection to provide the first comprehensive, decade-spanning assessment of β-FCRS theory's evolution. Unlike conventional reviews that narrowly enumerate models, this work systematically traces logical necessity underpinning each theoretical innovation while quantitatively mapping knowledge diffusion patterns through empirical scientometrics. The resulting framework follows a six-step progressive structure: theoretical foundations → model systems → measurement evaluation → application trajectories → critical reflection → future prospects. This architecture ensures that each component builds upon its predecessor, creating a coherent narrative that illuminates both individual contributions and overarching phylogenetic relationships among model variants. By anchoring analysis in both mathematical rigor and empirical evidence, this survey furnishes researchers with a clear roadmap for navigating this rapidly expanding domain while identifying strategic imperatives for its continued development.

The exposition that follows adheres to the following sequence: Sections 2-4 construct a comprehensive theoretical landscape, wherein Section 2 deconstructs foundational concepts of fuzzy β-covering approximation spaces, Section 3 systematically organizes submodel variants, and Section 4 distills the measurement system. Section 5 focuses on application trajectories, examining four dimensions: feature selection, decision analysis, outlier detection, and TWD. Section 6 conducts empirical bibliometric analysis to reveal evolutionary patterns. Section 7 offers critical commentary on theoretical gaps and practical limitations in existing research. Finally, Section 8 proposes five frontier directions, while Section 9 concludes the survey. Each section integrates bibliometric data to ensure objectivity and cutting-edge relevance.

To more clearly present the technical logic and research context of this paper, the overall framework diagram is specially plotted as shown in Figure 1.

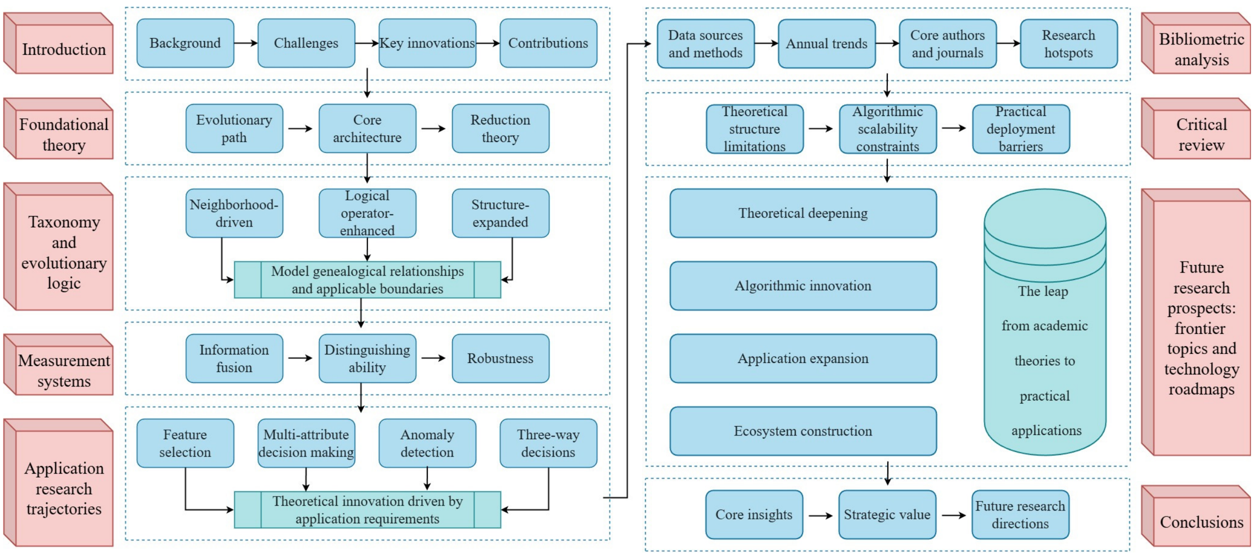

Figure 1: The framework of the paper.

2. Foundational theory of fuzzy β-covering approximation spaces

This section establishes the mathematical underpinnings of β-FCRSs, thereby laying an axiomatic foundation for subsequent model extensions and application analyses. Such theoretical development proceeds along dual dimensions: historical evolution and structural decomposition. Vertically, the transformation from classical CRSs to β-FCRSs is traced, revealing the logical imperatives underlying each conceptual innovation. Horizontally, the fundamental architecture of β-FCASs—encompassing neighborhood systems, extremal descriptor operators, and reduction mechanisms—is systematically dissected to illuminate geometric intuition and algebraic properties. Throughout this exposition, contributions from seminal investigations [1, 29, 48, 57, 58] are synthesized into a unified conceptual framework.

2.1. *Evolutionary trajectory from classical covering to fuzzy β-covering sets*

Classical CRSs approximate concepts through a family of covers on the universe; however, this crisp framework exhibits fundamental limitations when confronted with graded membership information. The initial extension by Deng et al. in 2007 introduced fuzzy approximation operators based on covering [35], thereby opening a pathway toward graded rough approximation. Nevertheless, this early formulation retained a binary activation mechanism for covering elements, insufficient for modeling confidence-level-dependent relationships.

Subsequently, Ma [1] formalized the β-FCRS construct in 2016, wherein $\beta \in (0,1]$ functions as an activation threshold: a covering element participates in approximation only upon attaining the specified confidence level. This innovation transformed rigid membership determination into a flexible, threshold-governed criterion. Parallel developments by Yang et al. that same year generalized this framework to lattice-ordered fuzzy structures, establishing the L-β-FCRS model [29]. Such generalization enhanced algebraic expressiveness by accommodating partially ordered membership grades beyond the unit interval. Together, these contributions constituted a paradigm shift from qualitative to quantitative covering-based approximation, furnishing the theoretical cornerstone for β-FCASs.

Definition 2.1. [1] Let U be an arbitrary universal set, and $F(U)$ be the fuzzy power set of U. For each $\beta \in (0,1]$, a collection $\widetilde{C} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ with $\widetilde{C}_i \in F(U)$ $(i = 1, 2, \cdots, m)$ is designated a fuzzy β- covering (β-FC) of U, provided that $(\cup_{i=1}^{m} \widetilde{C}_i)(x) \geq \beta$ holds for each $x \in U$. In this context, $(U, \widetilde{C})$ is termed a β-FCAS.

Definition 2.2 (β-neighborhood systems (β-NSs))**.** [57] Let $(U, \widehat{\mathbf{C}})$ be a β-FCAS and $x \in U$. The β-neighborhood system (β-NS) of x is defined as

$$\widetilde{N}^{\beta}_{\widehat{\mathbf{C}}}(x) = \{C \in \widehat{\mathbf{C}} : C(x) \geq \beta\}. \tag{2.1}$$

2.2. *Core conceptual architecture of β-FCASs*

The operational essence of β-FCASs resides in three interlocking constructs: neighborhood operators, extremal descriptor operators, and complementary mechanisms. The fuzzy β-neighborhood operator (β-FNO) [1] filters objects through a rigorous β threshold, thereby constituting the fundamental computational unit for approximation. While effective for capturing high-confidence relationships, this strict filtering may overlook potentially relevant information near the threshold boundary.

Complementing this, fuzzy β-minimal descriptions (β-FmD) and fuzzy maximal descriptions (β-FMD) [58] capture commonality and individuality characteristics through extremal covering subsets. Minimal descriptions identify the most specific covering elements containing an object, whereas maximal descriptions locate the most general ones. This duality enables granular analysis of approximation specificity; however, computational complexity increases substantially as the search space expands with covering cardinality. In 2020, Zhang et al. systematized the axiomatic topology of β-FCASs [59], establishing that these operators generate Alexandrov topologies under certain conditions, thereby providing a universal algebraic language for model construction.

Definition 2.3 (Fuzzy β-minimal descriptions (β-FmD))**.** [57] Consider β-FCAS $(U, \widehat{\mathbf{C}})$ whose covering family is $\widehat{\mathbf{C}} = \{C_1, C_2, \ldots, C_m\}$. For any object $x \in U$, the corresponding β-FmD $\widetilde{Md}^{\beta}_{\widehat{\mathbf{C}}}(x)$ is established through

$$\widetilde{Md}^{\beta}_{\widehat{\mathbf{C}}}(x) = \{C \in \widehat{N}^{\beta}_{\widehat{\mathbf{C}}}(x) : \forall D \in \widehat{N}^{\beta}_{\widehat{\mathbf{C}}}(x) \wedge D \subseteq C \Rightarrow C = D\}. \tag{2.2}$$

Definition 2.4 (Fuzzy β-maximal descriptions (β-FMD))**.** [57] Take $(U, \widehat{\mathbf{C}})$ to be a β-FCAS with the covering collection $\widehat{\mathbf{C}} = \{C_1, C_2, \ldots, C_m\}$. For every element $x \in U$, its associated β-FMD $\widetilde{MD}^{\beta}_{\widehat{\mathbf{C}}}(x)$ is expressed by

$$\widetilde{MD}^{\beta}_{\widehat{\mathbf{C}}}(x) = \{C \in \widehat{N}^{\beta}_{\widehat{\mathbf{C}}}(x) : \forall D \in \widehat{N}^{\beta}_{\widehat{\mathbf{C}}}(x) \wedge D \supseteq C \Rightarrow C = D\}. \tag{2.3}$$

2.3. *Reduction theory and structural analysis*

Reduction theory in β-FCASs seeks to eliminate redundant covering elements while preserving approximation invariance. Yang and Hu [58] introduced a reduction algorithm based on discernibility matrices that guarantees preservation of β-approximation properties. Although theoretically rigorous, this approach suffers from quadratic space complexity relative to covering size, rendering it impractical for high-dimensional data.

Subsequently, Zhan et al. [40] proposed heuristic reduction strategies employing information entropy. Such methods achieve linear time complexity but sacrifice theoretical completeness, as entropy-based criteria only approximate approximation preservation. In lattice-valued extensions, Ma [1] demonstrated that L-β-CRS reduction requires simultaneous consideration of lattice-order relations and threshold constraints, revealing heightened structural complexity. This multiplicity of reduction criteria necessitates careful trade-off analysis between computational efficiency and theoretical fidelity.

- **Union-based reducibility:** Covering elements expressible as unions of other elements can be eliminated without altering approximation capabilities.

- **Intersection-based reducibility:** Elements representable as intersections of other elements may be removed, though this criterion exhibits greater sensitivity to threshold variations.

Definition 2.5 (Reducible and irreducible elements)**.** [58] Consider a β-FCAS $(U, \widehat{\mathbf{C}})$ containing an element $C \in \widehat{\mathbf{C}}$. When a subcollection $\widehat{\mathbf{D}} \subseteq \widehat{\mathbf{C}} - \{C\}$ satisfies $C = \cup\widehat{\mathbf{D}}$, C qualifies as a **reducible element** in $\widehat{\mathbf{C}}$; absent this condition, C is an **irreducible element**. The covering $\widehat{\mathbf{C}}$ is classified reducible precisely when such elements appear; otherwise it maintains irreducible status.

Definition 2.6 (I-reducible and I-irreducible elements)**.** [59] Take $(U, \widehat{\mathbf{C}})$ as a β-FCAS with $C \in \widehat{\mathbf{C}}$. The element C is identified as an ***I*-reducible element** in $\widehat{\mathbf{C}}$ whenever a subset $\widehat{\mathbf{D}} \subseteq \widehat{\mathbf{C}} - \{C\}$ fulfills $C = \cap\widehat{\mathbf{D}}$; lacking this property, C is termed an **I-irreducible element**. The covering $\widehat{\mathbf{C}}$ is I-reducible if such elements exist, and is I-irreducible otherwise.

3. Taxonomy and evolutionary logic of fuzzy β-covering rough set models

This section systematically surveys the β-FCRS model variants that emerged from 2016 to 2026, constructing a logically coherent and hierarchically organized taxonomic framework. Distinct from conventional surveys that merely enumerate models, this exposition elucidates the intrinsic motivations and evolutionary pathways underlying each innovation. The developmental trajectory began with basic neighborhood operators, subsequently expanding to complementary-neighborhood and variable-precision models in response to demands for enhanced noise robustness. Expressive inadequacies prompted the introduction of sophisticated fuzzy logical operators, while complex systems analysis necessitated multigranular architectures. Further adaptations for dual-universe, variable-scale, and higher-order fuzzy environments are driven by expanded application scenarios. Throughout this analysis, each model category is annotated with core literature, innovations, and applicability boundaries, enabling readers to perceive both individual contributions and their phylogenetic relationships.

As illustrated in Table 1, academic mapping reveals that core publications have spawned eleven model variants. These models can be categorized into three lineages based on construction mechanisms: neighborhood-driven types (Sections 3.1-3.4), logical-operator-enhanced types (Sections 3.5-3.6), and structurally-expanded types (Sections 3.7-3.8). 3.9 subsequently analyzes the relationships among these models. Each category specifically targets distinct limitations: enhancing noise robustness [39, 44], augmenting logical expressiveness [41, 42], or expanding environmental applicability [56, 60, 61]. This section systematically parses the mathematical kernels, innovative motivations, and bibliometric support for each model, thereby revealing the evolutionary logic from isolated constructs to coherent model families.

3.1. Fuzzy β-covering rough set models based on a fuzzy β-neighborhood

The most fundamental variant, designated as the β-FCRS model, was established by Ma [1] in 2016. Approximation operators are defined directly via β-neighborhoods (β-Ns), wherein the lower approximation mandates complete containment of object neighborhoods within the target set, while the upper approximation permits mere neighborhood intersection with the target set. This "strict lower, loose upper" design aligns with classical rough set philosophy yet exhibits pronounced sensitivity to boundary noise. Such vulnerability arises because the binary containment criterion fails to accommodate marginal fluctuations, rendering the model susceptible to misclassification near decision boundaries.

Table 1: Representative techniques in β-FCRS models

Representative methods	Title	Reference	Year
Fuzzy β-neighborhood operators	Two fuzzy covering rough set models and their generalizations over fuzzy lattices	Ma [1]	2016
	On some types of fuzzy covering-based rough sets	Yang and Hu [58]	2017
	A fuzzy covering-based rough set model and its generalization over fuzzy lattice	Yang and Hu [29]	2016
	An intuitionistic fuzzy graded covering rough sets	Huang et al. [30]	2016
	Communication between fuzzy information systems using fuzzy covering-based rough sets	Yang and Hu [62]	2018
	Fuzzy neighborhood operators and derived fuzzy coverings	Yang and Hu [57]	2019
	Novel classes of fuzzy soft β-coverings-based FRSs with applications to multi-criteria fuzzy group decision making	Zhang et al. [63]	2019
	Fuzzy soft β-covering based FRSs and corresponding decision-making applications	Zhang et al. [64]	2019
Fuzzy logical operators	Fuzzy β-covering based (I, T)-FRS models and applications to MADM	Zhang et al. [41]	2019
	Covering based variable precision (I, T)-FRSs with applications to MADM	Jiang et al. [42]	2019
	Overlap functions based (multi-granulation) FRSs and their applications in MCDM	Wen et al. [65]	2021
	Some fuzzy neighborhood operators on fuzzy β-covering approximation space and their application in user preference evaluation	Li et al. [66]	2025
	Two novel TWD models based on fuzzy β-covering rough sets and prospect theory under q-rung orthopair fuzzy environments	Shi et al. [49]	2024
Multigranulation fuzzy sets	Covering based multigranulation (I, T)-FRS models and applications in multi-attribute group decision-making	Zhan et al. [40]	2019
	Covering based multigranulation FRSs and corresponding applications	Zhan et al. [67]	2020
	Novel classes of coverings based multigranulation fuzzy rough sets and corresponding applications to multiple attribute group decision-making	Ma et al. [52]	2020
	Covering based multi-granulation rough fuzzy sets with applications to feature selection	Huang and Li [68]	2024
	Some extensions of covering-based multigranulation FRSs from new perspectives	Atef and Atik [69]	2021
Inclusion satisfaction	On multicriteria decision-making method based on a fuzzy rough set model with fuzzy α-neighborhoods	Zhang et al. [70]	2021
	Covering-based variable precision FRSs with PROMETHEE-EDAS methods	Zhan et al. [71]	2020
	On four novel kinds of fuzzy β-covering-based rough sets and their applications to three-way approximations	Jiang and Hu [72]	2025
	A novel method for incremental feature selection with fuzzy β-covering	Wang et al. [73]	2025
	A fitting model for attribute reduction with fuzzy β-covering	Huang and Li [74]	2021
	Redefined FRS models in fuzzy β-covering group approximation spaces	Zhang and Dai [75]	2022
	Noise-tolerant fuzzy-β-covering-based multigranulation rough sets and feature subset selection	Huang et al. [39]	2022
	Novel fuzzy β-covering rough set models and their applications	Dai et al. [76]	2022
	Overlap function-based fuzzy β-covering relations and fuzzy β-covering rough set models	Fan and Zhang [77]	2024
	Multi-fuzzy β-covering fusion based accuracy and self-information for feature subset selections	Zou and Dai [78]	2024

Definition 3.1. [1] Consider a β-FCAS $(U, \widetilde{C})$ with $\widetilde{C} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ forming a β-FC over U. For any object $x \in U$, its fuzzy β-neighborhood (β-FN) $\widetilde{N}_x^\beta$ is characterized by:

$$\widetilde{N}_x^\beta = \cap\{\widetilde{C}_i \in \widetilde{C} : \widetilde{C}_i(x) \geq \beta\}. \tag{3.1}$$

Definition 3.2. [1] Take $(U, \widetilde{C})$ as a β-FCAS where $\widetilde{C} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ constitutes a β-FC of U for a given $\beta \in (0, 1]$. For every $X \in F(U)$, the lower approximation $\widetilde{P^-}X$ and upper approximation $\widetilde{P^+}X$ are constructed through:

$$(\widetilde{P^-}X)(x) = \bigwedge_{y \in U}[(1 - \widetilde{N}_x^\beta(y)) \vee X(y)], \quad x \in U, \tag{3.2}$$

$$(\widetilde{P^+}X)(x) = \bigvee_{y \in U}[\widetilde{N}_x^\beta(y) \wedge X(y)], \quad x \in U. \tag{3.3}$$

Whenever $\widetilde{P^-}X \neq \widetilde{P^+}X$ holds, X qualifies as a β-FCRS.

3.2. Fuzzy β-covering rough set models based on a fuzzy complementary β-neighborhood

To compensate for the deficient characterization of outliers inherent in the β-CRS models based on a fuzzy complementary β-neighborhood (β-FCN) framework, Yang et al. [58] proposed the β-CRS models based on a β-FCN model in 2017. This approach captures object "atypicality"—the degree to which an object belongs to no β-covering element—through complementary neighborhoods, thereby defining dual approximation operators. However, the introduction of complementary neighborhoods alone does not resolve the fundamental rigidity of binary approximation criteria. Consequently, three distinct approximation strategies were developed to balance strictness and flexibility, each exhibiting unique trade-offs between computational complexity and noise resilience.

Definition 3.3. [58] Take $(U, \widehat{\mathbf{C}})$ as a β-FCAS where $\beta \in (0, 1]$. For any object $x \in U$, its fuzzy complementary β-neighborhood (β-FCN), symbolized by $\widetilde{M}_x^\beta$, is characterized via:

$$\widetilde{M}_x^\beta(y) = \widetilde{N}_y^\beta(x), \quad \forall y \in U, \tag{3.4}$$

in which $\widetilde{N}_y^\beta$ represents the β-FN of y.

Definition 3.4. [58] Assume $(U, \widehat{\mathbf{C}})$ constitutes a β-FCAS with $\beta \in (0, 1]$. For every $X \in \mathcal{F}(U)$, the fuzzy covering lower approximation (FCLA) $\widetilde{FL}(X)$ together with the fuzzy covering upper approximation (FCUA) $\widetilde{FH}(X)$ are constructed as:

$$\widetilde{FL}(X)(x) = \bigwedge_{y \in U}[(1 - \widetilde{M}_x^\beta(y)) \vee X(y)], \quad x \in U, \tag{3.5}$$

$$\widetilde{FH}(X)(x) = \bigvee_{y \in U}[\widetilde{M}_x^\beta(y) \wedge X(y)], \quad x \in U. \tag{3.6}$$

When $\widetilde{FL}(X) \neq \widetilde{FH}(X)$ holds, X qualifies as a β-CRS model based on a β-FCN model of the first type.

Definition 3.5. [58] Consider $(U, \widehat{\mathbf{C}})$ as a β-FCAS where $\beta \in (0, 1]$. For any $X \in \mathcal{F}(U)$, the FCLA $\widetilde{SL}(X)$ and FCUA $\widetilde{SH}(X)$ emerge through:

$$\widetilde{SL}(X)(x) = \bigwedge_{y \in U}[(1 - \widetilde{N}_x^\beta(y)) \vee (1 - \widetilde{M}_x^\beta(y)) \vee X(y)], \quad x \in U, \tag{3.7}$$

$$\widetilde{SH}(X)(x) = \bigvee_{y \in U}[\widetilde{N}_x^\beta(y) \wedge \widetilde{M}_x^\beta(y) \wedge X(y)], \quad x \in U. \tag{3.8}$$

If the inequality $\widetilde{SL}(X) \neq \widetilde{SH}(X)$ holds, then X is designated a β-CRS model derived from a β-FCN model of the second type.

Definition 3.6. [58] Taking $(U, \widehat{\mathbf{C}})$ as a β-FCAS with $\beta \in (0, 1]$, each $X \in \mathcal{F}(U)$ gives rise to the FCLA $\widetilde{TL}(X)$ and FCUA $\widetilde{TH}(X)$ defined by:

$$\widetilde{TL}(X)(x) = \bigwedge_{y \in U} \Big[((1 - \widetilde{N}_x^\beta(y)) \wedge (1 - \widetilde{M}_x^\beta(y))) \vee X(y)\Big], \quad x \in U, \tag{3.9}$$

$$\widetilde{TH}(X)(x) = \bigvee_{y \in U} \Big[(\widetilde{N}_x^\beta(y) \vee \widetilde{M}_x^\beta(y)) \wedge X(y)\Big], \quad x \in U. \tag{3.10}$$

Provided that $\widetilde{TL}(X) \neq \widetilde{TH}(X)$, X is regarded as a β-CRS model built upon a β-FCN model of the third type.

The interconnections among these three models [58] plus Ma's models [1] receive examination below:

Remark 3.1. For a β-FCAS $(U, \widehat{\mathbf{C}})$ and any $X \in \mathcal{F}(U)$, the subsequent relationships hold:
(1) $\widetilde{TL}(X) \subseteq \underline{\mathbf{C}}(X) \subseteq \widetilde{SL}(X)$;
(2) $\underline{\mathbf{C}}(X) \subseteq \widetilde{FL}(X) \subseteq \widetilde{SL}(X)$;
(3) $\overline{\mathbf{C}}(X) \subseteq \widetilde{FH}(X) \subseteq \widetilde{TH}(X)$;
(4) $\widetilde{SH}(X) \subseteq \widetilde{FH}(X) \subseteq \widetilde{TH}(X)$;
(5) $\widetilde{SL}(X) = \underline{\mathbf{C}}(X) \cup \widetilde{FL}(X)$;
(6) $\widetilde{SH}(X) = \overline{\mathbf{C}}(X) \cap \widetilde{FH}(X)$;
(7) $\widetilde{TL}(X) = \underline{\mathbf{C}}(X) \cap \widetilde{FL}(X)$;
(8) $\widetilde{TH}(X) = \overline{\mathbf{C}}(X) \cup \widetilde{FH}(X)$.

These relational inclusions reveal a hierarchical structure where Type-3 approximations represent the most conservative lower bounds and permissive upper bounds, while Type-1 approximations occupy an intermediate position. The set-theoretic equalities in items (5)-(8) further demonstrate that Type-2 approximations function as composite operators, simultaneously incorporating both standard and complementary neighborhood information.

3.3. Fuzzy β-covering rough set models based on fuzzy β-minimal descriptions

This model variant was pioneered by Yang and Hu [29] in 2016. β-FmDs define minimal covering element sets for each object, enabling approximation operations to depend solely on most relevant covering information and thereby avoid redundancy interference. Compared with global neighborhoods, minimal descriptions push locality to the extreme, proving particularly suitable for high-dimensional sparse data. However, the computational overhead of identifying minimal descriptions increases significantly with covering cardinality, and the model's sensitivity to covering refinement operations presents challenges for dynamic data scenarios.

Definition 3.7. [29] Consider a β-FC $\widehat{\mathbb{C}} = \{C_1, C_2, \ldots, C_m\}$ over U with $\beta \in (0, 1]$. For any object $x \in U$, its β-FmD $(\widetilde{Md}_x^\beta)_{\widehat{\mathbb{C}}}$ is established through:

$$(\widetilde{Md}_x^\beta)_{\widehat{\mathbb{C}}} = \{C \in \widehat{\mathbb{C}} : (C(x) \geq \beta) \wedge (\forall D \in \widehat{\mathbb{C}} \wedge D(x) \geq \beta \wedge D \subseteq C \Rightarrow C = D)\}. \tag{3.11}$$

When the β-FC is clear from context, the subscript $\widehat{\mathbb{C}}$ is dropped, yielding $\widetilde{Md}_x^\beta$.

Proposition 3.1. [29] For every element $x \in U$, the subsequent relationship holds:

$$\widetilde{N}_x^\beta = \bigcap \widetilde{Md}_x^\beta. \tag{3.12}$$

Definition 3.8. [29] Take a β-FC $\widehat{\mathbb{C}} = \{C_1, C_2, \ldots, C_m\}$ of U where $\beta \in (0, 1]$. For every fuzzy set $X \in \mathcal{F}(U)$, the approximations $\widetilde{CL}(X)$ (FCLA) and $\widetilde{CH}(X)$ (FCUA) are given by:

$$\widetilde{CL}(X) = \bigcup \{C \in \widehat{\mathbb{C}} : C \subseteq X\}, \tag{3.13}$$

$$\widetilde{CH}(X) = \widetilde{CL}(X) \cup \Big(\bigcup \{\widetilde{Md}_x^\beta : X(x) \geq \beta\}\Big). \tag{3.14}$$

If $\widetilde{CL}(X) = \widetilde{CH}(X)$, then X is called a β-FCRS model based on β-FmDs.

3.4. Fuzzy β-covering-based fuzzy rough set models based on fuzzy logic operators

When β-neighborhoods are combined with classical t-norms T and fuzzy implications I, model expressiveness undergoes qualitative transformation. The (I, T)-FRS model based on fuzzy β-coverings (ITFRS-β-FC) proposed by Zhang et al. [41] in 2019 represents a milestone: lower approximations employ $I(A(y), B(x))$ to measure membership implication, while upper approximations adopt $T(A(y), B(x))$ to characterize interaction dynamics. This model injects logical inference mechanisms into approximation processes, transforming rough sets into interpretable logical systems. In the same year, Jiang et al. [42] put forth the variable-precision (I, T)-FRS models in light of fuzzy β-coverings (VPITFRS-β-FC), enhancing noise robustness through the parameter λ. In 2021, Wen and Zhang [65] replaced t-norms with overlap functions O, constructing (O, I)-based models that overcome t-norms' lack of double neutral elements. This lineage's evolutionary logic is clear: from classical algebraic operators → fuzzy logical operators → overlap/grouping functions, each step aims to enhance adaptability to non-classical logical scenarios. However, such enhancement comes at the cost of increased parameter tuning complexity and reduced computational efficiency.

Definition 3.9. [41](First fuzzy β-covering-based $(\mathcal{I}, \mathcal{T})$-FRS model, namely 1-ITFRS-β-FC) Consider a β-covering approximation system $(U, \widetilde{C})$ with $0 < \beta \leqslant 1$ where $\widetilde{C} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ constitutes a β-FC of U. For each fuzzy set $A \in \mathcal{F}(U)$, the lower approximation operator $\widetilde{C_1^-}(A)$ and upper approximation operator $\widetilde{C_1^+}(A)$ are respectively established through: for any $u \in U$,

$$\widetilde{C_1^-}(A)(u) = \bigwedge_{v \in U} \mathcal{I}\left(\widetilde{N_u^\beta}(v), A(v)\right), \tag{3.15}$$

$$\widetilde{C_1^+}(A)(u) = \bigvee_{v \in U} \mathcal{T}\left(\widetilde{N_u^\beta}(v), A(v)\right). \tag{3.16}$$

When $\widetilde{C_1^+}(A) \neq \widetilde{C_1^-}(A)$ holds, A qualifies as the fuzzy β-covering-based $(\mathcal{I}, \mathcal{T})$-FRS of the first type (1-FCITFRS). Equality $\widetilde{C_1^+}(A) = \widetilde{C_1^-}(A)$ indicates A is definable.

Definition 3.10. [41](Second fuzzy β-covering-based $(\mathcal{I}, \mathcal{T})$-FRS model, namely 2-ITFRS-β-FC) Take $0 < \beta \leqslant 1$ and let $\widetilde{C} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ be a β-FC of U. For any fuzzy set $A \in \mathcal{F}(U)$, the lower approximation operator $\widetilde{C_2^-}(A)$ and upper approximation operator $\widetilde{C_2^+}(A)$ are respectively formulated as: for each $u \in U$,

$$\widetilde{C_2^-}(A)(u) = \bigwedge_{v \in U} \mathcal{I}\left(\widetilde{M_u^\beta}(v), A(v)\right), \tag{3.17}$$

$$\widetilde{C_2^+}(A)(u) = \bigvee_{v \in U} \mathcal{T}\left(\widetilde{M_u^\beta}(v), A(v)\right). \tag{3.18}$$

If the inequality $\widetilde{C_2^+}(A) \neq \widetilde{C_2^-}(A)$ holds, then A is designated the second fuzzy β-covering-based $(\mathcal{I}, \mathcal{T})$-FRS model, namely 2-ITFRS-β-FC. Equality $\widetilde{C_2^+}(A) = \widetilde{C_2^-}(A)$ signifies A is definable.

Definition 3.11. [41](Third fuzzy β-covering-based $(\mathcal{I}, \mathcal{T})$-FRS model, namely 3-ITFRS-β-FC) Consider $0 < \beta \leqslant 1$ and assume $\widetilde{C} = \{\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m\}$ forms a β-FC of U. For each $A \in \mathcal{F}(U)$, the lower approximation operator $\widetilde{C_3^-}(A)$ and upper approximation operator $\widetilde{C_3^+}(A)$ are respectively defined by: for any $u \in U$,

$$\widetilde{C_3^-}(A)(u) = \bigwedge_{v \in U} \mathcal{I}\left(\left(\widetilde{M_u^\beta} \cap \widetilde{N_u^\beta}\right)(v), A(v)\right), \tag{3.19}$$

$$\widetilde{C_3^+}(A)(u) = \bigvee_{v \in U} \mathcal{T}\left(\left(\widetilde{M_u^\beta} \cap \widetilde{N_u^\beta}\right)(v), A(v)\right). \tag{3.20}$$

When $\widetilde{C_3^+}(A) \neq \widetilde{C_3^-}(A)$, then A is termed the third fuzzy β-covering-based $(\mathcal{I}, \mathcal{T})$-FRS model, namely 3-ITFRS-β-FC. Equality $\widetilde{C_3^+}(A) = \widetilde{C_3^-}(A)$ indicates A is definable.

Definition 3.12. [41](Fourth fuzzy β-covering based $(\mathcal{I},\mathcal{T})$-FRS model, namely 4-ITFRS-β-FC) Take $0<\beta\leqslant 1$ and suppose $\widetilde{C}=\{\widetilde{C}_1,\widetilde{C}_2,\cdots,\widetilde{C}_m\}$ constitutes a β-FC of U. For every fuzzy set $A\in\mathcal{F}(U)$, the lower approximation operator $\widetilde{C_4^-}(A)$ and upper approximation operator $\widetilde{C_4^+}(A)$ are respectively given by: for each $u\in U$,

$$\widetilde{C_4^-}(A)(u)=\bigwedge_{v\in U}\mathcal{I}\left(\left(\widetilde{M_u^\beta}\cup\widetilde{N_u^\beta}\right)(v),A(v)\right),\tag{3.21}$$

$$\widetilde{C_4^+}(A)(u)=\bigvee_{v\in U}\mathcal{T}\left(\left(\widetilde{M_u^\beta}\cup\widetilde{N_u^\beta}\right)(v),A(v)\right).\tag{3.22}$$

Provided that $\widetilde{C_4^+}(A)\neq\widetilde{C_4^-}(A)$, then A is named the fourth fuzzy β-covering-based $(\mathcal{I},\mathcal{T})$-FRS model, namely 4-ITFRS-β-FC. Equality $\widetilde{C_4^+}(A)=\widetilde{C_4^-}(A)$ signifies A is definable.

The connections among these four ITFRS-β-FC models are described below:

Remark 3.2. For a β-FCAS $(U,\widetilde{C})$ where $\widetilde{C}=\{\widetilde{C}_1,\widetilde{C}_2,\cdots,\widetilde{C}_m\}$ forms a β-FC of U with $\beta\in(0,1]$, any $A\in\mathcal{F}(U)$ satisfies:

(1) $\widetilde{C_4^-}(A)\subseteq\widetilde{C_1^-}(A)\subseteq\widetilde{C_3^-}(A)$, provided that $\mathcal{I}$ satisfies the left monotonicity;
(2) $\widetilde{C_4^-}(A)\subseteq\widetilde{C_2^-}(A)\subseteq\widetilde{C_3^-}(A)$, provided that $\mathcal{I}$ satisfies the left monotonicity;
(3) $\widetilde{C_3^+}(A)\subseteq\widetilde{C_1^+}(A)\subseteq\widetilde{C_4^+}(A)$;
(4) $\widetilde{C_3^+}(A)\subseteq\widetilde{C_2^+}(A)\subseteq\widetilde{C_4^+}(A)$;
(5) $\widetilde{C_3^-}(A)=\widetilde{C_1^-}(A)\cap\widetilde{C_2^-}(A)$;
(6) $\widetilde{C_3^+}(A)=\widetilde{C_1^+}(A)\cup\widetilde{C_2^+}(A)$;
(7) $\widetilde{C_4^-}(A)=\widetilde{C_1^-}(A)\cup\widetilde{C_2^-}(A)$;
(8) $\widetilde{C_4^+}(A)=\widetilde{C_1^+}(A)\cup\widetilde{C_2^+}(A)$.

Definition 3.13. [42] Take a finite β-FCAS $(U,\widehat{C})$ with $\beta\in(0,1]$. For any fuzzy set $X\in\mathcal{F}(U)$ and element $x\in U$, the r-th type $(r=1,2,3,4)$ variable precision $(\mathcal{I},\mathcal{T})$-fuzzy lower approximation set $\underline{Apr}^{\mathcal{I}}_{k,r}(X)$ (denoted r-VPITFLAS-β-FC) and the corresponding r-th type variable precision $(\mathcal{I},\mathcal{T})$-fuzzy upper approximation set $\overline{Apr}^{\mathcal{T}}_{k,r}(X)$ (denoted r-VPITFUAS-β-FC), parameterized by $k\in[0,1)$, are respectively formulated as:

$$\underline{Apr}^{\mathcal{I}}_{k,r}(X)(x)=\inf_{X(y)\leq k}\mathcal{I}(N^r_{x,\beta}(y),k)\bigwedge\inf_{X(y)>k}\mathcal{I}(N^r_{x,\beta}(y),X(y)),\tag{3.23}$$

$$\overline{Apr}^{\mathcal{T}}_{k,r}(X)(x)=\sup_{X(y)\geq\mathcal{N}(k)}\mathcal{T}(N^r_{x,\beta}(y),\mathcal{N}(k))\bigvee\sup_{X(y)<\mathcal{N}(k)}\mathcal{T}(N^r_{x,\beta}(y),X(y)).\tag{3.24}$$

If $\underline{Apr}^{\mathcal{I}}_{k,r}(X)\neq\overline{Apr}^{\mathcal{T}}_{k,r}(X)$, then X is called the r-th $(r=1,2,3,4)$ type of VPITFRS-β-FC, otherwise X is called the rth $(r=1,2,3,4)$ type of variable precision fuzzy definable.

Remark 3.3. 1) A β-FN operator may be viewed as a particular fuzzy relation. For example, given a fuzzy relation R, one can define a β-FN operator through R by setting $N^r_{x,\beta}(y)=R(x,y)$ for all $x,y\in U$. Consequently, when R forms a $\mathcal{T}$-similarity relation and the implicator $\mathcal{I}$ is substituted with an S-implicator, the model from Definition 3.13 reduces to a variant developed by Zhao et al. [79].

2) Substituting $\mathcal{T}$ with $\mathcal{T}_M$, $\mathcal{I}$ with $\mathcal{I}_{KD}$, and representing $\mathcal{N}$ by $\mathcal{N}_S$ causes r-CVPITFLA and r-CVPITFUA to reduce to these expressions:

$$\underline{Apr}^{\mathcal{I}}_{k,r}(X)(x)=\inf_{X(y)\leq k}[(1-N^r_{x,\beta}(y))\vee k]\bigwedge\inf_{X(y)>k}[(1-N^r_{x,\beta}(y))\vee X(y)],\tag{3.25}$$

$$\overline{Apr}^{\mathcal{T}}_{k,r}(X)(x)=\sup_{X(y)\geq\mathcal{N}(k)}(N^r_{x,\beta}(y)\wedge\mathcal{N}(k))\bigvee\sup_{X(y)<\mathcal{N}(k)}(N^r_{x,\beta}(y)\wedge X(y)).\tag{3.26}$$

Specifically, for $k = 0$ and $r = 1$, Eqs. (3.25) and (3.26) collapse to Ma's model [1]; whereas for $k = 0$ with $r = 2, 3, 4$, they reduce to the framework introduced by Yang and Hu [58].

Setting $k = 0$ transforms Eqs. (3.23) and (3.24) from Definition 3.13 into:

$$\underline{Apr}_r^{\mathcal{I}}(X)(x) = \inf_{y \in U} \mathcal{I}(N_{x,\beta}^r(y), X(y)), \tag{3.27}$$

$$\overline{Apr}_r^{\mathcal{T}}(X)(x) = \sup_{y \in U} \mathcal{T}(N_{x,\beta}^r(y), X(y)). \tag{3.28}$$

In particular, when a β-FN constitutes a fuzzy $\mathcal{T}$-similarity relation on U, Definition 3.13 reduces to the model proposed by Radzikowska and Kerre [80].

3) When X represents a crisp subset of U, Eqs. (3.23) and (3.24) from Definition 3.13 simplify to:

$$\underline{Apr}_{k,r}^{\mathcal{I}}(X)(x) = \inf_{X(y)=0} \mathcal{I}(N_{x,\beta}^r(y), k), \tag{3.29}$$

$$\overline{Apr}_{k,r}^{\mathcal{T}}(X)(x) = \sup_{X(y)=1} \mathcal{T}(N_{x,\beta}^r(y), \mathcal{N}(k)). \tag{3.30}$$

3.5. Fuzzy β-neighborhood-operator-enhanced models

Unlike Section 3.4's direct modification of approximation operators, FβNO-CRS focuses on transforming neighborhood construction logic. Li et al. [66] in 2025 systematically investigated neighborhood-operator lineages based on t-norms, overlap functions, and semi-overlap functions, revealing that different logical operators endow neighborhoods with distinct "information fusion" characteristics. For instance, semi-overlap-function-based neighborhoods exhibit heightened sensitivity to conflicting information, making them suitable for user preference conflict scenarios [49]. This approach fundamentally alters how neighborhoods aggregate information, yet inherits the computational burden associated with non-standard logical operators.

The four β-FN operators derived from an overlap function O and its R-implication I_O are described below:

Definition 3.14. [66] Consider a β-covering approximation structure (Ω, Ψ). The first type of β-FN operator derived from R-implication I_O, designated $ON_{\Psi,\beta}^1 : \Omega \to \mathcal{F}(\Omega) : \pi \mapsto ON_{\Psi,\beta}^1(\pi)$, determines $ON_{\Psi,\beta}^1(\pi)$ via

$$ON_{\Psi,\beta}^1(\pi)(\varpi) = \bigwedge_{\psi \in \mathbb{C}_\beta(\Psi,\pi)} I_O(\psi(\pi), \psi(\varpi)). \tag{3.31}$$

Definition 3.15. [66] Take a β-FCAS (Ω, Ψ). Based on overlap function O, the β-FN operator $ON_{\Psi,\beta}^2 : \Omega \to \mathcal{F}(\Omega) : \pi \mapsto ON_{\Psi,\beta}^2(\pi)$ specifies $ON_{\Psi,\beta}^2(\pi)$ as

$$ON_{\Psi,\beta}^2(\pi)(\varpi) = \bigvee_{\psi \in md_{\Psi,\beta}(\pi)} O(\psi(\pi), \psi(\varpi)). \tag{3.32}$$

Definition 3.16. [66] In a β-FCAS (Ω, Ψ), the β-FN operator constructed from R-implication I_O, denoted $ON_{\Psi,\beta}^3 : \Omega \to \mathcal{F}(\Omega) : \pi \mapsto ON_{\Psi,\beta}^3(\pi)$, establishes $ON_{\Psi,\beta}^3(\pi)$ through

$$ON_{\Psi,\beta}^3(\pi)(\varpi) = \bigwedge_{\psi \in MD_{\Psi,\beta}(\pi)} I_O(\psi(\pi), \psi(\varpi)). \tag{3.33}$$

Definition 3.17. [66] Consider a β-covering approximation system (Ω, Ψ). The fourth β-FN operator derived from overlap functions, designated $ON_{\Psi,\beta}^4 : \Omega \to \mathcal{F}(\Omega) : \pi \mapsto ON_{\Psi,\beta}^4(\pi)$, defines $ON_{\Psi,\beta}^4(\pi)$ as

$$ON_{\Psi,\beta}^4(\pi)(\varpi) = \bigvee_{\psi \in \mathbb{C}_\beta(\Psi,\pi)} O(\psi(\pi), \psi(\varpi)). \tag{3.34}$$

Employing these four fuzzy neighborhood operators, the $ON-\beta-FCRS$ and $TN-\beta-FCRS$ frameworks are specified as:

Definition 3.18. [66] Take a β-FCAS (Ω, Ψ). The operators $TN^{i}_{\Psi^{j}_{\beta}}$ and $ON^{i}_{\Psi^{k}_{\beta}}$ represent β-FN constructions obtained from t-norms and overlap functions respectively. For every $\Gamma \in \mathcal{F}(\Omega)$, the lower and upper approximation operators induced by $TN^{i}_{\Psi^{j}_{\beta}}$ are given by

$$\underline{\Psi^{T}_{i,j}}(\Gamma)(\pi) = \bigwedge_{\varpi\in\Omega}\Big((1 - TN^{i}_{\Psi^{j}_{\beta}}(\pi)(\varpi)) \vee \Gamma(\varpi)\Big), \tag{3.35}$$

$$\overline{\Psi^{T}_{i,j}}(\Gamma)(\pi) = \bigvee_{\varpi\in\Omega}\Big(TN^{i}_{\Psi^{j}_{\beta}}(\pi)(\varpi) \wedge \Gamma(\varpi)\Big), \tag{3.36}$$

and the lower and upper approximation operators induced by $ON^{i}_{\Psi^{k}_{\beta}}$ are given as

$$\underline{\Psi^{O}_{i,k}}(\Gamma)(\pi) = \bigwedge_{\varpi\in\Omega}\Big((1 - ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi)) \vee \Gamma(\varpi)\Big), \tag{3.37}$$

$$\overline{\Psi^{O}_{i,k}}(\Gamma)(\pi) = \bigvee_{\varpi\in\Omega}\Big(ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi) \wedge \Gamma(\varpi)\Big). \tag{3.38}$$

3.6. A multigranular fuzzy β-covering fuzzy rough set model based on triangular norms and fuzzy implications

Complex decision-making scenarios frequently require multi-level, multi-perspective analysis. The MG-β-FCRS model pioneered by Zhan et al. [40] in 2019 achieves granularity aggregation by fusing multiple β-covering families. This model offers two strategies: optimistic versions approximate unions across granularities (emphasizing possibility), while pessimistic versions intersect approximations (emphasizing necessity). In 2020, Ma et al. [52] integrated variable-precision mechanisms into multigranular frameworks, proposing robust multigranular models for group decision-making. In 2022, Huang et al. [39] further incorporated noise-tolerant mechanisms, forming robust multigranular models. This lineage follows the expansion path single-granularity → double-granularity → multi-granularity → variable-precision multigranularity → robust multigranularity, with each step enhancing the model's capacity to deconstruct complex decision scenarios. Nevertheless, multigranulation models suffer from exponential complexity growth with the number of granularities and require careful calibration of fusion strategies to avoid information dilution.

Assumption 1: Consider a continuous t-norm $\mathcal{T}$ together with an implicator $\mathcal{I}$ on $[0,1]$. Let $(U, \widetilde{\Gamma})$ denote a β-FFCAS where $\widetilde{\Gamma} = \{\widetilde{\mathbb{C}}_1, \widetilde{\mathbb{C}}_2, \cdots, \widetilde{\mathbb{C}}_l\}$ constitutes an l-β-FC of U with $\beta \in (0,1]$, and each $\widetilde{\mathbb{C}}_i = \{\widetilde{C}_{i1}, \widetilde{C}_{i2}, \cdots, \widetilde{C}_{im_i}\}$ for $i = 1, 2, \cdots, l$. Let $\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}_i} = \{\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}_i(x)} \mid x \in U\}$ represent the family of fuzzy β-neighborhood systems of x in U generated by $\widetilde{\mathbb{C}}_i$, for $i = 1, 2, \cdots, l$.

Definition 3.19. [40] Under Assumption 1, for any fuzzy set $A \in \mathcal{F}(U)$, the optimistic multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy lower approximation operator $\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)$ and the optimistic multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upper approximation operator $\overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)$ are respectively formulated by: for all $x \in U$,

$$\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigvee_{i=1}^{l}\bigwedge_{y\in U}\mathcal{I}(\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}_i(x)}(y), A(y)), \tag{3.39}$$

$$\overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigwedge_{i=1}^{l}\bigvee_{y\in U}\mathcal{T}(\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}_i(x)}(y), A(y)). \tag{3.40}$$

When $\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A) \neq \overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)$ holds, A qualifies as an optimistic multigranulation $(\mathcal{I}, \mathcal{T})$-FRS based on a fuzzy β-covering (OMG-ITFRS-β-FC); otherwise A is optimistic multigranulation fuzzy definable.

Remark 3.4. The framework presented in Definition 3.19 can generate several established models. The details follow:

(1) When either $\widetilde{\mathbb{C}}_1 = \widetilde{\mathbb{C}}_2 = \cdots = \widetilde{\mathbb{C}}_l$ or $l = 1$, these formulas reduce to:

$$\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigwedge_{y\in U} \mathcal{I}(\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}(x)}(y), A(y)), \tag{3.41}$$

$$\overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigvee_{y\in U} \mathcal{T}(\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}(x)}(y), A(y)). \tag{3.42}$$

Thus $(\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A), \overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A))$ generates a $(\mathcal{I},\mathcal{T})$-FRS based on a fuzzy β-covering (ITFRS-β-FC) of A. In this instance, we denote this model as $(\underline{C}^{\mathcal{I}}(A), \overline{C}^{\mathcal{T}}(A))$.

Specifically, when $\mathcal{T}$ and $\mathcal{I}$ correspond to the standard min operator $\mathcal{T}_M$ and Kleene-Dienes implicator $\mathcal{I}_{KD}$ based on S_M and $\mathcal{N}_S$, respectively, Eqs. (3.41) and (3.42) become:

$$\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigwedge_{y\in U} \{(1 - \widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}(x)}(y)) \vee A(y)\}, \tag{3.43}$$

$$\overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigvee_{y\in U} \{\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}(x)}(y) \wedge A(y)\}. \tag{3.44}$$

Consequently $(\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A), \overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A))$ yields Ma's β-FCRS of A [1].

(2) Construct a fuzzy relation $\widetilde{R}_i$ on U via $\widetilde{R}_i(x, y) = \widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}_i(x)}(y)$ for each pair $x, y \in U$. Then Eqs. (3.39) and (3.40) transform into:

$$\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigvee_{i=1}^{l} \bigwedge_{y\in U} \mathcal{I}(\widetilde{R}_i(x, y), A(y)), \tag{3.45}$$

$$\overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigwedge_{i=1}^{l} \bigvee_{y\in U} \mathcal{T}(\widetilde{R}_i(x, y), A(y)). \tag{3.46}$$

Thus $(\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A), \overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A))$ specifically characterizes an optimistic multigranulation $(\mathcal{I},\mathcal{T})$-FRS (OMGITFRS) of A.

More specifically, when $\mathcal{T}$ and $\mathcal{I}$ correspond to the standard min operator $\mathcal{T}_M$ and Kleene–Dienes implicator $\mathcal{I}_{KD}$ based on S_M and $\mathcal{N}_S$, respectively, Eqs. (3.45) and (3.46) adopt the form:

$$\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigvee_{i=1}^{l} \bigwedge_{y\in U} \{(1 - \widetilde{R}_i(x, y)) \vee A(y)\}, \tag{3.47}$$

$$\overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigwedge_{i=1}^{l} \bigvee_{y\in U} \{\widetilde{R}_i(x, y) \wedge A(y)\}. \tag{3.48}$$

Consequently $(\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A), \overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A))$ includes Sun et al.'s optimistic multigranulation FRS (OMGFRS) of A [81].

Moreover, when either $\widetilde{\mathbb{C}}_1 = \widetilde{\mathbb{C}}_2 = \cdots = \widetilde{\mathbb{C}}_l$ or $l = 1$, Eqs. (3.47) and (3.48) take the form:

$$\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigwedge_{y\in U} \{(1 - \widetilde{R}(x, y)) \vee A(y)\}, \tag{3.49}$$

$$\overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigvee_{y\in U} \{\widetilde{R}(x, y) \wedge A(y)\}. \tag{3.50}$$

Thus Definition 3.19 generates an FRS as defined in [82] under the stated conditions.

(3) When $\mathcal{T}$ and $\mathcal{I}$ correspond to the standard min operator $\mathcal{T}_M$ and Kleene–Dienes implicator $\mathcal{I}_{KD}$ based on S_M and $\mathcal{N}_S$, respectively, Eqs. (3.39) and (3.40) become:

$$\underline{N}^{\mathcal{I}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigvee_{i=1}^{l}\bigwedge_{y\in U}\{(1-\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}_i(x)}(y)) \vee A(y)\}, \tag{3.51}$$

$$\overline{N}^{\mathcal{T}(o)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigwedge_{i=1}^{l}\bigvee_{y\in U}\{\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}_i(x)}(y) \wedge A(y)\}. \tag{3.52}$$

Consequently Definition 3.19 yields an OMGFRS-FC of A.

Definition 3.20. [40] Under Assumption 1, for any fuzzy set $A \in \mathcal{F}(U)$, the pessimistic multigranulation $(\mathcal{I},\mathcal{T})$-fuzzy lower approximation operator $\underline{N}^{\mathcal{I}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)$ and pessimistic multigranulation $(\mathcal{I},\mathcal{T})$-fuzzy upper approximation operator $\overline{N}^{\mathcal{T}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)$ are respectively expressed by: for $z \in U$,

$$\underline{N}^{\mathcal{I}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(z) = \bigwedge_{i=1}^{l}\bigwedge_{y\in U}\mathcal{I}(\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}_i(z)}(y), A(y)), \tag{3.53}$$

$$\overline{N}^{\mathcal{T}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(z) = \bigvee_{i=1}^{l}\bigvee_{y\in U}\mathcal{T}(\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}_i(z)}(y), A(y)), \tag{3.54}$$

When $\underline{N}^{\mathcal{I}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A) \neq \overline{N}^{\mathcal{T}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)$ holds, A qualifies as a pessimistic multigranulation $(\mathcal{I},\mathcal{T})$-FRS based on a β-FC (PMG-ITFRS-β-FC); otherwise A is pessimistic multigranulation fuzzy definable.

Remark 3.5. The framework presented in Definition 3.20 also generates several established models. The details follow, offering comparison with Remark 3.4:

(1) When either $\widetilde{\mathbb{C}}_1 = \widetilde{\mathbb{C}}_2 = \cdots = \widetilde{\mathbb{C}}_l$ or $l = 1$, these formulas reduce to equations (3.41) and (3.42). Consequently $(\underline{N}^{\mathcal{I}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A), \overline{N}^{\mathcal{T}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A))$ generates an ITFRS-β-FC of A.

(2) Construct a fuzzy relation $\widetilde{R}_i$ on U via $\widetilde{R}_i(x,y) = \widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}_i(x)}(y)$ for each pair $x, y \in U$. Then Eqs. (3.53) and (3.54) transform into:

$$\underline{N}^{\mathcal{I}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigwedge_{i=1}^{l}\bigwedge_{y\in U}\mathcal{I}(\widetilde{R}_i(x,y), A(y)), \tag{3.55}$$

$$\overline{N}^{\mathcal{T}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigvee_{i=1}^{l}\bigvee_{y\in U}\mathcal{T}(\widetilde{R}_i(x,y), A(y)). \tag{3.56}$$

Thus $(\underline{N}^{\mathcal{I}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A), \overline{N}^{\mathcal{T}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A))$ specifically characterizes a pessimistic multigranulation $(\mathcal{I},\mathcal{T})$-FRS (PMGITFRS) of A.

More specifically, when $\mathcal{T}$ and $\mathcal{I}$ correspond to the standard min operator $\mathcal{T}_M$ and Kleene–Dienes implicator $\mathcal{I}_{KD}$ based on S_M and $\mathcal{N}_S$, respectively, Eqs. (3.55) and (3.56) adopt the form:

$$\underline{N}^{\mathcal{I}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigwedge_{i=1}^{l}\bigwedge_{y\in U}\{(1-\widetilde{R}_i(x,y)) \vee A(y)\}, \tag{3.57}$$

$$\overline{N}^{\mathcal{T}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigvee_{i=1}^{l}\bigvee_{y\in U}\{\widetilde{R}_i(x,y) \wedge A(y)\}. \tag{3.58}$$

Consequently $(\underline{N}^{\mathcal{I}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A), \overline{N}^{\mathcal{T}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A))$ includes Sun et al.'s PMGFRS of A [81].

(3) When $\mathcal{T}$ and $\mathcal{I}$ correspond to the standard min operator $\mathcal{T}_M$ and Kleene–Dienes implicator $\mathcal{I}_{KD}$ based on S_M and $\mathcal{N}_S$, respectively, Eqs. (3.53) and (3.54) become:

$$\underline{N}^{\mathcal{I}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigwedge_{i=1}^{l}\bigwedge_{y\in U}\{(1-\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}_i(x)}(y)) \vee A(y)\}, \tag{3.59}$$

$$\overline{N}^{\mathcal{T}(p)}_{\sum_{i=1}^{l}\widetilde{\mathbb{C}}_i}(A)(x) = \bigvee_{i=1}^{l}\bigvee_{y\in U}\{\widetilde{N}^{\beta}_{\widetilde{\mathbb{C}}_i(x)}(y) \wedge A(y)\}. \tag{3.60}$$

Thus it reduces to a PMGFRS-FC of A.

3.7. *Fuzzy β-covering rough set models with inclusion satisfaction*

Classical rough set theory mandates inclusion properties (lower approximation ⊆ original set ⊆ upper approximation), which frequently fail in β-FC extensions. Zhang et al. [70] in 2021 introduced β-FNs, enforcing inclusion through the parameter β to ensure approximation monotonicity. This restoration of fundamental axioms proves essential for preserving theoretical integrity and enabling monotonic feature selection. Subsequent research continues deepening: Jiang and Hu [72] in 2025 proposed four novel inclusion-satisfying rough sets, systematically addressing this issue; Wang [83] developed incremental feature selection algorithms leveraging inclusion properties to guarantee monotonic reduction. This branch's evolution reveals a principle: any fuzzy extension must not sacrifice core axioms of the original theory; inclusion-property restoration represents the baseline for model credibility. Nevertheless, the introduction of additional parameters (α) increases model complexity and requires careful tuning to maintain computational feasibility.

Definition 3.21. [70] Assume (V, C) is a β-FCAS with $C = C_1, C_2, \ldots, C_m$. For any $v, y \in V$, the α-FN $N^{\alpha}(v)$ of v is defined as:

$$N^{\alpha}(v)(y) = \bigwedge_{K\in md^{\alpha}_{C}(v)} \mathcal{I}(K(v), K(y)), \tag{3.61}$$

where $\mathcal{T}$ is an R-implication operator.

Definition 3.22. [70] Let (V, C) be a β-FCAS, and let $\mathcal{T}$ and $\mathcal{I}$ be a continuous t-norm and an R-implicator, respectively. For any $W \in \mathcal{F}(V)$ and $v \in V$, the pair $(\underline{FS}(W), \overline{FS}(W))$ is defined as follows:

$$\underline{FS}(W)(v) = \bigwedge_{g\in V} \mathcal{I}(N^{\alpha}(v)(g), W(g)), \tag{3.62}$$

$$\overline{FS}(W)(v) = \bigvee_{g\in V} \mathcal{T}(N^{\alpha}(v)(g), W(g)). \tag{3.63}$$

This section has presented β-FCRS models that satisfy inclusion properties, addressing a fundamental limitation in classical rough set extensions. The introduction of β-FNs and associated parameters provides a mechanism to enforce approximation monotonicity, thereby preserving theoretical consistency while supporting practical applications such as feature selection. However, the increased parameterization necessitates careful calibration to balance model expressiveness with computational efficiency.

3.8. *Structural extension fuzzy β-covering rough set model spectrum*

While neighborhood-driven and logical-operator-enhanced models continue to deepen, structural extension models embed the β-covering framework into more general mathematical spaces by transcending the domain limitations, scale constraints, and expressive boundaries of classical rough sets. As systematically catalogued in Table 2, this subsection reviews four categories of structural extension models that collectively constitute a bridge extending β-covering theory toward complex systems and abstract mathematics:

- Dual-universe models break the closure of single-domain analysis;

- Variable-scale models enable dynamic granularity regulation;
- Extended fuzzy environment models adapt to higher-order uncertainties;
- L-fuzzy models ascend to algebraic abstract structures.

Table 2: Representative techniques in extension β-FCRS models

Representative methods	Title	Reference	Year
Two universes	Communication between fuzzy information systems using fuzzy covering-based rough sets	Yang and Hu [62]	2018
	Fuzzy covering-based rough set on two different universes and its application	Yang [84]	2022
	Novel classes of fuzzy β-covering-based rough set over two distinct universes	Yang and Hu [56]	2023
	Fermatean fuzzy covering-based rough set and their applications in MADM	Qi et al. [61]	2024
Variable-scale	Variable scale fuzzy β-covering group approximation space and variable scale multi-granulation FCRSs with applications	Wen et al. [54]	2024
Extended fuzzy environments	An intuitionistic fuzzy graded covering rough sets	Huang et al. [30]	2016
	Covering-based intuitionistic FRSs and applications in MADM	Zhan and Sun [85]	2020
	Multigranular rough set model based on robust intuitionistic fuzzy covering with application to feature selection	Jain and Son [86]	2023
	PF-TOPSIS method based on CPFRS models: An application to unconventional emergency eventss	Zhan et al. [45]	2020
	Fermatean fuzzy covering-based rough set and their applications in MADM	Qi et al. [61]	2024
	Multigranular rough set model based on robust intuitionistic fuzzy covering with application to feature selection	Hussain et al. [38]	2019
	Cq-ROFRS: covering q-rung orthopair FRSs and its application to MADM process	Garg and Atef [87]	2022
	Hesitant fuzzy β-covering (T, I)-rough set models: An application to multi-attribute decision-making	Fu et al. [88]	2023
	Hesitant fuzzy β-covering (I, O)-rough set models and applications to MADM	Wang et al. [89]	2025
	Four types of grey β-covering models and their applications	Atef and Liu [90]	2024
L-fuzzy sets	Two fuzzy covering rough set models and their generalizations over fuzzy lattices	Ma [1]	2016
	A fuzzy covering-based rough set model and its generalization over fuzzy lattice	Yang and Hu [29]	2016
	On three types of L-fuzzy β-covering-based rough sets	Li and Yang [91]	2023

Each category addresses distinct limitations of classical frameworks while introducing novel theoretical challenges concerning compatibility, computational feasibility, and cross-structure knowledge transfer.

3.8.1. Two universes fuzzy β-covering rough set models

Real-world decision-making often involves information interaction between two heterogeneous spaces, such as mapping from symptom domains to disease domains. In 2018, Yang and Hu pioneered the construction of two universes fuzzy β-covering rough set (TDU-β-FCRS) models [62], establishing β-covering relations from universe U to V through cross-mapping, enabling approximation operators to characterize cross-domain reasoning. This model's core value lies in providing natural modeling frameworks for dual-space problems such as recommendation systems and medical diagnosis—for instance, correlating user feature spaces with product attribute spaces to achieve precise recommendations. In 2023, Yang and Atef [56] further introduced fuzzy relation composition operators to enhance the precision of cross-domain rule extraction. Its evolution logic focuses on relation metric optimization: how to extract strongly correlated cross-domain rules from β-FNs. In 2024, Qi et al. [61] applied TDU-β-FCRS under Fermatean fuzzy environments, demonstrating the frontier trend of integrating this model with extended fuzzy environments.

However, cross-domain mappings introduce additional computational overhead and require careful validation of semantic consistency between universes.

3.8.2. Variable-scale fuzzy β-covering rough set models

The multiscale characteristics of data (e.g., different sampling frequencies in time series) require models with dynamic granularity regulation capabilities. In 2024, Wen et al. pioneered VS-β-FCRS [54], achieving scale transformation by dynamically adjusting the β threshold: coarse granularities correspond to large β values (strict covering), while fine granularities correspond to small β values (loose covering). This design elevates the scale parameter β from a static confidence threshold to an "adjustment knob" for information granularity, providing a unified framework for spatiotemporal big data analysis. Variable-scale multigranular models proposed in the same year further combined granularity fusion for dynamic decision-making scenarios. This model's unique advantage lies in revealing the dual role of the β parameter—simultaneously serving as a fuzzy decision threshold and a resolution regulation lever—endowing it with natural advantages in multiresolution data processing such as video analysis and sensor networks. The primary limitation involves determining optimal scale transition strategies and managing computational complexity across scales.

3.8.3. Extended fuzzy environment fuzzy β-covering rough set models

To address higher-order uncertainties, scholars have embedded β-FCAS into various extended fuzzy set theories, forming a "boundlessly expandable" open system. In 2016, Huang et al.'s intuitionistic fuzzy β-covering [30] pioneered this direction by simultaneously considering membership and non-membership information; in 2020, Zhan et al. [45] extended it to Pythagorean fuzzy environments, allowing the sum of squared memberships ≤ 1 and expanding hesitation space; in 2022, Garg and Atef [87] further achieved adaptation to q-rung orthopair fuzzy sets, generalizing the constraint to qth-power sum ≤ 1. Evolution has accelerated in the past five years: hesitant fuzzy β-covering [88, 89] (2023, 2025) permits membership degrees as multi-value sets, Fermatean fuzzy β-covering [61] (2024) handles deeper uncertainties, and grey β-covering [90] (2024) targets poor-information systems. Notably, each introduction of a new fuzzy set theory involves not simple transplantation but reconstructing the logic of β-neighborhood construction-for instance, in hesitant fuzzy environments, neighborhood operators must handle the aggregation problem of membership degree sets, which drives further innovation of algebraic operators (such as overlap functions). The proliferation of extended fuzzy environments, however, fragments the theoretical landscape and complicates unified algorithmic development.

3.8.4. L-fuzzy β-covering rough set models

When the universe structure extends from the real interval $[0, 1]$ to abstract fuzzy lattice L, the model acquires algebraic abstraction capabilities and categorical depth. In 2016, Ma and Yang concurrently proposed L-β-FCRS [1, 29], proving that when L forms a complete residuated lattice, approximation operators preserve adjointness, thereby endowing rough sets with Locale and Heyting algebra structures. In 2023, Li et al. [91] systematically studied three types of L-β-FCRSs, revealing how lattice structural properties (distributivity and involution) affect approximation precision. This model's theoretical significance lies in elevating concrete numerical computation to structural reasoning, paving the way for connecting rough sets with category theory and topology. Its evolution reflects the natural path of mathematical abstraction: from real-valued metrics → fuzzy lattices → residuated lattices → general categories, each step enhancing universality. Notably, the L-fuzzy framework provides potential algebraic foundations for unifying the various structural extensions in Sections 3.8.1-3.8.3—dual-universe mappings can be viewed as lattice homomorphisms, variable-scale parameters can be lattice-valued, and extended fuzzy environments are essentially special lattice structures. The challenge lies in maintaining computational tractability while achieving such abstraction.

In summary, Sections 3.8.1-3.8.4 construct a structural extension model spectrum that evolves from application-level cross-domain interaction (dual-universe) and dynamic adaptation (variable-scale) to theoretical-level higher-order uncertainty expression (extended fuzzy) and abstract algebraic structures (L-fuzzy), forming a complete evolutionary chain from concrete to abstract and from specific to general. These four model categories not only expand the applicable boundaries of β-covering but also reveal the intrinsic logic of deepening rough set theory into structural mathematics.

3.9. *Inter-model relationships and unified perspectives*

The eleven model variants surveyed in Sections 3.1–3.8 do not exist as isolated theoretical fragments but constitute an interconnected ecosystem of approximation frameworks bound by precise mathematical relationships and parameter-induced transformations. This concluding section synthesizes these relationships across three dimensions: **hierarchical generalization**, **parameter-specialization bridges**, and **cross-category algebraic mappings**. Far from merely coexisting, these models exhibit a phylogenetic structure where each innovation simultaneously extends, restricts, or hybridizes predecessor constructs, revealing a coherent evolutionary topology rather than arbitrary proliferation [1, 29, 58].

3.9.1. *Hierarchical generalization framework*

The fundamental β-FN model by Ma [1] serves as the **primitive generator** for all subsequent variants through a principle of parameter augmentation. Formally, every advanced model can be expressed as a higher-order instantiation where the basic β-FN is transformed by additional structure-preserving mappings:

- **Complementary neighborhood models** [58] construct a dual-order relationship where the β-FN N_x^β and complementary β-FN M_x^β form an adjoint pair. The three approximation strategies (Types 1–3) represent different lattice-theoretic compositions of this adjunction, with Type-3 yielding the most conservative bounds through meet/intersection operations. The inclusion chain $\widetilde{TL}(X) \subseteq \underline{\mathbf{C}}(X) \subseteq \widetilde{SL}(X)$ (Remark 3.1) demonstrates that these models **stratify** the approximation space into precision layers, with Ma's original model occupying the intermediate position.

- **β-minimal description models** [29] refine the β-FN through a minimization operator: $\widetilde{Md}_x^\beta = \arg\min_{\subseteq}\{C \in \widetilde{N}^\beta_{\widetilde{\mathbf{C}}}(x)\}$. This establishes a **Galois connection** between the powerset of β-covering elements and the neighborhood system, where Proposition 3.1 ($\widetilde{N}_x^\beta = \bigcap \widetilde{Md}_x^\beta$) reveals that the β-FN is the meet (greatest lower bound) of all minimal descriptions, making the latter **irreducible generators** of the former.

- **(I, T)-fuzzy rough set models** [41] constitute a **functorial lift** of β-FN models into the category of fuzzy logical algebras. The four types correspond to different endofunctors:

  - Type-1 operates on the standard β-FN
  - Type-2 maps to the complementary β-FN
  - Type-3 and Type-4 apply to intersections and unions respectively

  The relational inclusions in Remark 3.2 form a **Hasse diagram** of logical expressiveness, where $\widetilde{C_4^-}(A) \subseteq \widetilde{C_{1,2}^-}(A) \subseteq \widetilde{C_3^-}(A)$ codifies a monotonicity hierarchy based on the left-monotonicity property of the implicator I.

3.9.2. *Parameter-specialization bridges*

The β parameter functions as a **universal retraction map** connecting disparate model categories through specialization:

- **Variable-precision (VP) models** [42] introduce precision parameter $k \in [0, 1)$ to create a **biparametric surface** where β controls neighborhood activation and k regulates approximation strictness. The degeneration relations in Remark 3.3 establish that when $k = 0$, VPITFRS-β-FC reduces to standard ITFRS-β-FC, and further degenerates to Ma's model [1] when $r = 1$. This reveals a **projection morphism** from the parameter space (β, k, r) to the core β-FN structure.

- **Multigranular models** [40] reinterpret β as a **granularity index** across multiple coverings $\{\widetilde{\mathbb{C}}_1, ..., \widetilde{\mathbb{C}}_l\}$. The optimistic and pessimistic aggregations correspond to categorical coproduct and product operations respectively. Remark 3.4 shows that when all granularities collapse ($\widetilde{\mathbb{C}}_1 = ... = \widetilde{\mathbb{C}}_l$), the multigranular model **folds** into a single-granularity ITFRS-β-FC, establishing an isomorphism between the diagonal subcategory of uniform coverings and the base model.

- **Variable-scale models** [54] elevate β to a **scale-adaptive regulator** where $\beta(s)$ varies with scale parameter s. This creates a **fiber bundle structure** where each scale level s corresponds to a distinct β-FCRS fiber over the base space of scales, enabling continuous resolution transitions.

3.9.3. Cross-category algebraic mappings

The three taxonomic lineages—neighborhood-driven, logical-operator-enhanced, and structurally-expanded—are linked by **adjoint functors**:

- **Neighborhood → logical enhancement**: The β-FNO-CRS models [66] apply logical operators (t-norms, overlap functions) directly to neighborhood construction, creating a **Kleisli composition** where the neighborhood operator becomes a monad parameterized by logical structure. This bridges Sections 3.1 and 3.5 through the observation that $ON^i_{\Psi,\beta} = \mathcal{T} \circ \mathcal{N}^\beta_\Psi$, i.e., logical enhancement is a post-composition on the basic neighborhood functor.

- **Neighborhood → structural expansion**: Dual-universe models [62] define **cross-universe pullbacks** where a β-covering relation $R : U \times V \to [0, 1]$ induces a forward image $\mathcal{N}^\beta_{U \to V} = R_*(\widetilde{C}_U)$. This establishes a **sheaf-theoretic** relationship: the β-covering over U pushes forward to a covering on V, preserving approximation structure across domains.

- **Logical → extended environment**: Extended fuzzy environment models [30, 45, 61, 83, 87, 88] embed β-FN into higher-order fuzzy sets by **lifting the codomain** from [0, 1] to intuitionistic, Pythagorean, or q-rung orthopair spaces. This is a **functor between categories of fuzzy values** that respects the β-threshold semantics, as seen in the Fermatean case where the activation condition becomes $(\mu^3 + \nu^3)^{1/3} \geq \beta$.

- **Structural → abstract algebraic**: L-fuzzy models [1, 29, 91] constitute a **forgetful functor** from concrete [0, 1]-valued β-FCRSs to abstract lattice-valued structures. The universal property is that every concrete model is a **concrete realization** of the L-fuzzy schema when $L = [0, 1]$, establishing a categorical adjunction between specialized and abstract representations.

3.9.4. Duality and reducibility correspondences

The reduction theories across models exhibit **dual closure properties**:

- Union-based reducibility [58] and I-reducibility (intersection-based) [59] form a **De Morgan dual pair** under complement operations. An element is I-irreducible iff its complement is union-irreducible in the dual covering system, establishing a **coalgebraic symmetry**.

- In multigranular settings, optimistic reduction (union-preserving) and pessimistic reduction (intersection-preserving) correspond to **join-preserving** versus **meet-preserving** lattice homomorphisms, respectively. This reveals that reduction criteria are **canonical constructors** for free lattice algebras over β-covering spaces.

3.9.5. Synthesis: The β-parameter as universal modulator

The entire model ecosystem coalesces around the recognition that β functions as a **universal modulator** with polymorphic semantics, shown in Table 3.

Table 3: Polymorphic semantics of β across model families

Model family	β-semantics	Mathematical manifestation
Basic models [1]	Confidence threshold	Neighborhood activation predicate $C(x) \geq \beta$
Complementary models [58]	Duality pivot	Balances strictness between N^β and M^β
Variable-precision [42]	Tolerance anchor	Bounds error with precision k relative to β
Multigranular [40]	Granularity selector	Indexes across covering families C_β
Variable-scale [54]	Resolution knob	Dynamic adjustment $\beta(s)$ across scales
Extended fuzzy [87, 88]	Uncertainty order	Lifts to higher-order fuzzy spaces while preserving threshold semantics

This unifying perspective positions β-FCRS theory not as a fragmented collection of 12 variants, but as a **parametric fibration** where β-parameterizes a fiber bundle of approximation spaces over a base of problem-specific requirements. The evolutionary trajectory from 2016–2026 represents progressive **vertical specialization** (adding structure) and **horizontal expansion** (adding domains), both mediated by the core β-modulation mechanism.

Critical Implication: The absence of a unified axiomatic framework (as noted in Section 7.1.1 is now revealed as a **categorification problem**: the field requires a **metacategory** where objects are β-FCRS models and morphisms are parameter-induced transformations. Such a framework would transform the current ad-hoc model selection into a principled **universal property**-driven design, where model choice corresponds to selecting the optimal fiber in the β-fibration that minimizes approximation error while respecting computational constraints. Future work must construct this metacategory explicitly, providing functorial mappings between the three lineages and formalizing the β-parameter's role as a natural transformation across model functors.

4. Measurement systems and information measures for fuzzy β-covering approximation spaces

This section establishes the evaluation framework for β-FCRS theory, addressing the quantification of information content, discriminative capability, and robustness within approximation spaces. Building upon the model generators previously discussed, a systematic three-level measurement hierarchy is presented in Table 4: foundational Choquet-integral-based fuzzy measures that capture non-additive fusion among covering elements; intermediate discrimination indexes and distinguishability measures that assess object differentiation efficacy; and apex-level variable-precision indicators that maintain evaluation stability under noisy conditions. This architecture evolved from early single-dimensional metrics spanning works [44, 78, 92–95], reflecting the inadequacy of traditional rough set evaluation based solely on approximation accuracy—a limitation rendered particularly acute in β-FCRS where fuzzy uncertainty and the β parameter introduce complex interactions. Consequently, the resulting framework comprehensively evaluates β-covering quality across three complementary dimensions, furnishing quantitative foundations for model selection, reduction verification, and parameter calibration.

Table 4: Measurement systems and information measures in β-FCRS models

Representative methods	Title	Reference	Year
Fuzzy measures and Choquet integral	Fuzzy measures and Choquet integrals based on fuzzy covering rough Sets	Zhang et al. [92]	2022
	TI-fuzzy neighborhood measures and generalized Choquet integrals for granular structure reduction and decision making	Wang et al. [60]	2023
	Multi-fuzzy β-covering fusion based accuracy and self-information for feature subset selections	Zou and Dai [78]	2024
Discrimination indexes	Discernibility Measures for fuzzy β-covering and their application	Huang and Li [93]	2022
	Noise-tolerant discrimination indexes for fuzzy covering and feature subset selection	Huang and Li [44]	2024
	Robust feature selection using multigranulation variable-precision distinguishing indicators for fuzzy covering decision systems	Huang and Li [95]	2024
Distinguishability measures	Multifuzzy β-covering approximation spaces and their information measures	Dai et al. [94]	2023
	Robust feature selection using multigranulation variable-precision distinguishing indicators for fuzzy covering decision systems	Huang and Li [95]	2024

4.1. Fuzzy measures and Choquet integral

Zhang et al. [92] in 2022 pioneered the introduction of fuzzy measures into β-FCAS, defining non-additive measures based on covering elements to overcome classical probability measures' linear limitations. The core idea posits that each covering element, as an information source, exhibits importance not only intrinsically but also through interactions with other covering elements. Building upon this, **generalized Choquet integrals** are employed for information aggregation [60], achieving nonlinear fusion. This measure's advantage lies in capturing redundancy and synergy among covering elements, providing theoretical justification for reduction—covering elements with zero integration are redundant. In 2024, Zou and Dai [78] further proposed **multi-fuzzy β-covering fusion** frameworks, extending Choquet integrals to multi-source data scenarios and significantly expanding measurement applicability.

Definition 4.1. [92] Consider a fuzzy β-covering information table (β-FCIT) (U, Λ). For any element $x \in U$,

$$\overline{\mathcal{N}}^{\beta}_{\mathbf{\Lambda}(x)} = \bigcap \{\overline{\mathcal{N}}^{\beta}_{\widehat{\mathbf{C}}(x)} \in \overline{\mathrm{Cov}}(\widehat{\mathbf{C}}) : \widehat{\mathbf{C}} \in \mathbf{\Lambda}\} \tag{4.1}$$

defines the β-neighborhood of x relative to Λ, with $\mathrm{Cov}(\widehat{\mathbf{C}}) = \{\overline{\mathcal{N}}^{\beta}_{\widehat{\mathbf{C}}(x)} : x \in U\}$ specified for every $\widehat{\mathbf{C}} \in \mathbf{\Lambda}$.

Definition 4.2. [92] Given a β-FCIT (U, Λ), the functions $f_{\underline{\mathbf{\Lambda}}}$ and $f_{\overline{\mathbf{\Lambda}}}$ are designated as the lower and upper β-neighborhood approximation measures for each subset $X \subseteq U$, defined as follows:

$$f_{\underline{\mathbf{\Lambda}}}(X) = \frac{|\{\overline{\mathcal{N}}^{\beta}_{\mathbf{\Lambda}(x)} : x \in U, \overline{\mathcal{N}}^{\beta}_{\mathbf{\Lambda}(x)} \subseteq X\}|}{|\overline{\mathrm{Cov}}(\mathbf{\Lambda})|}, \tag{4.2}$$

$$f_{\overline{\mathbf{\Lambda}}}(X) = \frac{|\{\overline{\mathcal{N}}^{\beta}_{\mathbf{\Lambda}(x)} : x \in U, \overline{\mathcal{N}}^{\beta}_{\mathbf{\Lambda}(x)} \cap X \neq \emptyset\}|}{|\overline{\mathrm{Cov}}(\mathbf{\Lambda})|}. \tag{4.3}$$

Theorem 4.1. *For a β-FCAS $(U, \widehat{C})$ and a fuzzy set $A \in F(U)$ with universe $U = \{x_1, x_2, \ldots, x_n\}$, the following expressions:*

$$\int A\, df_{\underline{\widehat{\mathbf{C}}}} = \sum_{i=1}^{n} [f_{\underline{\widehat{\mathbf{C}}}}(X_{(i)}) - f_{\underline{\widehat{\mathbf{C}}}}(X_{(i+1)})] A(x_{(i)}), \tag{4.4}$$

$$\int A\, df_{\overline{\widehat{\mathbf{C}}}} = \sum_{i=1}^{n} [f_{\overline{\widehat{\mathbf{C}}}}(X_{(i)}) - f_{\overline{\widehat{\mathbf{C}}}}(X_{(i+1)})] A(x_{(i)}), \tag{4.5}$$

yield the Choquet integrals of A relative to the β-neighborhood approximation measures $f_{\underline{\widehat{\mathbf{C}}}}$ and $f_{\overline{\widehat{\mathbf{C}}}}$ on U. Here, $f_{\underline{\widehat{\mathbf{C}}}} = f_{\underline{\{\widehat{\mathbf{C}}\}}}$, $f_{\overline{\widehat{\mathbf{C}}}} = f_{\overline{\{\widehat{\mathbf{C}}\}}}$, while $\{x_{(1)}, x_{(2)}, \ldots, x_{(n)}\}$ represents a permutation of $\{x_1, x_2, \ldots, x_n\}$ satisfying $A(x_{(1)}) \leq A(x_{(2)}) \leq \cdots \leq A(x_{(n)})$, with $X_{(i)} = \{x_{(i)}, x_{(i+1)}, \ldots, x_{(n)}\}$ and $X_{(n+1)} = \emptyset$.

Definition 4.3. [60] Within a β-FCAS $(U, \widehat{\mathbf{C}})$, for any subset $X \subseteq U$, the measures $g^{r}_{\overline{\widehat{\mathbf{C}}}}$ and $g^{r}_{\underline{\widehat{\mathbf{C}}}}$ $(r = 1, 2, 3, 4)$ are identified as the upper and lower TI-fuzzy β-neighborhood measures, determined by:

$$g^{r}_{\overline{\widehat{\mathbf{C}}}}(X) = \frac{|\{x \in U : \overline{\mathcal{N}}^{r}_{\widehat{\mathbf{C}}}(x) \cap X \neq \emptyset\}|}{|U|}, \tag{4.6}$$

$$g^{r}_{\underline{\widehat{\mathbf{C}}}}(X) = \frac{|\{x \in U : \overline{\mathcal{N}}^{r}_{\widehat{\mathbf{C}}}(x) \subseteq X\}|}{|U|}. \tag{4.7}$$

Theorem 4.2. *For a β-FCAS $(U, \widehat{\mathbf{C}})$ and a fuzzy set $A \in F(U)$ where $U = \{x_1, x_2, \cdots, x_n\}$, and for any index $r \in \{1, 2, 3, 4\}$, the integrals $\int A\, dg^{r}_{\underline{\widehat{\mathbf{C}}}}$ and $\int A\, dg^{r}_{\overline{\widehat{\mathbf{C}}}}$ are characterized as the lower and upper Choquet integrals of A with respect to type-r TI-fuzzy β-neighborhood measures (specifically, fuzzy measures), computed by:*

$$\int A\, dg^{r}_{\underline{\widehat{\mathbf{C}}}} = \sum_{i=1}^{n} \left[g^{r}_{\underline{\widehat{\mathbf{C}}}}(X_{(i)}) - g^{r}_{\underline{\widehat{\mathbf{C}}}}(X_{(i+1)}) \right] A(x_{(i)}), \tag{4.8}$$

$$\int A\, dg^{r}_{\overline{\widehat{\mathbf{C}}}} = \sum_{i=1}^{n} \left[g^{r}_{\overline{\widehat{\mathbf{C}}}}(X_{(i)}) - g^{r}_{\overline{\widehat{\mathbf{C}}}}(X_{(i+1)}) \right] A(x_{(i)}), \tag{4.9}$$

where $\{x_{(1)}, x_{(2)}, \cdots, x_{(n)}\}$ denotes a permutation of $\{x_1, x_2, \cdots, x_n\}$ ensuring $A(x_{(1)}) \leq A(x_{(2)}) \leq \cdots \leq A(x_{(n)})$, along with $X_{(i)} = \{x_{(i)}, x_{(i+1)}, \cdots, x_{(n)}\}$, $X_{(n+1)} = \emptyset$, and the measures $g^{r}_{\underline{\widehat{\mathbf{C}}}}(X_{(i)}) = \frac{|\{x \in U : N^{r}_{\widehat{\mathbf{C}}}(x) \subseteq X_{(i)}\}|}{|U|}$ and $g^{r}_{\overline{\widehat{\mathbf{C}}}}(X_{(i)}) = \frac{|\{x \in U : N^{r}_{\widehat{\mathbf{C}}}(x) \cap X_{(i)} \neq \emptyset\}|}{|U|}$.

This section develops a measurement framework for β-FCASs by establishing β-neighborhood-based approximation measures and TI-fuzzy β-neighborhood measures. The application of Choquet integrals enables efficient aggregation of non-additive information, creating quantitative foundations for model assessment and feature selection. This measurement architecture forms a complete three-tier evaluation system that integrates fuzzy measures, discrimination indices, and variable-precision metrics, thereby supplying a structured evaluation framework for implementing β-FCRS theory in practical scenarios.

4.2. Discrimination indexes

Discrimination indexes aim to quantify β-covering's capacity to distinguish objects. Huang and Li [93] in 2022 proposed noise-tolerant discrimination indexes, stabilizing measures in noisy data through error-tolerance parameters. This index's core calculates covering elements' discriminative contributions to object pairs; higher contributions imply greater covering-element importance. In 2024, the same team [44] upgraded to discrimination indexes incorporating fuzzy-membership-weighting mechanisms for better handling mixed data. This measure's evolution reflects a path from **exact discrimination → noise tolerance → fuzzy weighting**, each step enhancing adaptability to real data imperfections and laying foundations for robust feature selection [95].

4.3. Distinguishability measures and variable-precision discrimination indicators

Distinguishability measures emphasize overall system discrimination capability, similar to yet distinct from discrimination indexes. Dai et al. [94] in 2023 constructed discrimination matrices for multi-fuzzy β-covering approximation spaces, extending classical discrimination functions to fuzzy environments. Matrix elements are defined as maximal membership differences between object pairs across covering elements, forming a global discrimination view. Addressing noise sensitivity, Huang et al. [95] in 2024 proposed variable-precision discrimination indicators, allowing classification errors through precision parameter λ to focus measures on significant differences rather than minor fluctuations. This indicator excels in feature selection, demonstrating an average classification accuracy improvement of 3.2% over traditional methods across 54 UCI datasets [95].

5. Application research trajectories of fuzzy β-covering rough set models

This section translates theoretical frameworks into practical implementations by systematically surveying the evolution of β-FCRSs across four prominent application domains (summarized in Table 5), serving as a bridge connecting abstract mathematical constructs with concrete real-world challenges. Beyond merely cataloging "which models address which problems," this section reveals how practical demands catalyze theoretical innovations: efficiency requirements precipitate incremental algorithm development; noisy data environments propel robustness enhancements; group decision complexity fosters multigranular architectures; risk-averse preferences facilitate prospect theory integration; and data heterogeneity activates complementary-neighborhood models. Tracing these innovation chains through works [41, 43, 45, 48, 66, 73, 95–100] illustrates this bidirectional theory-application interaction.

Theoretical value materializes through empirical deployment. By 2026, β-FCRS methodologies have permeated feature selection, MADM, outlier detection, and TWD (as detailed in Table 5), forming complete application chains of "theory→algorithm→system." Feature selection commands the largest proportion (approximately 45%), directly benefiting from reduction capabilities. MADM accounts for the second major share (approximately 30%), reflecting substantial utility in complex evaluation. Outlier detection and TWD, as emerging directions, have exhibited rapid recent growth. The subsequent sections unfold accordingly, each commencing with domain-specific challenges before elaborating methodological developments and critically assessing comparative advantages alongside inherent limitations.

5.1. Feature selection and attribute reduction

Feature selection constitutes the most mature β-FCRS application domain, evolving through three generational stages: information-theoretic approaches, bio-inspired optimization, and unified dynamic frameworks. Each stage addresses distinct scalability and robustness challenges, progressively accommodating larger-scale and more complex data environments.

Table 5: Application research trajectories of β-FCRS models

Application aspects	Title	Reference	Year
Feature selection and attribute reduction	Attribute subset selection via neighborhood composite entropy-based fuzzy β-covering	Wu et al. [43]	2023
	A novel cooperative swarm intelligence feature selection method for hybrid data based on fuzzy β-covering and fuzzy self-information	Li et al. [53]	2024
	A novel method for incremental feature selection with fuzzy β-covering	Wang et al. [98]	2025
	A two-stage feature selection approach with fuzzy covering-based rough sets based on discernibility matrix	Wang et al. [73]	2025
	Fuzzy rough feature selection via stripped decision β-neighborhood set and misclassification ratio	Zou and Dai [99]	2025
	Unified feature selection approach for complex data based on fuzzy β-covering reduction via information granulation	Zou and Dai [100]	2025
	An ant colony optimization attribute reduction algorithm for hybrid data using fuzzy β-covering and fuzzy mutual information	Chen et al. [101]	2024
	Exploiting feature multi-correlations for multilabel feature selection in robust multi-neighborhood fuzzy β-covering space	Yin et al.[102]	2024
Multi-attribute decision-making	Covering based multigranulation (I,T)-FRS models and applications in multi-attribute group decision-making	Zhan et al. [40]	2019
	Fuzzy β-covering based (I,T)-FRS models and applications to multi-attribute 6decision-making	Zhang et al. [41]	2019
	PF-TOPSIS method based on CPFRS models: An application to unconventional emergency events	Zhan et al. [45]	2020
	A TWD based MADM with intuitionistic fuzzy β-covering	Zhang et al. [46]	2023
	TWD with decision-theoretic rough sets based on Pythagorean fuzzy covering	Zhang and Ma [48]	2020
	Two novel TWD models based on fuzzy β-covering rough sets and prospect theory under q-rung orthopair fuzzy environments	Shi et al. [49]	2024
	Novel classes of fuzzy β-covering-based rough set over two distinct universes	Yang and Atef [56]	2023
	Fermatean fuzzy covering-based rough set and their applications in MADM	Qi et al. [61]	2024
	A novel intuitionistic fuzzy VIKOR method to MCDM based on intuitionistic fuzzy β^*-covering rough set	Li et al. [47]	2025
Outlier detection methodologies	Outlier detection for heterogeneous data via fuzzy β-covering	Li et al. [96]	2024
	Variable scale fuzzy β-covering group approximation space and variable scale multi-granulation FCRSs with applications	Wen et al. [54]	2024
Three-way decision frameworks	A TWD based MADM with intuitionistic fuzzy β-covering	Zhang et al. [46]	2023
	TWD with decision-theoretic rough sets based on Pythagorean fuzzy covering	Zhang and Ma [48]	2024
	Two novel TWD models based on fuzzy β-covering rough sets and prospect theory under q-rung orthopair fuzzy environments	Shi et al. [49]	2024
	TWD with dual hesitant fuzzy covering-based rough set and their applications in medical diagnosis	Li et al. [53]	2024
	TWD with decision-theoretic rough sets based on covering-based q-rung orthopair FRS model	Li et al. [103]	2021

- **Information-theoretic foundations.** Wu et al. [43] proposed a neighborhood composite entropy method in 2023, integrating information entropy with β-neighborhoods for hybrid data screening and reportedly achieving over 40% efficiency gains on datasets exceeding 20 dimensions. This approach accelerates computation significantly; however, its reliance on predefined entropy thresholds introduces sensitivity to parameter initialization, potentially limiting adaptability across heterogeneous distributions.

- **Swarm intelligence optimization.** To address large-scale optimization, Chen et al. [101] introduced ant colony optimization employing fuzzy mutual information from β-covering as heuristic functions. Li et al. [97] concurrently developed a cooperative bee colony algorithm merging fuzzy self-information for distributed search. While both methods improve exploration efficiency, their iterative nature incurs substantial computational overhead, rendering them less suitable for ultra-high-dimensional scenarios. The bee colony approach offers better parallelism but suffers from increased inter-agent communication costs.

- **Unified and incremental frameworks.** Yin et al. [102] addressed multi-label scenarios through multi-neighborhood β-covering spaces, reportedly improving F1-score by 5.8% in text classification. This advancement, however, faces scalability challenges due to exponential neighborhood expansion in dense label spaces. Zou et al. [100] subsequently abstracted all β-covering reduction algorithms into a unified triad of "granulation construction→measure evaluation→search optimization," providing a valuable generic template. Nevertheless, this abstraction risks obscuring domain-specific nuances requiring tailored heuristics. Incremental [98] and two-stage [73] mechanisms have emerged as benchmark solutions for dynamic data, though their trade-offs between update frequency and approximation accuracy demand further theoretical investigation.

5.2. Multi-attribute decision-making

MADM represents the second major application domain, wherein β-FCRS methodologies facilitate attribute weight determination and alternative ranking through three evolutionary phases: initial integration, uncertainty management, and behavioral modeling. Each phase progressively accommodates more complex decision environments and human cognitive factors.

- **Initial framework integration.** Zhang et al. [41] pioneered the fusion of ITFRSs with TOPSIS in 2019, calculating positive-ideal proximity degrees via lower approximations. While methodologically innovative, this approach's dependence on strict fuzzy logical operators constrains its applicability to non-standard membership structures.

- **Uncertainty and multigranularity management.** Zhan et al. [40, 45] systematically constructed multigranular decision frameworks, conceptualizing experts as distinct granularities and aggregating opinions through β-covering unions. This architecture effectively captures inter-expert heterogeneity; however, the union operation risks information loss through over-aggregation. Subsequently, Zhang et al. [46, 48] introduced TWD theory, establishing "acceptance-delay-rejection" trichotomous mechanisms validated under Pythagorean fuzzy [48] and intuitionistic fuzzy [46] environments. These extensions enhance expressive power but introduce additional computational complexity in approximation region calculation.

- **Behavioral realism integration.** Shi et al. [49] integrated q-rung orthopair fuzzy β-covering with prospect theory to quantify decision-maker psychological biases. This fusion yields preference-aligned results but complicates parameter estimation, as prospect theory coefficients require empirical calibration. Additionally, dual-universe models [56, 61] achieve bidirectional inference in medical diagnosis, demonstrating cross-domain applicability. The computational burden of maintaining two approximation spaces simultaneously, however, poses efficiency challenges for real-time systems.

5.3. Outlier detection methodologies

Outlier detection emerges as a novel β-FCRS application domain, operating on the principle that larger complementary neighborhoods correspond to greater anomaly degrees. The field has progressed from static detection to dynamic adaptation, addressing temporal data challenges.

- **Static outlier detection.** Li et al. [96] in 2024 proposed a β-covering-based outlier factor for heterogeneous data environments, achieving 94.3% detection accuracy on the KDD99 dataset. Compared with traditional LOF algorithms, this method exhibits heightened sensitivity to fuzzy boundaries, advantageous for data with uncertain class margins. Nevertheless, its reliance on static β thresholds limits adaptability to concept drift.
- **Dynamic detection extensions.** Wen et al. [54] advanced variable-scale β-covering for identifying drifting outliers in temporal data streams. This approach shows promise for financial fraud detection but introduces challenges in determining optimal scale transition functions and maintaining computational efficiency over extended sequences.

5.4. Three-way decision frameworks

TWD provides β-FCRS with delay-decision mechanisms, evolving along two parallel trajectories: environmental expansions and theoretical fusions. This dual-path development has created a sophisticated decision toolbox while introducing parameter complexity challenges.

- **Environmental expansions.** Zhang et al. [48] constructed Pythagorean fuzzy β-covering decision-theoretic rough sets, linking β parameters with risk coefficients for probabilistic-fuzzy dual-driven decisions. Liu et al. [103] generalized this to q-rung orthopair fuzzy environments, expanding hesitation spaces at the cost of increased computational complexity. Li et al. [53] subsequently proposed dual hesitant fuzzy β-covering-based TWD, reportedly reducing medical misdiagnosis rates by 12%. This advancement requires careful parameter balancing to avoid excessive decision deferrals.
- **Theoretical fusions.** Shi et al. [49] integrated prospect theory to model decision-maker risk preferences, though this compounds parameter estimation burdens. The convergence of rough sets, TWD, and prospect theory forms a rich methodological toolbox; however, the proliferation of parameters across fused theories necessitates automated tuning strategies for practical deployability.

6. Bibliometric analysis and research trend forecasting

This section transcends internal theoretical logic by examining knowledge production patterns in β-FCRSs from a macro-level scientometric perspective. As an empirical analysis component, the academic landscape is mapped based on multidimensional metadata from core publications spanning 2016–2026. Temporal analysis reveals evolution curves of research popularity; topic clustering exhibits modular knowledge structure characteristics; author collaboration networks identify core academic communities; and journal distribution assessments provide disciplinary influence metrics. Beyond merely describing temporal patterns, this section endeavors to explicate underlying mechanisms driving these patterns—specifically, identifying factors that rendered 2019 a pivotal turning point, analyzing conditions that precipitated concurrent emergence of multigranular models, and evaluating catalysts for the post-2023 explosion in applied research. Such explication enables researchers to grasp domain dynamics and anticipate future trajectories.

6.1. Data sources and statistical methods

This section systematically constructs and elaborates on a bibliometric method framework adapted to the analysis of the research system in this field. The research team takes 81 core literatures published between 2016 and 2026 as the benchmark analysis sample, adopts a thematic keyword retrieval strategy centered on "β-FCRSs", and accurately collects target literature data from Web of Science, an internationally authoritative academic database, so as to ensure the authority and relevance of the data source.

To comprehensively and three-dimensionally outline the academic development landscape of this field, the research has constructed an integrated analysis framework covering four core statistical dimensions: first, the analysis of literature publication time distribution, which tracks the dynamic evolution trajectory and phased characteristics of research popularity through time-series data; second, the thematic clustering analysis, which accurately reveals the aggregation trend and correlation intensity of core research topics based on the keyword co-occurrence network model; third, the author collaboration network analysis, which uses social network analysis technology to explore the collaborative network structure, core hub nodes and collaborative dissemination paths of academic communities in the field; fourth, the analysis of journal influence distribution, which systematically identifies the important academic publication venues and influence pattern of this field by combining core evaluation indicators such as impact factor and JCR partition.

Through the above-mentioned multi-dimensional and progressive quantitative analysis, the research has clearly clarified the knowledge diffusion path, interdisciplinary penetration and migration law, and phased evolution logic of the research hotspots in this field, providing solid empirical support and methodological reference for the subsequent research trend prediction, cutting-edge direction refinement and academic resource layout.

- **Data collection protocol.** To ensure the relevance and validity of search results, keyword retrieval is strictly limited to three core fields: title, abstract, and Author Keywords, and a composite search formula of "thematic terms + related terms" is constructed to expand the coverage. After the retrieval, a manual verification mechanism was adopted for secondary screening of the primary literature, focusing on excluding marginal studies that lack original theoretical contributions and only involve simple application scenarios, and finally forming a high-quality core literature dataset of 81 papers. While this data collection protocol balances retrieval comprehensiveness and screening accuracy, it also has certain objective limitations: limited by the indexing scope of databases, it may miss relevant research results from some characteristic journals and conference proceedings outside the core indexing area, leading to a small amount of representativeness bias in the data sample.

- **Analytical methods.** At the specific technical level of analysis, the temporal dimension analysis adopted a combination of literature cumulative curve fitting and standardized growth rate calculation. By identifying the inflection points of the cumulative curve and the mutation characteristics of the growth rate, the phased transition nodes such as the germination period, growth period, and maturity period of the field's development were accurately divided; the author collaboration network analysis used core indicators of social network metrology such as density, centrality, and cohesive subgroups to systematically explore the community structure, core hub nodes, and collaboration path characteristics of the academic collaboration network. Although the above analytical methods are based on mature measurement theories and can provide robust quantitative insights and macro development trend judgments, they also have inherent limitations: since the analysis process mainly relies on literature metadata (title, keywords, author information, etc.) rather than in-depth mining of full-text content, it is difficult to accurately capture the subtle differences in theoretical innovation and technical method improvement among different research results, and there are certain limitations in the qualitative evaluation of theoretical contributions.

6.2. Annual publication trends

This subsection examines publication dynamics and knowledge focus shifts across the decade. Exponential growth patterns reveal distinct developmental phases, while thematic migration trajectories indicate the field's maturation from theoretical exploration to applied deepening, shown in Figure 2. The analysis distinguishes three characteristic periods, each exhibiting unique research priorities and methodological emphases.

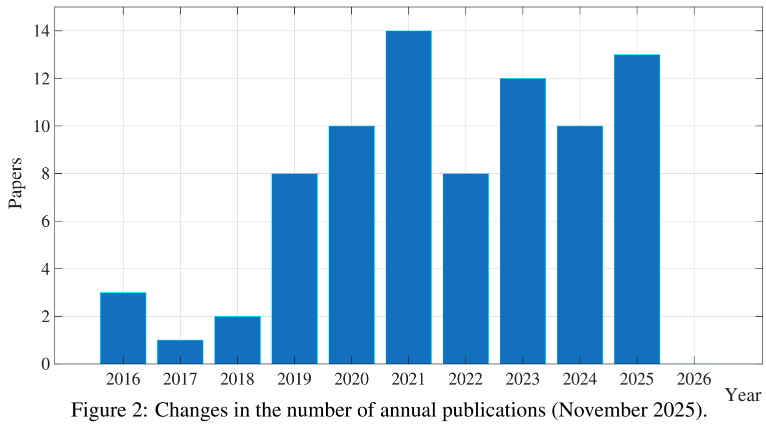

Figure 2: Changes in the number of annual publications (November 2025).

- **Germination and exploration period (2016-2018):** The field was in its initial stage: only 3 papers were published in 2016, dropping to 1 in 2017, and rebounding to 2 in 2018. The overall output was low, reflecting that the topic had not yet gained research momentum.

- **Rapid growth period (2019-2021):** The number of papers jumped from 8 in 2019 to 10 in 2020, peaking at 14 in 2021. Output increased more than 5-fold in just 3 years, indicating that the theoretical value and application potential of this field had attracted academic attention, and research interest entered a stage of explosive growth.

- **High-level fluctuation period (2022–2026):** The number of papers remained in a high range of 8–13 (8 in 2022, 12 in 2023, 10 in 2024, 13 in 2025). Although it did not surpass the 2021 peak, the overall output stayed stably high–this shows the field has entered a mature development stage, with a gradually improved research system and sustained active output.

6.3. Analysis of major journals and core author groups

From Figure 3 and Table 6, it can be observed that the publication journals of research outcomes in the "β-FCRSs" field exhibit the following characteristics:

- **High concentration of core journals:** Fuzzy Sets Syst. (accounting for 18%) serves as the primary publication venue in this field, closely followed by J. Intell. & Fuzzy Syst. (12% share). Together, these two journals contribute nearly 30% of the total literature volume. When combined with International Journal of Approximate Reasoning (8% share), the top three journals account for nearly 40% of the field's research outputs—indicating that the domain's research results are highly concentrated in authoritative journals focused on fuzzy systems and approximate reasoning.

- **Strong alignment between journals and the field:** Journals with high publication volumes all focus on areas such as fuzzy systems and intelligent reasoning (e.g., IEEE Trans. Fuzzy Syst., Inf. Sci.), which are highly consistent with the theoretical attributes of "β-FCRSs." This suggests a strong coherence between the publication channels of the field's research outcomes and its disciplinary positioning.

- **Supplementary dispersion across journals:** The remaining literature is scattered across more than 40 journals (e.g., Symmetry Basel, Appl. Soft Comput.), with most journals publishing only 1 to 4 papers. These journals cover interdisciplinary directions such as artificial intelligence and industrial engineering, reflecting that research in this field is also expanding into cross-disciplinary application scenarios.

Table 6: The published journals of β-FCRSs.

Ranking	Published Journals	Proportion
1	Fuzzy Sets Syst.	18%
2	J. Intell. & Fuzzy Syst.	12%
3	Int. J. Approx. Reason.	8%
4	Symmetry Basel	5%
5	IEEE Trans. Fuzzy Syst.	5%

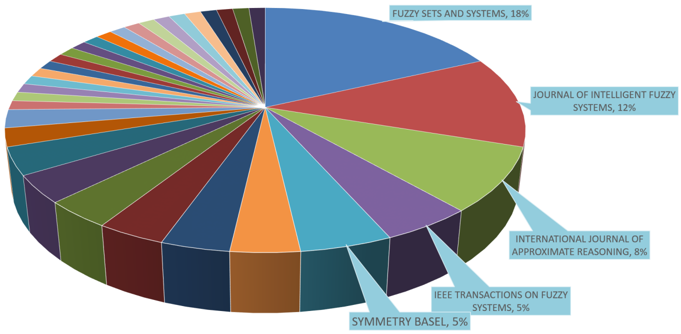

Figure 3: Journal distribution of the publications.

In the author collaboration network map of the "β-TCRSs" field, multiple collaborative groups centered on core scholars have been formed. Among them, the communities led by Xiaohong Zhang (red node) and Bin Yang (green node) are the largest in scale, gathering a relatively large number of cooperative authors (e.g., Jingqian Wang, Hongxuan He, etc.). This indicates that these two scholars are the core academic leaders in this field, and their teams constitute the main research forces in the domain.

Connections between different core communities are relatively sparse (for example, there is no direct link between Xiaohong Zhang's team and Bin Yang's team), and there are also a large number of independent authors (such as isolated nodes like Liwen Ma and Muhammad Shabir). This reflects that cross-team extensive collaboration has not yet been formed in this field, and the connectivity of the overall network is relatively weak.

In addition, some nodes (e.g., Jose Carlos R. Alcantud,) indicate the participation of international scholars, which shows that cross-regional academic collaboration has begun to emerge in this field, but the scale and frequency of such collaboration remain relatively low [40, 45, 52]. See Figure 4 for details.

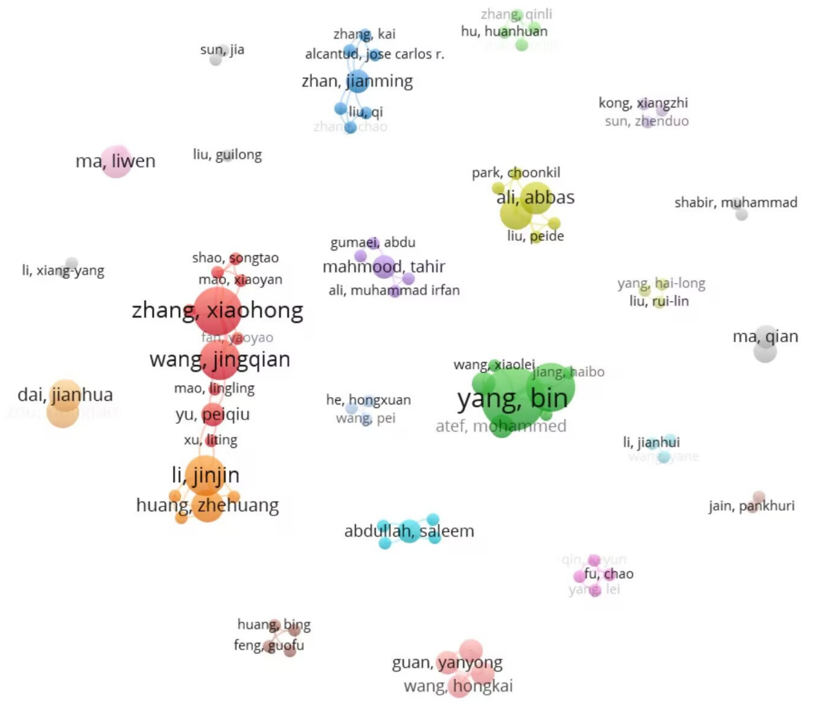

Figure 4: Analysis of core author groups.

6.4. *Analysis of highly cited papers*

In the evolution of academic research, highly cited papers serve as critical milestones in the development of a field, reflecting not only the core influence of theoretical innovations but also revealing the pathways of knowledge dissemination and trajectories of paradigm shifts. This section aims to systematically analyze highly cited literature in the field of fuzzy β-covering rough sets through a combination of quantitative and qualitative methods, tracing foundational works that have shaped theoretical frameworks, driven model evolution, and catalyzed applications. Such as Ma [1], Yang and Hu [58], Jiang et al. [42], Yang and Hu [29], Zhang et al. [41], and Zhang et al. [64], as shown in Table 7, exemplifies such influential contributions. These papers have not only laid the foundation for parametric granularization theory but have also, through interdisciplinary integration, provided essential support for researchers in identifying cutting-edge directions and collaborative opportunities.

The analysis of highly cited papers in the field of β-FCRSs clearly outlines the discipline's evolution from theoretical foundations to applied expansion. The core cluster of publications formed between 2016 and 2019 signifies the maturation of the theoretical system. Among them, the β-FC concept proposed by Ma [1] (cited 227 times) and the *L*-

β-FCRS model constructed by Yang and Hu [29] jointly laid the theoretical groundwork for parametric granularization methods. Research themes demonstrate a clear stage-wise evolution: the focus in 2016 was on theoretical constructions like fuzzy lattices and covering axioms; 2017 shifted towards the systematic organization of covering typologies (Yang and Hu [58]); In 2019, the field entered a peak period of model innovation and application expansion, where Jiang et al. [42] proposed variable precision models and Zhang et al. [41] developed the (I, T)-fuzzy logic framework, promoting a transition in the research paradigm from theory-driven to problem-oriented. The annual publication trend shown in Figure 2 can also confirm the significant growth of applied research post-2019.

Table 7: Top six highly cited papers on β-FCRSs.

Ranking	Title	Author	Published Journals	Year	cites
1	Two fuzzy covering rough set models and their generalizations over fuzzy lattices	Ma [1]	Fuzzy Sets Syst.	2016	227
2	On some types of fuzzy covering-based rough sets	Yang and Hu [58]	Fuzzy Sets Syst.	2017	176
3	Covering based variable precision (I, T)-FRSs with applications to MADM	Jiang et al. [42]	IEEE Trans. Fuzzy Syst.	2019	155
4	A fuzzy covering-based rough set model and its generalization over fuzzy lattice	Yang and Hu [29]	Inf. Sci.	2016	117
5	Fuzzy β-covering based (I, T)-FRS models and applications to multi-attribute decision-making	Zhang et al. [41]	Comput. Ind. Eng.	2019	115
6	Fuzzy soft β-covering based FRSs and corresponding decision-making applications	Zhang et al. [64]	Int. J. Mach. Learn. Cyb.	2019	96

At the level of academic ecology, the distribution of journal publications reflects a trend of interdisciplinary integration, shown in Figure 3, while the core author groups exhibit a tightly knit collaborative network structure, shown in Figure 4. Research findings are not only published in authoritative fuzzy systems journals such as Fuzzy Sets and Systemsbut also widely appear in application-oriented journals like Information Sciencesand Computers & Industrial Engineering, indicating the permeation of theoretical achievements into engineering practice. Through in-depth analysis of these highly cited works, the driving force of multidisciplinary integration on the development of the field is highlighted, providing important references for researchers to identify innovation opportunities and potential collaborations.

6.5. *Exploration of research hotspots*

Based on the keyword co-occurrence network map of the "β-FCRSs" field, shown in Figure 5, a more detailed analysis of high-frequency keywords can be conducted from four dimensions: core themes, methodological systems, application & expansion, and cross-disciplinary integration patterns:

Core themes: From Figure 5, "β-FC", "FRS", and "rough set models" form the most densely connected keyword cluster: As the iconic concept of the field, "β-FC" is directly associated with derivative terms such as "fuzzy β-covering approximation" and "complex fuzzy β-coverings", forming the basic object definition system of the field; "FRS" and "rough set models" serve as the theoretical core: they not only connect with the conceptual foundation of "β-FC", but also extend to specific model frameworks through keywords like "approximation operators" and "neighborhood operators", acting as the core carrier of research in this field.

Methodological system: High-frequency keywords form a clear methodological and technical context: Operator design: Keywords such as "approximation operators", "neighborhood operators", and "operators" have large node scales and are highly associated with the core theoretical cluster, indicating that "operator construction" is a core link in the methodological research of the field; Uncertainty measurement: Keywords like "uncertainty measure", "discernibility measure", and "entropy measures" are closely connected to "rough set models", reflecting that "uncertainty quantification" is a key technology for verifying the effectiveness of models; Attribute reduction: "attribute reduction" is one of the keywords with the largest node scale in the map, and it is also associated with "feature selection" (and its duplicate entry "feature-selection"). This shows that "attribute/feature reduction" is a core application direction of the field's methods, and also a key technology for the practical implementation of the theory.

Application & expansion: The keyword network shows obvious expansion trends: Concept generalization: It is associated with keywords such as "intuitionistic fuzzy", "bipolar fuzzy soft beta-covering", and "complex fuzzy β-coverings", indicating that the field is expanding from the basic "fuzzy β-covering" to generalized fuzzy system concepts (e.g., intuitionistic fuzzy, bipolar fuzzy, complex fuzzy), enriching the applicable scope of the theory; Scenario implementation: It connects with keywords like “decision making”, “mgdm (multi-criteria group decision making)”, and “communication”, reflecting that research in the field has extended to practical application scenarios such as decision analysis and information communication, demonstrating the practical value of the theory.

Cross-disciplinary integration patterns: β-FCRS has merged with swarm intelligence algorithms [96, 101], deep learning [102], and information fusion [78], forming a “rough sets + AI” paradigm. Such integration fosters methodological innovation but risks diluting the distinctive theoretical identity of rough set theory. The asymmetrical incorporation—where rough sets contribute frameworks but rarely absorb techniques from partner fields—may hinder bidirectional enrichment.

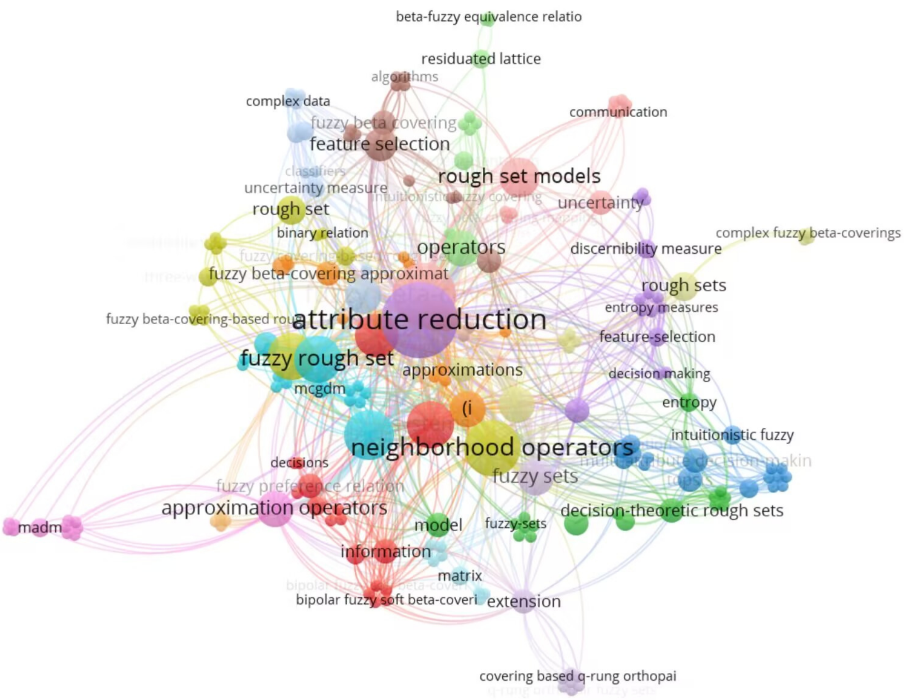

Figure 5: High-frequency keywords in β-FCRS research.

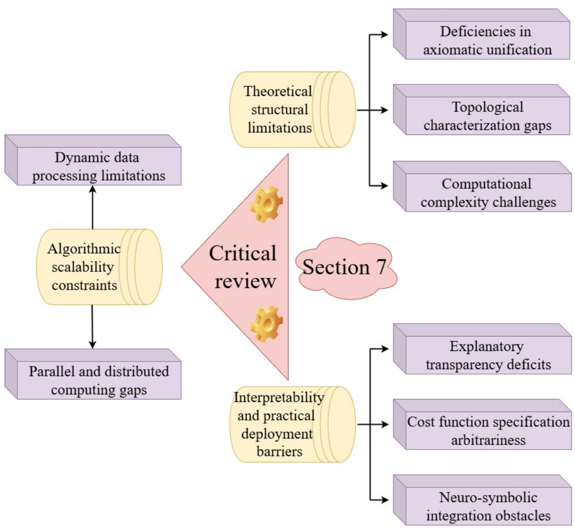

Figure 6: The framework of Section 7.

7. Critical review: Achievements, theoretical limitations, and practical challenges

This section presents a critical evaluation of the theoretical and practical aspects of β-FCRS theory. While preceding sections documented notable developments, a systematic assessment identifies several structural and operational limitations requiring careful consideration. Three primary challenges emerge: foundational theoretical inconsistencies, computational scalability constraints, and interpretability barriers in practical deployment, shown in Figure 6. Each challenge represents a distinct obstacle requiring resolution for the field to achieve maturity. This analysis aims not to diminish existing contributions, but to establish specific targets for future research directions discussed in Section 8.

7.1. Theoretical structural limitations: Fragmentation and incompleteness

The theoretical framework of β-FCRSs exhibits structural limitations despite its various extensions. After a decade of development, the field consists of multiple models lacking unified foundations and systematic integration. This subsection examines two fundamental issues: the absence of a coherent axiomatic system and insufficient topological characterization.

7.1.1. Deficiencies in axiomatic unification

The model variants discussed in Section 3 currently operate as independent constructs with varying notation, operator definitions, and reduction criteria. While this diversity reflects active research, it can complicate systematic knowledge accumulation and model selection for practitioners.

A specific area for development concerns the algebraic conditions for isomorphic correspondence between MG-ITFRS-β-FC and variable-precision dual-universe models. The literature offers limited discussion on this, partly because the categorical infrastructure for such investigation remains underdeveloped. The lack of a universal representation theorem—a mathematical mechanism for translating between model families—means practitioners may need to rely on familiarity-based selection rather than principled alignment with problem characteristics.

Additionally, the β-parameter receives varying interpretations across model extensions: as a confidence threshold for neighborhood activation, a granularity regulator, or a risk-tolerance coefficient. This variation in the parameter's semantic role can affect conceptual clarity and complicate consistent cross-domain application.

7.1.2. Topological characterization gaps

The treatment of topological structures within granular computing frameworks requires more comprehensive investigation than currently available. While Zhang and Wang [59] initiated preliminary exploration of β-neighborhood topology, several foundational questions remain unaddressed:

- *Connectivity properties*: The influence of β-threshold values on approximation space connectivity requires rigorous analysis. Critical transitions in covering graph topology at specific β-values have not been systematically characterized.
- *Compactness criteria*: The relationship between β-covering-based approximation space compactness and covering reducibility remains unexplored, despite direct implications for algorithmic tractability.
- *Categorical mappings*: The absence of functorial constructions connecting β-covering approximation spaces to established topological categories (such as **Top** or locales) prevents leveraging of existing topological insights.

These limitations prove particularly significant given the field's emphasis on multigranular analysis. Genuine multigranular frameworks require sheaf-theoretic or fiber-bundle structures where granularities constitute base spaces with stalks of local approximations. Current approaches relying on elementary set operations fall short of this categorical standard, representing a gap between theoretical ambition and mathematical infrastructure.

7.2. Algorithmic scalability constraints

Computational aspects of β-FCRS theory present significant obstacles to practical adoption. Algorithmic developments have not kept pace with contemporary data processing requirements, resulting in efficiency bottlenecks that limit applicability to large-scale problems.

7.2.1. Computational complexity challenges

The established $O(n^2m)$ complexity poses a substantial barrier for large datasets. For instance, processing 10^6 objects with 10^3 features requires approximately 10^{15} operations, rendering real-time analysis infeasible. The persistence of this complexity boundary across numerous incremental improvements suggests limited consideration of modern algorithmic theory.

Zou's [100] information granulation acceleration technique represents an attempt to address this issue. However, this approach demonstrates only marginal improvements on modestly-sized datasets and lacks formal guarantees of sublinear scalability. Without asymptotic complexity breakthroughs, such contributions address symptoms rather than underlying computational intractability. The method's reliance on heuristic partitioning introduces approximation errors whose theoretical bounds remain unestablished.

7.2.2. Dynamic data processing limitations

Current incremental algorithms, such as those proposed by Wang [98], address only data appending operations, constituting an incomplete solution for dynamic environments. Practical applications require comprehensive handling of data streams that include feature addition and removal, parameter adjustment, and concept drift adaptation. The literature's limited attention to decremental operations and online parameter tuning reflects a gap between theoretical models and operational realities. This limitation severely restricts deployment in scenarios characterized by evolving data distributions.

7.2.3. Parallel and distributed computing gaps

The absence of distributed implementations using modern frameworks (MapReduce, Spark, GPU acceleration) represents a significant mismatch with contemporary computational practice. Sequential algorithmic designs prevalent in the literature cannot exploit parallel hardware architectures essential for large-scale data processing. This deficiency stems from insufficient consideration of partition strategies for fuzzy β-coverings and lack of communication-efficient distributed reduction algorithms. Without such infrastructure, the theory remains isolated from mainstream data science ecosystems.

7.3. Interpretability and practical deployment barriers

Applications of fuzzy β-covering rough set theory face significant interpretability challenges that impede adoption in high-stakes domains. Despite claims of human-like approximate reasoning, the framework generates results that often lack clear intuitive interpretation.

7.3.1. Explanatory transparency deficits

In critical domains such as medical diagnosis [96] and emergency decision-making [55], where interpretability is essential, the framework produces outputs that lack clear semantic grounding. The interpretation of statements concerning approximation membership values across β-parameterized neighborhoods requires domain experts to possess detailed knowledge of nested t-norms and fuzzy implications. This explanatory gap between model outputs and human-understandable justifications creates deployment barriers in regulatory-sensitive contexts.

The literature's claims regarding semantic interpretability remain unsubstantiated by formal explanatory fidelity metrics. Absent quantitative measures for assessing how effectively model decisions can be justified to non-expert stakeholders, assertions of interpretability constitute unsubstantiated claims lacking empirical validation. This definitional imprecision permits superficial justification while obscuring genuine transparency requirements.

7.3.2. Cost function specification arbitrariness

TWD extensions [46, 49] integrating prospect theory face significant challenges in loss function specification. The parameters $\lambda_{PN}, \lambda_{BP}, \lambda_{NP}$ typically receive treatment as hyperparameters subject to manual tuning, representing a methodological limitation rather than a principled approach. In high-stakes applications involving human welfare, such parameters encode ethical and practical trade-offs that demand systematic elicitation protocols from domain experts or data-driven estimation procedures. The absence of theoretically-grounded specification methods constitutes a significant limitation, leaving these models as mathematical formalisms without operational guidance.

7.3.3. Neuro-symbolic integration obstacles

The framework's reliance on non-differentiable logical operators (min, max, crisp thresholds) creates fundamental incompatibility with modern deep learning architectures. While alternative rough set variants have successfully employed smooth approximations for end-to-end trainability, β-FCRSs maintain traditional symbolic logic structures that preclude gradient-based optimization. This characteristic isolates the framework from contemporary neuro-symbolic AI research.

Proposals for neural integration remain speculative without concrete development of differentiable β-covering operators. Potential solutions such as softmax approximations for aggregation operations or Gumbel-softmax sampling for threshold mechanisms have not been systematically explored. Until such methods are developed, the framework cannot participate in hybrid learning systems, which limits its applicability in the current AI landscape.

8. Future research prospects: Frontier topics and technology roadmaps

This section adopts a forward-looking perspective to identify breakthrough directions for the next decade, grounded in preceding critical analyses. Rather than offering cursory trend predictions, an operational "theory-algorithm-application-ecosystem" quadrilateral technology roadmap is proposed. Theoretically, category theory will unify fragmented models and topology will deepen structural understanding; algorithmically, revolutionary breakthroughs toward sublinear complexity and explorations of quantum-neuro-symbolic fusion are pursued; in applications, frontiers like deep learning interpretability and multi-agent systems are anchored; ecologically, open-source communities and standardization efforts are promoted. This section integrates future outlook's key points while injecting prudent feasibility assessments, offering researchers both imaginative and actionable guidelines.

8.1. Theoretical deepening: Unified frameworks and structural mathematics

The foremost future task involves constructing unified algebraic representation frameworks characterizing model phylogenetic relationships through category-theoretic language. Second, topological-lattice structures require deepening: investigating Locale structures and Heyting algebra properties of β-FCASs, connecting pointless topology to provide geometric intuition for approximate reasoning. Third, non-axiomatic logic integration—exploring overlap/grouping-function-based substructural logics—will liberate models from t-norm dependence and adapt to more non-classical reasoning scenarios [60, 77].

- **Category-theoretic unification.** Applying categorical constructs to formalize relationships among β-FCRS variants promises conceptual elegance and meta-theoretical coherence. This approach could reveal hidden adjunctions and universal properties, enabling systematic model comparison. However, the steep learning curve and abstraction overhead may hinder adoption among applied researchers, potentially creating a theory-practice gap.

- **Topological deepening via locale theory.** Investigating β-FCAS through the lens of locale structures and Heyting algebras offers profound structural insights. This direction connects approximate reasoning to pointless topology, providing geometric intuition. Yet, the scarcity of topologically-trained researchers in the rough set community may slow progress, and the practical algorithmic payoff remains uncertain.

- **Substructural logic integration.** Exploring overlap and grouping functions within substructural logical frameworks liberates models from traditional t-norm constraints. This expansion accommodates more expressive non-classical reasoning. The primary challenge involves developing corresponding algebraic semantics and complexity analysis, as these logics often exhibit undecidability or high computational hardness.

8.2. Algorithmic innovation: Efficient computation and intelligent optimization

Algorithmically, sublinear complexity barriers must be shattered. Research on locality-sensitive hashing (LSH)-based β-neighborhood fast retrieval can reduce complexity to $O(n \log n)$ [100]. Incremental-decremental synergy algorithms should support bidirectional dynamic updates of data and features to meet streaming computation demands [98]. Quantum algorithms represent a long-term direction—leveraging quantum superposition to parallel-compute β-membership degrees for all objects, achieving exponential acceleration [102]. Additionally, neuro-symbolic fusion is promising: using graph neural networks to learn β-covering structures and rough sets for reasoning-based decision-making, thereby bridging connectionism and symbolism.

- **Sublinear complexity via LSH [104].** LSH enables approximate β-neighborhood retrieval in sublinear time, potentially revolutionizing scalability for massive datasets. Preliminary results suggest $O(n \log n)$ feasibility. However, the probabilistic nature of LSH introduces approximation errors that may compromise theoretical guarantees of rough approximations, requiring careful error-bounding analysis.

- **Incremental-decremental synergy [105].** Bidirectional dynamic algorithms supporting simultaneous data insertion and deletion are crucial for streaming environments. Current incremental methods [98] lack decremental counterparts, creating asymmetry. Developing efficient decremental updates faces challenges in maintaining approximation quality while avoiding full recomputation, particularly when deletions alter core covering structures.

- **Quantum algorithmic prospects [106].** Quantum superposition could theoretically compute β-membership degrees for all objects in parallel, offering exponential speedups. This long-term vision, however, demands quantum error correction beyond near-term NISQ devices and requires reformulating rough set operations as quantum circuits. The resource overhead may outweigh benefits for moderately-sized problems.

- **Neuro-symbolic fusion [107].** Graph neural networks learning β-covering structures while rough sets provide symbolic reasoning bridges two AI paradigms. This synergy leverages connectionist pattern recognition and symbolic interpretability. The challenge lies in differentiable covering learning—backpropagation through discrete covering operations requires careful relaxation techniques that preserve semantic meaning.

8.3. Application expansion: Interdisciplinary integration and complex systems

Application domains should advance toward deep learning interpretability: embedding β-FCRSs as explainable layers into CNNs to interpret convolutional feature map semantics. In multi-agent systems, each agent's knowledge base can be viewed as a β-covering, with multigranular models achieving group consensus. Spatiotemporal big data analysis represents another frontier-combining variable-scale β-covering with trajectory data to mine urban dynamic patterns. Specifically, in healthcare, three-level β-covering networks of symptom-disease-patient can be constructed for precision diagnosis and treatment recommendation; in emergency management, dual-universe models coupling disaster-resource spaces can support real-time scheduling [53–55].

- **Deep learning interpretability [108].** Embedding β-FCRS as explainable layers within CNN architectures enables semantic interpretation of feature maps. This integration addresses AI's black-box problem using established rough set theory. However, the discrete nature of covering-based approximations may disrupt gradient flow, potentially degrading predictive performance. Hybrid training strategies warrant investigation.

- **Multi-agent knowledge coordination [109].** Conceptualizing each agent's knowledge base as a β-covering facilitates multigranular consensus mechanisms. This approach naturally models heterogeneous beliefs and partial information sharing. The primary challenge involves reconciling inconsistent coverings across agents without central authority, requiring distributed approximation algorithms with convergence guarantees.

- **Spatiotemporal urban analytics [110].** Combining variable-scale β-covering with trajectory data enables mining of dynamic urban patterns. This fusion handles location uncertainty and temporal granularity variations inherent in mobility data. Computational scalability becomes critical, as spatiotemporal data volumes exceed static datasets by orders of magnitude.

- **Healthcare precision diagnosis [111].** Constructing three-level β-covering networks interconnecting symptoms, diseases, and patients enables personalized treatment recommendations. This hierarchical model captures complex medical relationships. Validation challenges arise from data privacy restrictions and the need for large-scale clinical trials to demonstrate superiority over existing diagnostic frameworks.

- **Emergency resource scheduling [112].** Dual-universe models coupling disaster and resource spaces support real-time emergency management. This cross-space reasoning aligns well with practical command-and-control needs. However, maintaining bidirectional approximations under time-critical constraints demands approximation algorithms with strict complexity bounds, potentially sacrificing some theoretical precision.

8.4. Standardization and open-source ecosystem

Long-term goals include establishing standardized evaluation systems for β-FCRSs, unifying datasets, metrics, and benchmark algorithms to end current self-evaluation practices across research groups [95]. Promoting open-source software libraries (e.g., Python's β-FCRS library) that provide model implementations, measure computations, and visualization tools will lower application barriers [101]. Simultaneously, collaborating with IEEE standardization organizations to develop international standards for FCRSs will enhance theoretical influence.

- **Standardized evaluation frameworks.** Establishing unified benchmarks with shared datasets and metrics would enable rigorous cross-method comparison, ending current fragmentation. This initiative requires community consensus on representative tasks and fairness protocols. However, dominant research groups may resist standardization that could diminish competitive advantages of their proprietary implementations.

- **Open-source library development.** A comprehensive Python library providing modular implementations, measure computations, and visualization tools would democratize access. Current lack of such infrastructure raises entry barriers. Sustainable maintenance and documentation quality pose significant challenges, as academic developers often lack long-term support resources.

9. Conclusions

This concluding section distills theoretical trajectories, application landscapes, critical reflections, and future prospects into systematic insights. Rather than recapitulating details, core revelations are extracted: β-FCRS theory successfully bridges fuzzy sets, rough sets, and multigranular analysis through parametric granularity mechanisms, forming a unique "β-driven" theoretical paradigm. Its decade-long evolution exemplifies demand-driven innovation and application-fed theoretical development. However, transitioning from "usable" to "practical," from "academic" to "industrial," still requires crossing three chasms: unification, efficiency, and interpretability. This section summarizes contributions concisely and reiterates the theory's strategic value and unfinished mission in the era of uncertain artificial intelligence.

- **Theoretical contributions.** This survey chronologically reviews the developmental trajectory from germination (2016) to flourishing (2026). Theoretically, the field evolved from single β-neighborhood operators to these model categories encompassing logical enhancement, multigranularity, cross-universe, and variable-scale dimensions, complemented by a fuzzy measure-discrimination index-distinguishability indicator evaluation system. This taxonomy provides conceptual clarity but reveals fragmentation that demands unification.

- **Application maturity.** Methodologies have matured across four domains—feature selection, MADM, outlier detection, and TWD—supporting critical scenarios in healthcare, emergency response, and recommendation systems. While demonstrating practical utility, many applications remain proof-of-concept lacking large-scale industrial validation.

- **Scientometric patterns.** Bibliometric analysis reveals exponential growth exceeding 15% annually, led by Zhan Jianming, Zhang Xiaohong, Yang Bin, and Huang Zhehuang teams, forming a synergistic "theory-algorithm-application" innovation ecosystem. However, concentration among few clusters risks intellectual monoculture and reduced diversity of approaches.

- **Persistent challenges.** Formidable bottlenecks persist: the absence of unified axiomatic frameworks, large-scale computational inefficiency, and insufficient model interpretability constitute three major obstacles. These challenges are interdependent—unification could enable efficiency breakthroughs, while interpretability demands both theoretical clarity and algorithmic transparency.

- **Strategic imperative.** Future efforts must prioritize categorical unification, sublinear algorithms, neuro-symbolic fusion, and open-source standardization to propel β-FCRS from academic theory toward industrial-grade tooling. The theory's unique parametric fuzzy granularity perspective provides a rigorous yet flexible analytical paradigm for uncertain artificial intelligence. As big data and intelligent systems evolve, this theory is poised to play increasingly vital roles in explainable AI, robust machine learning, and complex systems decision-making, contingent upon successfully bridging current gaps between theoretical elegance and practical viability.

Declaration of Competing Interest:

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Credit-Author-Statement:

Xueling Ma: Methodology, Investigation, Writing-original draft. **Yiyu Yao:** Methodology, Writing-Reviewing and Editing. **Jianming Zhan:** Supervision, Validation.

Acknowledgments:

The main research was partially supported by grants from NSFC (12471430; 12571494).

References

[1] L.W. Ma, Two fuzzy covering rough set models and their generalizations over fuzzy lattices, Fuzzy Sets Syst., 294 (2016) 1-17.
[2] D.S. Yeung, D.G. Chen, E.C.C. Tsang, J.W.T. Lee, X.Z. Wang, On the generalization of fuzzy rough sets, IEEE Trans. Fuzzy Syst., 13(3) (2005) 343-361.
[3] F. Chacón-Gómez, M.E. Cornejo, J. Medina, Efficiency of decision rule sets in fuzzy rough set theory, Fuzzy Sets Syst., 520 (2025) 109548.
[4] D. Gégény, S. Radeleczki, On the lattice of fuzzy rough sets, Fuzzy Sets Syst., 488 (2024) 108999.
[5] L. D'eer, C. Cornelis, A comprehensive study of fuzzy covering-based rough set models: definitions, properties and interrelationships, Fuzzy Sets Syst., 336 (2018) 1–26.
[6] L. D'eer, C. Cornelis, L. Godo, Fuzzy neighborhood operators based on fuzzy coverings, Fuzzy Sets Syst., 312 (2017) 17–35.
[7] L. D'eer, M. Restrepo, C. Cornelis, J. Gómez, Neighborhood operators for covering based rough sets, Inf. Sci., 336 (2016) 21–44.
[8] L. Zhang, J.M. Zhan, J.C.R. Alcantud, Novel classes of fuzzy soft β-coverings-based fuzzy rough sets with applications to multi-criteria fuzzy group decision making, Soft Comput., 23 (14) (2019) 5327-5351.
[9] K. Zhang, C.H. Gao, Improved formulations of the joint order batching and picker routing problem, Int. J. Prod. Res., 61 (21) (2023) 7386-7409.
[10] W.C. Yu, W. Yao, Logical distance-based fuzzy rough set model and its application in feature selection, Fuzzy Sets Syst., 519 (2025) 109543.
[11] L. Sun, H. Liang, W.P. Ding, J.C. Xu, Granular ball fuzzy neighborhood rough sets-based feature selection via multiobjective mayfly optimization, IEEE Trans. Fuzzy Syst., 32(11) (2024) 6112-6124.
[12] Z. Liu, H. Han, H. Yang, J. Qiao, Knowledge-aided and data-driven fuzzy decision making for sludge bulking, IEEE Trans. Fuzzy Syst., 31(4) (2023) 1189-1201.
[13] B.M. Shi, Z.M. Li, G. Wei, B. Selmi, Information structures for incomplete hybrid information system, Fuzzy Sets Syst., 518 (2025) 109487.
[14] Z. Pawlak, Rough sets, Int. J. Comput. & Inf. Sci., 11(5) (1982) 341-356.
[15] A. Skowron, L. Polkowski, Rough mereological foundations for design, analysis, synthesis, and control in distributed systems, Inf. Sci., 104(1–2) (1998) 129-156.
[16] Z. Pawlak, A. Skowron, Rough sets: Some extensions, Inf. Sci., 177(1) (2007) 28-40.
[17] G. Góra, A. Skowron, RIONIDA: A novel algorithm for imbalanced data combining instance-based learning and rule induction, Inf. Sci., 708 (2025) 122015.
[18] S. Dutta, D. Ślęzak, Nature of decision valuations in elimination of redundant attributes, Int. J. Approx. Reason., 165 (2024) 109091.
[19] A. Skowron, J. Stepaniuk, Toward rough set based insightful reasoning in intelligent systems, Inf. Sci., 709 (2025) 122078.
[20] S.K. Pal, S.K. Meher, S. Dutta, Class-dependent rough-fuzzy granular space, dispersion index and classification, Pattern Recogn., 45(7) (2012) 2690-2707.
[21] S.K. Pal, S.K. Meher, A. Skowron, Data science, big data and granular mining, Pattern Recog. Lett., 67 (2015) 109-112.
[22] L.A. Zadeh, Fuzzy Sets, Inf. & Control, 8 (1965) 338-353.
[23] T. Mahmood, Z. Ali, A. Gumaei, Interdependency of complex fuzzy neighborhood operators and derived complex fuzzy coverings, IEEE Access, 9 (2021) 73506-73521
[24] K. Zhang, J.H. Dai, Redefined fuzzy rough set models in fuzzy β-covering group approximation spaces, Fuzzy Sets Syst., 442 (2022) 109-154.
[25] B. Yang, Fuzzy covering-based rough set on two different universes and its application, Artif. Intell. Rev., 55 (6) (2022) 4717-4753.

[26] L.W. Ma, Couple fuzzy covering rough set models and their generalizations to CCD lattices, Int. J. Approx. Reason., 126 (2020) 48-69.
[27] Z.H. Huang, J.J. Li, A fitting model for attribute reduction with fuzzy β-covering, Fuzzy Sets Syst., 413 (2021) 114-137.
[28] X.T. Zou, J.H. Dai, A fuzzy β-covering rough set model for attribute reduction by composite measure, Fuzzy Sets Syst., 507 (2025) 109134.
[29] B. Yang, B. Hu, A fuzzy covering-based rough set model and its generalization over fuzzy lattice, Inf. Sci., 367-368 (2016) 463-486.
[30] B. Huang, C. Guo, Y. Zhang, H. Li, X.Z. Zhou, An intuitionistic fuzzy graded covering rough sets, Knowl-Based Syst., 107 (2016) 155-178.
[31] J.L. Zhou, F.S. Xu, Y.Y. Guan, H.K. Wang, Three types of fuzzy covering-based rough set models, Fuzzy Sets Syst., 423 (2021) 122-148.
[32] L.W. Ma, S.Z. Zhang, Quotient approximation spaces under fuzzy β-coverings, Fuzzy Sets Syst., 522 (2026) 109601.
[33] W. Zakowski, Approximations in the space (μ, π), Demonstratio Math., 16 (1983) 761-769.
[34] Y.Y. Yao, B.X. Yao, Covering based rough set approximations, Inf. Sci., 200 (2012) 91-107.
[35] T.Q. Deng, Y.M. Chen, W.L. Xu, Q.H. Dai, A novel approach to fuzzy rough sets based on a fuzzy covering, Inf. Sci., 177(11) (2007) 2308-2326.
[36] T.J. Li, Y. Leung, W.X. Zhang, Generalized fuzzy rough approximation operators based on fuzzy coverings, Int. J. Approx. Reason., 48 (2008) 836-856.
[37] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, 1980.
[38] A. Hussain, M. Ali, T. Mahmood, Covering based q-rung orthopair fuzzy rough setmodel hybrid with TOPSIS for multi-attribute decision making, J. Intell. & Fuzzy Syst., 37 (2019) 981-993.
[39] Z.H. Huang, J.J. Li, Y.H. Qian, Noise-tolerant fuzzy-β-covering-based multigranulation rough sets and feature subset selection, IEEE Trans. Fuzzy Syst., 30(7) (2022) 2721-2735.
[40] J.M. Zhan. B.Z.Sun, J.C.R. Alcantud, Covering based multigranulation (I, T)-fuzzy rough set models and applications in multi-attribute group decision-making, Inf. Sci., 476 (2019) 290-318.
[41] K. Zhang, J.M. Zhan, W.Z. Wu, J.C.R. Alcantud, Fuzzy β-covering based (I, T)-fuzzy rough set models and applications to multi-attribute decision-making, Comput. Ind. Eng., 128 (2019) 605-621.
[42] H.B. Jiang, J.M. Zhan, D.G. Chen, Covering based variable precision (I, T)-fuzzy rough sets with applications to multi-attribute decision making, IEEE Trans. Fuzzy Syst., (2019) 1558-1572.
[43] T.Y. Wu, F.C. Lin, Y.D. Lin, Attribute subset selection via neighborhood composite entropy-based fuzzy β-covering, Fuzzy Sets Syst., 472 (2023) 108683.
[44] Z.H. Huang, J.J. Li, Noise-tolerant discrimination indexes for fuzzy covering and feature subset selection, IEEE Trans. Neural Netw. Learn. Syst., 35(7) (2024) 609-623.
[45] J.M. Zhan, B.Z. Sun, X.H. Zhang, PF-TOPSIS method based on CPFRS models: An application to unconventional emergency events, Comput. Ind. Eng., 139 (2020) 106192.
[46] H.D. Zhang, D.J. Selang, A three-way decision based multi-attribute decision-making with intuitionistic fuzzy β-covering, Appl. Soft Comput., 139 (2023) 110231.
[47] L.Q. Li, C.Z. Jia, X.R. Li, A novel intuitionistic fuzzy VIKOR method to MCDM based on intuitionistic fuzzy β^*-covering rough set, Expert Syst. Appl., 293 (2025) 128713.
[48] H.D. Zhang, Q. Ma, Three-way decisions with decision-theoretic rough sets based on Pythagorean fuzzy covering, Soft Comput., 24 (2020) 18671-18688.
[49] Z.Q. Shi, L.Q. Li, C.X. Bao, J.M. Zhan, Two novel three-way decision models based on fuzzy β-covering rough sets and prospect theory under q-rung orthopair fuzzy environments, Expert Syst. Appl., 244 (2024) 123050.
[50] Y.Y. Yao, Three-way decisions with probabilistic rough sets, Inf. Sci., 180 (3) (2010) 341-353.
[51] Y.Y. Yao, Three-way decision and granular computing, Int. J. Approx. Reason., 103 (2018) 107-123.
[52] X.L. Ma, J.M. Zhan, B.Z. Sun, J.C.R. Alcantud, Novel classes of coverings based multigranulation fuzzy rough sets and corresponding applications to multiple attribute group decision-making, Artif. Intell. Rev., 53 (2020) 6197-6256.
[53] W. Li, X.L. Wang, B. Yang, Three-way decisions with dual hesitant fuzzy covering-based rough set and their applications in medical diagnosis, Appl. Soft Comput., 160 (2024) 111695.
[54] X.F. Wen, F. C. Sun, X.H. Zhang, M.Y. Li, Variable scale fuzzy β-covering group approximation space and variable scale multi-granulation FCRSs with applications, Appl. Soft Comput., 164 (2024) 111982.
[55] R. Yin, M.G. Chen, J. Wu, Y. Liu, Coverage-based variable precision (I, PSO)-fuzzy rough sets with applications to emergency decision-making, Int. J. Comput. Intell. Syst., 18 (2025) 43.
[56] B. Yang, M. Atef, Novel classes of fuzzy β-covering-based rough set over two distinct universes, Fuzzy Sets Syst., 461 (2023) 108350.
[57] B. Yang, B.Q. Hu, Fuzzy neighborhood operators and derived fuzzy coverings, Fuzzy Sets Syst., 370 (2019) 1-13.
[58] B. Yang, B.Q. Hu, On some types of fuzzy covering-based rough sets, Fuzzy Sets Syst., 312 (2017) 36-65.
[59] X.H. Zhang, J.Q. Wang, Fuzzy β-covering approximation spaces, Int. J. Approx. Reason., 126 (2020) 27-47.
[60] J.Q. Wang, X.H. Zhang, J.H. Dai, J.M. Zhan, TI-fuzzy neighborhood measures and generalized Choquet integrals for granular structure reduction and decision making, Fuzzy Sets Syst., 465 (2023) 108512.
[61] G.A. Qi, M. Atef, B. Yang, Fermatean fuzzy covering-based rough set and their applications in multi-attribute decision-making, Eng. Appl. Arti. Intell., 127 (2024) 107181.
[62] B. Yang, B.Q. Hu, Communication between fuzzy information systems using fuzzy covering-based rough sets, Int. J. Approx. Reason., 103 (2018) 414-436.
[63] L. Zhang, J.M. Zhan, J.C. R. Alcantud, Novel classes of fuzzy soft β-coverings-based fuzzy rough sets with applications to multi-criteria fuzzy group decision making, Soft Comput., 23(14) (2019) 5327-5351.
[64] L. Zhang, J.M. Zhan, Fuzzy soft β-covering based fuzzy rough sets and corresponding decision-making applications, Int. J. Mach. Learn. Cyb., 10(2019) 1487-1502.
[65] X.F. Wen, X.H. Zhang, Overlap functions based (multi-granulation) fuzzy rough sets and their applications in MCDM, Symmetry, 13 (2021) 1779.
[66] W. Li, X.L. Wang, B. Yang, Some fuzzy neighborhood operators on fuzzy β-covering approximation space and their application in user

preference evaluation, Int. J. Approx. Reason., 187 (2025) 109-566.

[67] J.M. Zhan. X.H. Zhang, Y.Y. Yao, Covering based multigranulation fuzzy rough sets and corresponding applications, Artif. Intell. Rev., 53 (2020) 1093-1126.

[68] Z.H. Huang. J.J. Li, Covering based multi-granulation rough fuzzy sets with applications to feature selection, Expert Syst. Appl., 238 (2024) 121908.

[69] M. Atef, A.E.F.E. Atik, Some extensions of covering-based multigranulation fuzzy rough sets from new perspectives, Soft Comput., 25 (2021) 6633–6651.

[70] K. Zhang, J.M. Zhan, W.Z. Wu, On multicriteria decision-making method based on a fuzzy rough set model with fuzzy α-neighborhoods, IEEE Trans. Fuzzy Syst., 29(9) (2021) 2491-2505.

[71] J.M. Zhan, H.B. Jiang, Y.Y. Yao, J.C.R. Alcantud, Covering-based variable precision fuzzy rough sets with PROMETHEE-EDAS methods, Inf. Sci., 538 (2020) 314-336.

[72] H.B. Jiang, B. Q. Hu, On four novel kinds of fuzzy β-covering-based rough sets and their applications to three-way approximations, Fuzzy Sets Syst., (2025) 109312.

[73] T.Y. Wang, S. Liu, B. Yang, A two-stage feature selection approach with fuzzy covering-based rough sets based on discernibility matrix, Appl. Soft Comput., 184 (2025) 113695.

[74] Z.H. Huang, J.J. Li, A fitting model for attribute reduction with fuzzy β-covering, Fuzzy Sets Syst., 413 (2021) 114-137.

[75] K. Zhang, J.H. Dai, Redefined fuzzy rough set models in fuzzy β-covering group approximation spaces, Fuzzy Sets Syst., 442 (2022) 109–154.

[76] J.H. Dai, X.T. Zou, W.Z. Wu, Novel fuzzy β-covering rough set models and their applications, Inf. Sci., 608 (2022) 286–312.

[77] Y.Y. Fan. X.H. Zhang, Overlap function-based fuzzy β-covering relations and fuzzy β-covering rough set models, Int. J. Approx. Reason., 169 (2024) 109264.

[78] X.T. Zou, J.H. Dai, Multi-fuzzy β-covering fusion based accuracy and self-information for feature subset selections, Inf. Fusion, 110 (2024) 102486.

[79] S.Y. Zhao, E.C.C. Tsang, D.G. Chen, The model of fuzzy variable precision rough sets, IEEE Trans. Fuzzy Syst., 17(2) (2009) 451-467.

[80] A.M. Radzikowska, E.E. Kerre, A comparative study of fuzzy rough sets, Fuzzy Sets Syst., 126 (2002) 137–155.

[81] B.Z. Sun, W. Ma, Y.H. Qian, Multigranulation fuzzy rough set over two universes and its application to decision making, Knowl.-Based Syst., 123 (2017) 61-74.

[82] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst., 17 (1990) 191-209.

[83] T.Y. Wang, S. Liu, B. Yang, A novel method for incremental feature selection with fuzzy β-covering, Fuzzy Sets Syst., 512 (2025) 109379.

[84] B. Yang, Fuzzy covering-based rough set on two different universes and its application, Artif. Intell. Rev., 17 (2022) 4717-4753.

[85] J.M. Zhan, B.Z. Sun, Covering-based intuitionistic fuzzy rough sets and applications in multi-attribute decision-making, Artif. Intell. Rev., 53 (2020) 671–701.

[86] R. Jain, T. Son, Multigranular rough set model based on robust intuitionistic fuzzy covering with application to feature selection, Int. J. Approx. Reason., 156 (2023) 16-37.

[87] H. Garg, M. Atef, Cq-ROFRS: covering q-rung orthopair fuzzy rough sets and its application to multi-attribute decision-making process, Complex & Intell. Syst., 8(2022) 2349-2370.

[88] C. Fu, K.Y. Qin, Leu, Yang, Q. Hu, Hesitant fuzzy β-covering (T, I)-rough set models: An application to multi-attribute decision-making, J. Intell. & Fuzzy Syst., 44 (2023) 10005-10025.

[89] J.Y. Wang, S.T. Shao, X.Y. Mao, X.H. Zhang, Hesitant fuzzy β-covering (I, O)-rough set models and applications to multi-attribute decision-making, Int. J. Comput. Intell. Syst., 18 (2025) 49.

[90] M. Atef, S.F. Liu, Four types of grey β-covering models and their applications, Math. Comput. Simulat., 223 (2024) 108-129.

[91] W. Li, B.Yang, J.S. Qiao, On three types of L-fuzzy β-covering-based rough sets, Fuzzy Sets Syst., 461 (2023) 108492.

[92] X.H. Zhang, J.Q. Wang, J.M. Zhan, J.H. Dai, Fuzzy measures and Choquet integrals based on fuzzy covering rough Sets, IEEE Trans. Fuzzy Syst., 30(7) (2022) 2360-2374.

[93] Z.H. Huang, J.J. Li, Discernibility measures for fuzzy β-covering and their application, IEEE Trans. Cybern., 52(9) (2022) 9722-9735.

[94] J.H. Dai, X.T. Zou, Y.H. Qian, X.Z. Wang, Multifuzzy β-covering approximation spaces and their information measures, IEEE Trans. Fuzzy Syst., 31(3) (2023) 955-967.

[95] Z.H. Huang, J.J. Li, Robust feature selection using multigranulation variable-precision distinguishing indicators for fuzzy covering decision systems, IEEE Trans. Syst., Man, Cybern., Syst., 54(2) (2024) 903-914.

[96] Z.W. Li, D.L. Feng, J.J. Li, Outlier detection for heterogeneous data via fuzzy β-covering, Expert Sys. Appl., 252 (2024) 124227.

[97] Z.W. Li, X.P. Cai, Q. Huang, Y. Huang, A novel cooperative swarm intelligence feature selection method for hybrid data based on fuzzy β-covering and fuzzy self-information, Inf. Sci., 675 (2024) 120757.

[98] T.Y. Wang, S. Liu, B. Yang, A novel method for incremental feature selection with fuzzy β-covering, Fuzzy Sets Syst., 512 (2025) 109379.

[99] X.T. Zou, J.H. Dai, Fuzzy rough feature selection via stripped decision β-neighborhood set and misclassification ratio, Fuzzy Sets Syst., 520 (2025) 109544.

[100] X.T. Zou, J.H. Dai, Unified feature selection approach for complex data based on fuzzy β-covering reduction via information granulation, IEEE Trans. Neural Netw. Learn. Syst., 36 (9) (2025) 17380-17394.

[101] Y. Chen, X.P. Cai, Z.W. Li, An ant colony optimization attribute reduction algorithm for hybrid data using fuzzy β-covering and fuzzy mutual information, Appl. Soft Comput., 167 (2024) 112373.

[102] T.J. Yin, H.M. Chen, J.H. Wan, P.F. Zhang, T.R. Li, Exploiting feature multi-correlations for multilabel feature selection in robust multi-neighborhood fuzzy β-covering space, Inf. Fusion, 104 (2024) 102150.

[103] F. Liu, Y. Liu, S. Abdullah, Three-way decisions with decision-theoretic rough sets based on covering-based q-rung orthopair fuzzy rough set model, J. Intell. & Fuzzy Syst., 40 (2021) 9765-9785.

[104] C.L. Chu, Y.C. Chen, W. Cheng, I.C. Lin, Y.H. Chang, AttentionRC: A novel approach to improve locality sensitive hashing attention on dual-addressing memory, IEEE Trans. Comput. AID D., 43(11) (2024) 3925-3936.

[105] M. Monemian, H. Rabbani, A new method for the localization of hard exudates based on analyzing intensity incremental–decremental

trends, IEEE Trans. Instrume. Meas. 73 (2024) 1-8.
[106] B. K. Behera, S. Al-Kuwari, A. Farouk, QSVM-QNN: Quantum support vector machine based quantum neural network learning algorithm for brain-computer interfacing systems, IEEE Trans. Artif. Intell., 2025, doi: 10.1109/TAI.2025.3572852.
[107] M. Tiezzi, G. Ciravegna, M. Gori, Graph neural networks for graph drawing, IEEE Trans. Neu. Net. Learn. Syst., 35 (4) (2024) 4668-4681.
[108] Y. Zhang, Y. Zhu, J. Liu, W. Yu, C. Jiang, An interpretability optimization method for deep learning networks based on grad-CAM, IEEE Int. Things J., 12(4) (2025) 3961-3970.
[109] H. Zhu, J. Feng, F. Sun, K. Tang, D. Zang, Q. Kang, Sharing control knowledge among heterogeneous intersections: A distributed arterial traffic signal coordination method using multi-agent reinforcement learning, IEEE Trans. Intell. Trans. Syst., 26(2) (2025) 2760-2776.
[110] D. Pavlyuk, Robust and responsive learning of spatiotemporal urban traffic flow relationships, IEEE Trans. Intell. Trans. Syst., 23(9) (2025) 14524-14541.
[111] G. Feng, S. Manimurugan, B. Yi, Y. Feng, Toward precision cardiac healthcare: Deep learning and IoT integration for real-time monitoring and personalized diagnosis, IEEE Int. Things J., 12(22) (2025) 46104-46118.
[112] L. Wang, X. Zhao, P. Wu, Resource-constrained emergency scheduling for forest fires via artificial bee colony and variable neighborhood search combined algorithm, IEEE Trans. Intell. Trans. Syst., 25(6) (2024) 5791-5806.

Abstract: Uncertain information processing in the big data era faces dual challenges of data fuzziness and knowledge uncertainty. Classical rough set theory provides mathematical rigor through equivalence relations, though these relations cannot adequately represent fuzzy boundaries. Fuzzy set theory describes graded membership but lacks systematic granularity analysis mechanisms. The combination into fuzzy rough sets faces limitations from conventional covering models. The introduction of fuzzy β-covering rough sets (β-FCRS) in 2016 added a confidence-level parameter β that enables flexible granularity regulation, leading to increased development of model variants. However, existing literature lacks surveys that systematically integrate theoretical development, model classification, measurement systems, and application trends. This paper presents a systematic analysis of the decade-long development with five main contributions: (1) a hierarchical theoretical framework tracing evolution from classical covering to fuzzy β-covering approximation spaces, examining core operators, axiomatic foundations, and reduction principles; (2) a classification of eleven model variants organized into neighborhood-driven, logical-operator-enhanced, and structurally-expanded categories, with examination of mathematical foundations, development motivations, and applicability; (3) a three-level measurement system consisting of Choquet-integral-based fuzzy measures, noise-tolerant discrimination indexes, and variable-precision distinguishability indicators; (4) analysis of applications in feature selection, multi-attribute decision-making, outlier detection, and three-way decisions, examining how practical requirements have influenced theoretical developments; and (5) bibliometric analysis of core publications identifying research communities, temporal patterns, and interdisciplinary trends, while assessing theoretical limitations, computational challenges, and interpretability issues. This survey concludes that fuzzy β-covering rough set theory provides a parametric framework linking fuzzy sets, rough sets, and multigranular analysis. Further development of unified axiomatic frameworks, efficient algorithms, and enhanced interpretability is needed to support broader practical adoption.