# Higher dimensional Menger curvature as a tool for proving regularity of sets

Sławomir Kolasiński

Institute of Mathematics
University of Warsaw

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# Outline

# The Menger curvature

## Definition (Menger 1930)

The Menger curvature of three points $x$, $y$ and $z$ in $\mathbb{R}^n$ is given by the formula

$$c(x,y,z) := \frac{1}{R(x,y,z)} = \frac{4\mathcal{H}^2(\blacktriangle(x,y,z))}{|x-y||y-z||z-x|},$$

where $R(x,y,z)$ is the radius of a smallest circle passing through the points $x$, $y$ and $z$ and $\blacktriangle(x,y,z)$ denotes the convex hull of the set $\{x,y,z\}$.

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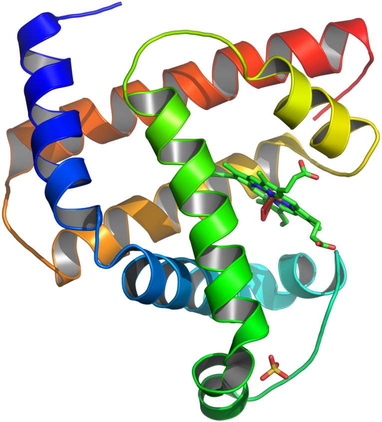
*Note:* Since it is defined in terms of distances and measures, it may be studied on very general metric, measure spaces!

# Motivation

More recently the Menger curvature turned out to be a useful tool (see Banavar et al. 2003 and Sutton, Balluffi, 1997) for modeling long, entangled objects like DNA molecules, protein structures or polymer chains.



*"The goal is to find analytically tractable notion of thickness for curves that does not rely on additional smoothness assumptions."*

[Strzelecki et al. 2010]

# Thickness for curves

Let $\gamma : S^1 \to \mathbb{R}^3$ be a continuous, rectifiable curve and let $\Gamma : S_L = \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ be its arclength parameterization.

## Definition (Gonzalez and Maddocks, 1999)

The thickness of a curve $\gamma$ is defined by

$$\Delta[\gamma] := \inf\{R(\Gamma(s), \Gamma(t), \Gamma(\sigma)) : s, t, \sigma \in S_L\} .$$

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**Theorem (Gonzalez et al. 2003)**

$\Delta[\gamma]$ *is positive if and only if the arclength parameterization is injective of class* $C^{1,1} \simeq W^{2,\infty}$.

# Thickness in variational problems

## Theorem (Gonzalez et al. 2002)

*The minimization problem*

$$\int_I |\gamma'| \to Min!\,, \quad \gamma \in W^{1,q}\,, q \in (1,\infty)\,, I = [a,b]\,,$$

*with the constraints*

$$\gamma(a) = \gamma(b)\,, \qquad \Delta[\gamma] > \theta\,,$$

$$\gamma(I) \text{ isotopic to some fixed reference curve } \tilde{\gamma}(I)\,,$$

*has a solution* $\gamma_*$ *and the arclength parameterization* $\Gamma_* \in C^{1,1}$.

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This proves the existence of so called *ideal knots*, which minimize the ratio of the length to the thickness!

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   - Curve thickness
   - Integral curvatures for curves
2. Higher dimensional Menger type curvatures
   - Dimension 2
   - Arbitrary dimension and codimension

# “Soft” curve energies

Strzelecki, Szumańska and von der Mosel suggested a different approach. The authors studied “soft” knot energies in the form of an integral of Menger curvature in some power.

## Definition

$$\mathscr{M}_p(\gamma) := \int_{S_L}\int_{S_L}\int_{S_L} \frac{ds\, dt\, d\sigma}{R(\Gamma(s),\Gamma(t),\Gamma(\sigma))^p},$$

$$\mathscr{S}_p(\gamma) := \int_{S_L}\int_{S_L} \frac{ds\, dt}{\inf_{\sigma\in S_L} R(\Gamma(s),\Gamma(t),\Gamma(\sigma))^p},$$

$$\mathscr{U}_p(\gamma) := \int_{S_L} \frac{ds}{\inf_{t,\sigma\in S_L} R(\Gamma(s),\Gamma(t),\Gamma(\sigma))^p}.$$

# Morrey-Sobolev imbeddings

**Theorem (Strzelecki, Szumańska and von der Mosel, 2008)**

*If the curve $\gamma$ satisfies*

$$\mathscr{S}_p(\gamma) = \int_{S_L}\int_{S_L} \frac{ds\,dt}{\inf_{\sigma\in S_L} R(\Gamma(s),\Gamma(t),\Gamma(\sigma))^p} < \infty$$

*for some $p \in (2,\infty]$ then the arclength parameterization $\Gamma$ is injective and of class $C^{1,1-\frac{2}{p}}$.*

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An analogue of the following Morrey-Sobolev imbedding

$$W^{2,p}(\mathbb{R}^2) \subset C^{1,1-\frac{2}{p}}(\mathbb{R}^2)\,, \quad \text{where} \quad p > 2\,.$$

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*for some $p \in (3,\infty]$ and the arclength parameterization $\Gamma$ is a local homeomorphism, then $\Gamma \in C^{1,1-\frac{3}{p}}$ and the image $\Gamma(S_L)$ is diffeomorphic to the circle $S^1$.*

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An analogue of

$$W^{2,p}(\mathbb{R}^3) \subset C^{1,1-\frac{3}{p}}(\mathbb{R}^3)\,, \quad \text{whenever} \quad p > 3\,.$$

# Application in variational problems

Let $L > 0$ and let $k$ be some fixed closed curve. We set

$$C_{L,k} := \left\{ \gamma \in C^0(S^1, \mathbb{R}^3) : \begin{array}{c} \text{length}(\gamma) = L \\ \text{and } \gamma \text{ is isotopic to } k \end{array} \right\} .$$

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**Theorem (Strzelecki, Szumańska and von der Mosel, 2007)**

*Let $p > 2$. In any given isotopy class represented by a closed curve $k$ there is an arclength parameterized curve $\Gamma \in C^{1,(p-2)/(p+4)}(S_L, \mathbb{R}^3) \cap C_{L,k}$ such that*

$$\mathscr{I}_p(\Gamma) = \int_{S_L} \int_{S_L} \frac{ds\, dt}{\inf_{\sigma \in S_L} R(\Gamma(s), \Gamma(t), \Gamma(\sigma))^p} = \inf_{\gamma \in C_{L,k}} \mathscr{I}_p(\gamma) .$$

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**Theorem (Strzelecki, Szumańska and von der Mosel, 2008)**

*Let $p > 3$. In any given isotopy class represented by a closed curve k there is an arclength parameterized curve $\Gamma \in C^{1,(p-3)/(p+6)}(S_L, \mathbb{R}^3) \cap C_{L,k}$ such that*

$$\mathscr{M}_p(\Gamma) = \int_{S_L} \int_{S_L} \int_{S_L} \frac{ds\, dt\, d\sigma}{R(\Gamma(s), \Gamma(t), \Gamma(\sigma))^p} = \inf_{\gamma \in C_{L,k}} \mathscr{M}_p(\gamma) .$$

# Digression: Total Menger curvature in dimension 1

**Remark.** One can define Menger curvature of 1-dimensional Borel sets. $\mathscr{M}_2(E)$ was used to characterize removable singularities of bounded analytic functions (David, Melnikov, Tolsa, Verdera, ...).

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**Theorem (David, Léger)**

*E is countably rectifiable if and only if*

$$\mathscr{M}_2(E) = \int_E \int_E \int_E \frac{1}{R^2(x,y,z)}\, d\mathcal{H}^1_x\, d\mathcal{H}^1_y\, d\mathcal{H}^1_z < \infty\,.$$

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Similar criteria of rectifiabilty of $d$-dimensional subsets of Hilbert spaces: Lerman, Whitehouse (2008)

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# Wrong generalization

## Definition

For any 4 points $x, y, z, \xi$ in $\mathbb{R}^n$ let us define

$$K_R(x, y, z, \xi) = R(x, y, z, \xi)^{-1} ,$$

where $R(x, y, z, \xi)$ is the radius of the smallest sphere passing through the points $x$, $y$, $z$ and $\xi$.

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## Example

Choose three vectors $v_1$, $v_2$, $v_3$ in the plane $\mathbb{R}^2$ such that each two of them span $\mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x) = \langle x, v_1^\perp \rangle \langle x, v_2^\perp \rangle \langle x, v_3^\perp \rangle$ and let $M = \text{graph}(f) \subset \mathbb{R}^3$. Then $M$ is a smooth, embedded manifold and one can easily find points $x, y, z, \xi \in M$ such that $K_R(x, y, z, \xi)$ is arbitrary big.

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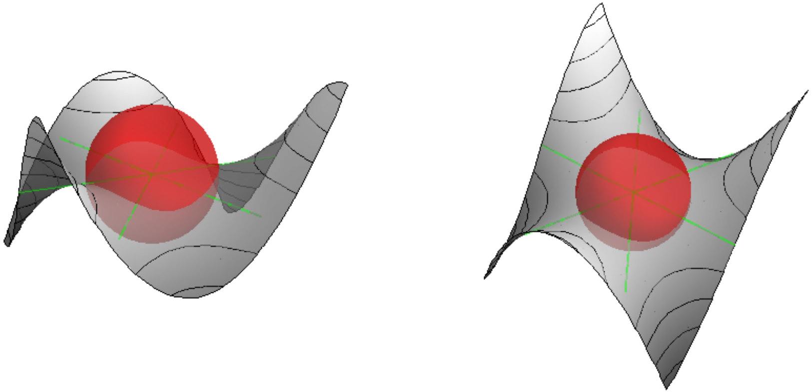

$$f(x) = \langle x, v_1^{\perp} \rangle \langle x, v_2^{\perp} \rangle \langle x, v_3^{\perp} \rangle$$

# Menger curvature for surfaces

## Definition (Strzelecki and von der Mosel, 2010)

The discrete curvature of a set of four points $x, y, z, \xi$ in $\mathbb{R}^3$ is

$$\mathcal{K}_{SvdM}(x, y, z, \xi) = \frac{\mathrm{Volume}(\blacktriangle(x, y, z, \xi))}{\mathrm{Area}(\blacktriangle(x, y, z, \xi))\,\mathrm{diam}(\{x, y, z, \xi\})^2}\,.$$

## Definition

Let $\Sigma \subset \mathbb{R}^3$ be any compact, 2-dimensional set. We define

$$\mathscr{M}_p(\Sigma) = \int_\Sigma \int_\Sigma \int_\Sigma \int_\Sigma \mathcal{K}_{SvdM}^p(x, y, z, \xi)\, d\mathcal{H}^2_x\, d\mathcal{H}^2_y\, d\mathcal{H}^2_z\, d\mathcal{H}^2_\xi\,.$$

# Regularity

**Theorem (Strzelecki and von der Mosel, 2010)**

*Any closed, compact and connected Lipschitz surface* $\Sigma$ *in* $\mathbb{R}^3$ *with* $\mathscr{M}_p(\Sigma) \le E < \infty$ *for some* $p > 8$ *is an orientable manifold of class* $C^{1,1-(8/p)}$.
*Moreover there exist constants* $R = R(E,p)$ *and* $C = C(E,p)$ *such that for each* $x \in \Sigma$ *the set* $\Sigma \cap \mathbb{B}(x,R)$ *is a graph of some function* $f$ *which satisfies*

$$|Df(y) - Df(z)| \le C|y-z|^{1-\frac{8}{p}} .$$

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*Note:* Since $R$ and $C$ depend only on $E$ and $p$, this theorem is useful for proving compactness results for the class of surfaces with uniformly bounded energy.

# Variational applications

Let $M_g$ be a surface of genus $g$ smoothly embedded in $\mathbb{R}^3$. Consider the classes $\mathcal{C}_E(M_g)$ and $\mathcal{C}_A(M_g)$ of closed, compact and connected Lipschitz surfaces $\Sigma \subset \mathbb{R}^3$ ambiently isotopic to $M_g$ with $\mathscr{M}_p(\Sigma) \le E$ or $\mathcal{H}^m(\Sigma) \le A$ respectively.

**Theorem (Strzelecki and von der Mosel, 2010)**

*For each $g \in \mathbb{N}$, $E > 0$ and each fixed reference surface $M_g$ the class $\mathcal{C}_E(M_g)$ contains a surface of least area.*

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**Theorem (Strzelecki and von der Mosel, 2010)**

*For each $g \in \mathbb{N}$, $E > 0$, there exists a surface $\Sigma \in \mathcal{C}_A(M_g)$ with*

$$\mathscr{M}_p(\Sigma) = \inf_{\mathcal{C}_E(M_g)} \mathscr{M}_p \, .$$

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# Discrete curvature and the *p*-energy

Let $0 < m < n$ and let $x_0, \ldots, x_{m+1}$ be some points in $\mathbb{R}^n$.

## Definition

The discrete curvature of $T = (x_0, \ldots, x_{m+1})$ is given by

$$\mathcal{K}(T) := \frac{\mathcal{H}^{m+1}(\blacktriangle T)}{\operatorname{diam}(T)^{m+2}}\,.$$

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## Definition

Let $\Sigma \subset \mathbb{R}^n$ be any $m$-dimensional set. We define the $p$-energy of $\Sigma$

$$\mathscr{E}_p(\Sigma) = \int_{\Sigma^{m+2}} \mathcal{K}(T)^p \, d\mu(T) ,$$

where $\mu = \mathcal{H}^m \otimes \cdots \otimes \mathcal{H}^m$.

# The class of sets under consideration

We are considering a restricted class of *m-fine sets*, which satisfy some mild conditions regarding their structure. These are compact, $m$-dimensional[1] subsets of $\mathbb{R}^n$ *without holes*. To formalize what we mean by "without holes" we use the notions of $\beta$ and $\theta$ numbers[2] introduced by Peter Jones.

[1] More precisely: we only need to have a lower bound on the measure of $\Sigma \cap \mathbb{B}(x, r)$ for $r$ small enough.

[2] $\theta$ numbers are also called *bilateral* $\beta$ numbers.

# Examples of fine sets

**Example**

Let $M$ be an $m$-dimensional, compact, closed and smooth manifold. Let $f : M \to \mathbb{R}^n$ be an immersion. Then the image $\Sigma = f(M)$ is an $m$-fine set. Any finite union of such immersions is also an $m$-fine set.

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Let $M$ be an $m$-dimensional, compact, closed and smooth manifold. Let $f : M \to \mathbb{R}^n$ be bi-Lipschitz. Then the image $\Sigma = f(M)$ is an $m$-fine set.

**Example**

Let $\Sigma \subset \mathbb{R}^2$ be the Koch snowflake. Then $\Sigma \in \mathcal{F}(1)$.

# Regularity result for $\mathscr{E}_p$

**Theorem (K. 2011)**

*Let $\Sigma \in \mathcal{F}(m)$ be an m-fine set such that $\mathscr{E}_p(\Sigma) \le E < \infty$ for some $p > m(m+2)$. Then there exists a constant $R > 0$ such that for each $x \in \Sigma$ the set $\Sigma \cap \mathbb{B}(x, R)$ is a graph of some function $F_x \in C^{1,\alpha}(T_x\Sigma, T_x\Sigma^{\perp})$, where $\alpha = 1 - \frac{m(m+2)}{p}$. Moreover the radius $R$ and the Hölder norm of $DF_x$ depend only on $E$, $m$ and $p$.*

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*Note:* As in the case of $2D$ surfaces, this also gives hope for compactness results for the class of surfaces with uniformly bounded energy.

# Conjecture

## Definition

$$\mathcal{A}_{p,E} := \{\Sigma \in \mathcal{F}(m) : \mathscr{E}_p(\Sigma) \leq E\,, 0 \in \Sigma \text{ and } \mathcal{H}^m(\Sigma) \leq 1\}.$$

## Conjecture

Let $E > 0$ and $p > m(m+2)$. There exist a constant $N = N(E, m, p)$ and $N$ sets $\Sigma_1, \ldots, \Sigma_N$ in $\mathcal{A}_{p,E}$ such that each other set $\Sigma \in \mathcal{A}_{p,E}$ is $C^1$-diffeomorphic to one of the sets $\Sigma_i$ for some $i$.

# Intermediate energies

## Definition

Let $k \in \{1, 2, \dots, m+2\}$. We set

$$\mathscr{E}_p^k := \int_{\Sigma^k} \sup_{x_k, \dots, x_{m+1}} K(x_0, \dots, x_{m+1})^p \, d\mathcal{H}^{mk}_{x_0, \dots, x_{k-1}} .$$

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## Theorem ([K., Strzelecki, von der Mosel, 2011])

*Let $p > m$. Then $\mathscr{E}_p^1(\Sigma) < \infty$ if and only if $\Sigma$ is locally a graph of a $W^{2,p}$ function.*

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## Theorem ([Blatt, K. 2011])

*Let $k \geq 2$ and $p > mk$ and $s = 1 - \frac{m(k-1)}{p}$. Then $\mathscr{E}_p^k(\Sigma) < \infty$ if and only if $\Sigma$ is locally a graph of a $W^{1+s,p}$ function.*

# Work in progress, plans

1. Finiteness theorems for $C^1$ manifolds with ‘energy bounds’.
2. Optimal shapes and higher regularity of minimizers. . . ?
3. Menger curvature in metric spaces and on varifolds.
4. Flows.

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