17.08.2016 6. Lemma 1 $m, n, k \in \mathbb{N}$, $m \geq n+k$, $n \geq 1$

(1) $A \subseteq \mathbb{R}^m$ is $\mathcal{H}^{n+k}$-measurable, $\mathcal{H}^{n+k}(A) < \infty$

$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $\operatorname{Lip}(f) < \infty$

Then

$$\int_{\mathbb{R}^n} \mathcal{H}^k(A \cap f^{-1}\{y\})\, d\mathcal{L}^n(y) \leq \frac{\alpha(k)\alpha(n)}{\alpha(n+k)} (\operatorname{Lip} f)^n \mathcal{H}^{n+k}(A).$$

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Proof. Let $\delta > 0$. (Destilled from [AFP00, 2.95])

Let $\mathcal{F}$ be a family of closed subsets of $\mathbb{R}^m$ such that

$$A \subseteq \bigcup \mathcal{F}, \quad \forall F \in \mathcal{F} \ \operatorname{diam} F < \delta, \text{ and}$$

$$\sum_{F \in \mathcal{F}} \zeta^{n+k}(F) \leq \mathcal{H}^{n+k}(A) + \delta$$

Set

$$u = \sum_{F \in \mathcal{F}} \zeta^k(F) \cdot \chi_{f[F]}$$

Observe that

$$f^{-1}\{y\} \cap A \subseteq \bigcup \{F \in \mathcal{F} : F \cap f^{-1}\{y\} \neq \emptyset\}$$

(*) Hence, $u(y) \geq \mathcal{H}^k_\delta(A \cap f^{-1}\{y\}) \quad \forall y \in \mathbb{R}^n$

Moreover,

$$\int u \, d\mathcal{L}^n = \sum_{F \in \mathcal{F}} \zeta^k(F) \mathcal{L}^n(f[F])$$

(isodiametric)

$$\leq \sum_{F \in \mathcal{F}} \alpha(k) \left(\frac{\operatorname{diam} F}{2}\right)^k \alpha(n) \left(\frac{\operatorname{diam}(f[F])}{2}\right)^n$$

$$\leq \frac{\alpha(n)\alpha(k)}{\alpha(n+k)} (\operatorname{Lip} f)^n \sum_{F \in \mathcal{F}} \zeta^{n+k}(F)$$

$$\leq \frac{\alpha(k)\alpha(n)}{\alpha(k+n)} (\operatorname{Lip} f)^n \left(\mathcal{H}^{n+k}(A) + \delta\right)$$

Thus, using (*) + definition of $\int^*$,

☺ $$\int^* \mathcal{H}^k_\delta(A \cap f^{-1}\{y\})\, d\mathcal{L}^n(y) \leq \frac{\alpha(k)\alpha(n)}{\alpha(n+k)} (\operatorname{Lip} f)^n \left(\mathcal{H}^{n+k}(A) + \delta\right)$$

Observation:

If $\mathcal{H}^{n+k}(A) = 0$, then passing to the limit $\delta \to 0$ in ☺ shows that

$$\mathcal{H}^k(A \cap f^{-1}\{y\}) = 0 \quad \text{for } \mathcal{L}^n \text{ almost all } y.$$

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(2)

[see 2.10.26]

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In particular, $\mathcal{H}^k_\infty(A\cap f^{-1}\{y\})=0$ for $\mathcal{L}^m$-almost all y.

Claim: $y \longmapsto \mathcal{H}^k_\delta(A\cap f^{-1}\{y\})$ is $\mathcal{L}^m$ measurable $\forall \delta>0$ whenever A is compact.

Proof of claim:

$$\{y\in\mathbb{R}^m : \mathcal{H}^k_\delta(A\cap f^{-1}\{y\}) \le t\} = \bigcap V_j$$

where

$$V_j = \Big\{y\in\mathbb{R}^m : \exists\, \mathcal{A} \text{ finite family of open sets},\ \forall S\in\mathcal{A}\ \operatorname{diam} S\le\delta,\ \sum_{S\in\mathcal{A}} \zeta^k(S) < t+1/j\Big\}$$

are open in $\mathbb{R}^m$.

Choose a sequence of compact sets $K_i\subseteq\mathbb{R}^{m}$ such that $\forall i$ $K_i\subseteq K_{i+1}\subseteq A$, $\mathcal{H}^{m+k}(A\sim\bigcup_{i=1}^{\infty} K_i)=0$.

Then RHS $= \lim\limits_{i\to\infty}\lim\limits_{\delta\to 0} \dfrac{\alpha(k)\alpha(m)}{\alpha(m+k)}(\mathrm{Lip} f)^m\big(\mathcal{H}^{m+k}(K_i)+\delta\big)$

$$\ge \lim_{i\to\infty}\lim_{\delta\downarrow 0}\int \mathcal{H}^k_\delta(K_i\cap f^{-1}\{y\})\,d\mathcal{L}^m(y)$$

(monotone convergence) $$= \int \mathcal{H}^k\Big(\bigcup K_i\cap f^{-1}\{y\}\Big)\,d\mathcal{L}^m(y)$$

(observation) $$= \int \mathcal{H}^k(A\cap f^{-1}\{y\})\,d\mathcal{L}^m(y) = \text{LHS}.$$

∎

Remark. See [Fed69, 2.10.25] for more general version.

---

1. Lemma 2 [Fed69, 2.10.11], $A\subseteq\mathbb{R}^m$ is $\mathcal{H}^m$-measurable, $f:\mathbb{R}^m\to\mathbb{R}^n$, $\mathrm{Lip}(f)<\infty$, $0\le m\le n$ integers.

Then
$$\int_{\mathbb{R}^n} N(f|A,y)\,d\mathcal{H}^m(y) \le (\mathrm{Lip} f)^m\,\mathcal{H}^m(A)$$

Proof. Consider Borel partitions H_j of A such that

- $\lim\limits_{j\to\infty}\sup\{\operatorname{diam} S : S\in H_j\}=0$
- each element of H_j is a union of elements of H_{j+1}

Then
$$\int_{\mathbb{R}^n} N(f|A,y)\,d\mathcal{H}^m(y) \overset{\text{monotone convergence}}{=} \lim_{j\to\infty}\int_{\mathbb{R}^n}\sum_{S\in H_j}\chi_{f[S]}(y)\,d\mathcal{H}^m(y)$$

$\chi_{f[S]}(y)=1 \Leftrightarrow f^{-1}\{y\}\cap S\neq\emptyset$

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$$= \lim_{j\to\infty} \sum_{S\in H_j} \mathcal{H}^m(f[S]) \le (\operatorname{Lip} f)^m \lim_{j\to\infty}\sum_{S\in H_j} \mathcal{H}^m(S) = (\operatorname{Lip} f)^m \mathcal{H}^m(A). \quad \square$$

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2. **Lemma 3** [Fed 69, 3.2.2]

$f: \mathbb{R}^m \to \mathbb{R}^n$, $m \le n$, f continuous, $\lambda > 1$

There exists a countable family $\mathcal{G}$ of Borel sets s.t.

(a) $\{x \in \mathbb{R}^m : Df(x) \text{ is a monomorphism}\} \subseteq \bigcup \mathcal{G}$

(b) $\forall E \in \mathcal{G}\ \exists s \in GL(m,\mathbb{R})$

- $f|_E$ is injective
- $\operatorname{Lip}(f|_E \circ s^{-1}) \le \lambda$ and $\operatorname{Lip}(s \circ (f|_E)^{-1}) \le \lambda$
- $\lambda^{-1}|s(v)| \le |Df(x)v| \le \lambda |s(v)| \quad \forall x \in E\ \forall v \in \mathbb{R}^m$
- $\lambda^{-m}|\det s| \le J_m f(x) \le \lambda^m |\det s| \quad \forall x \in E.$

Proof. Let $\varepsilon > 0$ be such that $\frac{1}{\lambda} + \varepsilon < 1 < \lambda - \varepsilon$.

Let $S \subseteq GL(m,\mathbb{R})$ be countable and dense.

For $s \in S$ and $i \in \mathbb{N}_+$ set

$$Z(s,i) = \left\{ a \in \mathbb{R}^m : \begin{array}{l} \left(\frac{1}{\lambda}+\varepsilon\right)|s(v)| \le |Df(a)v| \le (\lambda-\varepsilon)|s(v)| \quad \forall v \in \mathbb{R}^m \\ |f(b)-f(a)-Df(a)(b-a)| \le \varepsilon |s(b-a)| \quad \forall b \in B(a,\tfrac{1}{i}) \end{array} \right\}$$

Let $\mathcal{Z}(s,i)$ be a countable covering of $Z(s,i)$ by sets of diameter $\le \frac{1}{i}$.

Define $\mathcal{G} = \bigcup \{\mathcal{Z}(s,i) : s \in S,\ i \in \mathbb{N}_+\}$

Claim: $\mathcal{G}$ satisfies (a) and (b).

Remark: $Df(a)$ is mono $= h \circ g$ for some $g \in GL(m,\mathbb{R})$, $h \in O(m,n)$ $\square$

---

3. **Theorem 1** [Fed 69, 3.2.3]

$f: \mathbb{R}^m \to \mathbb{R}^n$, $\operatorname{Lip}(f) < \infty$, $m \le n$

1) If $A \subseteq \mathbb{R}^m$ is $\mathcal{L}^m$-measurable, then

$$\int_A J_m f \, d\mathcal{L}^m = \int_{\mathbb{R}^n} N(f|A, y)\, d\mathcal{H}^m(y)$$

2) If $u: \mathbb{R}^m \to \mathbb{R}$ is $\mathcal{L}^m$-integrable, then

$$\int_{\mathbb{R}^m} u(x) J_m f(x) \, d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \sum_{x \in f^{-1}\{y\}} u(x)\, d\mathcal{H}^m(y)$$

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Proof. We only proove 1. Assertion 2 follows by standard techniques.

$\boxed{\text{Step 1}}$: Assume $\mathcal{L}^m(A) = 0$.

Then, by Lemma 2, $0 = LHS = RHS = 0$

Hence, we may assume f is differentiable at all points $a \in A$ and $\mathcal{L}^m(A) < \infty$. (Rademacher)

$\boxed{\text{Step 2}}$: Assume $A \subseteq \{x : Df(x) \text{ is mono}\}$.

Employing Lemma 3 with some $\lambda > 1$ we get a Borel partition H of A s.t. elements of H satisfy (a) and (b) of Lemma 3.

Then,

$\forall B \in H \ \exists s \in GL(m, \mathbb{R})$

- $\lambda^{-m} |\det s| \, \mathcal{L}^m(B) \le \int_B J_m f \, d\mathcal{L}^m \le \lambda^m |\det s| \, \mathcal{L}^m(B)$
- $\lambda^{-m} \mathcal{H}^m(s[B]) \le \mathcal{H}^m(f[B]) \le \lambda^m \mathcal{H}^m(s[B])$

since $f[B] = (f|_B \circ s^{-1})[s[B]]$

and $\mathcal{H}^m(s[B]) = \mathcal{L}^m(s[B]) = |\det s| \, \mathcal{L}^m(B)$

Hence,

$$\lambda^{-2m} \mathcal{H}^m(f[B]) \le \int_B J_m f \, d\mathcal{L}^m \le \lambda^{2m} \mathcal{H}^m(f[B])$$

Summing over $B \in H$ we obtain

$$\lambda^{-2m} \int N(f|A, y) \, d\mathcal{H}^m(y) \le \int_A J_m f \, d\mathcal{L}^m \le \lambda^{2m} \int N(f|A, y) \, d\mathcal{H}^m(y)$$

Passing to the limit $\lambda \to 1$ we get the conclusion.

$\boxed{\text{Step 3}}$: Assume $A \subseteq \{x : J_m f(x) = 0\}$

Define $g : \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$, $\quad p : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$g(x) = (f(x), \varepsilon x)$, $\quad p(y, z) = y$

Note

- $f = p \circ g$, $\quad$ • $Dg(x)$ is mono $\ \forall x \in \mathbb{R}^m$
- $\| Dg(x) \| \le \mathrm{Lip}(f) + \varepsilon$
- g is injective
- $J_m g \le \varepsilon (\mathrm{Lip} f + \varepsilon)^{m-1}$ because $|Dg(x) v| = \varepsilon v$ whenever $v \in \ker Df(x)$

Employing Step 2 to g we obtain:

$$\mathcal{H}^m(f[A]) = \mathcal{H}^m(p \circ g[A]) \le \mathcal{H}^m(g[A]) = \int_A J_m g \, d\mathcal{L}^m$$

$$\le \varepsilon (\mathrm{Lip} f + \varepsilon)^{m-1} \mathcal{L}^m(A) \longrightarrow 0 \implies RHS = 0 \ \square$$

18.01.2016 3. **Lemma 4**: $A \subseteq \mathbb{R}^m$ $\mathcal{L}^m$ measurable

[Fed69, 3.2.4] $f : \mathbb{R}^m \to \mathbb{R}^m$, $\operatorname{Lip} f < \infty$.

There exists $B \subseteq A$ a Borel set such that

- $\forall x \in B$ $\quad \underset{\sim}{J}_m f(x) > 0$
- $f|_B$ is injective
- $\mathcal{H}^m(f[A] \sim f[B]) = 0$

Proof: If $\mathcal{L}^m(A) = 0$, then $B = \emptyset$.

If $\mathcal{L}^m(A) > 0$, assume A is Borel.

Set $P = \{x \in A : J_m f(x) > 0\}$

Employ Lemma 3 to find $G = \{E_i : i \in \mathbb{N}_+\}$, $P \subseteq \bigcup G$.

Define
$$F_i = P \cap E_i \sim \bigcup_{j=1}^{i-1} f^{-1}\left[f[P \cap E_j]\right]$$

$$B = \bigcup_{i=1}^{\infty} F_i \subseteq P \qquad , \qquad B \text{ is Borel}$$

Recall that $f|_{E_i}$ is injective $\forall i \in \mathbb{N}_+$.

So $f|_B$ is injective and $f[B] = f[P]$ by def.

Hence,
$$\mathcal{H}^m(f[A] \sim f[B]) = \mathcal{H}^m(f[A] \sim f[P]) = \mathcal{H}^m(f[A \sim P])$$
$$\leq \int_{A \sim P} J_m f \, d\mathcal{L}^m = 0 . \qquad \square$$

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4. **Lemma 5** [Fed69, 3.2.9]

$f : \mathbb{R}^m \to \mathbb{R}^n$ continuous, $m \geq n$.

There exists a countable family G of Borel sets s.t.

(a) $\{x \in \mathbb{R}^m : \operatorname{im} Df(x) = \mathbb{R}^n\} \subseteq \bigcup G$

(b) $\forall E \in G \;\exists p \in \mathbf{O}^*(m, m-n) \;\exists v : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^m$

$\operatorname{Lip}(v) < \infty$ and $v(u(x)) = x$ for $x \in E$

where $u : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^{m-n}$

$u(x) = (f(x), p(x))$

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Proof. For $\lambda \in \Lambda(m, m-n)$ define

$$p_\lambda(x_1, \dots, x_m) = (x_{\lambda(1)}, \dots, x_{\lambda(m-n)}), \quad p: \mathbb{R}^m \to \mathbb{R}^{m-n}$$

$$u_\lambda(x) = (f(x), p_\lambda(x)), \quad u_\lambda: \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^{m-n}$$

$$A_\lambda = \{x \in \mathbb{R}^m : Du_\lambda(x) \text{ is mono}\}$$

Observe

- $\ker Du_\lambda(x) = \ker Df(x) \cap \ker p_\lambda$
- $\{x \in \mathbb{R}^m : \operatorname{im} Df(x) = \mathbb{R}^n\} = \bigcup_{\lambda \in \Lambda(m, m-n)} A_\lambda$ ← by dim comparison

Apply Lemma 3 to u_λ to find a covering $\mathcal{G}_\lambda$ of A_λ s.t.

$$\forall E \in \mathcal{G}_\lambda \quad u_\lambda|_E \text{ is injective and bi-Lipschitz}$$

Next, employ Kirszbraun [2.10.43] to get $v: \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^m$ s.t. $v|_{u_\lambda[E]} = (u_\lambda|_E)^{-1}$. □

---

5. **Lemma 6** [Fed69, 3.2.10]

Let $f, m, n, \mathcal{G}, E \in \mathcal{G}, p, v, u$ be as in Lemma 5. Then

(a) $$\int_B J_n f \, d\mathcal{L}^m = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(B \cap f^{-1}\{y\}) \, d\mathcal{L}^n(y).$$

for any $B \subseteq E$ $\mathcal{L}^m$-measurable

(b) $$E \cap f^{-1}\{y\} = v[\{y\} \times p[E \cap f^{-1}\{y\}]] \quad \text{for } y \in \mathbb{R}^n$$

Proof. First, we show (b).

Set $v_y: \mathbb{R}^{m-n} \to \mathbb{R}^m$, $v_y(z) = v(y, z)$

Since $v \circ u|_E = \mathrm{id}_E$ we see that $v_y|_{p[E \cap f^{-1}\{y\}]}$ is injective.

and $v_y[p[E \cap f^{-1}\{y\}]] = E \cap f^{-1}\{y\}$.

Next, since $Du(x)^{-1} = Dv(u(x))$, we obtain the commutative diagram.

$$\begin{tikzcd}
\mathbb{R}^{m-n} \arrow[r, hook, "j"] \arrow[d, "L_x"'] & \mathbb{R}^n \times \mathbb{R}^{m-n} \arrow[r, two heads, "q"] \arrow[d, bend right, "Dv(u(x))"'] & \mathbb{R}^n \\
\operatorname{im} L_x \arrow[r, leftrightarrow, "g"'] & \mathbb{R}^m \arrow[u, "Du(x)"'] \arrow[ur, "Df(x)"] \arrow[r, leftrightarrow, "h"] & W \arrow[u]
\end{tikzcd}$$

$$\mathbb{R}^m \cong \operatorname{im} L_x \overset{\perp}{\oplus} W$$

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where $q(y,z) = y$

$$L_x = D\,v_{f(x)}(p(x)) = Dv(f(x), p(x)) \circ j = Dv(u(x)) \circ j$$

$$\operatorname{im} L_x = Du(x)^{-1}[\{0\} \times \mathbb{R}^{m-n}] = \ker Df(x)$$

$$W = (\ker Df(x))^{\perp}$$

$$q \circ Du(x)\eta = q(Df(x)\eta, p(\eta)) = Df(x)\eta \qquad \forall \eta \in \mathbb{R}^m$$

Hence,

$$Du(x) = q^* \circ q \circ Du(x) + j \circ j^* \circ Du(x) = q^* \circ Df(x) + j \circ L_x^{-1} \circ q^*$$

Choosing an o.n.b. $\eta_1 \dots \eta_m$ of $\mathbb{R}^m$ s.t.

$\ker Df(x) = \operatorname{span}\{\eta_1 \dots \eta_{m-n}\}$ we obtain

$$\|\Lambda_m Du(x)\| = |Du(x)\eta_1 \wedge \dots \wedge Du(x)\eta_m|$$

$$= |Df(x)\eta_{m-n+1} \wedge \dots \wedge Df(x)\eta_m| \cdot |L_x^{-1}\eta_1 \wedge \dots \wedge L_x^{-1}\eta_{m-n}|$$

$$= \|\Lambda_n Df(x)\| \cdot \|\Lambda_{m-n} L_x\|^{-1}$$

$$J_n f(x) = J_{m-n} V_{f(x)}(p(x)) \; J_m u(x)$$

Employing the Area-formula (Thm 1 improved, [Fed69, 3.2.5])

$$\int_B J_n f \, d\mathcal{L}^m = \int_B J_{m-n} V_{f(x)}(p(x)) \, J_m u(x) \, d\mathcal{L}^m(x)$$

$$\overset{\text{(Area)}}{=} \int_{u[B]} J_{m-n} V_y(z) \, d(\mathcal{L}^n \times \mathcal{L}^{m-n})(y,z)$$

$$\overset{\text{(Fubini)}}{=} \int_{\mathbb{R}^n} \int_{p[B \cap f^{-1}\{y\}]} J_{m-n} V_y(z) \, d\mathcal{L}^{m-n}(z) \, d\mathcal{L}^n(y)$$

$$\overset{\text{(Area)}}{=} \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(B \cap f^{-1}\{y\}) \, d\mathcal{L}^n(y)$$

because $V_y[p[E \cap f^{-1}\{y\}]] = E \cap f^{-1}\{y\}$. $\square$

---

7. Theorem 2 [Fed 69, 3.2.11, 3.2.12]

$f: \mathbb{R}^m \to \mathbb{R}^n$, $\operatorname{Lip}(f) < \infty$, $m \geq n$,

(a) If $A \subseteq \mathbb{R}^m$ is $\mathcal{L}^m$ measurable, then

$$\int_A J_n f \, d\mathcal{L}^m = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(f^{-1}\{y\} \cap A) \, d\mathcal{L}^n(y)$$

(b) If $g: \mathbb{R}^m \to \overline{\mathbb{R}}$ is $\mathcal{L}^m$-integrable, then

$$\int_{\mathbb{R}^m} g \cdot J_n f \, d\mathcal{L}^m = \int_{\mathbb{R}^n} \int_{f^{-1}\{y\}} g(x) \, d\mathcal{H}^{m-n}(x) \, d\mathcal{L}^n(y)$$

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Proof. Clearly (b) follows from (a) by standard techniques.

$\boxed{\text{Step 1}}$ If $\mathcal{L}^m(A) = 0$, then LHS $= 0$ and, by Lemma 1, also RHS $= 0$.

$\boxed{\text{Step 2}}$ If $A \subseteq \{x \in \mathbb{R}^m : \operatorname{im} Df(x) = \mathbb{R}^n\}$, then conclusion follows by Lemma 6.

$\boxed{\text{Step 3}}$ Assume $A \subseteq \{x \in \mathbb{R}^m : J_n f(x) = 0\} = \{x : \dim \operatorname{im} Df(x) < n\}$

Let $\varepsilon > 0$. Define

$$g : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad g(x,z) = f(x) + \varepsilon z$$

$$p : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad p(x,z) = z$$

Observe that for $x \in A$, $z \in \mathbb{R}^n$

$$Dg(x,z)(v,w) = Df(x)v + \varepsilon w$$

$$\operatorname{im} Dg(x,z) = \mathbb{R}^n, \quad \|Dg(x,z)\| \le \operatorname{Lip} f + \varepsilon$$

$$J_n g(x,z) \le \varepsilon (\operatorname{Lip} f + \varepsilon)^{n-1}$$

Hence,

$$\varepsilon(\operatorname{Lip} f + \varepsilon)^{n-1} \mathcal{L}^m(A)\,\alpha(n) \geq \int_{A \times B(0,1)} J_n g \, d(\mathcal{L}^m \times \mathcal{L}^n)$$

$$\stackrel{\text{(co-area)}}{=} \int_{\mathbb{R}^n} \mathcal{H}^m\big(\overbrace{A \times B(0,1)}^{Q} \cap g^{-1}\{y\}\big)\, d\mathcal{L}^n(y)$$

$$\stackrel{\substack{\text{(Lemma 1)}\\ \operatorname{Lip} p = 1}}{\geq} c(m,n) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{H}^{m-n}\big(Q \cap g^{-1}\{y\} \cap p^{-1}\{w\}\big)\, d\mathcal{L}^n(w)\, d\mathcal{L}^n(y)$$

$$= c(m,n) \int_{\mathbb{R}^n} \int_{B(0,1)} \mathcal{H}^{m-n}\big(A \cap f^{-1}\{y - \varepsilon w\}\big)\, d\mathcal{L}^n(w)\, d\mathcal{L}^n(y)$$

$$\stackrel{\substack{\text{Fubini}\\ \text{+change of variables}}}{=} c(m,n) \int_{\mathbb{R}^n} \mathcal{H}^{m-n}\big(A \cap f^{-1}\{y\}\big)\, d\mathcal{L}^n(y)$$

$(x,z) \in Q \cap g^{-1}\{y\} \cap p^{-1}\{w\}$

$\Updownarrow$

$z = w$

$g(x,z) = y = f(x) + \varepsilon w$

$\Updownarrow$

$f(x) = y - \varepsilon w$

Letting $\varepsilon \downarrow 0$ we see that RHS $= 0$. $\square$

---

9. Lemma 7 [Fed69, 3.2.18]

$W \subseteq \mathbb{R}^n$ is $(\mathcal{H}^m, m)$ rectifiable and $\mathcal{H}^m$ measurable

$1 < \lambda < \infty$

Then $\exists$ $K_1, K_2, \ldots$ compact subsets of $\mathbb{R}^m$

$\exists$ $\psi_1, \psi_2, \ldots : \mathbb{R}^m \to \mathbb{R}^n$

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$$\psi_i[K_i] \cap \psi_j[K_j] = \emptyset \quad \forall i \neq j$$
$$\psi_i[K_i] \subseteq W \quad \forall i$$
$$\mathcal{H}^m\Big(W \sim \bigcup_{i=1}^{\infty} \psi_i[K_i]\Big) = 0$$
$$\mathrm{Lip}(\psi_i) \leq \lambda, \quad \psi_i|_{K_i} \text{ is injective}$$
$$\mathrm{Lip}(\psi_i|_{K_i}^{-1}) \leq \lambda$$
$$\lambda^{-1}|v| \leq |D\psi_i(a)v| \leq \lambda|v| \quad \forall a \in K_i \ \forall v \in \mathbb{R}^m$$

Proof. Employ Lemmas 3 and 4 together with the def. of $(\mathcal{H}^m, m)$ rectifiable sets, Borel regularity of $\mathcal{H}^m$ and Kirszbraun theorem. □

Remark: Lemma 7 provides an analogue of an atlas of a smooth manifold.

---

10. Lemma 8 [Fed69, 3.2.19]

If W is $(\mathcal{H}^m, m)$ rect and $\mathcal{H}^m$-measurable, then

$$\Theta^m(\mathcal{H}^m \llcorner W, w) = 1$$
$$\mathrm{Tan}^m(\mathcal{H}^m \llcorner W, w) \in G(n, m)$$

for $\mathcal{H}^m$ almost all $w \in W$.

Moreover, if $f : W \to \mathbb{R}^\nu$ is Lipschitz, then $(\mathcal{H}^m \llcorner W, m)\ \mathrm{ap}\, Df(w)$ exists for $\mathcal{H}^m$ almost all $w \in W$.

Proof. Use the „atlas" from Lemma 7 for $\lambda = 1 + \frac{1}{k}$ together with Rademacher theorem [3.1.6] and generic properties of densities [2.10.19] and [2.9.11]. □

Remark: Preiss, Ann of Math 125, 1987
Mattila, Trans AMS 205, 1975
Mattila, Geometry of sets and measures in Euclidean space, 1995
[15.19, 17.6, 17.8]

19.08.2016 | 11. Theorem 3 [Fed69, 3.2.20]

Handout precise formulation

$m \le \nu$, $W \subseteq \mathbb{R}^n$ $(\mathcal{H}^m, m)$-rect. and $\mathcal{H}^m$-measurable

$f : W \longrightarrow \mathbb{R}^\nu$, $\operatorname{Lip} f < \infty$

Then
$$\int_W g \circ f \ \operatorname{ap} J_m f \, d\mathcal{H}^m = \int_{\mathbb{R}^\nu} g(z) N(f, z) \, d\mathcal{H}^m(z)$$

whenever $g : \mathbb{R}^\nu \longrightarrow \overline{\mathbb{R}}$ [Here $\operatorname{ap} J_m f = (\mathcal{H}^m \llcorner W, m) \operatorname{ap} J_m f$]

Proof. Use the „atlas" from Lemma 7 and apply Theorem 1 to the composition $f \circ \psi_i$ on K_i for each i. □

Toulouse (10) Lecture 1

12. Theorem 4 [Fed69, 3.2.22]

$W \subseteq \mathbb{R}^n$ $(\mathcal{H}^m, m)$ rect, $\mathcal{H}^m$-meas

$Z \subseteq \mathbb{R}^\nu$ $(\mathcal{H}^\mu, \mu)$ rect, $\mathcal{H}^\mu$-meas $\quad m \ge \mu$

$f : W \longrightarrow Z$, $\operatorname{Lip}(f) < \infty$

Then
$$\int_W g \ \operatorname{ap} J_\mu f \, d\mathcal{H}^m = \int_Z \int_{f^{-1}\{z\}} g \, d\mathcal{H}^{m-\mu} \, d\mathcal{H}^\mu(z)$$

whenever $g : W \longrightarrow \overline{\mathbb{R}}$ is $(\mathcal{H}^m \llcorner W)$-integrable.

Proof. Applying Lemma 7 twice we get „atlases" $\{(\psi_i, K_i)\}_{i=1}^\infty$ for W and $\{(C_j, \chi_j)\}_{j=1}^\infty$ for Z.

Define
$$P_{i,j} = \psi_i[K_i] \cap f^{-1}[\chi_j[C_j]]$$
$$Q = Z \sim \bigcup_{j=1}^\infty \chi_j[C_j]$$

Then
$$W = f^{-1}[Q] \cup \bigcup_{i,j=1}^\infty P_{i,j}$$

On each $P_{i,j}$ one applies Theorems 1 & 2 to the composition $\chi_j^{-1} \circ f \circ \psi_i$ to obtain the conclusion.

On $f^{-1}[Q]$ one uses [3.2.21] to show that
$$\mathcal{H}^\mu(Q) = 0 \implies \dim \operatorname{im} \operatorname{ap} Df(w) < \mu \text{ for } \mathcal{H}^m \text{ a.e. } w \in f^{-1}[Q]$$

so $f^{-1}[Q]$ is negligible. □

19.08.2016

Toulouse

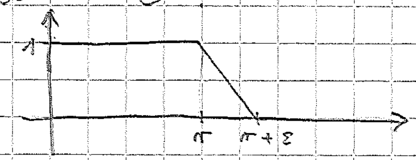
Lecture 2 (1)

1. Lemma [EG92, 5.7, Lemma 1] $\boxed{E \text{ has locally finite perimeter in } \mathbb{R}^n}$

$\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$, $x \in \mathbb{R}^n$

Then
$$\int_{E \cap B(x,r)} \operatorname{div} \varphi \, d\mathcal{L}^n = \int_{B(x,r)} \varphi \cdot \nu_E \, d\|\partial E\| + \int_{E \cap \partial B(x,r)} \varphi \cdot \nu_{B(x,r)} \, d\mathcal{H}^{n-1}$$
for $\mathcal{L}^1$ almost all $r > 0$

Proof Take $g_\varepsilon : \mathbb{R} \to \mathbb{R}$,



$h_\varepsilon : \mathbb{R}^n \to \mathbb{R}$, $h_\varepsilon(y) = g_\varepsilon(|y - x|)$

Compute
$$\int_{\mathbb{R}^n} h_\varepsilon \varphi \cdot \nu_E \, d\|\partial E\| = \int_E h_\varepsilon \operatorname{div} \varphi \, d\mathcal{L}^n - \frac{1}{\varepsilon} \int_{E \cap \{y : r < |x-y| < r+\varepsilon\}} \varphi \cdot \frac{y-x}{|y-x|} \, d\mathcal{L}^n(y)$$

Pass to the limit $\varepsilon \downarrow 0$. $\square$

2. Lemma [EG92, 5.7, Lemma 2] There exists $A = A(n) > 0$ s.t. for $x \in \partial^* E$

(i) $\Theta_*^n(\mathcal{L}^n \llcorner E, x) > A$

(ii) $\Theta_*^n(\mathcal{L}^n \llcorner (\mathbb{R}^n \sim E), x) > A$

(iii) $\Theta_*^{n-1}(\|\partial E\|, x) > A$

(iv) $\Theta^{*n-1}(\|\partial E\|, x) < 1/A$

(v) $\limsup_{r \downarrow 0} r^{-(n-1)} \|\partial(E \cap B(x,r))\|(\mathbb{R}^n) < 1/A$.

Proof $\boxed{\text{Step 1}}$ Using 1. we obtain

(*) $\|\partial(E \cap B(x,r))\|(\mathbb{R}^n) \le \|\partial E\|(B(x,r)) + \mathcal{H}^{n-1}(E \cap \partial B(x,r))$

Let $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ be such that $\varphi(y) = \nu_E(x)$ for $y \in B(x,r)$

Then $\int_{E \cap B(x,r)} \operatorname{div} \varphi \, d\mathcal{L}^n = 0$ so $\int_{B(x,r)} \nu_E(x) \cdot \nu_E \, d\|\partial E\| = -\int_{E \cap \partial B(x,r)} \nu_E(x) \cdot \nu_B \, d\mathcal{H}^{n-1}$

Thus there is $r_0 > 0$ s.t. for $\mathcal{L}^1$ almost all $r \in (0, r_0)$
$$1 \approx \frac{1}{\|\partial E\| B(x,r)} \nu_E(x) \cdot \int_{B(x,r)} \nu_E \, d\|\partial E\| = -\frac{1}{\|\partial E\| B(x,r)} \int_{E \cap \partial B(x,r)} \nu_E(x) \cdot \nu_B \, d\mathcal{H}^{n-1}$$
and

(**) $\|\partial E\| B(x,r) \le 2 \mathcal{H}^{n-1}(E \cap \partial B(x,r))$

for $\mathcal{L}^1$ almost all $r \in (0, r_0)$

13.08.2016

Toulouse Lecture 2 (2)

Isoperimetric inequality $\Downarrow$ lower density bound

Combining $\circledast$ and $\circledast\circledast$ we get

$(***)$ $\quad \|\partial(E \cap B(x,r))\|(\mathbb{R}^n) \leq 3\, \mathcal{H}^{n-1}(E \cap \partial B(x,r))$ for $\mathcal{L}^1$ a.e. $r \in (0,r_0)$

Step 2

Set $g(r) = \mathcal{L}^n(E \cap B(x,r))$

Then $g(r) = \int_0^r \mathcal{H}^{n-1}(E \cap \partial B(x,s))\, d\mathcal{L}^1(s)$ by co-area

$g'(r) = \mathcal{H}^{n-1}(E \cap \partial B(x,r))$ for $\mathcal{L}^1$ a.e. r

The isoperimetric inequality gives

$$g(r)^{1-1/n} \leq C\, \|\partial(E \cap B(x,r))\|(\mathbb{R}^n)$$

$$\overset{(***)}{\leq} \tilde{C}\, \mathcal{H}^{n-1}(E \cap \partial B(x,r)) = \tilde{C}\, g'(r) \qquad \text{for } \mathcal{L}^1 \text{ a.e. } r \in (0,r_0)$$

Thus

$$n\left(g^{1/n}\right)'(r) \geq \frac{1}{\tilde{C}}$$

Use [Fed69, 2.9.19] to integrate this and obtain

$$g(r) \geq \frac{1}{(\tilde{C}n)^n} \cdot r^n \quad \Rightarrow \quad (i)$$

Step 3

- Since $\|\partial E\| = \|\partial(\mathbb{R}^n \sim E)\|$ and $\nu_E = -\nu_{\mathbb{R}^n \sim E}$ (ii) follows from (i)
- (i) + (ii) + Relative Isoperimetric inequality $\Rightarrow$ (iii)
- $(**) \Rightarrow$ (iv)
- $\circledast$ + (iv) $\Rightarrow$ (v)

$\square$

3. Definition $\quad x \in \partial^* E$

$$H(x) = \{ y \in \mathbb{R}^n : \nu_E(x) \bullet (y-x) = 0 \}$$

$$H^+(x) = \{ y \in \mathbb{R}^n : \nu_E(x) \bullet (y-x) \geq 0 \}$$

$$H^-(x) = \{ y \in \mathbb{R}^n : \nu_E(x) \bullet (y-x) \leq 0 \}$$

$$E_r = \{ y \in \mathbb{R}^n : r(y-x) + x \in E \}$$

$$= \tau_x \circ \mu_{1/r} \circ \tau_{-x}[E]$$

4. Lemma [EG92, 5.7.2, Thm 1] $\quad x \in \partial^* E \Rightarrow \chi_{E_r} \xrightarrow[r \downarrow 0]{} \chi_{H^-(x)}$ in $L^1_{loc}(\mathbb{R}^n)$

Proof. Wlog $x = 0$, $\nu_E(x) = e_n = (0,0,\dots,0,1) \in \mathbb{R}^n$

Fix $L > 0$ and set $D_r = E_r \cap B(0,L)$

19.08.2016

Toulouse
Lecture 2

(3)

Step 1

For $\varphi \in \mathcal{C}^1_c(\mathbb{R}^n, \mathbb{R}^n)$ we have

$$\int_{D_r} \operatorname{div} \varphi \, d\mathcal{L}^n = \frac{1}{r^{n-1}} \int_{E \cap B(0,rL)} \operatorname{div}(\varphi \circ \mu_{1/r}) \, d\mathcal{L}^n$$

by Lemma 1

$$\le \frac{1}{r^{n-1}} \| \partial (E \cap B(0, rL)) \| (\mathbb{R}^n) \le C < \infty$$

Hence, $\| \partial D_r \| (\mathbb{R}^n) \le C < \infty$ for $r \in (0,1)$

$$\| \chi_{D_r} \|_{L^1} = \mathcal{L}^n(D_r) \le \mathcal{L}^n(B(0,L)) < \infty$$

$$\Rightarrow \| \chi_{D_r} \|_{BV(\mathbb{R}^n)} \le C < \infty \quad \forall r \in (0,1)$$

For any sequence $r_k \downarrow 0$ we can find a subsequence $s_j \downarrow 0$ (using the compactness theorem) such that

$$\chi_{E_j} \longrightarrow f \quad \text{in } L^1_{loc}(\mathbb{R}^n) \text{ and } f \in BV_{loc}(\mathbb{R}^n).$$

Choosing further subsequence we may assume

$$\chi_{E_j}(x) \longrightarrow f(x) \quad \text{for } \mathcal{L}^n \text{ almost all } x$$

so $f = \chi_F$ for some $F \subseteq \mathbb{R}^n$ of loc. finite perimeter.

$\boxed{\text{We claim } F = H^-(x)}$

$\boxed{\text{Step 2}}$ Observe $\forall \varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$

$$\int \varphi \cdot \nu_j \, d\|\partial E_j\| = \int_{E_j} \operatorname{div} \varphi \, d\mathcal{L}^n \xrightarrow[j \to \infty]{} \int_F \operatorname{div} \varphi \, d\mathcal{L}^n = \int \varphi \cdot \nu_F \, d\|\partial F\|$$

Thus, $\nu_j \|\partial E_j\| \longrightarrow \nu_F \|\partial F\|$ weakly as measures.

Whenever, $\|\partial F\| B(0,L) = 0$ we obtain

$$\int_{B(0,L)} \nu_j \, d\|\partial E_j\| \longrightarrow \int_{B(0,L)} \nu_F \, d\|\partial F\|$$

Since $E_j = \mu_{1/s_j}[E]$ we get by scaling

$$\circledast \quad \lim_{j \to \infty} \fint_{B(0,L)} \nu_j \, d\|\partial E_j\| = \lim_{j \to \infty} \fint_{B(0,s_j L)} \nu_E \, d\|\partial E\| = \nu_E(0) = e_n$$

since $0 \in \partial^* E$

By lower semi-continuity we get

$$\|\partial F\| B(0,L) \le \liminf_{j \to \infty} \|\partial E_j\| B(0,L)$$

$$\stackrel{\circledast}{=} \lim_{j \to \infty} \int_{B(0,L)} e_n \cdot \nu_j \, d\|\partial E_j\|$$

$\boxed{\|\partial F\| B(0,L) = 0}$

$$= \int_{B(0,L)} e_n \cdot \nu_F \, d\|\partial F\| \le \|\partial F\| B(0,L).$$

Since $|\nu_F| = 1$ $\|\partial F\|$ a.e., it follows that $\nu_F = e_n$ $\|\partial F\|$ almost everywhere.

19.02.2015

Toulouse Lecture 2 (4)

$\boxed{\text{Step 3}}$ Let $f^\varepsilon = \eta_\varepsilon * \chi_F$ where $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$,

$\eta \in \mathcal{C}_c^\infty(\mathbb{R}^n;\mathbb{R})$, $\int \eta = 1$, $\eta \geq 0$, $\eta(x) = \eta(-x)$

$\forall \varphi \in \mathcal{C}_c^1(\mathbb{R}^n, \mathbb{R}^n)$

$$\int f^\varepsilon \operatorname{div} \varphi \, d\mathcal{L}^n \overset{\text{Fubini}}{=} \int_F \operatorname{div}(\eta_\varepsilon * \varphi)\, d\mathcal{L}^n$$

$$= \int \eta_\varepsilon * \varphi \bullet e_n \, d\|\partial F\|$$

$$\int f^\varepsilon \operatorname{div} \varphi \, d\mathcal{L}^n = -\int \nabla f^\varepsilon \bullet \varphi \, d\mathcal{L}^n$$

Hence,

$$Df^\varepsilon(x) e_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, n-1 \quad \forall x \in \mathbb{R}^n$$

$$Df^\varepsilon(x) e_n \leq 0 \quad \text{for } x \in \mathbb{R}^n$$

Therefore,

$$F = \{ y \in \mathbb{R}^n : y \bullet e_n \leq \gamma \} \text{ for some } \gamma \in \mathbb{R}$$

because $f^\varepsilon \longrightarrow \chi_F$ pointwise.

$\boxed{\text{Step 4}}$ Assume $\gamma > 0$. Then

$$\alpha(n)\gamma^n = \mathcal{L}^n(\mathbb{B}(0,\gamma) \cap F) \overset{\chi_{E_j} \to \chi_F \text{ in } L^1_{loc}}{=} \lim_{j\to\infty} \mathcal{L}^n(\mathbb{B}(0,\gamma) \cap E_j)$$

$$= \lim_{j\to\infty} \frac{1}{s_j^n} \mathcal{L}^n(\mathbb{B}(0, \gamma s_j) \cap E)$$

Thus, $\Theta^n(\mathcal{L}^n \llcorner E, 0) = 1$ which contradicts 2.

If $\gamma < 0$, then do the same for $\mathbb{R}^n \sim E$.

Hence $\gamma = 0$

$\square$

5. Corollary [EG92, 5.7.2, Cor. 1] $x \in \partial^* E \Rightarrow$ (i) $\Theta^n(\mathcal{L}^n \llcorner H^+(x) \cap E, x) = 0$

$\Theta^n(\mathcal{L}^n \llcorner H^-(x) \sim E, x) = 0$

(ii) $\Theta^{n-1}(\|\partial E\|, x) = 1$

6. Lemma [EG92, 5.7.3, Lem. 3] There exists $C = C(n) > 0$ such that

$$\mathcal{H}^{n-1}(B) \leq C \|\partial E\|(B)$$

for any $B \subseteq \partial^* E$.

Proof. Simple application of Vitali's covering theorem and 2. $\square$

19.08.2016

Toulouse Lecture 2 (5)

7. Theorem [EG §5.7.3, Thm 2] (i) $\partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N$

$\|\partial E\|(N) = 0$

K_k is a compact subset of some C^1 hypersurface $S_k \subseteq \mathbb{R}^n$

(ii) $\nu_E(x) \perp \operatorname{Tan}(S_k, x)$ for $x \in K_k$ $\forall k$

(iii) $\|\partial E\| = \mathcal{H}^{n-1} \llcorner \partial^* E$.

Proof $\boxed{\text{Step 1}}$

Set $f_r^+(x) = \frac{1}{r^n} \mathcal{L}^n(\mathbb{B}(x,r) \cap H^+(x) \cap E)$

$f_r^-(x) = \frac{1}{r^n} \mathcal{L}^n(\mathbb{B}(x,r) \cap H^-(x) \sim E)$

By 5. we know $f_r^{\pm} \xrightarrow[r \downarrow 0]{} 0$ pointwise on $\partial^* E$.

Use Egoroff's theorem and then Lusin's theorem to find compact sets $\{K_k\}_{k=1}^{\infty}$ such that

- $\|\partial E\|(\partial^* E \sim \bigcup_{k=1}^{\infty} K_k) = 0$
- f_r^+, f_r^- converge uniformly to 0 as $r \downarrow 0$ on each K_k
- ν_E is continuous on each K_k.

$\boxed{\text{Step 2}}$ Define $\rho_k(\delta) = \sup \left\{ \frac{\nu_E(x) \bullet (y-x)}{|y-x|} : \begin{array}{l} 0 < |y-x| \le \delta \\ x, y \in K_k \end{array} \right\}$

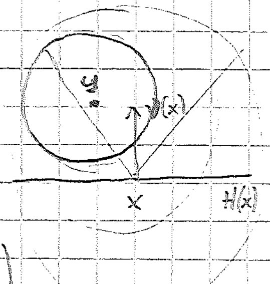
[To apply Whitney's Extension Theorem we need to show $\rho_k(\delta) \xrightarrow[\delta \downarrow 0]{} 0$ for each k.

Fix $\varepsilon > 0$. Let $\delta > 0$ be so small that

$\forall r \in (0, \delta)$ $\forall z \in K_k$

$$(*) \quad \begin{cases} \mathcal{L}^n(\mathbb{B}(z,r) \cap H^+(z) \cap E) \le \frac{\varepsilon^n}{2^{n+2}} \alpha(n) r^n \\ \mathcal{L}^n(\mathbb{B}(z,r) \cap H^-(z) \cap E) \ge \left(\frac{1}{2} - \frac{\varepsilon^n}{2^{n+2}}\right) \alpha(n) r^n \end{cases}$$

Assume $x, y \in K_k$, $0 < |x-y| < \delta$



Case 1: $\nu_E(x) \bullet (y-x) > \varepsilon |y-x|$

Then

$(**)$ $\mathbb{B}(y, \varepsilon|y-x|) \subseteq H^+(x) \cap \mathbb{B}(x, 2|x-y|)$

Applying $(*)$ with $z = x$ and then $(*)$ with $z = y$ we obtain a contradiction

18.02.2016

Follows Lecture 2 (6)

$$\mathcal{L}^n(\mathbb{B}(x, 2|x-y|) \cap H^+(x) \cap E) < \frac{\varepsilon^n}{4} \alpha(n) |x-y|^n$$

$$\mathcal{L}^n(\mathbb{B}(y, \varepsilon|x-y|) \cap E) \geq \mathcal{L}^n(\mathbb{B}(y, \varepsilon|x-y|) \cap E \cap H^-(y))$$
$$> \frac{\varepsilon^n}{4} \alpha(n) |x-y|^n$$

This contradicts (**).

<u>Case 2</u> $\nu_E(x) \cdot (y-x) \leq -\varepsilon|y-x|$ – the proof is the same.

$\boxed{\text{Step 3}}$ Apply Whitney Extension Theorem

to $f|_{K_k} \equiv 0$, $\nabla f|_{K_k} = \nu_E$

And set $S_k = \{ x \in \mathbb{R}^n : f_k(x) = 0 \text{ and } |\nabla f_k(x)| > \frac{1}{2} \}$

$\boxed{\text{Step 4}}$ Observe that

$$\lim_{r \downarrow 0} \frac{\|\partial E\| \, \mathbb{B}(x,r)}{(\mathcal{H}^{n-1} \llcorner K_k) \, \mathbb{B}(x,r)} = \lim_{r \downarrow 0} \frac{\|\partial E\| \, \mathbb{B}(x,r)}{\alpha(n-1) r^{n-1}} \frac{\alpha(n-1) r^{n-1}}{(\mathcal{H}^{n-1} \llcorner K_k) \mathbb{B}(x,r)} = 1$$

for $\mathcal{H}^{n-1}$ almost all $x \in K_k$

by 5. (ii) and standard properties of rectifiable sets.

Therefore, $\|\partial E\| \llcorner K_k = \mathcal{H}^{n-1} \llcorner K_k \quad \forall k$

Thus $\|\partial E\| = \mathcal{H}^{n-1} \llcorner \partial^* E$. by 6. and [Fed69, 2.9.7]

$\square$.

8. <u>Definition</u> $x \in \partial_* E \iff \Theta^{*n}(\mathcal{L}^n \llcorner E, x) > 0$ and $\Theta^{*n}(\mathcal{L}^n \llcorner \mathbb{R}^n \sim E, x) > 0$ Measure theoretic boundary.

9. <u>Lemma</u> [EG92, 5.8, Lem 1]

(i) $\partial^* E \subseteq \partial_* E$

(ii) $\mathcal{H}^{n-1}(\partial_* E \sim \partial^* E) = 0$

<u>Proof</u> (i) follows from 2.

(ii) If $x \in \partial_* E$, then $\Theta^{*n}(\mathcal{L}^n \llcorner E, x) = \alpha \in (0,1)$

By Relative Isoperimetric Inequality

$$\Theta^{*(n-1)}(\|\partial E\|, x) \geq \min\{\alpha, 1-\alpha\} = \gamma > 0$$

By [Fed 69, 2.10.19 (3)]

$$0 = \|\partial E\|(\partial_* E \sim \partial^* E) \geq \gamma \, \mathcal{H}^{n-1}(\partial_* E \sim \partial^* E)$$

$\square$

19.08.2016

Toulouse lecture 2 (7)

[EG92, 5.8, Thm 1]

10. Theorem $E \subseteq \mathbb{R}^n$ has locally finite perimeter

Then

$$\int_E \operatorname{div} \varphi \, d\mathcal{L}^n = \int_{\partial_* E} \varphi \cdot \nu_E \, d\mathcal{H}^{n-1} \qquad \forall \varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$$

Proof Simply because $\|\partial E\| = \mathcal{H}^{n-1} \llcorner \partial_* E$ $\square$

11. Theorem [EG92, 5.11, Thm 1]

$E \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable

Then E has locally finite perimeter if and only if

$$\mathcal{H}^{n-1}(K \cap \partial_* E) < \infty$$

whenever $K \subseteq \mathbb{R}^n$ is compact.

---

0. $U \subseteq \mathbb{R}^n$ open

$f \in BV(U) \iff f \in L^1(U)$ and $\|Df\|(U) < \infty$

where $\|Df\|(U) = \sup \left\{ \int_U f \operatorname{div} \varphi \, d\mathcal{L}^n : \varphi \in C^1_c(U, \mathbb{R}^n), |\varphi| \le 1 \right\}$

$V \subset\subset U \iff \operatorname{Clos} V \subseteq U$ and is compact

$f \in BV_{loc}(U) \iff \forall V \subset\subset U$ open $f|_V \in BV(V)$

$E \subseteq \mathbb{R}^n$ has finite perimeter in $U \iff \chi_E \in BV(U)$

$E \subseteq \mathbb{R}^n$ has locally finite perimeter in $U \iff \chi_E \in BV_{loc}(U)$.

[Fed 69, 2.5.12] $\Rightarrow$ $f \in BV_{loc}(U) \Rightarrow \exists\ \mu$ a Radon measure over U

$\sigma : U \to \mathbb{R}^n$ μ-measurable

$|\sigma(x)| = 1$ μ almost all x

$$\int_U f \operatorname{div} \varphi \, d\mathcal{L}^n = -\int \varphi \cdot \sigma \, d\mu$$

Notation $\|Df\| = \mu$

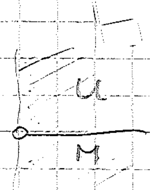
$$\chi_E \in BV_{loc}(U) \Rightarrow \begin{cases} \|\partial E\| = \|D\chi_E\| \\ \nu_E = -\sigma \quad \text{"outer unit normal"} \end{cases}$$

$\|f\|_{BV(U)} = \|f\|_{L^1(U)} + \|Df\|(U)$ the norm in $BV(U)$.

The goal is to show the Gauss-Green theorem for sets of locally finite perimeter, which amounts to showing $\boxed{\|\partial E\| = \mathcal{H}^{n-1} \llcorner \partial_* E}$

19.08.2016 **0.** M - smooth (open) submanifold of $\mathbb{R}^n$ s.t. $i : M \hookrightarrow U$ is proper, where $U \subseteq \mathbb{R}^n$ open

Ex:



$G_k(M) = \{ (x, T) : x \in M,\ T \subseteq \operatorname{Tan}(M, x),\ T \in G(n,k) \}$

$\mathbb{V}_k(M) = $ Radon measures over $G_k(M)$

(marginal) $\to$ $V \in \mathbb{V}_k(M) \longrightarrow \|V\|$ - a Radon measure on M, $\|V\|(A) = V(\{(x,S) \in G_k(M) : x \in A\})$

$F : M \to M'$ smooth (motivated by the area formula)

$$\Rightarrow\ F_{\#} V(\alpha) = \int \alpha(F(x), DF(x)[S]) \, \| \Lambda_k DF(x) \circ S_{\natural} \| \, dV(x,S)$$
$$\forall \alpha \in \mathcal{K}(G_k(M))$$

Toulouse Lecture 3 (1)

1. Definition / Lemma [3.3]

$V \in \mathbb{V}_k(M)$, $x \in M$, $\beta \in \mathcal{K}(G(n,k))$

$$V^{(x)} = \lim_{r \downarrow 0} \left(\|i_{\#} V\| B(x,r) \right)^{-1} \int_{B(x,r) \times G(n,k)} \beta(S) \, d(i_{\#} V)(y,S)$$

Then $V^{(\cdot)}$ is $\|V\|$-measurable function with values in Radon measures over $G(n,k)$ and for $\|V\|$ almost all $x \in M$

$V^{(x)}\left(G(n,k) \sim \{ S : S \subseteq \operatorname{Tan}(M,x) \} \right) = 0$

$V^{(x)}(G(n,k)) = 1$

$\int \alpha(x,S) \, d(i_{\#} V)(x,S) = \int V^{(x)}(\alpha_x) \, d\|V\|(x) \qquad \forall \alpha \in \mathcal{K}(G_k(U))$

where $\alpha_x(S) = \alpha(x,S)$

See [AFP00, 2.5] for details

Proof

For each fixed $\beta \in \mathcal{K}(G(n,k))$ consider the Radon measure over M given by

$$\mu_\beta(\gamma) = \int \gamma(x) \beta(S) \, d(i_{\#} V)(x,S)$$

- Employ the theory of symmetric derivation of measures [Fed69, 2.9.-] to obtain a $\|V\|$ measurable function $V_\beta : M \to \mathbb{R}$ given by $V_\beta(x) = V^{(x)}(\beta)$. such that
$$\int \gamma(x) V_\beta(x) \, d\|V\|(x) = \int \gamma(x) \beta(S) \, d(i_{\#} V)(x,S) \qquad \forall \gamma \in \mathcal{K}(M)$$
- Repeat that for a countable dense set of $\beta \in \mathcal{K}(G(n,k))$
- Approximate any function $\alpha \in \mathcal{K}(G_k(U))$ by functions of the form $\sum_i \beta_i(S) \gamma_i(x)$ for some $\beta_i \in \mathcal{K}(G(n,k))$, $\gamma_i \in \mathcal{K}(M)$ $\square$

19.08.2016 — **2. Definition** $V \in \mathbf{V}_k(M)$, $a \in M$, $j : \mathrm{Tan}(M,a) \hookrightarrow \mathbb{R}^n$

[3.4]
$$\mathrm{VarTan}(V,a) = \left\{ C \in \mathbf{V}_k(\mathrm{Tan}(M,a)) : \begin{array}{l} j_\# C \text{ is a weak limit} \\ \text{of } (\mu_{r_i} \circ \tau_{-a} \circ i)_\# V \\ \text{for some } r_i \to \infty \end{array} \right\}$$

Toulouse Lecture 3 (2)

3. Lemma [3.4(1)] Assume $\Theta^{*k}(i_\# \|V\|, a) < \infty$.

Then $\mathrm{VarTan}(V,a)$ is compact and non-empty.

Proof Let $f(r) = (\mu_r \circ \tau_{-a} \circ i)_\# V$

$f : (0,\infty) \to \mathbf{V}_k(\mathbb{R}^n)$

Observe that

$$j_\# \mathrm{VarTan}(V,a) = \{ C \in \mathbf{V}_k(\mathbb{R}^n) : \exists r_i \to \infty \ f(r_i) \to C \}$$

$$= \bigcap_{r>0} \mathrm{Clos}\, f[(r,\infty)] \quad \leftarrow \text{closed as a set } \text{~~of accumulation points of f~~}$$

Moreover, "=" whenever $\|j_\# C\|(\partial \mathbf{B}(0,R)) = 0$

$$\|j_\# C\| \, \mathbf{U}(0,R) \le \lim_{l\to\infty} \|(\mu_{r_l} \circ \tau_{-a} \circ i)_\# V\| \, \mathbf{U}(0,R)$$

(*)
$$= \lim_{l\to\infty} r_l^{\,k} \, i_\# \|V\| \, \mathbf{U}(a, R/r_l)$$

$$\le \Theta^{*k}(i_\# \|V\|, a) R^k < \infty$$

So $j_\# \mathrm{VarTan}(V,a)$ is a subset of the compact set

$$\{ W \in \mathbf{V}_k(\mathbb{R}^n) : \forall R \ \ \|W\| \, \mathbf{U}(0,R) \le \Theta^{*k}(i_\# \|V\|, a) R^k \}$$

□

4. Lemma [3.4(2)] Assume $\Theta^k(i_\# \|V\|, a) \in \mathbb{R}$ exists

$C \in \mathrm{VarTan}(V,a)$

Then (i) $\|C\|(\mathrm{Tan}(M,a) \cap \mathbf{B}(0,r)) = \Theta^k(i_\# \|V\|, a) \alpha(k) r^k$

(ii)
$$\int_{\mathbf{B}(0,1)\times G(n,k)} \beta(S) \, d(j_\# C)(x,S) = \Theta^k(i_\# \|V\|, a) \, \alpha(k) \, V^{(a)}(\beta)$$

$$\forall \beta \in \mathcal{K}(G(n,k))$$

Proof. For (i) proceed as in (*) which gives equality for $\mathcal{L}^1$ almost all R so actually for all R.

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(11)

For (ii) take $\gamma \in \mathcal{K}(\mathbb{R}^n)$ which approximates $\chi_{B(0,1)}$ and write

$$\int \gamma(x)\beta(S)\, d(i_{\#}C)(x,S) = \lim_{l\to\infty} r_l^k \int \gamma(r_l(x-a))\beta(S)\, dV(x,S)$$

$$= \lim_{l\to\infty} r_l^k \int \gamma(r_l(x-a)) \int \beta(S)\, dV^{(x)}(S)\, d\|V\|(x)$$

$$\cong \lim_{l\to\infty} \frac{1}{(1/r_l)^k} \int_{B(a,1/r_l)} V^{(x)}(\beta)\, d\|V\|(x) = \Theta^k(i_{\#}\|V\|, a)\,\alpha(k)\,V^{(a)}(\beta)$$

(since we know the limit exists)

$\square$

5. Definition

$E \subseteq M$, E is countably $(\mathcal{H}^k, k)$ rectifiable

$\mathcal{H}^k(E \cap K) < \infty$ for each compact $K \subseteq M$

$\mathbf{v}(E) \in \mathbf{V}_k(M)$

$$\mathbf{v}(E)(\alpha) = \int_E \alpha(x, \operatorname{Tan}^k(\mathcal{H}^k \llcorner E, x))\, d\mathcal{H}^k(x) \quad \forall \alpha \in \mathcal{K}(G_k(M))$$

$V \in \mathbf{RV}_k(M) \iff \exists c_i \in (0,\infty)\ \exists E_i$ as above

$$V = \sum c_i\, \mathbf{v}(E_i) \in \mathbf{V}_k(M)$$

$V \in \mathbf{IV}_k(M) \iff V \in \mathbf{RV}_k(M)$ and $c_i \in \mathbb{Z}$ $\forall i$

6. Theorem (i) $V \in \mathbf{RV}_k(M) \iff \Theta^k(i_{\#}\|V\|, a) \in (0,\infty)$

$$V^{(a)}(\beta) = \beta(\operatorname{Tan}^k(i_{\#}\|V\|, a))$$

for $\|V\|$ almost all $a \in M$.

(ii) $V \in \mathbf{RV}_k(M) \implies i_{\#}\|V\| = \mathcal{H}^k \llcorner \Theta^k(i_{\#}\|V\|, \cdot)$

$$V(\alpha) = \int_{\{x : \Theta^k(i_{\#}\|V\|, x) > 0\}} \alpha(x, \operatorname{Tan}^k(i_{\#}\|V\|, x))\, \Theta^k(i_{\#}\|V\|, x)\, d\mathcal{H}^k(x)$$

$$\operatorname{VarTan}(V, a) = \{\Theta^k(i_{\#}\|V\|, a)\, \mathbf{v}(\operatorname{Tan}^k(i_{\#}\|V\|, a))\}$$

(iii) $F : M \to M'$ smooth, then

$$F_{\#}\,\mathbf{RV}_k(M) \subseteq \mathbf{RV}_k(M')$$

$$F_{\#}\,\mathbf{IV}_k(M) \subseteq \mathbf{IV}_k(M')$$

$\square$

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4

0.
7. [All72, 4.1] $h : (-\varepsilon, \varepsilon) \times M \longrightarrow M$ smooth

$h_t(x) = h(t, x)$, $h_0 = \mathbb{1}_M$, h_t diffeomorphism

$C = \operatorname{Clos}\{x \in M : \exists t \in (-\varepsilon, \varepsilon) \ h_t(x) \neq x\}$

G open neighborhood of C in M

$V \in \mathbb{V}_k(M)$, $\|V\|(G) < \infty$

$\dot{h}_t(x) = \lim_{u \to 0} \frac{1}{u}\left(h_{t+u}(x) - h_t(x)\right)$; $\dot{h}_t \in \mathcal{X}(M)$

Goal: We want to compute

$$\circledast \quad \frac{d}{dt}\Big|_{t=0} \|h_{t\#} V\|(G) = \int \frac{d}{dt}\Big|_{t=0} \left|\Lambda_k Dh_t(x) \circ S\right| dV(x,S)$$

Write

$$h_t(x) = \underbrace{h_0(x)}_{=x} + \dot{h}_0(x) \cdot t + o(t)$$

$$\Rightarrow \circledast = \int \frac{d}{dt}\Big|_{t=0} \left|\Lambda_k\left(\mathbb{1}_{\operatorname{Tan}(M,x)} + t\, D\dot{h}_0(x) + o(t)\right) \circ S\right| dV(x,S)$$

formula from handout

$$= \int \left(D\dot{h}_0(x) \circ S\right) \bullet S \, dV(x,S)$$

1.
8. Definition [4.2]

$\delta V : \mathcal{X}(M) \longrightarrow \mathbb{R}$ first variation of V

$$\delta V(g) = \int \underbrace{\left(Dg(x) \circ S\right) \bullet S}_{= \operatorname{div}_S g = \sum Dg(x) e_i \bullet e_i \text{ for o.n.b. } e_1 \dots e_k \text{ of } S} dV(x,S)$$

Remark

- If $W \subseteq \mathbb{R}^n$ open and $g \in \mathcal{X}(W)$, then

$$\left(Dg(x) \circ S\right) \bullet S = Dg(x) \bullet S$$

- δV is a linear functional on $\mathcal{C}^1_c(M, \mathbb{R}^n)$

Definition total variation of V

$$\|\delta V\|(G) = \sup\{\delta V(g) : g \in \mathcal{X}(M),\ \operatorname{spt} g \subseteq G,\ |g| \leq 1\}$$

for $G \subseteq M$ open

$$\|\delta V\|(A) = \inf\{\|\delta V\|(G) : A \subseteq G,\ G \subseteq M \text{ open}\}$$

for $A \subseteq M$ arbitrary

Remark $\|\delta V\|$ is a Borel regular measure

$\delta V = 0 \Rightarrow V$ is stationary

$\|\delta V\|(G) = 0 \Rightarrow V$ is stationary in G

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2. 9. [All72, 4.3] Assume $\|\delta V\|$ is a Radon measure,

This is a regularity assumption!

i.e., $\|\delta V\|(K) < \infty$ whenever $K \subseteq M$ is compact

! δV is a linear functional on $\mathcal{E}^0_c(M, \mathbb{R}^n)$

Using [Fed69, 2.9, 2.5.12] we can then define

[normal vector] $\longrightarrow \eta(V, x) \in \mathrm{Tan}(M, x)$, $|\eta(V,x)| = 1$

[mean curvature vector] $\longrightarrow \mathbf{h}(V,x) = -\mathbf{D}(\|\delta V\|, \|V\|, x) \cdot \eta(V,x)$

[boundary measure] $\longrightarrow \|\delta V\|_{sing} = \|\delta V\| \llcorner \{x : \mathbf{D}(\|\delta V\|, \|V\|, x) = \infty\}$

by requireing that

$$\delta V(g) = -\int g(x) \bullet \mathbf{h}(V,x)\, d\|V\|(x) + \int g(x) \bullet \eta(V,x)\, d\|\delta V\|_{sing}(x)$$

whenever g is a Borel function with values in $\mathbb{R}^n$ such that $g(x) \in \mathrm{Tan}(M,x)$ for $x \in M$ and $\int |g|\, d\|\delta V\| < \infty$

$\left[\ g \in L^1(\|\delta V\|, \mathbb{R}^n) \text{ and } g(x) \in \mathrm{Tan}(M,x)\ \right]$

3. 10. Remark [4.4]

This reduces the study of varifolds in M to the study of varifolds in U!

$$\delta(i_\# V)(g) = \delta V(\mathrm{Tan}(M,g)) - \int \mathrm{Nor}(M,g)(x) \bullet \mathbf{h}(M,x,S)\, dV(x,S)$$

whenever $g \in \mathcal{X}(U)$ [by 2.5(3) def. of $\mathbf{h}(M,x,S)$]

4. 11. Example

$\Sigma \subseteq \mathbb{R}^n$ a smooth k-dim. submanifold with boundary

$\Theta : \Sigma \longrightarrow (0,\infty) \in \mathcal{E}^1$

$V \in \mathbf{V}_k(\mathbb{R}^n)$, $V(\alpha) = \int_\Sigma \alpha(x, \mathrm{Tan}(\Sigma,x))\, \Theta(x)\, d\mathcal{H}^k(x)$ $\quad \forall \alpha \in \mathcal{K}(\mathbb{R}^n \times G(n,k))$

Let $g \in \mathcal{X}(\mathbb{R}^n)$. Then

$g = \mathrm{Tan}(\Sigma, g) + \mathrm{Nor}(\Sigma, g)$

- $\mathbf{D}(\mathrm{Nor}(\Sigma,g))(x) \bullet \mathrm{Tan}(\Sigma,x) \overset{\text{by def. of } \mathbf{h}}{=} -g(x) \bullet \mathbf{h}(\Sigma,x)$
- $\mathbf{D}(\mathrm{Tan}(\Sigma,g))(x) \bullet \mathrm{Tan}(\Sigma,x)\ \Theta(x) =$

$$= \mathbf{D}(\Theta(x)\, \mathrm{Tan}(\Sigma,g)(x)) \bullet \mathrm{Tan}(\Sigma,x)$$
$$- (\mathbf{D}\Theta(x)\, g(x)) \bullet \mathrm{Tan}(\Sigma,x)$$

Unfolding all definitions

$$= \mathrm{div}_\Sigma(\Theta \cdot \mathrm{Tan}(\Sigma,g))(x)$$
$$- \mathrm{Tan}(\Sigma,x)_\natural(\mathrm{grad}\,\Theta(x)) \bullet g(x)$$

$$\delta V(g) = \delta V(\mathrm{Tan}(\Sigma,g)) + \delta V(\mathrm{Nor}(\Sigma,g))$$

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$$\delta V(g) = -\int_\Sigma g(x) \bullet \left(\mathbf{h}(\Sigma, x) + \operatorname{Tan}(\Sigma, x)_\natural \left(\operatorname{grad}(\log \theta)(x) \right) \right) \theta(x) \, d\mathcal{H}^k(x)$$

$$+ \int_{\partial \Sigma} g(x) \bullet \nu_\Sigma(x) \, \theta(x) \, d\mathcal{H}^{k}(x)$$

we use Stokes: Fed69, 4.1.31
co-dim = 1: [GT01, Lemma 16.1]

Hence:

$$\|\delta V\|_{sing} = \mathcal{H}^{k-1} \llcorner \partial \Sigma$$

$$\eta(V, x) = \nu_\Sigma(x) \quad \text{for} \quad x \in \partial \Sigma$$

$$\mathbf{h}(V, x) = \mathbf{h}(\Sigma, x) + \operatorname{Tan}(\Sigma, x)_\natural \left(\operatorname{grad}(\log \theta)(x) \right)$$

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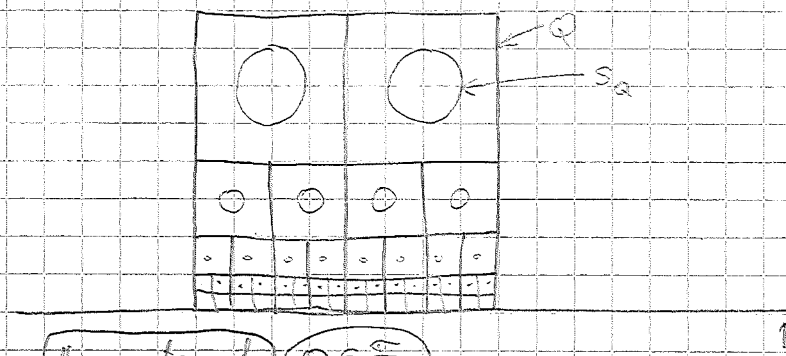
5. Example 12. $m = n = k+1$

Take the Whitney decomposition ($\mathcal{F}$) into dyadic cubes of $\{(x_1 \ldots x_n) \in \mathbb{R}^n : x_n > 0\}$ w.r.t. $\mathbb{R}^{n-1} \times \{0\}$.



In each cube of side-length $2^{-\ell}$ place a sphere S_Q (the center of $Q \in \mathcal{F}$) of radius r_ℓ. Let $\Sigma = \bigcup \{ S_Q : Q \in \mathcal{F} \}$.

If $r_\ell < \frac{1}{2} 2^{-\ell}$ then $\{ S_Q : Q \in \mathcal{F} \}$ is disjointed and Σ is a smooth manifold. Set $V = \mathbf{v}(S)$

Observe that

- $\operatorname{spt} \|V\| = (\mathbb{R}^{n-1} \times \{0\}) \cup \bigcup_{Q \in \mathcal{F}} S_Q$
- If $r_\ell < \left(\frac{1}{4}\right)^\ell$, then $\|V\|$ is Radon and $V \in \mathbf{V}_k(\mathbb{R}^n)$ because

$$\|V\|([0,1]^n) = \sum_{\ell=1}^{\infty} 2^{\ell k} \cdot k \alpha(k) r_\ell^k \le k\alpha(k) \sum_{\ell=1}^{\infty} \left(\frac{1}{2}\right)^\ell < \infty$$

- If $r_\ell < \left(\frac{1}{4}\right)^{\frac{\ell k}{k-p}}$ and $0 < p < k$, then

$$\mathbf{h}(V, \cdot) \in L^p_{loc}(\|V\|), \text{ because}$$

$$\int_{[0,1]^n} |\mathbf{h}(V,x)|^p \, d\|V\|(x) = k\alpha(k) \sum_{\ell=1}^{\infty} 2^{\ell k} \cdot r_\ell^k \cdot r_\ell^{-p}$$

$$\le k\alpha(k) \sum_{\ell=1}^{\infty} \left(\frac{1}{2}\right)^\ell < \infty$$

- $\Theta^k(\|V\|, x) = 1$ for $\|V\|$ almost all x

[cf. Menne, Adv. Calc. Var. 2009, Section 1]

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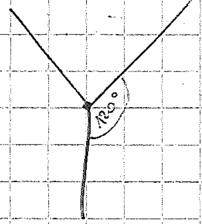
Example



$$e_j = \exp\left(2\pi i \tfrac{j}{3}\right) \in \mathbb{R}^2 \qquad j = 1,2,3$$

$$\Sigma = \{ t e_j : t \in [0,\infty),\ j = 1,2,3 \}$$

$$V \in \mathbf{v}(\Sigma) \in \mathbf{V}_1(\mathbb{R}^2)$$

$$\Rightarrow \delta V = 0$$

6.13. Remark: [4.11] Since sup of continuous functions is always l.s.c. we see that if $V_i \longrightarrow V$ in $\mathbf{V}_k(U)$, then

$$\|\delta V\|(G) \le \liminf_{i\to\infty} \|\delta V_i\|(G) \quad \text{for } G \subseteq U \text{ open.}$$

Remark: [4.12] If $k \ge 1$, $V \in \mathbf{V}_k(\mathbb{R}^n)$, then

$$\|\delta(\mu_{r\,\#} V)\| = r^{k-1} \mu_{r\,\#} \|\delta V\|$$

Recall

$$\|\mu_{r\,\#} V\| = r^k \mu_{r\,\#} \|V\|$$

Corollary: [4.12] If $\Theta^{k-1}(\|\delta V\|, a) = 0$ and $C \in \operatorname{VarTan}(V, a)$, then C is stationary, i.e., $\delta C = 0$.

In particular this holds if

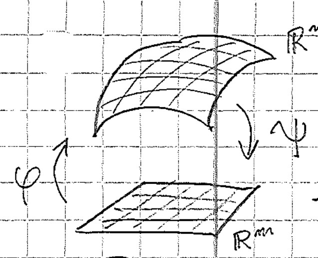
$$\left.\begin{array}{r} \Theta^k(\|V\|, a) \in (0,\infty) \\ \text{and } \mathbf{D}(\|\delta V\|, \|V\|, a) < \infty \end{array}\right\} \Rightarrow \Theta^k(\|\delta V\|, a) < \infty$$

If $V \in \mathbf{RV}_k(\mathbb{R}^n)$, then $\|V\|$ almost everywhere

---

22.08.2016 / 1. Lemma [All72, 4.5]

$W \subseteq U$ open, $Y \subseteq \mathbb{R}^m$ open, $\psi : W \to Y$, $\varphi : Y \to W$



ψ, φ - smooth, $\psi \circ \varphi = id_Y$, $W \cap \operatorname{im}\varphi = W \cap M$

$V \in \mathbb{V}_m(W \cap M)$, $\dim M = m$

Then (a) $\delta V(g) = \delta(\psi_* V)\big(|\Lambda_m D\varphi| \langle g, D\psi\rangle \circ \varphi\big)$ for $g \in \mathcal{X}(W \cap M)$

(b) $\int \langle v, D\beta(y)\rangle \, d\|\psi_* V\|(y) = \delta V\big(|\Lambda_m D\varphi|^{-1} \beta \langle v, D\varphi\rangle \circ \psi\big)$

for $v \in \mathbb{R}^m$, $\beta \in \mathcal{D}(Y, \mathbb{R})$.

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Proof. $\boxed{\text{Claim 1}}$ $\int_{W \cap M} \big(Dg(x) \circ \operatorname{Tan}(M,x)\big) \bullet \operatorname{Tan}(M,x) \, d\mathcal{H}^m(x) = 0$

$\forall g \in \mathcal{X}(M \cap W)$

$\|$

$\frac{d}{dt}\Big|_{t=0} \mathcal{H}^m(h_t[M \cap W]) = \frac{d}{dt}\Big|_{t=0} \|h_{t\#} \mathbf{v}(W \cap M)\|$ where h_t is a 1-param. family of diffeomorphisms generated by g (the flow of g)

$\|$

0 because g is tangential and has compact support.

$\boxed{\text{Claim 2}}$ $u_1, \dots, u_m \in \mathbb{R}^m$ a basis of $\mathbb{R}^m$

$\xi_i(x) = |\Lambda_m D\varphi(\psi(x))|^{-1} \langle u_i, D\varphi(\psi(x))\rangle$ for $x \in W \cap M$, $i = 1, 2, \dots, m$

Then $\big(D\xi_i(x) \circ \operatorname{Tan}(M,x)\big) \bullet \operatorname{Tan}(M,x) = 0$

If $\theta \in \mathcal{D}(W \cap M, \mathbb{R})$, then $\theta \xi_i \in \mathcal{X}(W \cap M)$ and we apply Claim 1 to get

$$(*) \quad 0 = \int_{W \cap M} \theta(x) \big(D\xi_i(x) \circ \operatorname{Tan}(M,x)\big) \bullet \operatorname{Tan}(M,x) \, d\mathcal{H}^m(x) + \underbrace{\int_{W \cap M} \langle \xi_i(x), D\theta(x)\rangle \, d\mathcal{H}^m(x)}$$

$\|$ by the area formula

$$\int_Y D(\theta \circ \varphi)(y) \, u_i \, d\mathcal{H}^m(y) = 0$$

Since $\circledast$ holds for all $\theta \in \mathcal{D}(W \cap M, \mathbb{R})$ the claim is proven.

$\boxed{\text{Proof of (a)}}$ Let $g \in \mathcal{X}(M \cap W)$. Since $\{\xi_i(x)\}_{i=1}^m$ form a basis of $\operatorname{Tan}(M,x)$ we can find $\zeta_1, \dots, \zeta_m \in \mathcal{D}(W \cap M, \mathbb{R})$ s.t.

$$g(x) = \sum_{i=1}^m \zeta_i(x) \, \xi_i(x) \quad \text{for } x \in W \cap M$$

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$$\delta V(g) = \sum_{i=1}^{m} \int \langle \xi_i(x), D\zeta_i(x)\rangle \, d\|V\|(x) \quad \leftarrow \boxed{\text{by Claim 2}}$$

$\boxed{x = \varphi\circ\psi(x),\ y = \psi(x)}$

$$= \sum_{i=1}^{m} \int \langle \xi_i(\varphi(y)), D\zeta_i(\varphi(y))\rangle \, d(\psi_\# \|V\|)(y)$$

$\boxed{\text{Expand } \xi_i}$

$$= \sum_{i=1}^{m} \int \langle u_i, D(\zeta_i\circ\varphi)(y)\rangle \, |\Lambda_m D\varphi(y)|^{-1} \, d(\psi_\# \|V\|)(y)$$

$D\psi(\varphi(y)) \circ \operatorname{Tan}(M,\varphi(y)) = D\varphi(y)^{-1}$

$$= \sum_{i=1}^{m} \int \langle u_i, D(\zeta_i\circ\varphi)(y)\rangle \, d\|\psi_\# V\|(y) \qquad ☺$$

(2)

Ex. $(\omega \wedge v) \bullet S = \langle Sv, \omega\rangle$ for $v \in \mathbb{R}^m$, $\omega \in \operatorname{Hom}(\mathbb{R}^m,\mathbb{R})$, $S \in G(m,k)$ [2.3(4)]

$$= \sum_{i=1}^{m} \int (D(\zeta_i\circ\varphi)(y) \cdot u_i) \bullet \mathbb{R}^m \, d\|\psi_\# V\|(y)$$

$\psi\circ\varphi = id_Y$

$$= \int D\Big(\sum_{i=1}^{m} (\zeta_i\circ\varphi) \overbrace{\langle D\varphi(\cdot)u_i, D\psi(\varphi(\cdot))\rangle}^{=u_i} \Big)(y) \bullet \mathbb{R}^m \, d\|\psi_\# V\|(y)$$

$\boxed{\text{Expand } \xi_i}$

$$= \int D\Big(\sum_{i=1}^{m} |\Lambda_m D\varphi| \langle (\zeta_i\circ\varphi)(\xi_i\circ\varphi), D\psi\circ\varphi\rangle \Big)(y) \bullet \mathbb{R}^m \, d\|\psi_\# V\|$$

$$= \int D\big(|\Lambda_m D\varphi| \langle g\circ\varphi, D\psi\circ\varphi\rangle \big)(y) \bullet \mathbb{R}^m \, d\|\psi_\# V\|(y)$$

$$= \delta(\psi_\# V)\big(|\Lambda_m D\varphi| \langle g\circ\varphi, D\psi\circ\varphi\rangle \big)$$

Proof of (6)

If $g = (|\Lambda_m D\varphi|^{-1} \beta \cdot \langle v, D\varphi\rangle)\circ\psi$

$v \in \mathbb{R}^m$, $v = u_1, u_2, \ldots, u_m$ is a basis of $\mathbb{R}^m$,

then

$$\zeta_1(x) = \beta(\psi(x)) \quad \text{and} \quad \zeta_i = 0 \text{ for } i = 2, \ldots, m$$

Plugging this into ☺ gives (6). □

Remark. 1. provides a link between distributional derivatives of $\|\psi_\# V\|$ and the first variation δV

2. Lemma [4.6(1)] $f \in \mathcal{D}(U,\mathbb{R})$, $V \in \mathbf{V}_k(U)$, $\operatorname{spt}\|V\| \subseteq f^{-1}\{0\}$, $\|\delta V\|$ is Radon

Then $S \subseteq \ker Df(x)$ for V almost all (x,S).

Proof

Set $g(x) = f(x)\nabla f(x) = 0$ for $\|V\|$ almost all x.

Then $Dg(x) \bullet S_\natural = (Df(x) \cdot \operatorname{grad} f(x)) \bullet S_\natural \overset{[2.3(4)]}{=} |S_\natural(\operatorname{grad} f(x))|^2$

Hence, $0 = \delta V(g) = \int |S_\natural(\operatorname{grad} f(x))|^2 \, dV(x,S)$ □

$\uparrow$ because $\|\delta V\|$ is Radon

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3. Corollary $\{V \in \mathbb{V}_k(U) : \operatorname{spt}\|V\| \subseteq M \text{ and } \|\delta V\| \text{ Radon}\} \subseteq i_{\#} \mathbb{V}_k(M)$.

Proof. One needs to show that for $V \in LHS$ one has

$$S \subseteq \operatorname{Tan}(M,x) \quad \text{for } V \text{ almost all } (x,S)$$

This follows from 2. by representing $\Omega \cap M = W \cap \bigcap_{i=1}^{n-m} f_i^{-1}\{0\}$, where W is a neighborhood of $x \in M$ and $f_i \in \mathcal{D}(U,\mathbb{R})$. □

4. Theorem [4.6(3)] CONSTANCY THEOREM

M - connected, $V \in \mathbb{V}_m(U)$, $\operatorname{spt}\|V\| \subseteq M$, $\|\delta V\|$ Radon

$\delta V(g) = 0$ for $g \in \mathscr{X}(U)$ with $\operatorname{Nor}(M,g) = 0$.

Then

$$\exists\, c \in [0,\infty) \quad V = c\,\mathbf{v}(M)$$

In fact $c = \dfrac{\|V\|(A)}{\mathcal{H}^m(A)}$ for any $A \subseteq M$ with $\mathcal{H}^m(A) < \infty$.

Proof.
- From 3. find $\bar V \in \mathbb{V}_m(M)$ s.t. $V = i_{\#} \bar V$
- Using (Lecture 3, 3.)[4.4] we see that $\delta \bar V(g) = 0$ for $g \in \mathscr{X}(M)$.
- Set $T = \|\psi_{\#} \bar V\| \in \mathcal{D}'(Y,\mathbb{R})$ as in 1.

  Using 1. we compute that $D_j T = 0$ for $j = 1, \dots, m$.
- The Constancy Theorem for distributions [Fed69, 4.1.4] gives

  $$\|\psi_{\#} \bar V\| = c\,\mathcal{L}^m \quad \text{for some } c \in \mathbb{R}.$$
- Hence, $\|\bar V\| \llcorner (M \cap W) = c\,\mathcal{H}^m \llcorner (M \cap W)$
- Since M is connected we get the conclusion □

5. Example [4.7] $E \subseteq M$, $b \in M$

- $N(M,E,b) = \{u \in \operatorname{Tan}(M,b) \cap \{v : |v| = 1\} :$

$$\Theta^m(\mathcal{H}^m \llcorner \{x \in M : (x-b)\bullet u > 0\} \cap E, b) = 0$$
$$\Theta^m(\mathcal{H}^m \llcorner \{x \in M : (x-b)\bullet u < 0\} \sim E, b) = 0 \}$$

- If $\mathcal{H}^0(N(M,E,b)) = 1$, let $n(M,E,b) \in N(M,E,b)$ and $n(M,E,b) = 0$ otherwise.
- $Q = \{x \in M : \Theta^m(\mathcal{H}^m \llcorner (M \sim E), x) = 0\}$

  $R = \{x \in M : \Theta^m(\mathcal{H}^m \llcorner (M \cap E), x) = 0\}$

  measure theoretic interior and exterior

  Assume

  $$\mathcal{H}^{m-1}(K \sim (Q \cup R)) < \infty \quad \text{for any } K \subseteq M \text{ compact.}$$

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Then $\mathbf{v}(E) \in \mathbf{V}_m(M)$

$\partial^* E = \{ b \in M : n(M, E, b) \neq 0 \}$

$K \cap \partial^* E$ is $(\mathcal{H}^{m-1}, m-1)$ rectifiable and $\mathcal{H}^{m-1}$ measurable for any $K \subseteq M$ compact

$$\delta \mathbf{v}(E)(g) = \int_{\partial^* E} g(b) \bullet n(M, E, b) \, d\mathcal{H}^{m-1}(b) \quad \text{for } g \in \mathscr{X}(M)$$

One has to check: (assume $E \subseteq W \cap M$, $\varphi: Y \to W$, $\psi: W \to Y$ los in 1.)

- $\psi[E]$ has locally finite perimeter in Y
- $n(M, E, b) \neq 0 \iff n(\mathbb{R}^m, \psi[E], \psi(b)) \neq 0$

  so that $\psi[\partial^* E] = \partial^* \psi[E]$,
- $$n(M, E, b) = \langle n(\mathbb{R}^m, \psi[E], \psi(b)), (D\varphi(\psi(b))^{-1})^* \rangle \cdot \frac{|\Lambda_m D\varphi(\psi(b))|}{|\Lambda_{m-1} D\varphi(\psi(b)) \circ \operatorname{Tan}^{m-1}(\mathcal{H}^{m-1} \llcorner \partial^* E, b)|}$$

Hint. $e_1, \dots, e_m$ an o.n.b. of $\mathbb{R}^m$ s.t. $e_m = n(\mathbb{R}^m, \psi[E], \psi(b))$

$p \in \mathbf{O}^*(n, m)$, $\operatorname{im} p^* = \operatorname{im} D\varphi(\psi(b))$

$L = p \circ D\varphi(\psi(b)) \in \operatorname{Aut}(\mathbb{R}^m) = GL(m, \mathbb{R})$

Then $(L^{-1})^* e_m \, |\Lambda_m L| = *(L e_1 \wedge \dots \wedge L e_{m-1})$

---

6. **Lemme** [4.10(1)] $V \in \mathbf{V}_k(U)$, $r \in \mathbb{R}$, $f: U \to \mathbb{R} \in C^0$, $g \in \mathscr{X}(U)$, $\|\delta V\|$ Radon

f is smooth in a neighborhood of $\operatorname{spt}\|V\| \cap f^{-1}\{r\} \cap \operatorname{spt} g$

Then

$$\delta V(\chi_{\{f > r\}} g) = \delta(V \llcorner \{(x, S) : f(x) > r\})(g) + \lim_{h \downarrow 0} \frac{1}{h} \int_{\{(x,S) : r < f(x) \le r+h\}} S_\natural(g(x)) \bullet \operatorname{grad} f(x) \, dV(x, S).$$

Remark If $E_r = \{x \in U : f(x) > r\}$, then

$$\underbrace{(\delta V) \llcorner E_r(g) - \delta(V \llcorner E_r \times G(n,k))(g)}_{\substack{\| \\ V \partial E_r(g)}} = \lim_{h \downarrow 0} \frac{1}{h} \int_{(E_r \sim E_{r+h}) \times G(n,k)} S_\natural(g(x)) \bullet \operatorname{grad} f(x) \, dV(x, S)$$

in the sense of Menne, Indiana Univ. Math. J., 2016, §5.

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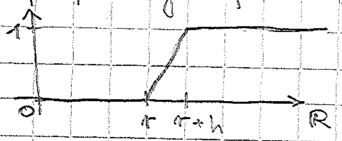
Proof. Let $\varphi \in \mathcal{D}(\mathbb{R},\mathbb{R})$ be such that f is smooth in a neighborhood of $\operatorname{spt}\|V\| \cap f^{-1}[\operatorname{spt}\varphi'] \sim \operatorname{spt} g$

- $D\big(((\varphi\circ f)\cdot g)\big)(x)\bullet S_\natural = (\varphi\circ f)(x)\, Dg(x)\bullet S + (\varphi'\circ f)(x)\,\operatorname{grad} f(x)\bullet S_\natural(g(x))$
- For $h>0$ set $\gamma_h : \mathbb{R}\to\mathbb{R}$



- Take $\varphi_j \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ s.t.

$$\|\varphi_j - \gamma_h\|_{L^\infty} \to 0$$

$$\varphi_j'(r) = 0, \quad \varphi_j'(r+h) = \tfrac{1}{h} \quad \text{for } j\in\mathbb{N}$$

$$\varphi_j'(s) \xrightarrow[j\to\infty]{} \gamma_h'(s) \quad \text{for } s\in\mathbb{R}\sim\{r, r+h\}.$$

- Use two definitions of δV (together with the dominated convergence theorem):

$$\lim_{h\downarrow 0}\lim_{j\to\infty} \delta V((\varphi_j\circ f)g) = \lim_{h\to 0}\lim_{j\to\infty}\int (\varphi_j\circ f)(x)\, g(x)\bullet \eta(V,x)\, d\|\delta V\|(x)$$

$$= \lim_{h\downarrow 0}\lim_{j\to\infty}\int D((\varphi_j\circ f)g)(x)\bullet S\, dV(x,S)$$

$$= \int \chi_{\{f>r\}}(x)\, g(x)\bullet\eta(V,x)\, d\|\delta V\|(x) = \delta V(\chi_{\{f>r\}}\, g) = \text{LHS}$$

$$\text{RHS} = \begin{cases} \int \chi_{\{f>r\}}(x)\, Dg(x)\bullet S\, dV(x,S) \\ + \lim_{h\downarrow 0}\frac{1}{h}\int_{\{r<f\le r+h\}\times G(n,k)} S_\natural(g(x))\bullet \operatorname{grad} f(x)\, dV(x,S)\end{cases}$$

$\square$.

7. Theorem [4.10(2)] $V\in \mathbb{V}_k(U)$, $-\infty\le a<b\le\infty$, $\|\delta V\|$ Radon, $f: U\to\mathbb{R}\in\mathcal{C}^0$ is smooth in a neighborhood of

$$\operatorname{spt}\|V\|\cap\{x : a<f(x)<b\}$$

Then for $\mathcal{L}^1$ almost all $r\in(a,b)$

(a) $\|\delta(V \llcorner \{(x,S) : f(x)>r\})\|$ is Radon

(b) $$\int_a^b \|\delta(V\llcorner\{(x,S): f(x)>r\})\|(B)\, d\mathcal{L}^1(r)$$

$$\le \int_{B\cap\{a<f<b\}\times G(n,k)} |S_\natural(\operatorname{grad} f(x))|\, dV(x,S)$$

$$+ \int_a^b \|\delta V\|(B\cap\{f>r\})\, d\mathcal{L}^1(r)$$

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Proof. • Wlog f is smooth around $f^{-1}\{a\}$

• Set $$T_r(g) = \int_{\{(x,S)\,:\, a < f(x) < r\}} S_\natural(g(x)) \bullet \operatorname{grad} f(x)\, dV(x,S)$$ for $g \in \mathcal{K}(U)$, $r \in (a,b)$

• Define

$$\mu_r(G) = \sup\{T_r(g) : g \in \mathcal{K}(U),\ \operatorname{spt} g \subseteq G,\ |g| \leq 1\}$$ for $G \subseteq U$ open

$$\mu_r(A) = \inf\{\mu_r(G) : A \subseteq G,\ G \subseteq U \text{ open}\}.$$

• Observe that for each $r, s \in (a,b)$ with $r \leq s$

* μ_r is a Radon measure over U

* $\mu_r \leq \mu_s$

• Employ [2.6(3)] to see that for $\mathcal{L}^1$ almost all $r \in (a,b)$ there exists a Radon measure μ_r' s.t.

$$\mu_r'(k) = \lim_{h \downarrow 0} \frac{1}{h}\left(\mu_{r+h}(k) - \mu_r(k)\right) \quad \text{for } k \in \mathcal{K}(U).$$

• Using 6. we see that for $K \subseteq U$

$$(*)\quad \|\delta(V \llcorner \{(x,S) : f(x) > r\})\|(K) \leq \|\delta V\|(K \cap \{f > r\}) + \mu_r'(K) < \infty$$ (if K is compact)

So (a) is proven

• To prove (b) integrate $(*)$:

$$\int_a^b \|\delta(V \llcorner \{(x,S) : f(x) > r\})\|(B)\, d\mathcal{L}^1(r)$$

$$\leq \int_a^b \|\delta V\|(B \cap \{f > r\})\, d\mathcal{L}^1(r) + \int_a^b \mu_r'(B)\, d\mathcal{L}^1(r)$$

__Warning:__ * We know $\mu_r'(k) = \frac{d}{dt}\Big|_{t=r} \mu_t(k)$ for $k \in \mathcal{K}(U)$

* For open sets $G \subseteq U$

$$\mu_r'(G) \leq \frac{d}{dt}\Big|_{t=r} \mu_r(G) \quad \text{by } [2.6(2)(c)]$$

* For $B \subseteq U$ Borel

$$\int_a^b \inf\{\mu_r'(G) : B \subseteq G,\ G\text{-open}\}\, d\mathcal{L}^1(r)$$

Fatou's Lemma

$$\leq \inf\left\{\int_a^b \mu_r'(G)\, d\mathcal{L}^1(r) : B \subseteq G,\ G\text{-open}\right\}$$

[2.9.19]

$$\leq \inf\{\mu_b(G) : B \subseteq G,\ G\text{-open}\} = \mu_b(B),$$

$$\leq \int |S_\natural(\operatorname{grad} f(x))|\, dV(x,S)$$

Lemme 15 **3.1.**

(1)

$m \le n$, M - an m-dim submani of $\mathbb{R}^n$ of class C^1

Y - a normed vectorspace

$f: M \to Y$ of class C^1 rel. M.

Then (1)

$$\rho(C,\delta) = \sup\Big(\{0\} \cup \{ |f(x) - f(a) - \langle \mathrm{Tan}(M,a)_\natural (x-a), Df(a)\rangle| / |x-a| \; : x, a \in C, \; 0 < |x-a| \le \delta \}\Big)$$

Then $\rho(C,\delta) \to 0$ as $\delta \to 0$

for any $C \subseteq M$ compact.

(2) $\exists\, U \subseteq \mathbb{R}^n$ open $M \subseteq U$

$\exists\, g: U \to Y \in C^1$ $g|_M = f$

$Dg(a) = Df(a) \circ \mathrm{Tan}(M,a)_\natural$ for $a \in M$.

Proof. (1) immediate

(2) Define $P_a: \mathbb{R}^n \to Y$ by

$$P_a(x) = f(a) + \langle \mathrm{Tan}(M,a)_\natural (x-a), Df(a)\rangle$$

for $a \in M$, $x \in \mathbb{R}^n$.

Apply Whitney extension theorem on each closed (in $\mathbb{R}^n$) set $A \subseteq M$, to obtain a function $g_A: \mathbb{R}^n \to Y \in C^1$, $g|_A = f|_A$.

and $Dg_A(a) = Df(a) \circ \mathrm{Tan}(M,a)_\natural$ for $a \in A$.

Then use a partition of unity. $\square$

Heure 15

(2)

3.2 $1 \le m \le n$, M - m-dim submani of class $\mathcal{C}^1$ in $\mathbb{R}^n$.

Then $\exists r : U \to M$ where $M \subseteq U$, U - open in $\mathbb{R}^n$

$$r[U] = M, \quad r(x) = x \text{ for } x \in M.$$

$$Dr(a) = \mathrm{Tan}(M, a) \quad \text{for } a \in M.$$

Proof. First extend $f = 1_M : M \to \mathbb{R}^n$ using [3.1] and then apply Whitney, p.121 to get a retraction $h : U \to M$ of class $\mathcal{C}^1$. Finally set $r = h \circ g$.

$$x \in M \Rightarrow Dr(x)u = Dh(g(x))\, Dg(x)u =$$

$$= Dh(g(x))(\mathrm{Tan}(M, x)u) = \mathrm{Tan}(M, x)u.$$

□

3.3 $U \subseteq \mathbb{R}^n$ open, μ Radon over U, $h : U \to \mathbb{R}$, $h \in \mathcal{C}^1$, $A = \{x : h(x) \ge 0\}$, $\varepsilon > 0$

Then

$$\exists\, g : U \to \mathbb{R}_+ \in \mathcal{C}^1$$

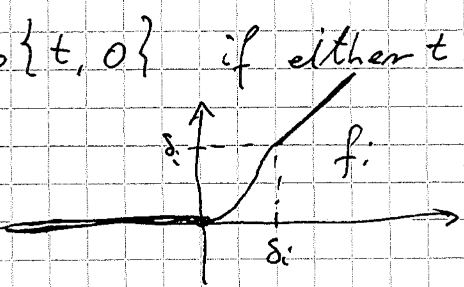
$$\mu(A \sim \{x : h(x) = g(x)\}) \le \varepsilon.$$

Proof. Take a partition of unity $\zeta_i \in \mathcal{D}(U, \mathbb{R})$ associated to $\{U\}$. Set $K_i = \operatorname{spt} \zeta_i$.

Choose $\delta_i > 0$ and functions $f_i : \mathbb{R} \to \mathbb{R}_+ \in \mathcal{C}^1$ s.t.

$$\mu(K_i \cap \{x : 0 < h(x) < \delta_i\}) \le 2^{-i}\varepsilon,$$

$$f_i(t) = \sup\{t, 0\} \text{ if either } t \le 0 \text{ or } t \ge \delta_i.$$



Observe: $A \sim \{x : h(x) = \sum_{i=1}^{\infty} \zeta_i(x)(f_i \circ h)(x)\} \subseteq \bigcup_{i=1}^{\infty} K_i \cap \{x : 0 < h(x) < \delta_i\}$

Memme 15

(3)

Therefore, $g = \sum_{i=1}^{\infty} \zeta_i (f_i \circ h)$ ~~satisfies~~

has the required properties $\square$.

$\boxed{3.4}$ $l, n \in \mathbb{Z}_+$, $U \subseteq \mathbb{R}^n$ open, $A \subseteq U$, $f: U \to \mathbb{R}^l$

$f \in C^1$, $\varepsilon > 0$

Then $\exists X \subseteq U$ open $\exists g: \mathbb{R}^n \to \mathbb{R}^l \in C^1$

$A \subseteq X$, $f|X = g|X$ and

$$\operatorname{Lip} g \leq \varepsilon + \sup\{\operatorname{Lip}(f|_A), \sup \|Df\|[A]\}.$$

Moreover, if $l = 1$, $f \geq 0$, then $g \geq 0$.

Proof. Assume

$\kappa = \max\{\operatorname{Lip}(f|_A), \sup \|Df\|[A]\}$.

A is rel. closed in U.

Step 1. $\exists G \subseteq U$ open

$A \subseteq G$, $\operatorname{Lip}(f|_G) \leq \frac{1}{2}\varepsilon + \kappa$

Define $\eta = 2^{-4} \varepsilon (\varepsilon + \kappa)^{-1}$

Note $\eta \leq \frac{1}{2}$

Choose $\delta: A \to (0, \infty)$ s.t.

$U(a, \delta(a)) \subseteq U$ and $\operatorname{Lip}(f|U(a,\delta(a))) \leq \frac{1}{2}\varepsilon + \kappa$

for $a \in A$.

Set

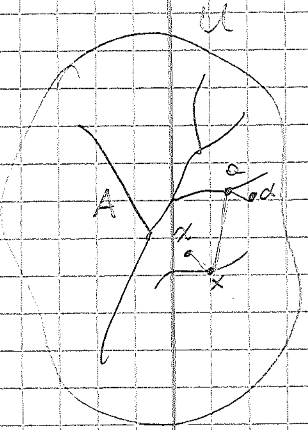
$G = \bigcup\{U(a, \eta\delta(a)) : a \in A\}$.

Suppose
$$\boxed{\begin{array}{l} a, x \in A, \quad \alpha, \chi \in \mathbb{R}^n, \\ |a - \alpha| < \eta\delta(a), \quad |x - \chi| < \eta\delta(x) \end{array}}$$

In case $\delta(a) + \delta(x) \leq 4|\chi - \alpha|$.

$$\underbrace{|a-\alpha| + |x-\chi|}_{\le \eta(\delta(a)+\delta(x))} \le 4\eta|\chi-\alpha|$$

$$|x-a| \le |x-\chi| + |\chi-\alpha| + |\alpha-a| \le (1+4\eta)|\chi-\alpha|$$



$$|f(\chi)-f(\alpha)| \le |f(a)-f(\alpha)| + |f(x)-f(a)| + |f(\chi)-f(x)|$$

$$\le \left(\tfrac{1}{2}\varepsilon + k\right)\left(|a-\alpha| + |x-\chi|\right) + k|x-a|$$

$$\le \left(\tfrac{1}{2}\varepsilon + k\right)4\eta|\chi-\alpha| + k\,(1+4\eta)|\chi-\alpha|$$

$$= |\chi-\alpha|\left(2\eta\varepsilon + 4\eta k + k + 4\eta k\right)$$

$$\le |\chi-\alpha|\left(8\eta(k+\varepsilon) + k\right)$$

$$\le |\chi-\alpha|\left(\tfrac{1}{2}\varepsilon + k\right)$$

In case $\delta(a) + \delta(x) > 4|\chi-\alpha|$ and $\delta(a) \ge \delta(x)$ we have

$$\Rightarrow 2\delta(a) \ge \delta(a)+\delta(x) \ge 2\delta(x)$$

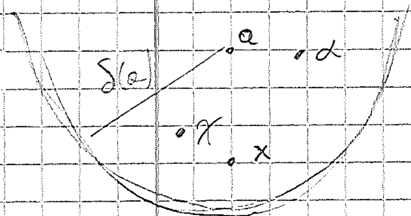
$$|\chi-a| \le |\chi-\alpha| + |\alpha-a| \le \left(\tfrac{1}{2}+\eta\right)\delta(a) \le \delta(a)$$

$$|\alpha-a| \le \delta(a)$$

$$\cancel{|f(\chi)-f(\alpha)| \le |f(\chi)-f(x)| + |f(x)-f(a)| + |f(a)-f(\alpha)|}$$

$$\cancel{\le \left(k+\tfrac{\varepsilon}{2}\right)\left(|\chi-x| + |a-\alpha|\right) + k|x-a|}$$

$$\cancel{\le \left(k+\tfrac{\varepsilon}{2}\right)\left(\eta\delta(a)+\eta\delta(x)\right)} + \cancel{(1+4\eta)|\chi-\alpha|}$$



$$\boxed{\text{Hence, } x, \alpha, \chi \in U(a, \delta(a)) \text{ so } |f(\chi)-f(\alpha)| \le \left(\tfrac{\varepsilon}{2} + k\right)|\chi-\alpha|.}$$

__Step 2.__ Choose $g_0 : \mathbb{R}^n \to \mathbb{R}^l$ s.t. $g_0|_G = f|_G$,

$g_0 \ge 0$ if ($l = 1$, and $f \ge 0$),

$\mathrm{Lip}(g_0) = \mathrm{Lip}(f|_G)$ Using Kirszbraun's theorem.

Hanne 15

(5)

Apply [3.1.13] (with $\{G, U \sim A\}$) to construct a partition of unity, i.e., functions $\varphi_0, \varphi_1, \ldots \in \mathcal{D}(U, \mathbb{R})$ nonnegative

$$\varphi_i \geq 0 \quad \forall i$$

s.t.

- $\mathcal{H}^0(\{i : K \cap \operatorname{spt} \cancel{\varphi_i}\, \varphi_i \neq \emptyset\}) < \infty$ for any compact set $K \subseteq U$.
- $A \subseteq \operatorname{Int}\{x : \varphi_0(x) = 1\}$
- $\operatorname{spt} \varphi_0 \subseteq G$
- $\operatorname{spt} \varphi_i \subseteq U \sim A$
- $\sum_{j=0}^{\infty} \varphi_i(x) = 1$ for $x \in U$.

Mollifying [redacted] g_0 we get $g_i : \mathbb{R}^n \to \mathbb{R}^l \in \mathcal{C}^1$ with

$$\operatorname{Lip} g_i \leq \operatorname{Lip} g_0, \quad \operatorname{Lip}(\varphi_i) \cdot (\sup \operatorname{im} |g_i - g_0|) \leq \frac{\varepsilon}{2^{i+1}}$$

if $l = 1$ and $f \geq 0$, then $g_i \geq 0$. for $i \in \mathcal{P}$.

Define $$g = \sum_{j=0}^{\infty} \varphi_j g_j \in \mathcal{C}^1 \leftarrow$$ of class $\mathcal{C}^1$ because $g_0|_G = f|_G$ and $f \in \mathcal{C}^1$

For $x, \chi \in U$ we obtain

$$g(x) - g(\chi) = \sum_{j=0}^{\infty} [redacted]\, \varphi_j(x)(g_j(x) - g_j(\chi)) + (\varphi_j(x) - \varphi_j(\chi))(g_j(\chi) - g_0(\chi)\, \cancel{\ldots})$$

$$\Rightarrow \operatorname{Lip} g \leq \sum_{j=0}^{\infty} \left(\operatorname{Lip}(g_j)\varphi_j + \operatorname{Lip} \varphi_j \cdot \sup \operatorname{im} |g_j - g_0|\right)$$

$$\leq \operatorname{Lip} g_0 + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \operatorname{Lip}(f|_G) \leq \varepsilon + \kappa.$$

Moreover, $g(x) = f(x)$ whenever $\varphi_0(x) = 1$, so we can take $X = \cancel{\varphi_0^{-1}\{1\}}\ \operatorname{Int}(\varphi_0^{-1}\{1\})$.

$\square$.

Menne 15

(6)

Using <u>Whitney extension</u> would not preserve the Lipschitz constant. Using <u>Kirszbraun</u> does not give $\mathcal{C}^1$. <u>Convolution</u> does not increase the Lip constant but spoils extension property.

$\boxed{3.5}$ $1 \leq m \leq n$, $U \subseteq \mathbb{R}^n$ open, $V \in \mathbf{RV}_m(U)$, $\varepsilon > 0$

<u>Then</u>

(1) $\exists M$ an m-dim. submani of $\mathbb{R}^n$ of class $\mathcal{C}^1$ with $\|V\|(U \sim M) < \varepsilon$.

(2) Y - fin. dim. vectorspace normed

$f : A \longrightarrow Y$ $\|V\|$ measurable

$A = \{x : (\|V\|, m) \operatorname{ap} Df(x) \text{ exists}\}$.

<u>Then</u> $\exists g : U \longrightarrow Y \in \mathcal{C}^1$

$\|V\|(A \sim \{x : f(x) = g(x)\}) \leq \varepsilon$.

The proof relies on [redacted] a similar property for $(\mathcal{L}^m, m)$ approximately diff. functions $\boxed{3.1.16}$ + [3.2.29] - charact. of. rect. in terms of $\mathcal{C}^1$ submanifolds

$\boxed{3.6}$ $l, m, n \in \mathscr{P}$, $1 \leq m \leq n$, $U \subseteq \mathbb{R}^n$ open

$V \in \mathbf{RV}_m(U)$, C - rel. closed in U

$f : U \to \mathbb{R}^l$ locally Lipschitz,

$\operatorname{spt} f \subseteq \operatorname{Int} C$, $\varepsilon > 0$.

<u>Then</u>

$\exists g : U \to \mathbb{R}^l \in \mathcal{C}^1$

$\operatorname{spt} g \subseteq C$, $\operatorname{Lip} g \leq \operatorname{Lip} f + \varepsilon$,

$\|V\|(\{x \in U : f(x) \neq g(x)\}) \leq \varepsilon$.

Moreover, if $l = 1$ and $f \geq 0$, then $g \geq 0$.

Use rectifiability of V and [3.1.16] Lusin approximation to obtain $h : X \to \mathbb{R}^l$ of class $\mathcal{C}^1$.
Use 3.2. to get the retraction r to obtain the bound $\|D(h\circ r)\| \le \operatorname{Lip} f$ on B
Use 3.4. to extend $h \circ r$.

Proof. Let $X = \operatorname{Int} C$

By [Menne, Indiana] 11.1(3)

using 3.5 we obtain $h : X \to \mathbb{R}^l \in \mathcal{C}^1$

and $M \subseteq \mathbb{R}^n$ an m-dim submanifold of class $\mathcal{C}^1$

such that

$$M \subseteq X$$
$$\|V\|\left(X \sim \{x \in M : f(x) = h(x)\}\right) < \varepsilon$$

Moreover, one may require $h \geq 0$ if $l = 1$, $f \geq 0$.

Note

$$D(f|_M)(x) = D(h|_M)(x) \quad \text{for } \mathcal{H}^m \text{ almost all } x \in M \text{ with } f(x) = h(x)$$

by standard GMT

Then setting

$$B = \{x \in M : f(x) = h(x),\ D(f|_M)(x) = D(h|_M)(x)\}$$

Satisfies

$$\|V\|(X \sim B) < \varepsilon.$$

Apply 3.2 to get a retraction $r \in \mathcal{C}^1$

$$r : G \longrightarrow M, \quad X \subseteq G, \quad G\text{ - open},$$
$$Dr(a) = \operatorname{Tan}(M, a)_\natural \quad \text{for } a \in M.$$

Hence, $\sup \|D(h \circ r)\|[B] \leq \operatorname{Lip} f$

Apply 3.4 with

$$(U \sim C) \cup G, \quad (U \sim C) \cup B, \quad ((U \sim C) \times \{0\}) \cup h \circ r$$

in place of

$$U \qquad A \qquad f$$

We get

$$g : U \longrightarrow \mathbb{R} \in \mathcal{C}^1 \text{ s.t.}$$

$$g|_{U \sim X} = 0, \quad g|_B = f|_B, \quad \operatorname{Lip} g \leq \operatorname{Lip} f + \varepsilon.$$

and $g \geq 0$ if $l = 1$ and $f \geq 0$

$\square$

Menne 15

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$\boxed{3.7}$ $l, m, n \in \mathscr{P}$, $1 \le m \le n$, $U \subseteq \mathbb{R}^n$ open
$V \in \mathbf{RV}_m(U)$, $K \subseteq U$ compact, $f : U \to \mathbb{R}^l$
$\operatorname{Lip}(f) < \infty$, $\operatorname{spt} f \subseteq \operatorname{Int} K$.

Then

$\exists f_i \in \mathscr{D}(U, \mathbb{R}^l)$

$f_i(x) \xrightarrow[i \to \infty]{} f(x)$ uniformly for $x \in \operatorname{spt}\|V\|$

$\| (\|V\|, m) \operatorname{ap} D(f_i - f) \| \xrightarrow[i \to \infty]{} 0$ in $\|V\|$ measure

$\operatorname{spt} f_i \subseteq K$,

$\limsup_{i \to \infty} \operatorname{Lip} f_i \le \operatorname{Lip} f$.

Moreover, if $l = 1$, $f \ge 0$, then $f_i \ge 0$.

Proof. By employing convolution
~~one can~~ ~~to f.o.g~~ satisfies to fing $f_i \in \mathscr{C}^1$.
Employing 3.6 + Arzelà-Ascoli theorem [2.10.21]
we get the conclusion
using [Indiana, 11.1(4)].

$\square$