UNIVERSITY OF WARSAW

# Geometric ellipticity

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$$\Phi : \{\text{geometric objects}\} \to \mathbf{R}$$

Goal: To study critical points of $\Phi$.

Examples:

1. The Plateau problem
2. The isoperimetric problem

$$M \subseteq U \subseteq \mathbf{R}^n\,, \quad F : \mathbf{G}(n,k) \to [a,b] \subseteq (0,\infty)\,,$$

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where $P_F(T) \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ is such that

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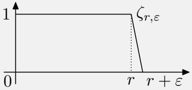
**Def.** $V$ is *$F$-stationary* iff. $\delta_F V \equiv 0$

Assume $\delta_F V = 0$, $\mathbf{B}(0, r+\varepsilon) \subseteq U$, and $g(x) = \zeta_{r,\varepsilon}(|x|)x$.



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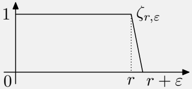


Compute $\delta_F V(g)$ and pass to the limit $\varepsilon \downarrow 0$ to get

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\|V_F\|\,\mathbf{B}(0,r)}{r^k}\right) = \frac{\mathrm{d}}{\mathrm{d}r}\left(\int_{\mathbf{B}(0,r)} \frac{\big(\mathrm{id}_{\mathbf{R}^n} - P_F(T)\big)x \bullet x}{|x|^{k+2}}\,\mathrm{d}V_F(x,T)\right).$$

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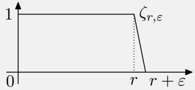


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A sufficient condition for LHS $\geq 0$ would be

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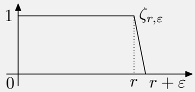


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Exercise. Let $Q \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ be a projection such that $Qx \bullet x \geq 0$ for all $x \in \mathbf{R}^n$. Show that $Q = Q^*$.

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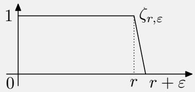


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More general result to be found in
W. Allard, *A characterization of the area integrand*, 1974.

$\mathcal{A}$ is a family of closed subsets of $U$ with finite $\mathscr{H}^k$-measure s.t. $\varphi[M] \in \mathcal{A}$ whenever $M \in \mathcal{A}$ and $\varphi : U \to U$ is $\mathscr{C}^1$ with $\operatorname{Clos}(W \cup \varphi[W]) \subseteq U$ compact, where $W = \{x : \varphi(x) \neq x\}$.

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If $F$ is *elliptic* then

$$T = \operatorname{Tan}(\Sigma, x) \text{ for } V \text{ a.a. } (x, T) \quad \Longrightarrow \quad \Phi_F(V) = \Phi_F(\Sigma)\,.$$

[E. R. Reifenberg F. Almgren, G. De Philippis, A. De Rosa, F. Ghiraldin, C. De Lellis, F. Maggi, Y. Fang, K., J. Harrison, H. Pugh, G. David ...]

**Def.** $(S, D)$ is a test pair if $D$ is a flat $k$-disc, $S \subseteq \mathbf{R}^n$ is $(\mathscr{H}^k, k)$-rectifiable and compact, $\partial D \subseteq S$ is not a Lipschitz retract of $S$, and $\mathscr{H}^k(S) > \mathscr{H}^k(D)$.

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- If $k = n - 1$, then $F \in \mathrm{UAE}$ iff. $F$ comes from a uniformly convex norm.

**Q.** Are there any non-trivial elliptic integrands?
[ e.g. a non-Euclidean norm $\nu$ on $\mathbf{R}^n$ generates $\mathscr{H}^k_\nu = \Phi_F$, with $F(T) = \boldsymbol{\alpha}(k)/\mathscr{L}^k(\mathbf{B}^\nu(0,1) \cap T)$. Does $F \in \mathrm{AE}$? ]

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**Q.** Is there a condition that can be easily checked for an integrand given by an explicit formula?

**Def.** Let $n = k + 1$. We say that $F : \mathbf{G}(n, k) \to \mathbf{R}$ is convex if the associated positively homogeneous function $N : \mathbf{R}^n \to \mathbf{R}$ such that $N(v) = F(\mathrm{span}\{v\}^\perp)$ is a norm.

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**Theorem** (D. Burago, S. Ivanov, GAFA 2004)

*An integrand is Almgren* **R***-Elliptic iff. it is extendibly convex.*

Desired property: $T = \operatorname{Tan}(\operatorname{spt}\|V\|, x)$ for $V$ almost all $(x, T)$ if $\delta_F V = 0$.

**Def.** $F \in$ BC iff. for all $V = (\mathscr{H}^k \llcorner T) \times \mu$

$$\delta_F V = 0 \quad \Longrightarrow \quad \mu = \operatorname{Dirac}(T)\,.$$

Desired property: $T = \mathrm{Tan}(\mathrm{spt}\,\|V\|, x)$ for $V$ almost all $(x, T)$ if $\delta_F V = 0$.

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G. De Philippis, A. De Rosa, F. Ghiraldin, *Rectifiability of varifolds with locally bounded first variation with respect to anisotropic surface energies*, CPAM 2017

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hence, $S = T$ for $\mu$ almost all $S$, i.e., $\mu = \operatorname{Dirac}(T)$. Consequently, $A_F(\mu) = P_F(T)^*$ and $\dim \ker A_F(\mu) = n - k$.

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Two points $S$ and $T$ involved in checking condition 2.

Uniform version of SAC (a.k.a. USAC)
(corresponding to "uniform convexity" of $\mathcal{G}$)

$$\exists \Gamma > 0 \quad \forall S, T \in \mathbf{G}(n,k) \qquad P_F(S)^* \bullet Q_F(T) \geq \Gamma \| S_\natural - T_\natural \|^2$$

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Moreover, USAC is stable under $\mathscr{C}^2$ perturbations.

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- condition 2 seems plausible if, e.g., $\mathrm{D}^2 f_T(T)(X,X) \geq 0$.

**Def.** We write $F \in \mathrm{WC}$ if $F$ is weakly convex.

**Conjecture** (A. De Rosa, K. in progress)

$WC \subseteq SAC \subseteq AC = BC \subseteq AE \subseteq WC.$

Thank you for listening.