Anisotropic integrands in geometric variational problems. Lecture notes

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Abstract

We investigate properties of anisotropic integrands and first vairation of varifolds with respect to such integrands.

1 Notation

The set of *non-negative integers* is denoted $\mathbb{N}$. We fix $n, d \in \mathbb{N}$ satisfying $1 \le d \le n$.

In principle we shall follow the notation of Federer; see [Fed69, pp. 669–671]. In particular, given two sets A, B, we denote with $A \sim B$ their *set-theoretic difference* and, for every $a \in \mathbf{R}^n$ and $s \in \mathbf{R}$ we define the functions $\boldsymbol{\tau}_a(x) = a + x$ and $\boldsymbol{\mu}_s(x) = sx$ for $x \in \mathbf{R}^n$; see [Fed69, 2.7.16, 4.2.8]. Concerning varifolds, we shall follow Allard [All72].

Additionally, we use the notation $[A \ni y \mapsto f(y)]$ to denote an *unnamed function* whose domain is A and which evaluates at y to $f(y)$. We also use standard abbreviations for *intervals* $(a, b) = \mathbf{R} \cap \{t : a < t < b\}$, $[a, b] = \mathbf{R} \cap \{t : a \le t \le b\}$ etc. The *identity map* on a set X is denoted id_X. The *characteristic function* of a set A is denoted by $\mathbb{1}_A$ and characterised by the requirement $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise. Whenever A, B are subset of a vectorspace we write $A + B$ for the set $\{a + b : a \in A,\ b \in B\}$. We also define *unit spheres* $\mathbf{S}^k = \mathbf{R}^{k+1} \cap \{x : |x| = 1\}$ for $k \in \mathbb{N}$. If $U \subseteq \mathbf{R}^n$ is open and $A \subseteq U$, then we say that A is a *d-set in U* if A is $\mathscr{H}^d$ measurable and $\mathscr{H}^d(A \cap K) < \infty$ for all compact sets $K \subseteq U$. If X is a set and $x \in X$, we denote by $\mathrm{Dirac}(x)$ the measure over X given for $A \subseteq X$ by

$$\mathrm{Dirac}(x)(A) = 1 \quad \text{if } x \in A \quad \text{and} \quad \mathrm{Dirac}(x)(A) = 0 \quad \text{if } x \in X \sim A\,.$$

2 Preliminaries

The space of homomorphisms between vectorspaces.

If X is a real topological vectorspace we write X^* for the space of continuous linear functionals on X. Assume X and Y are real finite dimensional inner product spaces (*Euclidean spaces*) and $f \in \mathrm{Hom}(X, Y)$. The natural isomorphisms $X \to X^*$ and $Y \to Y^*$ are used to identity $f^* \in \mathrm{Hom}(Y^*, X^*)$ given by $f^*(\omega) = \omega \circ f$ for $\omega \in Y^*$ with $f^* \in \mathrm{Hom}(Y, X)$ characterised by $f(x) \bullet y = x \bullet f^*(y)$ for $x \in X$ and $y \in Y$.

Recall also from [Fed69, 1.7.9] that the vectorspace $\mathrm{Hom}(X, Y)$ is equipped with a natural inner product given by $A \bullet B = \mathrm{trace}(A^* \circ B)$ and, thus, is itself a Euclidean space.

The Euclidean norm $|\cdot|$ of $A \in \operatorname{Hom}(X,Y)$ is

$$|A| = \sqrt{A \bullet A} = \operatorname{trace}(A^* \circ A)^{1/2}\,.$$

If Z and W are normed spaces, then we introduce the norm $\|\cdot\|$ on $\operatorname{Hom}(Z,W)$ by setting

$$\|A\| = \sup\{|Au| : u \in Z,\, |u| \le 1\} \quad \text{for } A \in \operatorname{Hom}(Z,W)\,.$$

The derivative and the gradient.

If X and Y are normed vectorspaces, $k \in \mathbb{N}$, $A \subseteq X$ is open, and $f : A \to Y$. Recall [Fed69, 3.1.1, 3.1.11] for the definition of the *k-th derivative of f* which is a map of the type

$$\mathrm{D}^k f : U \to \textstyle\bigodot^k(X,Y)\,.$$

In particular, if $k = 1$, we have $\mathrm{D}f : U \to \operatorname{Hom}(X,Y)$. In case $Y = \mathbf{R}$ and X is a Euclidean space, we define the *gradient of f at a* to be the vector $\operatorname{grad} f \in X$ characterised by

$$\operatorname{grad} f(a) \bullet v = \langle v,\, \mathrm{D}f(a)\rangle \quad \text{for } v \in X\,.$$

The trace.

We extend the definitions of *trace* given in [Fed69, 1.4.5, 1.7.10]. Let X be a Euclidean space, Y a finite dimensional vector space, $\phi : \operatorname{Hom}(X, \operatorname{Hom}(X,Y)) \to X \otimes X^* \otimes Y$ be the inverse of the composition of the natural isomorphisms (see [Fed69, 1.1.4])

$$X \otimes X^* \otimes Y \to X^* \otimes X^* \otimes Y \to X^* \otimes \operatorname{Hom}(X,Y) \to \operatorname{Hom}(X, \operatorname{Hom}(X,Y))\,,$$

and $\psi : X \otimes X^* \otimes \mathbf{R} \to X \otimes X^*$. Then we define $\operatorname{trace} : \operatorname{Hom}(X, \operatorname{Hom}(X,Y)) \to Y$ by requiring that

$$\omega \circ \operatorname{trace} = \operatorname{trace} \circ \psi \circ (\mathrm{id}_X \otimes \mathrm{id}_{X^*} \otimes \omega) \circ \phi \quad \text{for } \omega \in Y^*\,. \tag{1}$$

Given an orthonormal basis $u_1, \ldots, u_n$ of X and $f \in \operatorname{Hom}(X, \operatorname{Hom}(X,Y))$ we obtain

$$\operatorname{trace} f = \textstyle\sum_{i=1}^n f u_i u_i\,.$$

The Grassmannian.

We denote by $\mathbf{G}(n,d)$ the Grassmannian of d-dimensional linear subspaces of $\mathbf{R}^n$. Following [Alm68] and [Alm00], if $S \in \mathbf{G}(n,d)$, then $S_\natural \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ shall denote the *orthogonal projection* onto S. In particular, if $p \in \mathbf{O}^*(n,d)$ is such that $\operatorname{im} p^* = S$, then $S_\natural = p^* \circ p$; cf. [Fed69, 1.7.2, 1.7.4].

2.1 Definition. Set

$$\mathcal{G}_{n,d} = \big\{T_\natural : T \in \mathbf{G}(n,d)\big\} = \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \cap \big\{A : A^* = A,\, A \circ A = A,\, \operatorname{trace} A = d\big\}\,.$$

Let $X = \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ and $\Psi : X \to X \times X \times \mathbf{R}$ be given by

$$\Psi(T) = (T \circ T - T, T^* - T, \operatorname{trace} T - d) \quad \text{for } T \in X\,.$$

Clearly Ψ is a polynomial function (see [Fed69, 1.10.4]) and $\mathcal{G}_{n,d} = \Psi^{-1}\{(0,0,0)\}$. Moreover, if $T \in \mathbf{G}(n,d)$, then $A \in \ker \mathrm{D}\Psi(T_\natural)$ if and only if

$$A \circ T_\natural + T_\natural \circ A = A\,, \quad A^* = A\,, \quad \text{and} \quad \operatorname{trace} A = 0\,. \tag{2}$$

From the first conditions it follows that

$$T_\natural \circ A = T_\natural \circ A \circ T_\natural + T_\natural \circ A \quad \Rightarrow \quad T_\natural \circ A \circ T_\natural = 0\,,$$
$$A \circ T_\natural^\perp = T_\natural \circ A \circ T_\natural^\perp \quad \Rightarrow \quad T_\natural^\perp \circ A \circ T_\natural^\perp = 0\,.$$

In particular, the third condition of (2) follows from the first; hence, we have $A \in \ker \mathrm{D}\Psi(T_\natural)$ if and only if

$$A = T_\natural^\perp \circ A \circ T_\natural + T_\natural \circ A \circ T_\natural^\perp \quad \text{and} \quad A^* = A\,.$$

To compute the dimension of $\ker \mathrm{D}\Psi(T_\natural)$, we observe that any $A \in \ker \mathrm{D}\Psi(T_\natural)$ is completely determined by $T_\natural^\perp \circ A \circ T_\natural$ and, vice versa, for any $B \in \operatorname{Hom}(T, T^\perp)$ the map $T_\natural^\perp \circ B \circ T_\natural + (T_\natural^\perp \circ B \circ T_\natural)^*$ is an element of $\ker \mathrm{D}\Psi(T_\natural)$; hence,

$$\dim \ker \mathrm{D}\Psi(T_\natural) = \dim \operatorname{Hom}(T, T^\perp) = d(n-d) \quad \text{for any } T \in \mathbf{G}(n,d)\,.$$

Employing [Fed69, 3.1.19(2)] we see that $\mathcal{G}_{n,d}$ is a real analytic submanifold of $\operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ of dimension $d(n-d)$ and

$$\operatorname{Tan}(\mathcal{G}_{n,d}, T) = \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \cap \big\{A : A = T_\natural^\perp \circ A \circ T_\natural + (T_\natural^\perp \circ A \circ T_\natural)^*\big\}\,.$$

2.2 Definition. We define the map

$$\Pi : \mathbf{G}(n,d) \to \operatorname{Hom}(\operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n), \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n))$$
$$\text{by} \quad \Pi(S)L = S_\natural^\perp \circ L \circ S_\natural + (S_\natural^\perp \circ L \circ S_\natural)^* \quad \text{for } S \in \mathbf{G}(n,d) \text{ and } L \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)\,.$$

Observe that if $S \in \mathbf{G}(n,d)$, then $\Pi(S) \circ \Pi(S) = \Pi(S)$; hence, $\Pi(S)$ is a projection and $S \in \ker \Pi(S)$. Moreover, we have

$$\begin{aligned} \mathrm{D}\Pi(S)AL &= S_\natural^\perp \circ L \circ A_\natural - A_\natural \circ L \circ S_\natural + (S_\natural^\perp \circ L \circ A_\natural - A_\natural \circ L \circ S_\natural)^* \\ \text{and} \quad \Pi(S)^*L &= S_\natural^\perp \circ L \circ S_\natural + (S_\natural \circ L \circ S_\natural^\perp)^* \end{aligned} \tag{3}$$
$$\text{for } S \in \mathbf{G}(n,d),\ L \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n), \text{ and } A \in \operatorname{Tan}(\mathcal{G}_{n,d})\,.$$

In particular, $\Pi(S)$ is *not* an orthogonal projection.

2.3 Remark. Observe that for any $T \in \mathbf{G}(n,d)$ we have

$$|T_\natural|^2 = \operatorname{trace}(T_\natural^* \circ T_\natural) = \operatorname{trace} T_\natural = d\,,$$

hence, $\mathcal{G}_{n,d} \subseteq \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \cap \{A : |A| = \sqrt{d}\}$.

2.4 Exercise (cf. [All72, 8.9(1)(2)(3)]). For $S, T \in \mathbf{G}(n,d)$ we have

$$|S_\natural - T_\natural|^2 = 2 S_\natural \bullet T_\natural^\perp = 2 S_\natural^\perp \bullet T_\natural = 2|S_\natural \circ T_\natural^\perp|^2 = |S_\natural^\perp - T_\natural^\perp|^2$$
$$\text{and} \quad \|S_\natural - T_\natural\| = \|S_\natural^\perp \circ T_\natural\| = \|S_\natural \circ T_\natural^\perp\| = \|S_\natural^\perp - T_\natural^\perp\|\,.$$

Radon measures

Let X be a Polish space (i.e. separable topological space which is metrizable in a complete way). By a *measure over* X we mean any function $\phi : \mathbf{2}^X \to \overline{\mathbf{R}}$ such that $\phi(\varnothing) = 0$ and

$$\phi(A) \le \sum_{B \in F} \phi(B) \quad \text{whenever } F \subseteq \mathbf{2}^X,\ F \text{ is countable, and } A \subseteq \bigcup F\,.$$

We say that $A \subseteq X$ is *ϕ-measurable* if

$$\phi(T) = \phi(T \cap A) + \phi(T \sim A) \quad \text{for all } T \subseteq X\,.$$

A measure ϕ is said to be *Borel regular* if all Borel sets are ϕ-measurable and for any $A \subseteq X$ there exists a Borel set B such that $A \subseteq B$ and $\phi(A) = \phi(B)$.

Since X is a Polish space we may say that ϕ is a *Radon measure* if and only if ϕ is Borel regular and $\phi(K) < \infty$ for all compact sets $K \subseteq X$; cf. [Sch73, Chap. II, §3].

For a compact set $K \subseteq X$ we define $\mathscr{K}_K(X)$ the be the vectorspace of all continuous functions of the type $X \to \mathbf{R}$ supported in K. We equip $\mathscr{K}_K(X)$ with the supremum norm, i.e., if $f \in \mathscr{K}_K(X)$, then $\|f\| = \sup \operatorname{im} |f|$. Then $\mathscr{K}_K(X)$ becomes a Banach space. Next, we define $\mathscr{K}(X) = \bigcup \{\mathscr{K}_K(X) : K \subseteq X \text{ compact}\}$ and endow $\mathscr{K}(X)$ with the *locally convex topology* characterised by the following condition: for any locally convex topological vector space F a map $h : \mathscr{K}(X) \to F$ is continuous if and only if $h \circ j_K$ is continuous for all compact sets $K \subseteq X$, where $j_K : \mathscr{K}_K(X) \to \mathscr{K}(X)$ is the inclusion map.

Let $\lambda \in \mathscr{K}(X)^*$ be a continuous linear functional on $\mathscr{K}(X)$. We say that λ is *monotone* if

$$\lambda(f) \le \lambda(g) \quad \text{whenever } f, g \in \mathscr{K}(X) \text{ and } f \le g\,.$$

Referring to [Men16, §2] and [Fed69, 2.5.19] we see that the set of Radon measures over X may be identified with $\mathscr{K}(X)^* \cap \{\lambda : \lambda \text{ is monotone}\}$. We endow $\mathscr{K}(X)^*$ with the weak* topology, i.e., the topology generated by the sets

$$\mathscr{K}(X)^* \cap \{\phi : a < \phi(f) < b\}$$

corresponding to all choices of $a, b \in \mathbf{R}$ and $f \in \mathscr{K}(X)$. This topology is in fact the same as the topology inherited from the embedding $\mathscr{K}(X)^* \subseteq \mathbf{R}^{\mathscr{K}(X)}$, where the space $\mathbf{R}^{\mathscr{K}(X)}$ is a Cartesian product of infinitely many copies of $\mathbf{R}$ with the product topology (a.k.a. Tychonoff topology).

Norms in $\mathbf{R}^n$

2.5 Definition. We say that $F : \mathbf{R}^n \to \mathbf{R}$ is a *norm of class* $\mathscr{C}^k$ if

(a) F is convex and non-negative,

(b) $F^{-1}\{0\} = \{0\}$.

(c) $F(\lambda x) = |\lambda| F(x)$ for $x \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$,

(d) $F | \mathbf{R}^n \sim \{0\}$ is of class $\mathscr{C}^k$,

2.6 Definition. We say that F is a *strictly convex norm* if it is a norm and

$$F(x + y) < F(x) + F(y) \quad \text{whenever } x, y \in \mathbf{R}^n \text{ are linearly independent}\,.$$

2.7 Definition. We say F is a *uniformly convex norm* if it is a norm and there exists $c \in (0,\infty)$ such that $\big[\mathbf{R}^n \ni x \mapsto F(x) - c|x|\big]$ is convex.

2.8 Remark (cf. [Fed69, 5.1.3]). If F is a norm of class $\mathscr{C}^2$, then uniform convexity of F (with constant c) is equivalent to the condition

$$\big\langle (v,v),\, \mathrm{D}^2 F(u) \big\rangle \ge c\frac{|u \wedge v|^2}{|u|^3} = c\frac{|v|^2 - (v \bullet u/|u|)^2}{|u|} \quad \text{for } u \in \mathbf{R}^n,\ u \ne 0,\ v \in \mathbf{R}^n\,.$$

2.9 Definition. Let F be a norm. We define the *dual norm* F^* by setting

$$F^*(x) = \sup\big\{ x \bullet y : y \in \mathbf{R}^n,\ F(y) = 1 \big\}\,.$$

2.10 Remark. Note that F^* corresponds to the norm naturally induced by F on $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R})$ under the natural identification $\mathbf{R}^n \simeq \mathrm{Hom}(\mathbf{R}^n,\mathbf{R})$ coming from the choice of the scalar product on $\mathbf{R}^n$.

2.11 Definition. Let F be a norm. We define the *Wulff shape of* F to be the open unit ball with respect to the dual norm F^*, i.e., the set $\mathbf{R}^n \cap \{x : F^*(x) < 1\}$.

2.12. A quote from [BM94] (using notation therein):

> A round soap bubble solves the classical isoperimetric problem; that is, it minimises surface area for a given volume. From a physical point of view the bubble minimises total surface energy arising from surface tension in the soap film. On the other hand, the surface energy of a crystal depends on the surface orientation with respect to the underlying crystal lattice and is given by some norm (or more general integrand) Ψ applied to the unit normal $\mathbf{n}$. (The case of area is given by the Euclidean norm $\Psi(x) = |x|$, so that $\Psi(\mathbf{n}) = 1$.) In 1901, Wulff [Wul01] gave a construction for the surface-energy-minimising shape for a given volume of material now called the *Wulff shape* B_Ψ, most easily defined as the unit ball in the dual norm:
> $$B_\Psi = \{x : \Psi^*(x) \le 1\}\,.$$

2.13 Definition (cf. [Fed69, 4.5.5])**.** Let $A \subseteq \mathbf{R}^n$ and $b \in \mathbf{R}^n$. We say that u is an *exterior normal of* A *at* b if $u \in \mathbf{R}^n$, $|u| = 1$,

$$\begin{gathered}\boldsymbol{\Theta}^{n+1}(\mathscr{L}^n \llcorner \{x : (x-b) \bullet u > 0\} \cap A, b) = 0\,,\\ \text{and} \quad \boldsymbol{\Theta}^{n+1}(\mathscr{L}^n \llcorner \{x : (x-b) \bullet u < 0\} \sim A, b) = 0\,.\end{gathered}$$

We also set $\mathbf{n}(A,b) = u$ if u is the exterior normal of A at b and $\mathbf{n}(A,b) = 0$ if there exists no exterior normal of A at b.

2.14 Exercise. Let F be a uniformly convex norm of class $\mathscr{C}^2$,

$$\begin{gathered} W = \mathbf{R}^n \cap \{x : F(x) < 1\}\,, \quad W^* = \mathbf{R}^n \cap \{x : F^*(x) < 1\}\,,\\ G, G^* : \mathbf{R}^n \to \mathbf{R}^n \quad \text{be given by} \quad G = \operatorname{grad} F \quad \text{and} \quad G^* = \operatorname{grad} F^*\,.\end{gathered}$$

Prove the following:

(a) $F^*(G(x)) = 1$ and $F(G^*(x)) = 1$ for any $x \in \mathbf{R}^n \sim \{0\}$.

(b) $G|\partial W : \partial W \to \partial W^*$ is a Lipschitz homeomorphism.

(c) $F^*(x) = x \bullet G^*(x)$ and $F(x) = x \bullet G(x)$ for $x \in \mathbf{R}^n \sim \{0\}$.

(d) $F^{**} = F$.

(e) F^* is a strictly convex norm.

(f) $G^*|\partial W^* = (G|\partial W)^{-1}$.

(g) F^* is of class $\mathscr{C}^1$.

(h) F^* is of class $\mathscr{C}^2$ and $G|\partial W : \partial W \to \partial W^*$ is bilipschitz.

(i) For $x \in \partial W$ and $y \in \partial W^*$ there holds

$$\mathbf{n}(W, x) = G(x)F(\mathbf{n}(W, x)) \quad \text{and} \quad \mathbf{n}(W^*, y) = G^*(y)F(\mathbf{n}(W^*, y)) .$$

In particular, $G(\mathbf{n}(W^*, y)) = y$ for $y \in \partial W^*$ and $G^*(\mathbf{n}(W, x)) = x$ for $x \in \partial W$.

Hint. Proofs can be found in [DKS19, 2.36].

3 Varifolds

Let $U \subseteq \mathbf{R}^n$, $d \in \mathbb{N}$. A d-dimensional *varifold* in U is simply a Radon measure over $U \times \mathbf{G}(n, d)$. The space of all d-dimensional varifolds in U is denoted $\mathbf{V}_d(U)$.

3.1 Example. Let $M \subseteq U$ be a submanifold of class $\mathscr{C}^1$. We define $\mathbf{v}_d(M) \in \mathbf{V}_d(U)$ by

$$\mathbf{v}_d(M)(\alpha) = \int_M \alpha(x, \operatorname{Tan}(M, x)) \, d\mathscr{H}^d(x) \quad \text{for } \alpha \in \mathscr{K}(U \times \mathbf{G}(n, d)) .$$

3.2 Example. Let $E \subseteq U$ be a countably $(\mathscr{H}^d, d)$ rectifiable d-set in U and $\theta : E \to (0, \infty)$ be $\mathscr{H}^d \llcorner E$ measurable and such that $\int_{K \cap E} \theta \, d\mathscr{H}^d < \infty$ for any compact set $K \subseteq U$. We define $\mathbf{v}_d(E, \theta) \in \mathbf{V}_d(U)$ by

$$\mathbf{v}_d(E, \theta)(\alpha) = \int_E \alpha(x, \operatorname{Tan}^d(\mathscr{H}^d \llcorner E, x))\theta(x) \, d\mathscr{H}^d(x) \quad \text{for } \alpha \in \mathscr{K}(U \times \mathbf{G}(n, d)) .$$

Varifolds of this type are called *rectifiable varifolds*. The set of all rectifiable varifolds in U is denoted $\mathbf{RV}_d(U)$. In case $\theta(x) \in \mathbb{N}$ for $\mathscr{H}^d \llcorner E$ almost all x, then $\mathbf{v}_d(E, \theta)$ is an *integral varifold*. The set of all integral varifolds in U is denoted $\mathbf{IV}_d(U)$.

3.3 Example. Let $S, T \in \mathbf{G}(n, d)$ and set $V_1 = (\mathscr{L}^n \llcorner U) \times \operatorname{Dirac}(T)$ and $V_2 = (\mathscr{H}^d \llcorner (T \cap U)) \times \operatorname{Dirac}(S)$. Then $V_1, V_2 \in \mathbf{V}_d(U)$. Moreover, $V_2 \in \mathbf{RV}_d(U)$ if and only if $S = T$.

3.4 Definition. For $V \in \mathbf{V}_d(U)$ we define the *weight measure* $\|V\|$ of V to be the measure over U such that

$$\|V\|(A) = V(A \times \mathbf{G}(n, d)) \quad \text{for } A \subseteq U .$$

3.5 Remark (cf. [All72, 3.3], [Fed69, 2.5.20], [AFP00, 2.5]). Every $V \in \mathbf{V}_d(U)$ can be *disintegrated.* For $x \in \operatorname{spt} \|V\|$ and $\beta \in \mathscr{K}(\mathbf{G}(n,d))$ we set

$$V^{(x)}(\beta) = \lim_{r \downarrow 0} \fint_{\mathbf{B}(x,r) \times \mathbf{G}(n,d)} \beta(S) \, \mathrm{d}V(y,S) \,.$$

Then $[\operatorname{spt} \|V\| \ni x \mapsto V^{(x)}]$ is a $\|V\|$ measurable function with values in $\mathscr{K}(\mathbf{G}(n,d))^* \cap \{\mu : \mu(\mathbf{G}(n,d)) = 1\}$ such that

$$\int \alpha(x,S) \, \mathrm{d}V(x,S) = \int \int \alpha(x,S) \, \mathrm{d}V^{(x)}(S) \, \mathrm{d}\|V\|(x) \quad \text{for } \alpha \in \mathscr{K}(U \times \mathbf{G}(n,d)) \,.$$

4 The first variation of a varifold

4.1 Definition (cf. [All72, 3.2])**.** Let $W \subseteq \mathbf{R}^N$ be open, $\varphi : U \to W$ be of class $\mathscr{C}^1$, and $V \in \mathbf{V}_d(U)$. We define the *push-forward* $\varphi_\# V \in \mathbf{V}_d(W)$ by

$$\varphi_\# V(\alpha) = \int \alpha(\varphi(x), \mathrm{D}\varphi(x)[T]) J_T\varphi(x) \, \mathrm{d}V(x,T) \quad \text{for } \alpha \in \mathscr{K}(W \times \mathbf{G}(N,d)) \,,$$

where

$$J_T\varphi(x) = \|{\textstyle\bigwedge_d} \mathrm{D}\varphi(x) \circ T_\natural\| \quad \text{for } x \in U \text{ and } T \in \mathbf{G}(n,d)$$

and with the understanding that if $\dim \mathrm{D}\varphi(x)[T] < d$, then the whole integrand equals zero.

4.2 Exercise. If $B \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ and $\dim \operatorname{im} B \le d$, then

$$|{\textstyle\bigwedge_d} B| = \|{\textstyle\bigwedge_d} B\| \,.$$

4.3 Exercise. Let $x \in U$ be such that $S = \mathrm{D}\varphi(x)[T] \in \mathbf{G}(N,d)$. Then $\mathrm{D}\varphi(x)|T \in \operatorname{Hom}(T,S)$ and

$$J_T\varphi(x) = \|{\textstyle\bigwedge_d} \mathrm{D}\varphi(x) \circ T_\natural\| = \det\big((\mathrm{D}\varphi(x)|T)^* \circ \mathrm{D}\varphi(x)|T\big)^{1/2} \,.$$

Hint. First apply 4.2.

4.4 Exercise. If $V = \mathbf{v}_d(M)$ for some manifold $M \subseteq U$ of class $\mathscr{C}^1$, then

$$\varphi_\# V = \mathbf{v}_d(\varphi[M], N(\varphi, \cdot)) \,,$$

where $N(\varphi, x) = \mathscr{H}^0(\varphi^{-1}\{x\})$ for $x \in W$.

4.5 Exercise. Let $A \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ and $S \in \mathbf{G}(n,d)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \det(\mathrm{id}_{\mathbf{R}^n} + tA) = \operatorname{trace} A \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \big|{\textstyle\bigwedge_d}(\mathrm{id}_{\mathbf{R}^n} + tA) \circ S_\natural\big| = A \bullet S_\natural \,.$$

4.6 Exercise. Let $A \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$, $S \in \mathbf{G}(n,d)$, $f, g : \mathbf{R} \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ be given by

$$f(t) = \mathrm{id}_{\mathbf{R}^n} + tA \quad \text{and} \quad g(t) = \big(f(t)[S]\big)_\natural \quad \text{for } t \in \mathbf{R} \,.$$

Then

$$g'(0) = \Pi(S)A \,.$$

Hint. Differentiate the equation $g(t) \circ f(t) \circ S_\natural = f(t) \circ S_\natural$.

4.7 Definition. Let $U \subseteq \mathbf{R}^n$ be open. We say that F is a *d-dimensional integrand of class $\mathscr{C}^k$ over U* (or just an *integrand of class $\mathscr{C}^k$* if the choice of d and U is clear from the context) if $F : U \times \mathbf{G}(n,d) \to \mathbf{R}$ is positive, of class $\mathscr{C}^k$, and satisfies $\sup \operatorname{im} F / \inf \operatorname{im} F \in (0,\infty)$.

For $x \in U$ and $T \in \mathbf{G}(n,d)$ we also define

$$F_T : U \to \mathbf{R} \quad \text{and} \quad F_x : \mathscr{G}_{n,d} \to \mathbf{R} \quad \text{so that} \quad F_T(x) = F(x,T) = F_x(T_\natural)\,.$$

4.8 Remark. With any integrand F of class $\mathscr{C}^k$ we may associate a function $\tilde{F} : U \times \operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n) \to \mathbf{R}$ by first requiring that $\tilde{F}(x,T_\natural) = F(x,T)$ for $T \in \mathbf{G}(n,d)$ and then extending $\tilde{F}$ to $U \times \operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n)$ arbitrarily so that $\tilde{F}|U \times (\operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n) \sim \{0\})$ is of class $\mathscr{C}^k$. In the sequel we shall often tacitly identify F with $\tilde{F}$ and we shall use the name *integrand* also for the function $\tilde{F}$.

4.9 Definition. Let $U \subseteq \mathbf{R}^n$ be open and F be an integrand of class $\mathscr{C}^0$. For $V \in \mathbf{V}_d(U)$ we set

$$\Phi_F(V) = \int F(x,T)\,\mathrm{d}V(x,T) \in [0,\infty]\,.$$

4.10 Example. The *area integrand* is given by $F(x,T) = 1$ for $x \in U$ and $T \in \mathbf{G}(n,d)$. In this case $\Phi_F(V) = \|V\|(U)$.

4.11 Example (cf. [APT04, 4.1] and [BI12, §1]). Let $\nu : \mathbf{R}^n \to \mathbf{R}$ be a norm and set

$$W = \mathbf{R}^n \cap \{x : \nu(x) < 1\} \quad \text{and} \quad W^* = \mathbf{R}^n \cap \{x : \nu^*(x) < 1\}\,.$$

The *Busemann-Hausdorff integrand* F^{bh} and the *Holmes-Thompson integrand* F^{ht} (cf. [HT79]) are given for $x \in \mathbf{R}^n$ and $T \in \mathbf{G}(n,d)$ by

$$F^{\mathrm{bh}}(x,T) = \frac{\boldsymbol{\alpha}(d)}{\mathscr{H}^d(T \cap W)} \quad \text{and} \quad F^{\mathrm{ht}}(x,T) = \frac{\mathscr{H}^d(T_\natural[W^*])}{\boldsymbol{\alpha}(d)}\,.$$

4.12 Exercise. Define a metric ρ on $\mathbf{R}^n$ by setting $\rho(x,y) = \nu(x-y)$. Let $\mathscr{H}^d_\rho$ be the d-dimensional Hausdorff measure associated with the metric ρ; see [Fed69, 2.10.2(1)]. Show that $\Phi_{F^{\mathrm{bh}}}(\mathbf{v}_d(S)) = \mathscr{H}^d_\rho(S)$ for any $(\mathscr{H}^d, d)$ rectifiable d-set S; see [Bus47].

4.13 Definition. Let $U \subseteq \mathbf{R}^n$ be open, F be an integrand of class $\mathscr{C}^1$, and $V \in \mathbf{V}_d(U)$. The *first variation of V with respect to F*, denoted $\delta_F V$, is the linear functional on $\mathscr{X}(U)$ defined by the formula

$$\delta_F V(g) = \int \big\langle g(x), \mathrm{D}F_T(x) \big\rangle + \mathrm{D}g(x) \bullet B_F(x,T)\,\mathrm{d}V(x,T)\,, \tag{4}$$

where $B_F(x,T) \in \operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n)$ is characterzied by

$$B_F(x,T) \bullet L = F(x,T) T_\natural \bullet L + \big\langle \Pi(T)L,\, \mathrm{D}F_x(T_\natural) \big\rangle \quad \text{for } L \in \operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n)\,. \tag{5}$$

4.14 Lemma. *Let F, U, V, g be as in 4.13, and $h_t(x) = x + tg(x)$ for $t \in \mathbf{R}$ and $x \in U$. Then*

$$\delta_F V(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_F(h_{t\#}V)\,.$$

Proof. We compute using 4.6, 4.5, and 4.2.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Phi_F(h_{t\#}V) &= \int \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(h_t(x), \mathrm{D}h_t(x)[T])J_T h_t(x)\,\mathrm{d}V(x,T)\\ &= \int \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(x+tg(x), (\mathrm{id}_{\mathbf{R}^n} + t\mathrm{D}g(x))[T])J_T h_t(x)\,\mathrm{d}V(x,T)\\ &= \int \mathrm{D}F_T(g(x)) + \mathrm{D}F_x(T)(\Pi(T)\mathrm{D}g(x)) + F(x,T)T \bullet \mathrm{D}g(x)\,\mathrm{d}V(x,T)\,. \quad \Box\end{aligned}$$

4.15 Remark. Let $x \in U$ and $F : U \times \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}$ be an integrand of class $\mathscr{C}^1$; cf. 4.8. Assume $F_x(\lambda T) = |\lambda|^d F_x(T)$ for $T \in \mathcal{G}_{n,d}$ and $\lambda \in \mathbf{R}$. Then

$$\mathrm{D}F_x(T)T = d\,F_x(T) \quad \text{for } T \in \mathcal{G}_{n,d}\,.$$

For $T \in \mathcal{G}_{n,d}$ we have $|T|^2 = d$ and we define $\Omega(T) : \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \to \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ by

$$\Omega(T)L = \Pi(T)L + \tfrac{1}{d}(L \bullet T)\cdot T \quad \text{for } L \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)\,.$$

We obtain

$$\begin{aligned} &B_F(x,T) \bullet L = \big\langle \Pi(T)L,\, \mathrm{D}F_x(T)\big\rangle + (L \bullet \tfrac{T}{\sqrt{d}})\big\langle \tfrac{T}{\sqrt{d}},\, \mathrm{D}F_x(T)\big\rangle\\ &\quad = \big\langle \Omega(T)L,\, \mathrm{D}F_x(T)\big\rangle = \big\langle \mathrm{grad}\,F_x(T),\, \Omega(T)\big\rangle \bullet L \quad \text{for } T \in \mathcal{G}_{n,d} \text{ and } L \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)\,.\end{aligned}$$

Thus, defining $\mathrm{cone}\,\mathcal{G}_{n,d} = \{\lambda A : A \in \mathcal{G}_{n,d},\ \lambda \in (0,\infty)\}$, we get

$$B_F(x,T) = \big\langle \mathrm{grad}\,F_x(T),\, \Omega(T)\big\rangle \quad \text{for } T \in \mathcal{G}_{n,d}$$

and $\Omega(T)$ is a projection in $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ onto $\mathrm{Tan}(\mathrm{cone}\,\mathcal{G}_{n,d}, T)$.

4.16 Definition. For $x \in U$, $T \in \mathbf{G}(n,d)$, and $V \in \mathbf{V}_d(\mathbf{R}^n)$ we introduce

$$\tilde{B}_F(x,T) = \frac{B_F(x,T)}{F(x,T)}\,, \quad P(x,T) = \mathrm{im}\,\tilde{B}_F(x,T)\,, \quad \text{and} \quad V_F = F \cdot V\,.$$

4.17 Lemma. *For any $x \in U$ and $T \in \mathbf{G}(n,d)$ we have $\|T_\natural - P(x,T)_\natural\| < 1$ and*

$$\tilde{B}_F(x,T) \circ \tilde{B}_F(x,T) = \tilde{B}_F(x,T) \quad \textit{and} \quad \ker \tilde{B}_F(x,T) = T^\perp\,;$$

hence, $\tilde{B}_F(x,T)$ is the linear projection onto $P(x,T) \in \mathbf{G}(n,d)$ along $T^\perp$.

Proof. Let $x \in U$ and $T \in \mathbf{G}(n,d)$. Given $w \in \mathbf{R}^n$ we define $\omega_w \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R})$ so that $\omega_w(w) = 1$ and $\omega_w(u) = 0$ whenever $u \in \mathrm{span}\{w\}^\perp$. Define $\tilde{C}_F(x,T) \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ so that

$$F(x,T)\tilde{C}_F(x,T) \bullet L = \big\langle T_\natural^\perp \circ L \circ T_\natural + (T_\natural^\perp \circ L \circ T_\natural)^*,\, \mathrm{D}F_x(T_\natural)\big\rangle$$

for $L \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$. If $L = \omega_w \cdot v$, then $\tilde{C}_F(x,T)w \bullet v = \tilde{C}_F(x,T) \bullet L$. Hence, $\tilde{C}_F(x,T)w \bullet v = 0$ if either $v \in T$ or $w \in T^\perp$. Therefore,

$$\begin{gathered}\mathrm{im}\,\tilde{C}_F(x,T) \subseteq T^\perp \subseteq \ker \tilde{C}_F(x,T)\\ \text{so} \quad T_\natural \circ \tilde{B}_F(x,T) = T_\natural \quad \text{and} \quad \tilde{B}_F(x,T) \circ T_\natural = \tilde{B}_F(x,T)\,.\end{gathered}$$

We thus have

$$\tilde{B}_F(x,T) \circ \tilde{B}_F(x,T) = \tilde{B}_F(x,T) \circ T_\natural \circ \tilde{B}_F(x,T) = \tilde{B}_F(x,T) \circ T_\natural = \tilde{B}_F(x,T)\,.$$

Since $\tilde{B}_F(x,T) - \tilde{C}_F(x,T) = T_\natural$ and $\mathrm{im}\,\tilde{C}_F(x,T) \subseteq T^\perp$ we see also that $\ker \tilde{B}_F(x,T) = T^\perp$ and that $P(x,T)$ and T are not orthogonal, i.e., $\|T_\natural - P(x,T)_\natural\| < 1$. $\Box$

4.18 Corollary. *For $x \in U$ and $T \in \mathbf{G}(n,d)$ we have the representation*

$$\tilde{B}_F(x,T) = \big(T_\natural|_{P(x,T)}\big)^{-1} \circ T_\natural \quad \text{and} \quad \tilde{B}_F(x,T)^* = \big(P(x,T)_\natural|_T\big)^{-1} \circ P(x,T)_\natural \,.$$

In particular, $\tilde{B}_F(x,T)^*$ is the projection onto T along $P(x,T)^\perp$.

4.19 Definition. The *total variation measure with respect to F* of $V \in \mathbf{V}_d(U)$ is the Borel regular measure $\|\delta_F V\|$ over U characterised by

$$\begin{aligned} \|\delta_F V\|(G) &= \sup\{\delta_F V(g) : g \in \mathscr{X}(U),\, \operatorname{spt} g \subseteq G,\, |g| \le 1\} \quad \text{for } G \subseteq U \text{ open}\,, \\ \|\delta_F V\|(A) &= \inf\{\|\delta_F V\|(G) : G \subseteq U \text{ open},\, A \subseteq G\} \quad \text{for } A \subseteq U \text{ arbitrary}\,. \end{aligned}$$

4.20 Remark. In case $\|\delta_F V\|$ is Radon (which means that $\|\delta_F V\|(K) < \infty$ whenever $K \subseteq U$ is compact), then we may employ a general representation theorem [Fed69, 2.5.12] together with the theory of symmetric derivation of measures [Fed69, 2.8.18, 2.9] to obtain the following representation

$$\delta_F V(g) = \int \boldsymbol{\eta}_F(V,x) \bullet g(x) \, \mathrm{d}\|\delta_F V\|_{\mathrm{sing}}(x) - \int \mathbf{h}_F(V,x) \bullet g(x) F(x,T) \, \mathrm{d}V(x,T) \quad \text{for } g \in \mathscr{X}(U)\,,$$

where $\boldsymbol{\eta}_F(V,\cdot)$ is $\|\delta_F V\|$ measurable $\mathbf{S}^{n-1}$-valued function coming from application of [Fed69, 2.5.12], $\mathbf{h}_F(V,x) = -\mathbf{D}(\|\delta_F V\|, \|F \cdot V\|, x)$ for $\|V\|$ almost all x, and $\|\delta_F V\|_{\mathrm{sing}}$ is the singular, with respect to $\|V\|$, part of $\|\delta_F V\|$.

We call $\mathbf{h}_F(V,x)$ the *generalised F-mean curvature vector of V at x* or the *generalised anisotropic mean curvature vector of V at x* if the choice of F is clear from the context.

5 Anisotropic first variation of a submanifold of $\mathbf{R}^n$ of class $\mathscr{C}^2$.

Here we compute formulas for the anisotropic generalised mean curvature and normal vector in case V is associated to a submanifold of $\mathbf{R}^n$ of class $\mathscr{C}^2$ with $\mathscr{C}^2$ boundary.

Recall from [Fed69, 3.1.21] the definition of the tangent and normal cones for a subset of a vectorspace.

5.1 Setup. Let $U \subseteq \mathbf{R}^n$ be open, $M \subseteq U$ and $\partial M = \operatorname{Clos} M \sim M$ be submanifolds of $\mathbf{R}^n$ of class $\mathscr{C}^2$ and dimensions d and $d-1$ respectively, $F : U \times \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \sim \{0\} \to \mathbf{R}$ be of class $\mathscr{C}^2$ and such that

$$\operatorname{grad} F(x,\cdot)(T) \in \operatorname{Tan}(\mathcal{G}_{n,d}, T) \quad \text{for } x \in U \text{ and } T \in \mathcal{G}_{n,d}\,.$$

The last condition may be achieved by composing $F(x,\cdot)$ with the nearest point projection mapping certain open neighbourhood of $\mathcal{G}_{n,d}$ in $\operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n)$ onto $\mathcal{G}_{n,d}$. Assume there exists a retraction $\boldsymbol{\xi} : U \to M$ of class $\mathscr{C}^2$ such that $|x - \boldsymbol{\xi}(x)| = \operatorname{dist}(x, M)$ for $x \in U$. Define

$$
\begin{aligned}
F_x : \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \sim \{0\} \to \mathbf{R} \quad & \text{by} \quad F_x(A) = F(x, A) \quad \text{for } x \in U \text{ and } A \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)\,, \\
F_A : U \to \mathbf{R} \quad & \text{by} \quad F_A(x) = F(x, A) \quad \text{for } x \in U \text{ and } A \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)\,, \\
\tau : U \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \quad & \text{by} \quad \tau(x) = \mathrm{D}\boldsymbol{\xi}(x) \quad \text{for } x \in U\,, \\
\nu : U \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \quad & \text{by} \quad \nu(x) = \mathrm{id}_{\mathbf{R}^n} - \tau(x) \quad \text{for } x \in U\,, \\
C : M \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \quad & \text{by} \quad C(x) = \Pi(\tau(x))^* \operatorname{grad} F_x(\tau(x)) \quad \text{for } x \in M\,, \\
E : M \to \mathbf{R}^n \quad & \text{by} \quad E(x) = \operatorname{grad} F_{\tau(x)}(x) \quad \text{for } x \in M\,, \\
H : M \to \mathbf{R} \quad & \text{by} \quad H(x) = F(x, \tau(x)) \quad \text{for } x \in M\,, \\
B : M \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \quad & \text{by} \quad B(x) = C(x) + H(x)\tau(x) \quad \text{for } x \in M\,, \\
\boldsymbol{\eta} : \partial M \to \partial \mathbf{B}(0,1) \quad & \text{by} \quad -\boldsymbol{\eta}(x) \in \operatorname{Tan}(M, x) \cap \operatorname{Nor}(\partial M, x) \quad \text{for } x \in \partial M\,, \\
V \in \mathbf{V}_d(U) \quad & \text{by} \quad V = \mathbf{v}_d(M)\,.
\end{aligned}
$$

5.2 Remark. The map C is characterised by the requirement

$$C(x) \bullet L = \big\langle \Pi(\tau(x))L,\, \mathrm{D}F_x(\tau(x)) \big\rangle \quad \text{for } x \in M \text{ and } L \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$$

and we have

$$B(x) = B_F(x, \tau(x)) \quad \text{for } x \in M\,.$$

5.3 Remark. Let $x \in M$. Observe that

$$\tau(x) = \operatorname{Tan}(M, x)_\natural\,.$$

The reason for defining τ as the derivative of $\boldsymbol{\xi}$ is to be able to differentiate τ also in directions orthogonal to M. Moreover, for $u, v \in \mathbf{R}^n$ and $w \in \operatorname{Tan}(M, x)$ we obtain

$$\mathrm{D}\tau(x)uv = \big\langle u \odot v,\, \mathrm{D}^2\boldsymbol{\xi}(x) \big\rangle = \mathrm{D}\tau(x)vu\,, \quad \mathrm{D}\tau(x)w \in \operatorname{Tan}(\mathcal{G}_{n,d}, \tau(x))\,; \tag{6}$$

hence, for $u, v \in \operatorname{Tan}(M, x)$ and $\eta, \zeta \in \operatorname{Nor}(M, x)$ we have

$$\mathrm{D}\tau(x)uv \in \operatorname{Nor}(M, x) \quad \text{and} \quad \mathrm{D}\tau(x)u\eta \in \operatorname{Tan}(M, x)\,,$$

and, using (6), we also get

$$\mathrm{D}\tau(x)\eta u \in \operatorname{Tan}(M, x) \quad \text{and} \quad \mathrm{D}\tau(x)\eta\zeta = 0\,.$$

5.4 Remark. Let $x \in M$. Observe that if $\mathbf{b}(M, x)$ denotes the second fundamental form of M at x as defined, e.g., in [All72, 2.5(1)], then (see, e.g., [KM17, 3.1(1)])

$$\mathbf{b}(M, x)(u, v) = \mathrm{D}\tau(x)uv \quad \text{for } u, v \in \operatorname{Tan}(M, x)\,.$$

In particular, (recall (1) and [All72, 2.5(2)])

$$\mathbf{h}(M, x) = \operatorname{trace} \mathrm{D}\tau(x) \circ \tau(x)\,.$$

5.5 Lemma. *Let V, M, F, H, C, τ, ν, $\boldsymbol{\eta}$ be as in 5.1. Then*

$$\delta_F V(g) = \int_{\partial M} g(x) \bullet \boldsymbol{\eta}_F(M, x)\, \mathrm{d}\mathscr{H}^{d-1}(x) - \int_M \mathbf{h}_F(M, x) \bullet g(x) F(x, \tau(x))\, \mathrm{d}\mathscr{H}^d(x)\,, \tag{7}$$

where

$$\boldsymbol{\eta}_F(M,x) = \big\langle \boldsymbol{\eta}(x),\, B_F(x,\tau(x)) \big\rangle \quad \textit{for } x \in \partial M\,,$$

and $\mathbf{h}_F(M,x) \in \operatorname{Nor}(M,x)$ *for* $x \in M$ *is given by*

$$\begin{aligned} F(x,\tau(x))\mathbf{h}_F(M,x) &= \big\langle \operatorname{trace} \mathrm{D}B(x) - E(x),\, \nu(x) \big\rangle \\ &= F(x,\tau(x))\mathbf{h}(M,x) - \big\langle \operatorname{grad} F_{\tau(x)}(x) + \operatorname{grad}(F_x \circ \tau)(x),\, \nu(x) \big\rangle \\ &\quad + 2\big\langle \operatorname{trace} \mathrm{D}\big[\operatorname{grad} F_x \circ \tau + \operatorname{grad} F_{(\cdot)} \circ \tau(x)\big](x) \circ \tau(x),\, \nu(x) \big\rangle\,. \end{aligned}$$

Proof. Let $g \in \mathscr{X}(U)$. Formulas (4) and (5) together with (3) and 4.17 give

$$\delta_F V(g) = \int_M E(x) \bullet g(x) + B(x) \bullet \mathrm{D}g(x) \circ \tau(x)\, \mathrm{d}\mathscr{H}^d(x)\,. \tag{8}$$

If $x \in M$ and $u_1, \ldots, u_n$ is an orthonormal basis of $\mathbf{R}^n$ such that $u_1, \ldots, u_d$ spans $\operatorname{Tan}(M,x)$, then

$$\begin{aligned} B(x) \bullet \mathrm{D}g(x) \circ \tau(x) &= \sum_{i=1}^d B(x)u_i \bullet \mathrm{D}g(x)u_i \\ &= \tau(x) \bullet \mathrm{D}\big[M \ni y \mapsto \langle g(y),\, B(y)^* \rangle\big](x) - \sum_{i=1}^d \mathrm{D}B(x)u_iu_i \bullet g(x)\,. \end{aligned} \tag{9}$$

Plugging (9) into (8) and employing the Stokes theorem we obtain

$$\begin{aligned} \delta_F V(g) &= \int_{\partial M} g(x) \bullet \big\langle \boldsymbol{\eta}(x),\, B(x) \big\rangle\, \mathrm{d}\mathscr{H}^{d-1}(x) \\ &\quad - \int_M \big(\operatorname{trace} \mathrm{D}B(x) - E(x)\big) \bullet g(x)\, \mathrm{d}\mathscr{H}^d(x)\,. \end{aligned} \tag{10}$$

Fix $x \in M$. We shall now show that

$$\operatorname{trace} \mathrm{D}B(x) - E(x) \in \operatorname{Nor}(M,x)\,. \tag{11}$$

Let $u_1, \ldots, u_n$ be an orthonormal basis of $\mathbf{R}^n$ such that $u_1, \ldots, u_d$ spans $\operatorname{Tan}(M,x)$. We have

$$\begin{aligned} \sum_{i=1}^d \mathrm{D}(H \cdot \tau)(x)u_iu_i - E(x) &= \sum_{i=1}^d \mathrm{D}H(x)u_i \cdot u_i + H(x) \cdot \mathrm{D}\tau(x)u_iu_i - \operatorname{grad} F_{\tau(x)}(x) \\ &= \big\langle \operatorname{grad} F_{\tau(x)}(x),\, \tau(x) \big\rangle + \sum_{i=1}^d \mathrm{D}(F_x \circ \tau)(x)u_i \cdot u_i + H(x)\mathbf{h}(M,x) - \operatorname{grad} F_{\tau(x)}(x) \\ &= -\big\langle \operatorname{grad} F_{\tau(x)}(x),\, \nu(x) \big\rangle + \big\langle \operatorname{grad}(F_x \circ \tau)(x),\, \tau(x) \big\rangle + F(x,\tau(x))\mathbf{h}(M,x)\,. \end{aligned} \tag{12}$$

Now we only need to show that

$$\big\langle \operatorname{grad}(F_x \circ \tau)(x),\, \tau(x) \big\rangle + \operatorname{trace} \mathrm{D}C(x) \in \operatorname{Nor}(M,x)\,. \tag{13}$$

For $i, j \in \{1, \ldots, n\}$ define $L_{i,j} \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ so that $L_{i,j} u_i = u_j$ and $L_{i,j} u_k = 0$ if $k \neq i$. Note that $L_{i,j}^* = L_{j,i}$. For $j \in \{1, \ldots, d\}$ we have $L_{i,j}[\operatorname{Tan}(M, x)] \subseteq \operatorname{Tan}(M, x)$ so $\Pi(\tau(x)) L_{i,j} = 0$; hence,

$$\begin{aligned}
\sum_{i=1}^{d} \mathrm{D}C(x) u_i u_i \bullet u_j = \sum_{i=1}^{d} \mathrm{D}C(x) u_i \bullet L_{i,j} &= \sum_{i=1}^{d} \mathrm{D}\big[M \ni y \mapsto C(y) \bullet L_{i,j}\big](x) u_i \\
&= \sum_{i=1}^{d} \mathrm{D}\big[M \ni y \mapsto \big\langle \Pi(\tau(y)) L_{i,j},\, \mathrm{D}F_y(\tau(y)) \big\rangle\big](x) u_i \\
&= \sum_{i=1}^{d} \big\langle \mathrm{D}\big[M \ni y \mapsto \Pi(\tau(y)) L_{i,j}\big](x) u_i,\, \mathrm{D}F_x(\tau(x)) \big\rangle .
\end{aligned}$$

Recalling 2.2 and (3) we see that

$$\begin{aligned}
\big\langle L,\, \mathrm{D}(\Pi \circ \tau)(x) u \big\rangle = {}& \nu(x) \circ L \circ \mathrm{D}\tau(x) u - \mathrm{D}\tau(x) u \circ L \circ \tau(x) \\
&+ \big(\nu(x) \circ L \circ \mathrm{D}\tau(x) u - \mathrm{D}\tau(x) u \circ L \circ \tau(x)\big)^*
\end{aligned}$$

for $L \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ and $u \in \mathbf{R}^n$. Thus, for $j \in \{1, 2, \ldots, d\}$ we get

$$\sum_{i=1}^{d} \mathrm{D}C(x) u_i u_i \bullet u_j = - \sum_{i=1}^{d} \big\langle \mathrm{D}\tau(x) u_i \circ L_{i,j} + L_{j,i} \circ \mathrm{D}\tau(x) u_i,\, \mathrm{D}F_x(\tau(x)) \big\rangle .$$

However, for $j \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, d\}$ we have

$$\big\langle u_k,\, \textstyle\sum_{i=1}^{d} \mathrm{D}\tau(x) u_i \circ L_{i,j} \big\rangle = \mathrm{D}\tau(x) u_k u_j = \mathrm{D}\tau(x) u_j u_k$$

for $l \in \{1, \ldots, n\}$ and $j, k \in \{1, \ldots, d\}$ we obtain

$$\begin{aligned}
\big\langle u_l,\, \textstyle\sum_{i=1}^{d} L_{j,i} \circ \mathrm{D}\tau(x) u_i \big\rangle \bullet u_k &= \big\langle u_k,\, \textstyle\sum_{i=1}^{d} \mathrm{D}\tau(x) u_i \circ L_{i,j} \big\rangle \bullet u_l \\
&= \mathrm{D}\tau(x) u_k u_j \bullet u_l = \mathrm{D}\tau(x) u_j u_k \bullet u_l = u_k \bullet \mathrm{D}\tau(x) u_j u_l
\end{aligned}$$

and if $j \in \{1, \ldots, n\}$ and $l, k \in \{1, \ldots, d\}$, then

$$\begin{aligned}
\big\langle u_l,\, \textstyle\sum_{i=1}^{d} L_{j,i} \circ \mathrm{D}\tau(x) u_i \big\rangle \bullet u_k &= \mathrm{D}\tau(x) u_k u_j \bullet u_l \\
&= u_j \bullet \mathrm{D}\tau(x) u_k u_l = u_j \bullet \mathrm{D}\tau(x) u_l u_k = \mathrm{D}\tau(x) u_l u_j \bullet u_k = \mathrm{D}\tau(x) u_j u_l \bullet u_k .
\end{aligned}$$

Thus

$$\sum_{i=1}^{d} \mathrm{D}\tau(x) u_i \circ L_{i,j} + L_{j,i} \circ \mathrm{D}\tau(x) u_i = \mathrm{D}\tau(x) u_j \quad \text{for } j \in \{1, \ldots, n\} . \tag{14}$$

It follows that

$$\sum_{i=1}^{d} \mathrm{D}C(x) u_i u_i \bullet u_j = -\big\langle \mathrm{D}\tau(x) u_j,\, \mathrm{D}F_x(\tau(x)) \big\rangle = -\mathrm{D}(F_x \circ \tau)(x) u_j \quad \text{for } j \in \{1, \ldots, d\} .$$

Therefore,

$$\begin{aligned}\Big\langle \textstyle\sum_{i=1}^d \mathrm{D}C(x)u_iu_i,\, \tau(x)\Big\rangle &= \sum_{j=1}^d\sum_{i=1}^d \mathrm{D}C(x)u_iu_i \bullet u_j \cdot u_j\\ &= -\sum_{j=1}^d \mathrm{D}(F_x\circ\tau)(x)u_j\cdot u_j = -\big\langle \mathrm{grad}(F_x\circ\tau)(x),\, \tau(x)\big\rangle\,.\end{aligned}$$

which finishes the proof of (13) and, together with (12), shows that (11) holds.

For $j\in\{d+1,\dots,n\}$ we have

$$\begin{aligned}(15)\quad \sum_{i=1}^d \mathrm{D}C(x)u_iu_i\bullet u_j &= -\sum_{i=1}^d\big\langle \mathrm{D}\tau(x)u_i\circ L_{i,j}+L_{j,i}\circ\mathrm{D}\tau(x)u_i,\, \mathrm{D}F_x(\tau(x))\big\rangle\\ &+\sum_{i=1}^d\big\langle \mathrm{D}\tau(x)u_i\circ L_{j,i}+L_{i,j}\circ\mathrm{D}\tau(x)u_i,\, \mathrm{D}F_x(\tau(x))\big\rangle\\ &+\sum_{i=1}^d\big\langle (L_{i,j}+L_{j,i},\mathrm{D}\tau(x)u_i),\, \mathrm{D}^2F_x(\tau(x))\big\rangle\\ &+\sum_{i=1}^d\big\langle (0,L_{i,j}+L_{j,i})\odot(u_i,0),\, \mathrm{D}^2F(x,\tau(x))\big\rangle\,.\end{aligned}$$

The first term on the right-hand side can be transformed using (14) into

$$(16)\qquad -\sum_{i=1}^d\big\langle \mathrm{D}\tau(x)u_i\circ L_{i,j}+L_{j,i}\circ\mathrm{D}\tau(x)u_i,\, \mathrm{D}F_x(\tau(x))\big\rangle = -\mathrm{D}(F_x\circ\tau)(x)u_j\,.$$

The second term is

$$\begin{aligned}&\sum_{i=1}^d\big\langle \mathrm{D}\tau(x)u_i\circ L_{j,i}+L_{i,j}\circ\mathrm{D}\tau(x)u_i,\, \mathrm{D}F_x(\tau(x))\big\rangle\\ &\quad=\sum_{i=1}^d \mathrm{grad}\,F_x(\tau(x))\bullet\big(\mathrm{D}\tau(x)u_i\circ L_{j,i}+L_{i,j}\circ\mathrm{D}\tau(x)u_i\big)\\ &\quad=\sum_{i=1}^d \mathrm{grad}\,F_x(\tau(x))u_j\bullet\mathrm{D}\tau(x)u_iu_i+\sum_{k=1}^n(\mathrm{grad}\,F_x(\tau(x))u_k\bullet u_j)\cdot(\mathrm{D}\tau(x)u_iu_k\bullet u_i)\\ &\quad=\mathrm{grad}\,F_x(\tau(x))u_j\bullet\mathbf{h}(M,x)+u_j\sum_{i=1}^d\sum_{k=d+1}^n(u_k\bullet\mathrm{grad}\,F_x(\tau(x))^*u_j)\cdot(u_k\bullet\mathrm{D}\tau(x)u_iu_i)\\ &\quad=\mathbf{h}(M,x)\bullet\big(\mathrm{grad}\,F_x(\tau(x))u_j+\mathrm{grad}\,F_x(\tau(x))^*u_j\big)\,.\end{aligned}$$

Since $\mathrm{grad}\,F_x(\tau(x))\in\mathrm{Tan}(\mathcal{G}_{n,d},\tau(x))$ we know that $\mathrm{grad}\,F_x(\tau(x))=\mathrm{grad}\,F_x(\tau(x))^*$ and we obtain

$$\begin{aligned}(17)\quad \sum_{i=1}^d\big\langle \mathrm{D}\tau(x)u_i\circ L_{j,i}+L_{i,j}\circ\mathrm{D}\tau(x)u_i,\, \mathrm{D}F_x(\tau(x))\big\rangle &= 2\mathbf{h}(M,x)\bullet\mathrm{grad}\,F_x(\tau(x))u_j\\ &= 2\,\mathrm{grad}\,F_x(\tau(x))\mathbf{h}(M,x)\bullet u_j\,.\end{aligned}$$

Since $\mathbf{h}(M,x), u_j \in \operatorname{Nor}(M,x)$ and $\operatorname{grad} F_x(\tau(x)) \in \operatorname{Tan}(\mathcal{G}_{n,d}, \tau(x))$ maps $\operatorname{Nor}(M,x)$ to the tangent space $\operatorname{Tan}(M,x)$ we see that

$$\operatorname{grad} F_x(\tau(x))\mathbf{h}(M,x) \bullet u_j = 0\,. \tag{18}$$

The third term on the right-hand side of (15) is

$$\begin{aligned}
&\sum_{i=1}^{d} \big\langle (L_{i,j} + L_{j,i}, \mathrm{D}\tau(x)u_i),\, \mathrm{D}^2 F_x(\tau(x)) \big\rangle = \sum_{i=1}^{d} \mathrm{D}(\operatorname{grad} F_x \circ \tau)(x)u_i \bullet (L_{i,j} + L_{j,i}) \\
&\quad = \sum_{i=1}^{d} \mathrm{D}(\operatorname{grad} F_x \circ \tau)(x)u_i u_i \bullet u_j + \mathrm{D}(\operatorname{grad} F_x \circ \tau)(x)u_i u_j \bullet u_i \\
&\quad = \sum_{i=1}^{d} \mathrm{D}(\operatorname{grad} F_x \circ \tau)(x)u_i u_i \bullet u_j + u_j \bullet (\mathrm{D}(\operatorname{grad} F_x \circ \tau)(x)u_i)^* u_i \\
&\quad = 2u_j \bullet \sum_{i=1}^{d} \mathrm{D}(\operatorname{grad} F_x \circ \tau)(x)u_i u_i = 2u_j \bullet \operatorname{trace} \mathrm{D}(\operatorname{grad} F_x \circ \tau)(x) \circ \tau(x)\,.
\end{aligned} \tag{19}$$

The last term of (15) can be written as

$$\begin{aligned}
&\sum_{i=1}^{d} \big\langle (0, L_{i,j} + L_{j,i}) \odot (u_i, 0),\, \mathrm{D}^2 F(x, \tau(x)) \big\rangle \\
&= 2\sum_{i=1}^{d} \big\langle u_i,\, \mathrm{D}[U \ni y \mapsto \operatorname{grad} F_y \circ \tau(x)u_i \bullet u_j](x) \big\rangle = 2u_j \bullet \sum_{i=1}^{d} \mathrm{D}\big[\operatorname{grad} F_{(\cdot)} \circ \tau(x)\big](x)u_i u_i \\
&\qquad = 2u_j \bullet \operatorname{trace} \mathrm{D}\big[\operatorname{grad} F_{(\cdot)} \circ \tau(x)\big](x) \circ \tau(x)\,.
\end{aligned} \tag{20}$$

Combining (10), (12), (15), (16), (17), (18), (19), (20) we obtain (7). □

5.6 Remark. Another way to see that $\mathbf{h}_F(M,x) \in \operatorname{Nor}(M,x)$ for $x \in M$ is to consider $\delta_F V(g)$ for $g \in \mathscr{X}(U)$ such that $g|M \in \mathscr{X}(M)$, i.e., $g(x) \in \operatorname{Tan}(M,x)$ for $x \in M$ and $g(x) = 0$ in some neighbourhood of ∂M. Let h be the flow of g, i.e., $h : I \times U \to U$, where $I \subseteq \mathbf{R}$ is an open interval containing 0, $h(s+t,x) = h(s,h(t,x))$ whenever $s,t,s+t \in I$, and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} h(t,x) = g(x)$ for $x \in U$. Set $h_t = h(t,\cdot)$ for $t \in I$. We have $h_t[M] = M$ for $t \in I$; hence,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Phi_F(h_t[M]) = \delta_F V(g) = -\int_M g(x) \bullet \mathbf{h}_F(M,x)F(x,t)\,\mathrm{d}V(x,T)\,.$$

Since this holds for all $g \in \mathscr{X}(U)$ such that $g|M \in \mathscr{X}(M)$ we see that $\mathbf{h}_F(M,x) \in \operatorname{Nor}(M,x)$ for $x \in M$.

6 The case of codimension one

Assume $d = n-1$. Let $F : U \times \mathcal{G}_{n,d} \to \mathbf{R}$ be of class $\mathscr{C}^2$.

6.1 Definition. Define $\pi : \mathbf{R}^n \sim \{0\} \to \mathcal{G}_{n,d}$ by $\pi(v) = \operatorname{span}\{v\}^{\perp}_{\natural}$ for $v \in \mathbf{R}^n \sim \{0\}$.

6.2 Remark. We have

$$\pi(v)u = u - |v|^{-2}(u \bullet v)\cdot v \quad \text{for } v \in \mathbf{R}^n \sim \{0\} \text{ and } u \in \mathbf{R}^n\,.$$

Thus,

$$\mathrm{D}\pi(v)wu = -|v|^{-2}(u\bullet w)\cdot v - |v|^{-2}(u \bullet v)\cdot w + 2|v|^{-4}(v\bullet w)\cdot(u\bullet v)\cdot v \\ \text{for } v \in \mathbf{R}^n \sim \{0\},\ w, u \in \mathbf{R}^n\,.$$

In case $|v| = 1$, $w \perp v$, and $u \in \mathbf{R}^n$ we get

$$\mathrm{D}\pi(v)wu = -(u\bullet w)\cdot v - (u \bullet v)\cdot w\,.$$

6.3 Definition. Whenever $w, v \in \mathbf{R}^n$ we define

$$L_{w,v} \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \quad \text{by} \quad \langle u,\, L_{w,v}\rangle = (u\bullet w)\cdot v \quad \text{for } u \in \mathbf{R}^n\,.$$

6.4 Definition. Define $\bar{F} : U \times \mathbf{R}^n \sim \{0\} \to \mathbf{R}$ and $\bar{B}_F : U \times \mathbf{R}^n \sim \{0\} \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ by

$$\bar{F}(x, v) = |v|F(x, \pi(v)) \quad \text{and} \quad \bar{B}_F(x, v) = B_F(x, \pi(v)) \quad \text{for } x \in U \text{ and } v \in \mathbf{R}^n \sim \{0\}\,.$$

6.5 Remark. Of course one can also define F starting from $\bar{F}$. Note that if $\bar{F}_x$ is convex for each $x \in \mathbf{R}^n$, then it is a norm and $(\mathbf{R}^n, F)$ becomes a Finsler manifold.

6.6 Lemma. *For $x \in U$, $v \in \mathbf{R}^n \sim \{0\}$ with $|v| = 1$, and $u \in \mathbf{R}^n$ there holds*

$$\operatorname{grad} \bar{F}_x(v) = \bar{F}_x(v)\cdot v - \bar{B}_F(x, v)^* v \quad \textit{and} \quad \bar{B}_F(x, v)u = \bar{F}_x(v)u - v\cdot \mathrm{D}\bar{F}_x(v)u\,.$$

Proof. Fix $x \in U$ and $v \in \mathbf{R}^n \sim \{0\}$ with $|v| = 1$. Since $\bar{F}_x(\lambda v) = |\lambda|\bar{F}_x(v)$ for $\lambda \in \mathbf{R} \sim \{0\}$ we clearly have

$$\mathrm{D}\bar{F}_x(v)v = \bar{F}_x(v)\,.$$

Now let $w \in \mathbf{R}^n$ be such that $|w| = 1$ and $w \perp v$. Using 6.2 we get

$$\begin{aligned} \mathrm{D}\bar{F}_x(v)w = \mathrm{D}F_x(\pi(v))\circ \mathrm{D}\pi(v)w &= -\big\langle \big[\mathbf{R}^n \ni u \mapsto (u\bullet w)\cdot v + (u\bullet v)\cdot w\big],\, \mathrm{D}F_x(\pi(v))\big\rangle \\ &= -\big\langle L_{w,v} + L_{v,w},\, \mathrm{D}F_x(\pi(v))\big\rangle = -\big\langle \Pi(\pi(v))L_{w,v},\, \mathrm{D}F_x(\pi(v))\big\rangle \\ &= -B_F(x, \pi(v)) \bullet L_{w,v} = -\bar{B}_F(x, v)w \bullet v\,. \end{aligned}$$

Representing $u \in \mathbf{R}^n$ as $u = (u\bullet v)v + \pi(v)u$ we obtain

$$\mathrm{D}\bar{F}_x(v)u = (u\bullet v)\bar{F}_x(v) - \bar{B}_F(x, v)\circ \pi(v)u \bullet v$$

so, recalling that $v \in \ker \bar{B}_F(x, v)$, we get

$$\operatorname{grad} \bar{F}_x(v) = \bar{F}_x(v)v - \bar{B}_F(x, v)^* v\,.$$

Now, we know

$$\bar{B}_F(x, v)v = 0\,, \quad \bar{B}_F(x, v)\circ \pi(v)u \bullet v = -\mathrm{D}\bar{F}_x(v)u \\ \text{and} \quad \pi(v)\circ \bar{B}_F(x, v)u = \bar{F}(x, v)\pi(v)u\,;$$

hence, the conclusion follows. ☐

6.7 Lemma. *Let M be a submanifold of U of class $\mathscr{C}^2$ of dimension $d = n-1$, $G \subseteq U$ be open, $x \in G \cap M$, $\eta : G \to \mathbf{R}^n$ be a map of class $\mathscr{C}^1$ such that $|\eta(y)| = 1$, $\eta(y) \in \operatorname{Nor}(M, y)$, and $\langle \eta(y), \mathrm{D}\eta(y) \rangle = 0$ for $y \in G \cap M$, $u_1, \ldots, u_n$ be an orthonormal basis of $\mathbf{R}^n$. Then*

$$\begin{aligned} - \bar{F}(x, \eta(x)) \mathbf{h}_F(x) = \eta(x) \cdot \big(&\operatorname{trace} \mathrm{D}\big[\operatorname{grad} \bar{F}_x \circ \eta\big](x) \\ &+ \operatorname{trace}\big[\mathbf{R}^n \times \mathbf{R}^n \ni (u,v) \mapsto \big\langle (u,0) \odot (0,v),\, \mathrm{D}^2 \bar{F}(x, \eta(x)) \big\rangle\big]\big) \\ = \eta(x) \cdot \Big(&\textstyle\sum_{i=1}^n \big\langle u_i \odot \mathrm{D}\eta(x) u_i,\, \mathrm{D}^2 \bar{F}_x(\eta(x)) \big\rangle + \big\langle (u_i, 0) \odot (0, u_i),\, \mathrm{D}^2 \bar{F}(x, \eta(x)) \big\rangle \Big) . \end{aligned}$$

Proof. Let $\tau(y) = \pi(\eta(y))$ and $\bar{B}(y) = \bar{B}_F(y, \eta(y))$ for $y \in G$. Assume $u_n = \eta(x)$. From 5.5 we know that

$$-\bar{F}(x, \eta(x)) \mathbf{h}_F(x) = \eta(x) \cdot \eta(x) \bullet \big(\operatorname{grad} F_{\tau(x)}(x) - \operatorname{trace} \mathrm{D}B(x) \big) . \tag{21}$$

Let $u \in \operatorname{Tan}(M, x)$. Using 6.6 and the fact that $\mathrm{D}\eta(x) u \bullet \eta(x) = 0$ we obtain

$$\begin{aligned} - \mathrm{D}B(x) u u \bullet \eta(x) &= \mathrm{D}\big[G \ni y \mapsto \mathrm{D}\bar{F}_y(\eta(y)) u\big](x) u \bullet \eta(x) \\ &= \mathrm{D}\big[\operatorname{grad} \bar{F}_x \circ \eta\big](x) u \bullet u + \big\langle (u,0) \odot (0,u),\, \mathrm{D}^2 \bar{F}(x, \eta(x)) \big\rangle . \end{aligned} \tag{22}$$

Using 1-homogeneity of $\bar{F}_x$ we also get

$$\begin{aligned} &\operatorname{grad} F_{\tau(x)}(x) \bullet \eta(x) = \big\langle \eta(x),\, \mathrm{D}\big[G \ni y \mapsto \bar{F}(y, \eta(x))\big](x) \big\rangle \\ &= \big\langle \eta(x),\, \mathrm{D}\big[G \ni y \mapsto \langle \eta(x),\, \mathrm{D}\bar{F}_y(\eta(x)) \rangle\big](x) \big\rangle = \big\langle (\eta(x), 0) \odot (0, \eta(x)),\, \mathrm{D}^2 \bar{F}(x, \eta(x)) \big\rangle . \end{aligned} \tag{23}$$

Conclusion follows by summing over $i \in \{1, \ldots, d\}$ expression (22) with u_i in place of u, then adding (23) and plugging the result into (21). □

7 Ellipticity conditions

Almgren ellipticity

7.1 Definition. We say that (S, D) is a *(rectifiable) test pair* if S is compact and $(\mathscr{H}^d, d)$ rectifiable and there exists $T \in \mathbf{G}(n, d)$ such that $D = T \cap \mathbf{B}(0,1)$ and the $(d-1)$-dimensional sphere $B = T \cap \mathbf{S}^{n-1}$ is not a Lipschitz retract of S, i.e., for all Lipschitz maps $f : \mathbf{R}^n \to \mathbf{R}^n$ satisfying $f(x) = x$ for $x \in B$ there holds $f[S] \neq B$.

7.2 Definition. Let F be an integrand and $x \in U$. We define the integrand F^x so that

$$F^x(y, T) = F(x, T) \quad \text{for } y \in U \text{ and } T \in \mathbf{G}(n, d) .$$

7.3 Definition (cf. [Alm76, IV.1(7)])**.** Let F be an integrand and $x \in U$.

(a) We say that F is *strictly Almgren elliptic at x* and write $F \in \mathrm{AE}_x$ if

$$\Phi_{F^x}(S) - \Phi_{F^x}(D) > 0 \quad \text{for any test pair } (S, D) \text{ with } \mathscr{H}^d(S) > \mathscr{H}^d(D) . \tag{24}$$

(b) We say that F is *uniformly Almgren elliptic at x* and write $F \in \mathrm{AUE}_x$ if there is a number $c \in (0, \infty)$ such that

$$\Phi_{F^x}(S) - \Phi_{F^x}(D) > c\big(\mathscr{H}^d(S) - \mathscr{H}^d(D)\big) \quad \text{for all test pairs } (S, D) .$$

7.4 Remark. In [Fed69, 5.1.2] similar notion of ellipticity is defined in the setting of currents. In this case, an integrand is a function $\Psi : U \times \bigwedge_d \mathbf{R}^n \to \mathbf{R}$ satisfying $\Psi(x, r\alpha) = r\Psi(x,\alpha)$ for $r \in (0,\infty)$, $\alpha \in \bigwedge_d \mathbf{R}^n$, and $x \in U$. For $x \in U$ we define $\Psi_x : \bigwedge_d \mathbf{R}^n \to \mathbf{R}$ and $\Psi^x : U \times \bigwedge_d \mathbf{R}^n \to \mathbf{R}$ by requiring that $\Psi_x(\alpha) = \Psi(x,\alpha) = \Psi^x(y,\alpha)$ for $\alpha \in \bigwedge_d \mathbf{R}^n$ and $y \in U$. The integrand Ψ is said to be elliptic at $x \in U$ if there is a number $c \in (0,\infty)$ such that

$$\langle \Psi^x,\, R\rangle - \langle \Psi^x,\, S\rangle \geq c\big(\mathbf{M}(R) - \mathbf{M}(S)\big)$$

whenever R and S are d-dimensional rectifiable currents with $\partial R = \partial S$ and S is naturally associated to a subset of some $T \in \mathbf{G}(n,d)$.

If R, S, T are as above and $T = \mathbf{R}^n \cap \{v : \gamma \wedge v = 0\}$ for some $\gamma \in \bigwedge_d \mathbf{R}^n$, then

$$\int \vec{R}\, \mathrm{d}\|R\| = \int \vec{S}\, \mathrm{d}\|S\| = \mathbf{M}(S)\gamma\,.$$

This is true because $\partial(R-S) = 0$, $R - S = \partial(\boldsymbol{\delta}_0 \bowtie (R-S))$[1], and for every $\chi \in \operatorname{Hom}(\bigwedge_d \mathbf{R}^n, \mathbf{R})$ the differential form ϕ defined by $\phi(z) = \chi$ for all $z \in U$ has exterior derivative zero; hence,

$$\begin{aligned}\chi(\textstyle\int \vec{R}\, \mathrm{d}\|R\| - \int \vec{S}\, \mathrm{d}\|S\|) = \displaystyle\int \chi \circ \vec{R}\, \mathrm{d}\|R\| - \int \chi \circ \vec{S}\, \mathrm{d}\|S\| \\ = (R-S)\phi = (\boldsymbol{\delta}_0 \bowtie (R-S))\, \mathrm{d}\phi = 0\,.\end{aligned}$$

Let $c \in (0,\infty)$. Define

$$F(\alpha) = \Psi(x,\alpha) - c|\alpha| \quad \text{for } \alpha \in \textstyle\bigwedge_d \mathbf{R}^n$$

and assume F is convex. Then $F(\alpha+\beta) \leq F(\alpha) + F(\beta)$ for $\alpha, \beta \in \bigwedge_d \mathbf{R}^n$ and we get

$$\langle \Psi^x,\, S\rangle - c\mathbf{M}(S) = F(\mathbf{M}(S)\gamma) = F(\textstyle\int \vec{R}\, \mathrm{d}\|R\|) \leq \displaystyle\int F \circ \vec{R}\, \mathrm{d}\|R\| = \langle \Psi^x,\, R\rangle - c\mathbf{M}(R)\,;$$

hence, convexity of F suffices for ellipticity of Ψ at x.

7.5 Remark. Definition 7.3 should be understood as a geometric counterpart of quasiconvexity; see [Mor66]. Assume $F \in \mathrm{AUE}_x$ for all $x \in U$, $T \in \mathbf{G}(n,d)$, $f : T \to T^\perp$, $G = \{x + f(x) : x \in T\}$, $V = \mathbf{v}_d(G \cap U)$, and $\delta_F V = 0$. Then the condition $\delta_F V = 0$ can be translated into a system of PDE's satisfied by f and this system will be elliptic in the traditional sense; see [Fed69, 5.1, 5.2].

7.6 Remark. Checking whether $F \in \mathrm{AE}_x$ is difficult because of no algebraic restrictions on the family of test pairs. In case of currents every test pair consists of two rectifiable currents (R, S), one of them flat, with common boundary. This additional current structure enables cancellation of orienting d-vectors so that integrating these d-vectors over the sum $R+S$ yields zero. The definition of AE_x allows, a priori, for non-orientable test pairs or even test pairs that do not admit the structure of a rectifiable current with any coefficient group. The multitude of test pairs makes the problem hard.

7.7 Remark. Recall 4.11. Burago and Ivanov [BI12] proved that, in case $d = 2$, the Busemann-Hausdorff integrand is elliptic, in the sense of [Fed69, 5.1], and *conjecture* that this is also true for $d > 2$. On the other hand, Busemann, Ewald and Shephard [BES63] proved that the Holmes-Thompson integrand may fail to be elliptic.

[1] We denote by $R \bowtie S$ the *join of R and S* as defined in [Fed69, 4.1.11].

7.8 Remark. Assume $n = d+1$ and $\bar{F}$ is associated to F as in 6.4. Almgren observed in [Alm76, IV.1(7), p. 88] that $F \in \mathrm{AUE}_x$ *if and only if* $\bar{F}_x$ is a uniformly convex norm.

7.9 Conjecture. Recall 4.8 and assume $F_x : \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}$ is a uniformly convex norm, i.e., $F_x(\lambda A) = |\lambda| F_x(A)$ for $\lambda \in \mathbf{R}$ and $A \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ and there exists $c \in (0, \infty)$ such that $\big[\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \ni A \mapsto F_x(A) - c|A|\big]$ is convex. Then $F \in \mathrm{AUE}_x$.

The atomic condition and the class AC

7.10 Definition (cf. [DPDRG18, Definition 1.1])**.** Let F be an integrand of class $\mathscr{C}^1$ and $x \in U$. We say that F satisfies the *atomic condition at* x and write $F \in \mathrm{AC}_x$ if given any probability measure μ over $\mathbf{G}(n,d)$ and setting

$$A_\mu(x) = \int B_F(x,T) \, \mathrm{d}\mu(T) \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \,,$$

there holds

(a) $\dim \ker A_\mu(x) \le n - d$,

(b) if $\dim \ker A_\mu(x) = n - d$, then $\mu = \mathrm{Dirac}(T)$ for some $T \in \mathbf{G}(n,d)$.

We write $F \in \mathrm{AC}$ if $F \in \mathrm{AC}_x$ for all $x \in U$.

7.11 Remark. Recall that $\tilde{B}_F(x,T) = B_F(x,T)/F(x,T)$. Let us write $F \in \tilde{\mathrm{AC}}_x$ if F satisfies 7.10 but with $\tilde{A}_\mu(x)$ in place of $A_\mu(x)$, where

$$\tilde{A}_\mu(x) = \int \tilde{B}_F(x,T) \, \mathrm{d}\mu(T) \,.$$

For any probability measure μ over $\mathbf{G}(n,d)$ we define the probability measure $\tilde{\mu}$ by

$$\tilde{\mu}(f) = \frac{\int f(T) F(x,T)^{-1} \, \mathrm{d}\mu(T)}{\int F(x,T)^{-1} \, \mathrm{d}\mu(T)} \quad \text{for } f \in \mathscr{K}(\mathbf{G}(n,d))$$

and note that

$$A_{\tilde{\mu}}(x) \textstyle\int F(x,T)^{-1} \, \mathrm{d}\mu(T) = \tilde{A}_\mu(x) \,.$$

Hence, $F \in \tilde{\mathrm{AC}}_x$ if and only if $F \in \mathrm{AC}_x$.

7.12 Remark. Let us consider the set $K = \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \cap \{B_F(x,T) : T \in \mathbf{G}(n,d)\}$. Since $\mathbf{G}(n,d)$ is compact and B_F is continuous we see that K is also compact. Let C denote the convex hull of K in $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$. Clearly

$$A_\mu(x) \in C \quad \text{for any probability measure } \mu \text{ over } \mathbf{G}(n,d) \,.$$

Since $\dim \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) = n^2$ we may use the Caratheodory theorem [Roc70, §17] to see that for any probability measure μ over $\mathbf{G}(n,d)$ there exists a set $\{T_1, \ldots, T_N\} \subseteq \mathbf{G}(n,d)$ with $N \le n^2 + 1$ such that

$$A_\mu(x) \in \mathrm{conv}\{B_F(x,T_i) : i \in \{1, \ldots, N\}\} \,.$$

Therefore, it suffices to check the condition AC_x for probability measures μ which are convex combinations of at most $n^2 + 1$ Dirac deltas.

7.13 Remark. Fix $x \in U$ and define the map

$$\psi_{F,x} : \mathbf{G}(n,d) \to \mathbf{G}(n,d) \quad \text{given by} \quad \psi_{F,x}(T) = \operatorname{im} B_F(x,T) \quad \text{for } T \in \mathbf{G}(n,d)\,.$$

Given any map $\psi : \mathbf{G}(n,d) \to \mathbf{G}(n,d)$ we define the map $\hat{B}_\psi : \mathbf{G}(n,d) \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ by

$$\hat{B}_\psi(T) = \big(T_\natural|_{\psi(T)}\big)^{-1} \circ T_\natural \quad \text{for } T \in \mathbf{G}(n,d)\,.$$

Recalling 4.18 we see that

$$\hat{B}_{\psi_{F,x}}(T) = \hat{B}_F(x,T) \quad \text{for } T \in \mathbf{G}(n,d)\,.$$

7.14 Remark. Assume $n = d+1$ and $\bar{F}$ is associated to F as in 6.4. In [DPDRG18, §5] the authors prove that $F \in \mathrm{AC}_x$ *if and only if* $\bar{F}_x$ is a strictly convex norm.

7.15 Question. What one needs to assume about $\psi : \mathbf{G}(n,d) \to \mathbf{G}(n,d)$ to be able to find an integrand F and $x \in U$ such that $\psi = \psi_{F,x}$?

7.16 Question. Let $x \in U$ and F be an integrand of class $\mathscr{C}^1$. It is not hard to see that if $F \in \mathrm{AC}_x$ or $F_x : \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}$ is a strictly convex norm, then $\psi_{F,x}$ is a homeomorphism.

Assume that $\psi_{F,x}$ is a homeomorphism. Does it follow that $F \in \mathrm{AC}_x$?

The class BC

7.17 Definition (cf. [DK18, 4.8])**.** Let F be an integrand of class $\mathscr{C}^1$ and $x \in U$. We write $F \in \mathrm{BC}_x$ if given any varifold W of the form $W = (\mathscr{H}^d \llcorner T) \times \mu$, where $T \in \mathbf{G}(n,d)$ and μ is a probability measure over $\mathbf{G}(n,d)$, the following holds

$$\text{if } \delta_F W = 0\,, \text{ then } \mu = \mathrm{Dirac}(T)\,.$$

7.18 Lemma (cf. [De 19])**.** *Let μ be a probability measure over $\mathbf{G}(n,d)$, $k \in \mathbb{N}$, $T \in \mathbf{G}(n,k)$, $W = (\mathscr{H}^d \llcorner T) \times \mu$, $x \in U$, $F \in \mathrm{BC}_x$, $\delta_{F^x} W = 0$. Then $k \geq d$.*

Proof. If $d = n$, then $\mathbf{G}(n,d)$ contains only one element so there is only one probability measure over $\mathbf{G}(n,d)$ and there is nothing to prove.

Assume $1 \leq d < n$ and $k < d$. Choose $R \in \mathbf{G}(n, d-k)$ such that $R \perp T$ and set $V = (\mathscr{H}^d \llcorner (T+R)) \times \mu$. We get

$$\begin{aligned} \delta_{F^x} V(g) = \int_R \int_T \int_{\mathbf{G}(n,d)} & B_F(u+v, S) \bullet \mathrm{D}g(x) \,\mathrm{d}\mu(S) \,\mathrm{d}\mathscr{H}^k(u) \,\mathrm{d}\mathscr{H}^{d-k}(v) \\ &= \int_R \delta_{F^x} W(g(v+\cdot)) \,\mathrm{d}\mathscr{H}^{d-k}(v) = 0 \quad \text{for } g \in \mathscr{X}(\mathbf{R}^n)\,. \end{aligned}$$

Thus, $\delta_{F^x} V = 0$ and, since $F \in \mathrm{BC}_x$, we obtain $\mu = \mathrm{Dirac}(T+R)$. However, since R was chosen arbitrarily from $\mathbf{G}(n,d) \cap \{R : R \perp T\} \simeq \mathbf{G}(n-k, d-k)$ which contains more than one element, we reach a contradiction. □

7.19 Remark. In [DK18, 4.8] the definition of BC_x includes the condition that if $\delta_{F^x} W = 0$, then $k \geq d$. Lemma 7.18 shows that this condition is unnecessary.

7.20 Lemma (cf. [DK18, 7.1])**.** $\mathrm{AC}_x = \mathrm{BC}_x$.

Proof. Assume $F \in \mathrm{AC}_x$. Take $W = (\mathscr{H}^d \llcorner T) \times \mu$ as in 7.17 and assume $\delta_{F^x} W = 0$. Then

$$(25)\quad 0 = \delta_{F^x} W(g) = \int \int_T B_F(x,S) \bullet \mathrm{D}g(y)\, \mathrm{d}\mathscr{H}^d(y)\, \mathrm{d}\mu(S) = A_\mu(x) \bullet \int_T \mathrm{D}g(y)\, \mathrm{d}\mathscr{H}^d(y)$$
$$= A_\mu(x) \circ T_\natural^\perp \bullet \int_T \mathrm{D}g(y) \circ T_\natural^\perp\, \mathrm{d}\mathscr{H}^d(y) \quad \text{for } g \in \mathscr{X}(\mathbf{R}^n);$$

hence, $T^\perp \subseteq \ker A_\mu(x)$. From 7.10(a) we get $\dim \ker A_\mu(x) = n - d$ and 7.10(b) gives $\mu = \mathrm{Dirac}(S)$ for some $S \in \mathbf{G}(n,d)$. However, recalling (5) we see that $S^\perp \subseteq \ker B_F(x,S) = \ker A_\mu(x)$ so $S = T$.

Assume $F \in \mathrm{BC}_x$. Take μ as in 7.10 and define

$$T = \operatorname{im} A_\mu(x)^*, \quad k = \dim T, \quad W = (\mathscr{H}^k \llcorner T) \times \mu.$$

Note that $T^\perp = (\operatorname{im} A_\mu(x)^*)^\perp = \ker A_\mu(x)$; thus, repeating the computation (25) we get $\delta_{F^x} W = 0$. Therefore, $k = \dim T \geq d$, so $\dim \ker A_\mu(x) \leq n - d$ and if $k = d$, then 7.17 gives $\mu = \mathrm{Dirac}(T)$. □

7.21 Theorem (cf. [DPDRG18, Theorem 1.2])**.** *Let U be open and F be an integrand of class $\mathscr{C}^1$ over U. Define*

$$\mathscr{V}_F(U) = \mathbf{V}_d(U) \cap \left\{ V : \|\delta_F V\| \text{ is Radon and } \mathbf{\Theta}^d_*(\|V\|, x) > 0 \text{ for } \|V\| \text{ almost all } x \right\}.$$

(a) if $F \in \mathrm{AC}$, then $\mathscr{V}_F(U) \subseteq \mathbf{RV}_d(U)$.

(b) Assume $T \in \mathbf{G}(n,d)$ and $F = F^x$ for some $x \in U$. Then $\mathscr{V}_F(U) \subseteq \mathbf{RV}_d(U)$ if and only if $F \in \mathrm{AC}$.

7.22 Remark. Having in mind 7.20 one can summarize 7.21 the following way: if F is such that the counterpart of the Rectifiability Theorem [All72, 5.5(1)] holds for flat varifolds (i.e. of the type $(\mathscr{H}^d \llcorner T) \times \mu$), then it holds for all varifolds. Ellipticity is a condition that ensures compatibility between the Grassmannian part and the space part of a varifold whose first variation is a Radon measure.

8 The Plateau-type problems

8.1. The problem is formulated the following way:

among *surfaces* with given *boundary* find the one with least *area*.

This is not a precise formulation since one has to specify what is a surface, what is its boundary, and what is its area. There are tons of papers about the problem and we have no intention of summarising all the results that have been achieved till now. Very good surveys have been written recently by David [Dav14] and also by Harrison and Pugh [HP15]. A brief list of results most relevant to my research:

- Douglas [Dou31] and Radó [Rad30]: 2-dimensional surfaces in $\mathbf{R}^3$ parameterised by a disc.
- Reifenberg [Rei60]: arbitrary dimension and co-dimension; homological spanning, minimising Hausdorff measure.

- Federer and Fleming [FF60]: arbitrary dimension and co-dimension; integral currents; minimising the mass of a current.
- Almgren [Alm68]: arbitrary dimension and co-dimension; homological spanning; minimising Huasdorff measure.
- De Lellis, Ghiraldin, and Maggi [DLGM17]: co-dimension one; abstract class of competitors; minimising Huasdorff measure.
- De Lellis, De Rosa, and Ghiraldin [DPDRG16]: arbitrary co-dimension; abstract class of competitors; minimising Huasdorff measure.
- De Lellis, De Rosa, and Ghiraldin [DLDRG19]: co-dimension one; abstract class of competitors; anisotropic integrands.
- De Philippis, De Rosa, and Ghiraldin [DDG17]: arbitrary dimension and co-dimension; abstract class of competitors; anisotropic integrands. Existence result obtained using ideas from previous works and the fundamental rectifiability result 7.21.
- Harrison and Pugh [HP17], [Pug19]: arbitrary dimension and co-dimension; abstract class of competitors; anisotropic integrands. Existence result modelled on Reifenberg's approach [Rei60].
- Fang and Kolasiński [FK18]: arbitrary dimension and co-dimension; abstract class of competitors; anisotropic integrands. Existence result modelled on Almgren's approach [Alm68].

8.2 Definition. A map $f : U \to U$ of class $\mathscr{C}^1$ is called a *local deformation in U* if there exists a closed ball $B \subseteq U$ such that $f(x) = x$ for $x \in U \sim B$ and $f[B] \subseteq B$.

8.3 Definition. A family of sets $\mathcal{A}$ is said to be a *competitor class in U* if

- each element S of $\mathcal{A}$ is a relatively closed $(\mathscr{H}^d, d)$ rectifiable subset of U and
- $f[S] \in \mathcal{A}$ whenever $S \in \mathcal{A}$ and f is a local deformation in U.

We shall focus on the following abstract formulation of the problem.

8.4 Setup. Assume $U \subseteq \mathbf{R}^n$ is open (one can imagine that $\mathbf{R}^n \sim U$ is the boundary), F is a continuous integrand, and $\mathcal{A}$ is a competitor class in U.

We are interested in answering the following questions:

(I) Is there a relatively closed $(\mathscr{H}^d, d)$ rectifiable set $E \subseteq U$ such that

$$\Phi_F(E) \le \Phi_F(A) \quad \text{for all } A \in \mathcal{A}\,?$$

(II) $E \in \mathcal{A}$?

(III) What is the regularity of E?

8.5 Remark. It is rather easy to find a varifold V which minimises Φ_F in $\mathcal{A}$. Let $S_i \in \mathcal{A}$ be a minimising sequence such that $\Phi_F(S_{i+1}) \le \Phi_F(S_i)$ for $i \in \mathbb{N}$. We note that

$$\|\mathbf{v}_d(S_i)\|(U) \le (\inf \operatorname{im} F)^{-1} \Phi_F(S_i) \le (\inf \operatorname{im} F)^{-1} \Phi_F(S_0) \quad \text{for each } i \in \mathbb{N}\,.$$

Therefore, we can apply the Banach-Alouglu theorem and find a subsequence of S_i (still denoted S_i) such that the measures $\mathbf{v}_d(S_i)$ converge, in weak* sense, to some $V \in \mathbf{V}_d(U)$. Moreover, by the very definition of the weak* limit, we obtain

$$\Phi_F(V) = \lim_{i \to \infty} \Phi_F(\mathbf{v}_d(S_i)) = \inf\{\Phi_F(S) : S \in \mathcal{A}\}\,.$$

At this point regularity theory begins. We want to know whether V equals $\mathbf{v}_d(S)$ for some $S \subseteq U$ and, if so, how regular is S.

8.6 Theorem (cf. [FK18] based on [Alm68]). *Assume 8.4. There exists a varifold* $V \in \mathbf{V}_d(U)$*, a relatively closed set* $S \subseteq U$*, and a sequence* $\{S_i : i \in \mathbb{N}\} \subseteq \mathcal{A}$ *such that*

(a) S *is* $(\mathscr{H}^d, d)$ *rectifiable.*

(b) $\lim_{i\to\infty} \mathbf{v}_d(S_i) = V$.

(c) $\Phi_F(V) = \lim_{i\to\infty} \Phi_F(S_i) \le \Phi_F(A)$ *for all* $A \in \mathcal{A}$.

(d) $\operatorname{spt}\|V\| \subseteq S$ *and* $\mathscr{H}^d(S \sim \operatorname{spt}\|V\|) = 0$.

(e) $\|V\|$ *and* $\mathscr{H}^d \llcorner S$ *are mutually absolutely continuous.*

(f) $\lim_{i\to\infty} \sup\{|\operatorname{dist}(x, S_i) - \operatorname{dist}(x, S)| : x \in K\} = 0$ *for any compact set* $K \subseteq U$.

Moreover, if $F \in \mathrm{AUE}_x$ *for all* $x \in U$*, then* $V = \mathbf{v}_d(S)$*, which means that* $\mathbf{\Theta}^d(\|V\|, x) = 1$ *and* $\operatorname{Tan}^d(\|V\|, x) = T$ *for* V *almost all* (x, T).

8.7 Remark. Clearly $S \in \mathcal{A}$ assuming $\mathcal{A}$ has the following closure property: if $R_i \in \mathcal{A}$ and R_i converges locally in Hausdorff distance (or as varifolds, i.e., in weak* sense) to some $(\mathscr{H}^d, d)$ rectifiable set R, then $R \in \mathcal{A}$. This is the case, e.g., if $\mathcal{A}$ is defined to be the family of all sets that span a given boundary $B = \mathbf{R}^n \sim U$ in the homological sense of Reifenberg [Rei60], i.e., S is said to *span* B if the group homomorphism $\check{\mathbf{H}}_{d-1}(B) \to \check{\mathbf{H}}_{d-1}(S \cup B)$ induced by the inclusion map is zero (here, $\check{\mathbf{H}}_k$ stands for k^{th} Čech homology group with integer ocefficients); see [FK18, §12].

8.8 Remark. One can replace the condition "$F \in \mathrm{AUE}_x$ for all $x \in U$" with "$F \in \mathrm{BC}_x$ for all $x \in U$" as the following theorem shows.

8.9 Theorem (cf. [DK18, 6.7]). *Assume 8.4 and* S *and* V *are as in 8.6. If* $F \in \mathrm{BC}_x$ *for all* $x \in U$*, then* $V = \mathbf{v}_d(S)$.

9 AC implies AE

9.1 Remark. As emphasised in 7.6 checking the condition $F \in \mathrm{AE}_x$ is rather hard while, recalling 7.12, checking $F \in \mathrm{AC}_x$ might be easier. Therefore, it is useful to relate the two conditions.

9.2 Theorem (cf. [DK18, 8.8, 9.23]). $\mathrm{BC}_x \subseteq \mathrm{AE}_x$

Sketch of the proof. First note that one can equivalently define the class AE_x, by checking the condition (24) on test pairs (S, D), where D is not a disc but rather a d-dimensional cube of side-length 1 (we shall say that (S, D) is a *cubical test pair*). Assume there exists $F \in \mathrm{BC}_x \sim \mathrm{AE}_x$. Then there is a cubical test pair (S, Q) such that

$$\mathscr{H}^d(S) > \mathscr{H}^d(Q) \quad \text{but} \quad \Phi_{F^x}(S) \le \Phi_{F^x}(Q)\,.$$

Define the class of competitors

$$\mathcal{A} = \mathbf{2}^{\mathbf{R}^n} \cap \big\{S : (S, Q) \text{ is a cubical test pair}\big\}\,.$$

Observe that this class satisfies all the conditions of 8.4 and we can employ 8.9 to find a compact $(\mathscr{H}^d, d)$ rectifiable set $R \subseteq \mathbf{R}^n$ for which

$$\Phi_{F^x}(R) \le \Phi_{F^x}(A) \quad \text{whenever } A \in \mathcal{A}\,.$$

Prove that (R, Q) is a cubical test pair and choose P so that

$$P = R \text{ if } \Phi_{F^x}(R) < \Phi_{F^x}(Q) \quad \text{and} \quad P = S \text{ if } \Phi_{F^x}(R) = \Phi_{F^x}(Q) = \Phi_{F^x}(S)\,.$$

In any case,

$$\frac{\mathscr{H}^d(P)}{\mathscr{H}^d(Q)} = \vartheta > 1\,, \quad \Phi_{F^x}(P) < \Phi_{F^x}(Q)\,, \quad \text{and} \quad (P, Q) \text{ is a cubical test pair}\,.$$

Let $T \in \mathbf{G}(n, d)$ be such that $Q \subseteq T$. Define V to be the varifold obtained by translating P along vectors in T with integer coordinates. This gives kind of a *tiling* of T with copies of P. Then, for each $N \in \mathbb{N}$, define $W_N = \boldsymbol{\mu}_{2^{-N}\#} V$, where $\boldsymbol{\mu}_r(x) = rx$ for $r \in \mathbf{R}$ and $x \in \mathbf{R}^n$, that is, W_N is a rescaled copy of V. Using the characterisation of the measure $\mathscr{H}^d \llcorner T$ as the only non-zero, locally finite measure over T which is invariant under translations in T, one readily verifies that $\{W_N : N \in \mathbb{N}\}$ converges, in varifold sense, to some W and there exists a probability measure μ over $\mathbf{G}(n, d)$ such that

$$W = \vartheta(\mathscr{H}^d \llcorner T) \times \mu\,.$$

For $N \in \mathbb{N}$ let P_N denote the set obtained by tiling Q with 2^{Nd} copies of $\boldsymbol{\mu}_{2^{-N}}[P]$. Observe, that

$$\Phi_{F^x}(P_N) = \Phi_{F^x}(P) \quad \text{for } N \in \mathbb{N}\,;$$

hence,

$$\Phi_{F^x}(W \llcorner (Q \times \mathbf{G}(n, d))) = \lim_{N \to \infty} \Phi_{F^x}(P_N) = \Phi_{F^x}(P) = \inf\{\Phi_{F^x}(A) : A \in \mathcal{A}\}\,.$$

This shows that $W \llcorner (Q \times \mathbf{G}(n, d))$ minimises Φ_{F^x} in the class $\mathcal{A}$. Assume that (P_N, Q) is a cubical test pair for each $N \in \mathbb{N}$, i.e., that $P_N \in \mathcal{A}$. Then

$$\delta_F\big[\mathbf{v}_d(P_N)\big](g) = 0 \quad \text{if } g \in \mathscr{X}(\mathbf{R}^n \sim \partial Q)\,; \quad \text{hence,} \quad \delta_{F^x} W = 0$$

and condition BC_x yields $\mu = \mathrm{Dirac}(T)$; thus,

$$\Phi_{F^x}(Q) < \vartheta \Phi_{F^x}(Q) = \Phi_{F^x}(W \llcorner (Q \times \mathbf{G}(n, d))) = \Phi_{F^x}(P) \le \Phi_{F^x}(Q)$$

which gives the desired contradiction. □

9.3 Remark. To prove that (P_N, Q) is a cubical test pair for each $N \in \mathbb{N}$ we had to employ a quite involved argument; see [DK18, §9]. Very roughly speaking the procedure works as follows.

(a) With the help of a deformation theorem we reduce the problem to the case when P is a cubical complex so that it is possible to apply tools of algebraic topology to P. In particular, we are using the *obstruction theory* (a sophisticated version of the *Brouwer fixed-point theorem*).

(b) Using induction it suffices to show that (P_1, Q) is a cubical test pair (because $P_2 = (P_1)_1$).

(c) Let $B = \partial Q$ and note that B is homeomorphic to the sphere $\mathbf{S}^{d-1}$.

(d) Using some simple topological arguments (uniform continuity of continuous functions on compact sets, homotopy extension property, the Tietze extension theorem etc.) we verify that

$$\begin{gathered} B \text{ is not a Lipschitz retract of } P \\ \textit{if and only if} \\ \text{there exists no continuous map } f : P \to B \text{ such that } \deg(f|B) = 1. \end{gathered}$$

Clearly $B \subseteq P$ and then $f|B : B \to B$ so the topological degree of $f|B$ makes sense.

(e) Employing obstruction theory, we prove that

$$\begin{gathered} \text{if } f, g : P \to B \text{ are continuous, } \deg(f|B) = d_1,\ \deg(g|B) = d_2, \\ \text{then there exits a continuous map } h : P \to B \text{ such that } \deg(h|B) = \gcd(d_1, d_2). \end{gathered}$$

(f) In consequence, we obtain

$$D = \big\{\deg(f|B) : f : P \to B\big\} = \big\{km : k \in \mathbf{Z}\big\}, \quad \text{where } m = \min D \cap \{e : e > 0\}$$

and $m > 1$ because B is not a retract of P; see (d).

(g) We observe that the set P_1 is homotopy equivalent to a *wedge sum* (a.k.a. *bouquet*) of 2^d copies of P; hence, for every map $f : P_1 \to B$ there exist maps $f_1, \ldots, f_{2^d} : P \to B$ such that

$$\deg(f|B) = \sum_{j=1}^{2^d} \deg(f_j|B) \in D\,;$$

thus, $\deg(f|B) \neq 1$ since $1 \notin D$. Recalling (d) we see that B is not a retract of P_1 and (P_1, Q) is a cubical test pair.

9.4 Remark. To fully appreciate the problem consider the following example. A *triple Möbius strip* is a topological space homeomorphic to the space $Y \times [0,1]/\sim$, where

$$\begin{gathered} Y = \mathbf{C} \cap \{z : |z| \leq 1,\ z^3 \in \mathbf{R}\}\,, \\ (z,t) \sim (w,s) \quad \text{if and only if} \quad t = 0\,, \quad s = 1\,, \quad z = w\exp(2\pi\mathbf{i}/3)\,. \end{gathered}$$

Assume $n \geq 3$ and $d = 2$. Take a Möbius strip M Lipschitz-embedded in $\mathbf{R}^n$ so that its boundary coincides with the boundary of the cube $Q_1 = [0,1]^2 \times \{0\}^{n-2}$. Let N be a triple Möbius strip Lipschitz-embedded in $\mathbf{R}^n$ so that its boundary coincides with the boundary of the cube $Q_2 = [-1,0] \times [0,1] \times \{0\}^{n-2}$. Assume $M \cap N = \{0\} \times [0,1] \times \{0\}^{n-2}$.

Both M and N can be retracted onto their "middle circles" and, thus, are homotopic to a circle $\mathbf{S}^1$. However, the inclusion $j : \partial M \hookrightarrow M$ has topological degree 2, so given any continuous map $f : M \to \partial M$ we have $j \circ f = f|\partial M : \partial M \to \partial M$ and we see that $\deg(f|\partial M) = \deg(j)\deg(f)$ is an *even* integer which means that $f|\partial M$ cannot equal the identity on ∂M. Therefore, (M, Q_1) is a cubical test pair. A similar argument shows that also (N, Q_2) is a cubical test pair.

Observe that $A = M \cup N$ is homeomorphic to the Adams' surface; see [Rei60, Example 8 on p. 81]. By contracting the line segment $M \cap N$ to a point we see that A has the homotopy type of the wedge sum $M \vee N \approx \mathbf{S}^1 \vee \mathbf{S}^1$. The inclusion of the boundary of M into M has degree 2, the inclusion of the boundary of N into N has degree 3, these numbers are relatively prime, and A is homotopy equivalent to the wedge sum of two circles so, defining $f : A \to \mathbf{S}^1$ to be of degree -1 on M and of degree 1 on N, we get a map such that $f \circ j$ is of degree one, where $j : \mathbf{S}^1 \to A$ is a parameterisation of the boundary of A. One can then construct a Lipschitz retraction of A onto its boundary; cf. 9.3(d). Luckily for us, the situation is different if one puts together many copies of *the same* set X. We proved that if (X, Q) is a cubical test pair, then one cannot have two maps $f, g : X \to \partial Q$ such that $\deg(f|\partial Q)$ and $\deg(g|\partial Q)$ are relatively prime.

9.5 Remark. Recalling 7.20, 7.8 and 7.14 it is clear that $\mathrm{BC}_x \not\subseteq \mathrm{AUE}_x$.

10 The anisotropic isoperimetric problem

In this section we assume $n = d + 1$.

10.1 Definition (cf. [Fed69, 4.5.12] and [AFP00, Def. 3.60])**.** Let $A \subseteq \mathbf{R}^n$. The *essential boundary* $\partial^* E$ *of* E is the set of points $x \in \mathbf{R}^n$ for which

$$\mathbf{\Theta}^{*n}(\mathscr{L}^n \llcorner E, x) > 0 \quad \text{and} \quad \mathbf{\Theta}^{*n}(\mathscr{L}^n \llcorner (\mathbf{R}^n \sim E), x) > 0 \,.$$

10.2 Remark. A set $E \subseteq \mathbf{R}^n$ is a *set of finite perimeter* if and only if $\mathscr{H}^d(\partial^* E) < \infty$; see [Fed69, 4.5.11]. Moreover, $\mathscr{H}^d(\partial^* E) < \infty$ implies that $\partial^* E$ is $(\mathscr{H}^d, d)$ rectifiable and $\mathbf{n}(E, x) \in \mathbf{S}^d$ for $\mathscr{H}^d$ almost all $x \in \partial^* E$; cf. [Fed69, 4.5.9(16), 4.5.6].

10.3. The *anisotropic isoperimetric problem* is about minimising the anisotropic perimeter under a fixed volume constraint. More precisely, we are given an integrand F which does not depend on the space variable, i.e., $F = F^0$, and we want to minimise $\Phi_F(\partial^* E)$ among all finite perimeter sets $E \subseteq \mathbf{R}^n$ under the constraint $\mathscr{L}^n(E) = 1$. In case F is continuous and elliptic, then a minimiser must be, up to translation, the Wulff shape; cf. [Tay75, Tay74, FM91, MS86, BM94, Wul01]. Hence, this problem is completely solved. However, it is interesting to ask what are the minimal assumptions on E that make it the Wulff shape. There are various variational and geometric characterisations of the round sphere and we would like to also have characterisations of the unit sphere in different norms.

Considering deformations of E by one-parameter families of diffeomorphisms preserving the volume one derives variational conditions satisfied by the minimiser; namely, $\partial^* E$ must have constant *anisotropic mean curvature*. Hence, we are led to study *critical points* of the anisotropic isoperimetric problem, i.e., sets having constant anisotropic mean curvature. In case F is the area integrand, this is the content of the *Alexandrov Rigidity Theorem*. He, Li, Ma, and Ge [HLMG09] proved that in case F is smooth and one knows, a priori, that ∂E is smooth, then E must be a finite union of Wulff shapes of the same radius (equal to the inverse of the anisotropic mean curvature divided by $n - 1$). For the case when ∂E is only piecewise smooth see [Pal12] and [Koi19]. Delgadino and Maggi [DM19] dropped regularity assumptions on ∂E, i.e., they admit all finite perimeter sets as competitors, but consider only the area integrand; see also [Mag18]. Recently Santilli [San19a] proved a bit more general theorem which includes the result of [DM19] but does not employ the Allard Regularity Theorem [All72].

In the paper [DKS19], we were able to solve the problem almost in full generality: we consider F of class $\mathscr{C}^3$ and assume that E is such that $\partial^* E$ has constant anisotropic mean curvature (defined in varifold sense) and $\mathscr{H}^{n-1}(\partial E \sim \partial^* E) = 0$. The last condition is currently hard to drop because of the fundamental problem with anisotropic integrands; namely, the lack of monotonicity formula and, consequently, the lack of the Allard Regularity Theorem or a second order rectifiability theorem; see 10.31.

10.4 Setup. In the sequel we shall always assume that F is an integrand over $\mathbf{R}^n$ of class $\mathscr{C}^3$, $F = F^0$, $\bar{F} : \mathbf{R}^n \to \mathbf{R}$ is given by

$$\bar{F}(v) = |v|\, F(0, \operatorname{span}\{v\}^{\perp}) \quad \text{for } v \in \mathbf{R}^n ,$$

and $\bar{F}$ is a uniformly convex norm.

The smooth case

10.5 Remark. Let $W \subseteq \mathbf{R}^n$ be open, ∂W be a submanifold of $\mathbf{R}^n$ of class $\mathscr{C}^2$, $x \in \partial W$. Observe that from 6.7 it follows that

$$\bar{F}(\mathbf{n}(W,x))\mathbf{h}_F(\partial W, x) = \operatorname{trace} \mathrm{D}\big[\operatorname{grad} \bar{F} \circ \mathbf{n}(W,\cdot)\big](x) \quad \text{for } x \in \partial W .$$

Moreover, $\mathrm{D}\big[\operatorname{grad} \bar{F} \circ \mathbf{n}(W,\cdot)\big](x) = \mathrm{D} \operatorname{grad} \bar{F}(\mathbf{n}(W,x)) \circ \mathrm{D}\mathbf{n}(W,\cdot)(x)$ is a composition of two self-adjoint maps; hence, $\mathrm{D}\big[\operatorname{grad} \bar{F} \circ \mathbf{n}(W,\cdot)\big](x)| \operatorname{Tan}(\partial W, x)$ has exactly d real eigenvalues (see [Lan87, Chap. VIII, Thm. 4.3] and [DKS19, 2.30]).

10.6 Definition. Let $W \subseteq \mathbf{R}^n$ be open, ∂W be a submanifold of $\mathbf{R}^n$ of class $\mathscr{C}^2$, $x \in \partial W$. We define the *anisotropic principal curvatures of W at x*

$$\kappa^F_{W,1}(x) \le \ldots \le \kappa^F_{W,d}(x) .$$

to be the eigenvalues of the map $\mathrm{D}\big[\operatorname{grad} \bar{F} \circ \mathbf{n}(W,\cdot)\big](x)| \operatorname{Tan}(\partial W, x)$.

10.7 Remark. Let $W = \mathbf{R}^n \cap \{x : \bar{F}^*(x) < r\}$. Then 2.14(i) yields

$$\operatorname{grad} \bar{F}(\mathbf{n}(W,x)) = \frac{x}{r} \quad \text{and} \quad \bar{F}(\mathbf{n}(W,x))\mathbf{h}_F(\partial W, x) = \frac{d}{r} \quad \text{for } x \in \partial W .$$

The converse is also true as the following lemma shows.

10.8 Lemma (cf. [DKS19, 3.2])**.** *Assume $M \subseteq \mathbf{R}^n$ is a connected d dimensional submanifold of class $\mathscr{C}^{1,1}$ satisfying* $\operatorname{Clos} M \sim M = \varnothing$*, $\nu : M \to \mathbf{R}^n$ is Lipschitz, $\nu(x) \in \operatorname{Nor}(M,x)$ and $|\nu(x)| = 1$ for $x \in M$, $\kappa : M \to \mathbf{R}$ is such that*

$$\mathrm{D}\big[\operatorname{grad} \bar{F} \circ \nu\big](x)u = \kappa(x)u \quad \textit{for } x \in M \textit{ and } u \in \operatorname{Tan}(M,x) . \tag{26}$$

Then there exists $\lambda \in \mathbf{R}$ such that $\kappa(x) = \lambda$ for $x \in M$. Moreover, either $\lambda = 0$ and M is a hyperplane, or $\lambda \neq 0$ and $M = \mathbf{R}^n \cap \{x : \bar{F}^*(x-a) = |\lambda|^{-1}\}$ for some $a \in \mathbf{R}^n$.

10.9 Exercise. Try proving 10.8. First represent M locally as a graph of some function and then derive PDE's from (26). Use the fact that Lipschitz functions are absolutely continuous.

10.10 Definition. Let $E \subseteq \mathbf{R}^n$ be a set of finite perimeter. Define the *F-perimeter of E* by

$$\mathcal{P}_F(E) = \int_{\partial^* E} \bar{F}(\mathbf{n}(E,x))\, \mathrm{d}\mathscr{H}^d(x)\,.$$

10.11 Exercise (cf. [DKS19, 6.7]). Assume $E \subseteq \mathbf{R}^n$ is a set of finite perimeter and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathcal{P}_F(h_t[E])}{\mathscr{L}^n(h_t[E])} = 0$$

whenever $\{h_t : t \in (-\varepsilon, \varepsilon)\}$ is a flow of some vectorfield $g \in \mathscr{X}(\mathbf{R}^n)$. Define $V = \mathbf{v}_d(\partial^* E)$. Show that $\|\delta_F V\|_{\mathrm{sing}} = 0$ and

$$\bar{F}(\mathbf{n}(E,x))\mathbf{h}_F(V,x) = -\frac{d}{d+1}\frac{\mathcal{P}_F(E)}{\mathscr{L}^n(E)}\mathbf{n}(E,x) \quad \text{for } \|V\| \text{ almost all } x\,.$$

10.12 Definition. Let $A \subseteq \mathbf{R}^n$ be closed. We define

(a) the *distance function* $\boldsymbol{\delta}_A^F : \mathbf{R}^n \to \mathbf{R}$ by $\boldsymbol{\delta}_A^F(x) = \inf\{\bar{F}^*(x-y) : y \in A\}$ for $x \in \mathbf{R}^n$;

(b) the set $\mathrm{Unp}^F(A)$ consisting of points $x \in \mathbf{R}^n$ for which there exists a *unique nearest point*, i.e., a point $a \in A$ such that $\boldsymbol{\delta}_A^F(x) = \bar{F}^*(x-a) < \bar{F}^*(x-b)$ for $b \in A \sim \{a\}$;

(c) the *nearest point projection* $\boldsymbol{\xi}_A^F : \mathrm{Unp}^F(A) \to A$ by requiring that $\bar{F}^*(x - \boldsymbol{\xi}_A^F(x)) = \boldsymbol{\delta}_A^F(x)$ for $x \in \mathrm{Unp}^F(A)$;

(d) the *F-normal vector* $\mathbf{n}^F(A,x) = \operatorname{grad} \bar{F}(\mathbf{n}(A,x))$ whenever $\mathbf{n}(A,x) \neq 0$.

(e) the *normal bundle* $N^F(A) = A \times \mathbf{R}^n \cap \{(a,u) : \boldsymbol{\delta}_A^F(a+su) = s \text{ for some } s > 0\}$.

If $F \equiv 1$ is the area integrand, we shall omit the superscript in the notation.

10.13 Exercise (cf. [DKS19, 2.42] and [Fed59, 4.8]). Let $A \subseteq \mathbf{R}^n$ be closed.

(a) If $x, y \in \mathbf{R}^n$, then $|\boldsymbol{\delta}_A^F(x) - \boldsymbol{\delta}_A^F(y)| \le \bar{F}^*(x-y)$.

(b) If $x \in \mathrm{Unp}^F(A)$ and $y \in \{tx + (1-t)\boldsymbol{\xi}_A^F(x) : t \in [0,1]\}$, then $y \in \mathrm{Unp}^F(A)$.

(c) If $a = \boldsymbol{\xi}_A^F(x)$ for some $x \in \mathrm{Unp}^F(A)$, then $x - a \in \operatorname{span}\{\operatorname{grad} \bar{F}^*[\mathrm{Nor}(A,a)]\}$.

(d) $\boldsymbol{\xi}_A^F$ is continuous.

(e) If $x \in \mathbf{R}^n \sim A$ and $a \in A$ are such that $\boldsymbol{\delta}_A^F(x) = \bar{F}^*(x-a)$, then

$$\boldsymbol{\delta}_A^F(a + t(x-a)) = t\bar{F}^*(x-a) = t\boldsymbol{\delta}_A^F(x) \quad \text{for } t \in (0,1]\,.$$

(f) If $x \in \mathbf{R}^n \sim A$ and $a \in A$ are such that $\boldsymbol{\delta}_A^F(x) = \bar{F}^*(x-a)$ and $\mathrm{D}\boldsymbol{\delta}_A^F(x)$ exists, then

$$\operatorname{grad} \bar{F}\big(\operatorname{grad} \boldsymbol{\delta}_A^F(x)\big) = \frac{x-a}{\boldsymbol{\delta}_A^F(x)}\,.$$

In particular, a is uniquely determined by the formula

$$a = x - \operatorname{grad} \bar{F}\big(\operatorname{grad} \boldsymbol{\delta}_A^F(x)\big)\boldsymbol{\delta}_A^F(x)\,; \quad \text{hence,} \quad x \in \mathrm{Unp}^F(A)\,.$$

(g) $\mathscr{L}^n(\mathbf{R}^n \sim \operatorname{Unp}^F(A)) = 0$.

(h) $\boldsymbol{\xi}_A^F$ is Lipschitz continuous. (*Hint.* $\bar{F}$ is a uniformly convex norm.)

10.14 Remark. Assume $\Omega \subseteq \mathbf{R}^n$ is open and connected, $\partial\Omega$ is a submanifold of $\mathbf{R}^n$ of class $\mathscr{C}^2$.

$$\mathscr{H}^d(\partial^*\Omega) < \infty\,, \quad H \in \partial\Omega \to \mathbf{R}^n\,, \quad V = \mathbf{v}_d(\partial^*\Omega)\,,$$
$$\|\delta_F V\|_{\mathrm{sing}} = 0\,, \quad \bar{F}(\mathbf{n}(\Omega,x))\mathbf{h}_F(V,x) = -H(x)\mathbf{n}(\Omega,x) \quad \text{for } \|V\| \text{ almost all } x\,,$$

Set $C = \mathbf{R}^n \sim \Omega$ and $Q = \partial C$. Observe that

$$0 \le \frac{H(x)}{d} \le -\kappa_{C,1}^F(y) \le \frac{1}{\boldsymbol{\delta}_C^F(y)} \quad \text{for } y \in \operatorname{Unp}^F(C)\,, \tag{27}$$

because the closed $\bar{F}^*$-ball $\mathbf{R}^n \cap \{x : \bar{F}^*(x-y) \le \boldsymbol{\delta}_C^F(y)\}$ touches C exactly at one point x; see [DKS19, 2.38]. Define

$$Z = Q \times \mathbf{R} \cap \big\{(x,t) : 0 < t \le -\kappa_{C,1}^F(x)^{-1}\big\}$$
$$\text{and} \quad \zeta : Z \to \mathbf{R}^n \quad \text{by} \quad \zeta(x,t) = x + t\mathbf{n}^F(C,x)\,.$$

Set

$$J_n\zeta(x,t) = \|{\textstyle\bigwedge_n}(\mathscr{H}^n \llcorner Z, n)\operatorname{ap}\mathrm{D}\zeta(x,t)\|\,.$$

Recalling 10.5 we may choose a basis $\tau_1,\dots,\tau_d$ of $\operatorname{Tan}(Q,x)$ such that

$$\big\langle \tau_i, \mathrm{D}\mathbf{n}^F(C,\cdot)(x)\big\rangle = \kappa_{C,i}^F(x)\,\tau_i \quad \text{for } i \in \{1,2,\dots,d\}\,, \quad |\tau_1 \wedge \dots \wedge \tau_d| = 1\,.$$

Then it is rather easy to verify that

$$\begin{aligned} J_n\zeta(x,t) &= \big|\mathrm{D}\zeta(x,t)(\tau_1,0) \wedge \dots \wedge \mathrm{D}\zeta(x,t)(\tau_d,0) \wedge \mathrm{D}\zeta(x,t)(0,1)\big| \\ &= F(\mathbf{n}(C,x))\prod_{i=1}^n\big(1 + t\,\kappa_{Q,i}^F(x)\big) \quad \textit{for } (x,t) \in Z\,. \end{aligned}$$

Recalling 10.13 and (27) we get

$$0 = \mathscr{L}^n(\Omega \sim \operatorname{Unp}^F(C)) = \mathscr{L}^n(\Omega \sim (\boldsymbol{\xi}_C^F)^{-1}[Q]) = \mathscr{L}^n(\Omega \sim \zeta[Z])\,.$$

Next, we use the, so called, *Montiel-Ros argument*; cf. [MR91].

$$\begin{aligned} \mathscr{L}^n(\Omega) \le \mathscr{L}^n(\zeta(Z)) &\le \int_{\zeta(Z)} \mathscr{H}^0(\zeta^{-1}(y))\,\mathrm{d}\mathscr{L}^n(y) = \int_Z J_n\zeta\,\mathrm{d}\mathscr{H}^n \\ &= \int_Q F(\mathbf{n}(C,x)) \int_0^{-1/\kappa_{C,1}^F(x)} \prod_{i=1}^d \big(1 + t\kappa_{C,i}^F(x)\big)\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^d(x)\,. \end{aligned} \tag{28}$$

The standard inequality between the arithmetic and the geometric mean yields

$$\begin{aligned} \mathscr{L}^n(\Omega) &\le \int_Q F(\mathbf{n}(C,x)) \int_0^{-1/\kappa_{C,1}^F(x)} \Big(\frac{1}{n}\sum_{i=1}^d \big(1 + t\kappa_{C,i}^F(x)\big)\Big)^d\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^d(x) \\ &\le \int_Q F(\mathbf{n}(C,x)) \int_0^{n/H(x)} \Big(1 - t\frac{H(x)}{d}\Big)^d\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^d(x) = \frac{d}{n}\int_{\partial\Omega} \frac{F(\mathbf{n}(C,x))}{H(x)}\,\mathrm{d}\mathscr{H}^d(x)\,. \end{aligned} \tag{29}$$

Thus, we arrive at a *Heintze-Karcher type inequality*; cf. [HK78] or [Ros87].

$$\mathscr{L}^n(\Omega) \le \frac{d}{d+1} \int_{\partial\Omega} \frac{1}{|\mathbf{h}_F(V,x)|} \, \mathrm{d}\mathscr{H}^d(x) \,.$$

Now, in case $\partial\Omega$ is a critical point of the anisotropic isoperimetric problem, then recalling 10.11 we get

$$\frac{d}{d+1} \int_{\partial\Omega} \frac{F(\mathbf{n}(C,x))}{H(x)} \, \mathrm{d}\mathscr{H}^d(x) = \frac{d}{d+1} \mathcal{P}_E(\Omega) \frac{d+1}{d} \frac{\mathscr{L}^n(\Omega)}{\mathcal{P}_E(\Omega)} = \mathscr{L}^n(\Omega) \,;$$

hence, all inequalities in (28) and (29) turn into equalities and we have

$$\mathscr{L}^{n+1}(\zeta(Z) \sim \Omega) = 0 \,, \tag{30}$$

$$\mathscr{H}^0(\zeta^{-1}(y)) = 1 \quad \text{for } \mathscr{L}^n \text{ almost all } y \in \zeta(Z) \,, \tag{31}$$

$$-\kappa^F_{C,j}(z)^{-1} = \frac{d}{H(z)} \quad \text{for } \mathscr{H}^d \text{ almost all } z \in Q \text{ and all } j = 1, \ldots, d \,. \tag{32}$$

This, in particular, means that $\mathrm{D}\big[\operatorname{grad} \bar{F} \circ \mathbf{n}(\Omega, \cdot)\big]$ is as in 10.8 (we may say that Q is *totally F-umbilical*) and we conclude that $\Omega = \mathbf{R}^n \cap \{x : \bar{F}^*(x-a) < r\}$ for some $a \in \mathbf{R}^n$ and $r \in (0, \infty)$.

The non-smooth case

10.15 Definition (cf. [DDH19, Definition 3.1])**.** We say that $Z \subseteq \Omega$ is an *(n,h)-set with respect to F* if Z is relatively closed in Ω and for any open set $N \subseteq \Omega$ such that $\partial N \cap \Omega$ is smooth and $Z \subseteq \operatorname{Clos} N$ there holds

$$F(\mathbf{n}(N,p))\mathbf{h}_F(\mathbf{v}_n(\partial N), p) \bullet \mathbf{n}(N,p) \ge -h \quad \text{for } p \in Z \cap \partial N \cap \Omega \,.$$

10.16 Definition (cf. [San19b])**.** Suppose $\Omega \subseteq \mathbf{R}^n$ is open and $A \subseteq \mathbf{R}^n$ is closed. We say that A satisfies the *d dimensional Lusin (N) condition in* Ω if and only if the following implication holds

$$S \subseteq A \cap \Omega, \quad \mathscr{H}^d(S) = 0 \quad \Longrightarrow \quad \mathscr{H}^d(N^F(A)|S) = 0 \,.$$

10.17 Remark. According to Schneider [Sch15] a typical (in the sense of Baire category) compact convex body in $\mathbf{R}^n$ does not satisfy the d-dimensional Lusin (N) condition. However, it turns out that (d,h) sets satisfy the Lusin (N) condition as the following theorem shows.

10.18 Theorem (cf. [DKS19, 4.4, 5.4])**.** *Suppose $\Omega \subseteq \mathbf{R}^n$ is open, $0 \le h < \infty$, A is an (d,h) subset of Ω with respect to F that is a countable union of sets with finite $\mathscr{H}^d$ measure. Then $N^F(A)$ satisfies the d dimensional Lusin (N) condition in Ω.*

10.19 Lemma (cf. [DKS19, 4.5])**.** *Assume $\Omega \subseteq \mathbf{R}^n$ is open,*

$$V \in \mathbf{V}_d(\Omega) \,, \quad \bar{F}(\overline{\mathbf{h}}_F(V,x)) \le h \quad \text{for } \|V\| \text{ almost all } x \,, \quad \|\delta_F V\|_{\text{sing}} = 0 \,.$$

Then $\operatorname{spt}\|V\|$ *is an (d,h) subset of Ω with respect to F.*

10.20 Corollary. *If V is as in 10.19 and, additionally,* $\operatorname{spt}\|V\|$ *is a countable union of sets with finite $\mathscr{H}^d$ measure, then* $\operatorname{spt}\|V\|$ *satisfies the Lusin (N) condition.*

10.21 Exercise. The proof of 10.19 in [DKS19, 4.5] is indirect and relies on [DDH19, 3.4]. Prove 10.19 directly by modifying [Whi10].

10.22 Definition. Let $A \subseteq \mathbf{R}^n$, $k \in \mathbb{N}$, $\alpha \in [0,1]$. We say that $x \in A$ is a $\mathscr{C}^{k,\alpha}$ *regular point of* A if there exists an open set $U \subseteq \mathbf{R}^n$ such that $x \in U$ and $A \cap U$ is a d-dimensional submanifold of $\mathbf{R}^n$ of class $\mathscr{C}^{k,\alpha}$.

10.23 Definition. Let $A \subseteq \mathbf{R}^n$ be closed. The *anisotropic reach of* A is defined by

$$\begin{aligned}\operatorname{reach}^F(A) &= \inf\bigl\{\sup\{r : \{x : \bar{F}^*(x-a) < r\} \subseteq \operatorname{dmn} \boldsymbol{\xi}_A^F\} : a \in A\bigr\}\\ &= \sup\bigl\{r : \{x : \boldsymbol{\delta}_A^F(x) < r\} \subseteq \operatorname{dmn} \boldsymbol{\xi}_A^F\bigr\}.\end{aligned}$$

10.24 Definition. Assume $A \subseteq \mathbf{R}^n$ is closed. We define $\boldsymbol{\nu}_A^F : \operatorname{Unp}^F(A) \sim A \to \{x : \bar{F}^*(x) = 1\}$ and $\boldsymbol{\psi}_A^F : \operatorname{Unp}^F(A) \sim A \to A \times \{x : \bar{F}^*(x) = 1\}$ by the formulas

$$\boldsymbol{\nu}_A^F(z) = \boldsymbol{\delta}_A^F(z)^{-1}(z - \boldsymbol{\xi}_A^F(z)) \quad \text{and} \quad \boldsymbol{\psi}_A^F(z) = (\boldsymbol{\xi}_A^F(z), \boldsymbol{\nu}_A^F(z)) \quad \text{for } z \in \operatorname{Unp}^F(A) \sim A\,.$$

Sets of positive anisotropic reach can be detected by testing the following version of *Steiner formula.*

10.25 Theorem (cf. [DKS19, 5.9])**.** *Assume $A \subseteq \mathbf{R}^n$ is closed. Let $r > 0$ and suppose that for every $\mathscr{H}^d$ measurable bounded function $f : \mathbf{R}^n \times \{x : \bar{F}^*(x) = 1\} \to \mathbf{R}$ with compact support there are numbers $c_1(f), \dots, c_n(f) \in \mathbf{R}$ such that*

$$\int_{\mathbf{R}^n \sim A} f \circ \boldsymbol{\psi}_A^F \cdot \mathbb{1}_{\{x : \boldsymbol{\delta}_A^F(x) \le t\}} \, d\mathscr{L}^n = \sum_{j=1}^n c_j(f) t^j \quad \textit{for } 0 < t < r\,.$$

Then $\operatorname{reach}^F(A) \ge r$*.*

10.26 Exercise. Prove 10.25 by modifying [HHL04, Theorem 3].

10.27 Theorem (cf. [All86, The Regularity Theorem, p. 27])**.** *Assume*

$$\begin{gathered}\alpha \in (0,1)\,, \quad H \in \mathbf{R}\,, \quad U \subseteq \mathbf{R}^n \textit{ is open}\,,\\ V \in \mathbf{IV}_d(U)\,, \quad \boldsymbol{\Theta}^d(\|V\|, x) = 1 \quad \textit{for } \|V\| \textit{ almost all } x\,, \quad \|\delta_F V\| \le H\|V\|\,,\\ \textit{if } B \subseteq U \textit{ and } \|V\|(B) = 0, \textit{ then } \mathscr{H}^d(\operatorname{spt}\|V\| \cap B) = 0\,.\end{gathered}$$

Then $\mathscr{H}^d$ almost all $x \in \operatorname{spt}\|V\|$ are $\mathscr{C}^{1,\alpha}$ regular points of $\operatorname{spt}\|V\|$.

10.28 Remark. The crucial assumption, that cannot be easily dismissed, is that $\mathscr{H}^d \llcorner \operatorname{spt}\|V\|$ is absolutely continuous with respect to $\|V\|$. This is because of the *lack of the monotonicity formula* (see [All72, 5.1(1)] and [All74]) in case of anisotropic integrands.

10.29 Theorem (cf. [DKS19])**.** *Assume*

$$\begin{gathered}\Omega \subseteq \mathbf{R}^n \textit{ is open and connected}\,, \quad \mathscr{H}^d(\partial^*\Omega) < \infty\,, \quad \alpha \in (0,1)\,, \quad C \in (0,\infty)\,,\\ \mathscr{H}^d(\partial\Omega \sim \partial^*\Omega) = 0\,,\\ H : \partial^*\Omega \to [0, C] \textit{ is locally of class } \mathscr{C}^{1,\alpha} \textit{ on the } \mathscr{C}^{1,\alpha} \textit{ regular part of } \partial^*\Omega\,,\\ V = \mathbf{v}_d(\partial^*\Omega)\,, \quad \|\delta_F V\|_{\mathrm{sing}} = 0\,,\\ \bar{F}(\mathbf{n}(\Omega, x))\mathbf{h}_F(V, x) = -H(x)\mathbf{n}(\Omega, x) \quad \textit{for } \|V\| \textit{ almost all } x\,,\end{gathered} \tag{33}$$

Then $\Omega = \mathbf{R}^n \cap \{x : \bar{F}^*(x-a) < r\}$ for some $a \in \mathbf{R}^n$ and $r \in (0,\infty)$.

Sketch of the proof. We define

$$C = \mathbf{R}^n \sim \Omega\,, \quad Q = \partial C \cap \{x : x \text{ is a } \mathscr{C}^{2,\alpha}\text{-regular point of } \partial C\}\,.$$

(a) We first prove a Heintze-Karcher type inequality; cf. [HK78].

$$\mathscr{L}^n(\Omega) \le \frac{d}{d+1} \int_{\partial^*\Omega} \frac{1}{|\mathbf{h}_F(V,x)|}\, \mathrm{d}\mathscr{H}^d(x)\,. \tag{34}$$

The proof can be done as in 10.14 given

$$\mathscr{L}^n(\Omega \sim (\boldsymbol{\xi}_A^F)^{-1}[Q]) = 0\,. \tag{35}$$

To prove (35) we employ *standard regularity theory* for codimension one varifolds with bounded anisotropic mean curvature, i.e., theorem 10.27 together with [Fed69, 5.2.15]. This is the first point, where we need (33). We deduce that $\mathscr{H}^d$ almost all of $\partial^*\Omega$ is $\mathscr{C}^{2,\alpha}$ regular; hence,

$$\mathscr{H}^d(\partial C \sim Q) = 0\,.$$

Next, we use the *weak maximum principle* 10.19 together with 10.18 and, once again, (33) (see 10.20) to get

$$\mathscr{H}^d(\{x : \boldsymbol{\delta}_C^F(x) = r\} \sim (\boldsymbol{\xi}_C^F)^{-1}[Q]) = 0 \quad \text{for each } r \in (0,\infty)\,.$$

Recalling 10.13 we see that $F(\operatorname{grad} \boldsymbol{\delta}_C^F(x)) = 1$ for $x \in \operatorname{dmn} \mathrm{D}\boldsymbol{\delta}_C^F$; thus, there exits $C \in (0,\infty)$ depending only on F such that $|\operatorname{grad} \boldsymbol{\delta}_C^F(x)| \ge C$ for $x \in \operatorname{dmn} \mathrm{D}\boldsymbol{\delta}_C^F$. The coarea formula then yields

$$\begin{aligned} \frac{1}{C}\mathscr{L}^n(\Omega \sim (\boldsymbol{\xi}_C^F)^{-1}[Q]) &\le \int_{\Omega \sim (\boldsymbol{\xi}_C^F)^{-1}[Q]} |\operatorname{grad} \boldsymbol{\delta}_C^F(x)|\, \mathrm{d}\mathscr{L}^n(x) \\ &= \int_0^\infty \mathscr{H}^d\big(\{x : \boldsymbol{\delta}_C^F(x) = r\} \sim (\boldsymbol{\xi}_C^F)^{-1}[Q]\big)\, \mathrm{d}r = 0\,. \end{aligned}$$

(b) We assume that equality holds in (34) to get (30), (31), and (32). At this point we deduce that each point of the regular part Q is totally umbilical but we cannot conclude the proof as in 10.14 because we have no control of the position of different components of Q with respect to each other. To remedy this problem we consider level-sets of the anisotropic distance function $\boldsymbol{\delta}_C^F$

$$S^F(C,r) = \{x : \boldsymbol{\delta}_C^F(x) = r\}\,, \quad \text{where } r > 0\,.$$

Since $\boldsymbol{\xi}_C^F$ is Lipschitz continuous we immediately deduce from 10.13 that for $\mathscr{L}^1$ almost all $r > 0$ the set $S^F(C,r)$ is a *submanifold of* Ω *of class* $\mathscr{C}^{1,1}$. We check validity of the Steiner formula 10.25 to argue that C has positive F-reach which implies that

$$C_r = \{x : \boldsymbol{\delta}_C^F(x) \le r\} \subseteq \operatorname{Unp}^F(C) \quad \text{for some } r \in (0,1)\,. \tag{36}$$

We define

$$T = Q \cap \{x : \kappa_{C,j}^F(x) = -H(x)/d \text{ for } j = 1,2,\dots,d\}\,.$$

Again using the Lusin (N) property for ∂C we see that

$$\mathscr{H}^d(S^F(C,r) \sim (\boldsymbol{\xi}_C^F)^{-1}[T]) = 0\,.$$

Recall the definition of ζ from 10.14. From (36) we deduce that

$$\sigma = \boldsymbol{\xi}_C^F | S^F(C,r) \cap (\boldsymbol{\xi}_C^F)^{-1}[T] \quad \text{and} \quad \varphi = \zeta | T \times \{r\}$$

are well defined and inverse to each other. This allows us to compute

$$\begin{gathered}
\big\langle u,\, \mathrm{D}\varphi(x) \big\rangle = (1 - rH(x)/d)u \quad \text{for } x \in T \text{ and } u \in \operatorname{Tan}(T,x)\,, \\
\big\langle u,\, \mathrm{D}\sigma(z) \big\rangle = (1 - rH(\boldsymbol{\xi}_C^F(z))/d)^{-1} u \quad \text{for } z \in \operatorname{dmn}\sigma \text{ and } u \in \operatorname{Tan}(T, \boldsymbol{\xi}_C^F(z))\,, \\
\mathrm{D}\mathbf{n}^F(C_r,\cdot)(z)u = \frac{-H(\boldsymbol{\xi}_C^F(z))}{d - rH(\boldsymbol{\xi}_C^F(z))} u \\
\text{for } \mathscr{H}^d \text{ almost all } z \in S^F(C,r) \text{ and } u \in \operatorname{Tan}(T, \boldsymbol{\xi}_C^F(z))\,.
\end{gathered}$$

In consequence, we may use 10.8 to deduce that $S^F(C,r) = \mathbf{R}^n \cap \{x : \bar{F}^*(x-a) = \rho\}$ for some $a \in \mathbf{R}^n$ and $\rho \in (0,\infty)$ and conclude the proof by letting $r \to 0$.

□

10.30 Remark. We needed (33) to enable the use of 10.27 and to get $\mathscr{C}^{2,\alpha}$ regularity at $\mathscr{H}^d$ almost all points of $\partial\Omega$. This was necessary to be able to compute $\mathbf{h}_F(V,\cdot)$ by means of the formula 6.7. For this point it would suffice to have second order rectifiability V plus locality of the anisotropic mean curvature vector; see 10.31.

However, there is another point in the proof where the assumption (33) kicks in. We are using the Lusin (N) condition which is a consequence of being a (d,h) set but only if $\partial\Omega$ is a countable union of sets with finite $\mathscr{H}^d$ measure.

10.31 Conjecture. Assume $V \in \mathbf{IV}_d(\mathbf{R}^n)$, $H \in \mathbf{R}$, and $\|\delta_F V\| \le H\|V\|$. Then there exist a countable collection $\mathcal{A}$ of $\mathscr{C}^2$ submanifolds of $\mathbf{R}^n$ of dimension d, such that $\|V\|(\mathbf{R}^n \sim \bigcup \mathcal{A}) = 0$. Moreover,

$$\mathbf{h}_F(M,x) = \mathbf{h}_F(V,x) \quad \text{for } M \in \mathcal{A} \text{ and } \|V| \text{ almost all } x \in M\,.$$

10.32 Conjecture. Assume $V \in \mathbf{IV}_d(\mathbf{R}^n)$, $H \in \mathbf{R}$, and $\|\delta_F V\| \le H\|V\|$. Then $\operatorname{spt}\|V\|$ is a countable union of sets having finite $\mathscr{H}^d$ measure.

10.33 Remark. Proving 10.32 might actually be not easier than proving some kind of monotonicity formula for V which, for the time being, is the *Holy Grail* of geometric measure theory.

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