Integral Menger curvature for sets of arbitrary dimension and codimension

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Outline

The Menger curvature

Menger curvature is a notion introduced by Karl Menger back in 1930s [Menger, 1930]

Definition

The Menger curvature of three points x, y and z in $\mathbb{R}^n$ is given by the formula

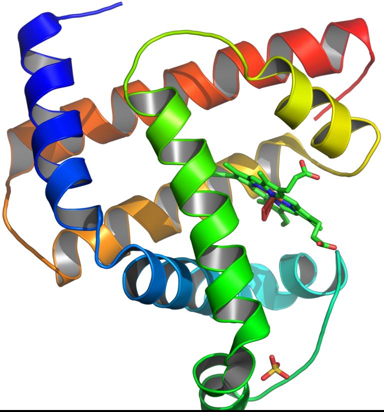
$$c(x,y,z) := \frac{1}{R(x,y,z)} = \frac{4\mathcal{H}^2(\blacktriangle(x,y,z))}{|x-y||y-z||z-x|},$$

where $R(x,y,z)$ is the radius of a smallest circle passing through the points x, y and z and $\blacktriangle(x,y,z)$ denotes the convex hull of the set $\{x,y,z\}$.

Since it is defined in terms of distances and measures, it may be studied on an arbitrary metric, measure space.

Motivation

More recently the Menger curvature turned out to be a useful tool (see [Banavar et al. 2003] and [Sutton, Balluffi. 1997]) for modeling long, entangled objects like DNA molecules, protein structures or polymer chains.



"The goal is to find analytically tractable notion of thickness for curves that does not rely on additional smoothness assumptions."

Thickness for curves

Let $\gamma : S^1 \to \mathbb{R}^3$ be a continuous, rectifiable curve and let $\Gamma : S_L = \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ be its arclength parameterization.

Definition (Gonzalez and Maddocks. 1999)

The thickness of a curve γ is defined by

$$\Delta[\gamma] := \inf\{R(\Gamma(s), \Gamma(t), \Gamma(\sigma)) : s, t, \sigma \in S_L\} .$$

Using this definition we obtain

Theorem (Gonzalez, Maddocks, Schuricht, von der Mosel. 2003)

$\Delta[\gamma]$ *is positive if and only if the arclength parameterization is injective of class* $C^{1,1} \simeq W^{2,\infty}$.

Thickness in variational problems

Theorem (Gonzalez et al. 2002)

The minimization problem

$$\int_I |\gamma'| \to Min! \,, \quad \gamma \in W^{1,q} \,, q \in (1,\infty) \,, I = [a,b] \,,$$

with the constraints

$$\gamma(a) = \gamma(b) \,, \qquad \Delta[\gamma] > \theta \,,$$

$$\gamma(I) \text{ isotopic to some fixed reference curve } \tilde{\gamma}(I) \,,$$

has a solution γ_* *and the arclength parameterization* $\Gamma_* \in C^{1,1}$.

This proves the existence of so called *ideal knots*, which minimize the ratio of the length to the thickness.

Outline

1. One dimensional curvatures
   - Curve thickness
   - Integral curvatures for curves
2. Higher dimensional Menger type curvatures
   - Dimension 2
   - Arbitrary dimension and codimension

“Soft” curve energies

Strzelecki, Szumańska and von der Mosel suggested a different approach. The authors studied “soft” knot energies in the form of an integral of Menger curvature in some power.

Definition

$$\mathscr{M}_p(\gamma) := \int_{S_L}\int_{S_L}\int_{S_L} \frac{ds\, dt\, d\sigma}{R(\Gamma(s),\Gamma(t),\Gamma(\sigma))^p},$$

$$\mathscr{I}_p(\gamma) := \int_{S_L}\int_{S_L} \frac{ds\, dt}{\inf_{\sigma\in S_L} R(\Gamma(s),\Gamma(t),\Gamma(\sigma))^p},$$

$$\mathscr{U}_p(\gamma) := \int_{S_L} \frac{ds}{\inf_{t,\sigma\in S_L} R(\Gamma(s),\Gamma(t),\Gamma(\sigma))^p}.$$

Morrey-Sobolev imbeddings

Theorem (Strzelecki, Szumańska and von der Mosel. 2008)

If the curve γ satisfies

$$\mathscr{S}_p(\gamma) = \int_{S_L} \int_{S_L} \frac{ds\, dt}{\inf_{\sigma \in S_L} R(\Gamma(s), \Gamma(t), \Gamma(\sigma))^p} < \infty$$

for some $p \in (2, \infty]$ then the arclength parameterization Γ is injective and of class $C^{1,1-\frac{2}{p}}$.

This is an analogue of the following

$$W^{2,p}(\mathbb{R}^2) \subset C^{1,1-\frac{2}{p}}(\mathbb{R}^2) \quad \text{if} \quad p > 2\,.$$

Morrey-Sobolev imbeddings

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for some $p \in (3, \infty]$ and the arclength parameterization Γ is a local homeomorphism, then $\Gamma \in C^{1,1-\frac{3}{p}}$ and the image $\Gamma(S_L)$ is diffeomorphic to the circle S^1.

This is an analogue of the following

$$W^{2,p}(\mathbb{R}^3) \subset C^{1,1-\frac{3}{p}}(\mathbb{R}^3) \quad \text{if} \quad p > 3\,.$$

Application in variational problems

Let $L > 0$ and let k be some fixed closed curve. We set $C_{L,k} := \{\gamma \in C^0(S^1, \mathbb{R}^3) : \text{length}(\gamma) = L \text{ and } \gamma \text{ is isotopic to } k\}$.

Theorem (Strzelecki, Szumańska and von der Mosel. 2007)

Let $p > 2$. In any given isotopy class represented by a closed curve k there is an arclength parameterized curve $\Gamma \in C^{1,(p-2)/(p+4)}(S_L, \mathbb{R}^3) \cap C_{L,k}$ such that $\mathscr{S}_p(\Gamma) = \inf_{\gamma \in C_{L,k}} \mathscr{S}_p(\gamma)$.

Theorem (Strzelecki, Szumańska and von der Mosel. 2008)

Let $p > 3$. In any given isotopy class represented by a closed curve k there is an arclength parameterized curve $\Gamma \in C^{1,(p-3)/(p+6)}(S_L, \mathbb{R}^3) \cap C_{L,k}$ such that $\mathscr{M}_p(\Gamma) = \inf_{\gamma \in C_{L,k}} \mathscr{M}_p(\gamma)$.

Outline

Wrong generalization

Definition

For any 4 points x, y, z, ξ in $\mathbb{R}^n$ let us define

$$K_R(x, y, z, \xi) = R(x, y, z, \xi)^{-1},$$

where $R(x, y, z, \xi)$ is the radius of the smallest sphere passing through the points x, y, z and ξ.

Example

Choose three vectors v_1, v_2, v_3 in the plane $\mathbb{R}^2$ such that each two of them span $\mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(v) = \langle v, v_1^\perp \rangle \langle v, v_2^\perp \rangle \langle v, v_3^\perp \rangle$ and let $M = \mathrm{graph}(f) \subset \mathbb{R}^3$. Then M is a smooth, embedded manifold and one can easily find points $x, y, z, \xi \in M$ such that $K_R(x, y, z, \xi)$ is arbitrary big.

Menger curvature for surfaces

Definition (Strzelecki and von der Mosel. 2010)

The discrete curvature of a set of four points x, y, z, ξ in $\mathbb{R}^3$ is

$$\mathcal{K}_{SvdM}(x, y, z, \xi) = \frac{\text{Volume}(\blacktriangle(x, y, z, \xi))}{\text{Area}(\blacktriangle(x, y, z, \xi))\,\text{diam}^2\{x, y, z, \xi\}} .$$

Definition

Let $\Sigma \subset \mathbb{R}^3$ be any compact, 2-dimensional set. We define

$$\mathscr{M}_p(\Sigma) = \int_\Sigma \int_\Sigma \int_\Sigma \int_\Sigma \mathcal{K}^p_{SvdM}(x, y, z, \xi)\, d\mathcal{H}^2_x\, d\mathcal{H}^2_y\, d\mathcal{H}^2_z\, d\mathcal{H}^2_\xi .$$

Regularity

Theorem (Strzelecki and von der Mosel. 2010)

Any closed, compact and connected Lipschitz surface Σ in $\mathbb{R}^3$ with $\mathscr{M}_p(\Sigma) \leq E < \infty$ for some $p > 8$ is an orientable $C^{1,1-(8/p)}$-manifold with local graph representations whose domain size is controlled solely in terms of E and p.

Variational applications

Let M_g be a surface of genus g smoothly embedded in $\mathbb{R}^3$. Consider the class $\mathcal{C}_E(M_g)$ of closed, compact and connected Lipschitz surfaces $\Sigma \subset \mathbb{R}^3$ ambiently isotopic to M_g with $\mathscr{M}_p(\Sigma) \le E$.

Theorem (Strzelecki and von der Mosel. 2010)

For each $g \in \mathbb{N}$, $E > 0$ and each fixed reference surface M_g the class $\mathcal{C}_E(M_g)$ contains a surface of least area.

Theorem (Strzelecki and von der Mosel. 2010)

For each $g \in \mathbb{N}$, $A > 0$, there exists a surface $\Sigma \in \mathcal{C}_A(M_g)$ with

$$\mathscr{M}_p(\Sigma) = \inf_{\mathcal{C}_A(M_g)} \mathscr{M}_p\,.$$

Outline

Discrete curvature and the p-energy

Let $0 < m < n$ and let $x_0, \ldots, x_{m+1}$ be some points in $\mathbb{R}^n$. Set $T = (x_0, \ldots, x_{m+1})$.

Definition

The discrete curvature of T is given by

$$\mathcal{K}(T) := \frac{\mathcal{H}^{m+1}(\blacktriangle T)}{\operatorname{diam}(T)^{m+2}}\,.$$

Definition

Let $\Sigma \subset \mathbb{R}^n$ be any m-dimensional set. We define the p-energy of Σ

$$\mathscr{E}_p(\Sigma) = \int_{\Sigma^{m+2}} \mathcal{K}(T)^p \, d\mu(T)\,,$$

where $\mu = \mathcal{H}^m \otimes \cdots \otimes \mathcal{H}^m$.

Flatness coefficients

Definition

$$\beta(x,r) := \frac{1}{r} \inf_{H \in G(n,m)} \sup_{z \in \Sigma \cap B(x,r)} \operatorname{dist}(z, x+H)\,,$$

$$\theta(x,r) := \frac{1}{r} \inf_{H \in G(n,m)} d_{\mathcal{H}}(\Sigma \cap B(x,r), (x+H) \cap B(x,r))\,.$$

Here $G(n,m)$ denotes the collection of all m-dimensional, linear subspaces of $\mathbb{R}^n$ and $d_{\mathcal{H}}$ stands for the Hausdorff distance.

β and θ numbers

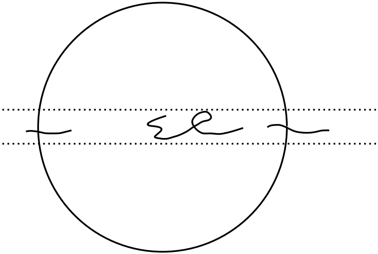

Figure: Small β and large θ. Small β numbers assure our set is flat, but it may have holes.

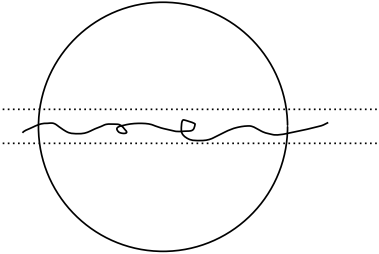

Figure: Small β and small θ.

The class of sets under consideration

Definition

Let $\Sigma \subseteq \mathbb{R}^n$ be a compact set. We call Σ an *m-fine set* and write $\Sigma \in \mathcal{F}(m)$ if there exist constants $A_\Sigma > 0$, $R_\Sigma > 0$ and $M_\Sigma \geq 2$ such that

1. **(Ahlfors regularity)** for all $x \in \Sigma$ and all $r \leq R_\Sigma$ we have
$$\mathcal{H}^m(\Sigma \cap B(x,r)) \geq A_\Sigma r^m \quad \text{and}$$
2. **(control of gaps)** for each $x \in \Sigma$ and each $r \leq R_\Sigma$ we have
$$\theta(x,r) \leq M_\Sigma \beta(x,r)\,.$$

Examples of fine sets

Example

Let M be an m-dimensional, compact, closed and smooth manifold. Let $f : M \to \mathbb{R}^n$ be an immersion. Then the image $\Sigma = f(M)$ is an m-fine set. Any finite union of such immersions is also an m-fine set.

Example

Let M be an m-dimensional, compact, closed and smooth manifold. Let $f : M \to \mathbb{R}^n$ be bi-Lipschitz. Then the image $\Sigma = f(M)$ is an m-fine set.

Example

Let $\Sigma \subset \mathbb{R}^2$ be the Koch snowflake. Then $\Sigma \in \mathcal{F}(1)$.

The main result

Theorem (K. 2011)

Let $\Sigma \in \mathcal{F}(m)$ be an m-fine set such that $\mathscr{E}_p(\Sigma) \le E < \infty$ for some $p > m(m+2)$. Then there exists a constant $R > 0$ such that that for each $x \in \Sigma$ the set $\Sigma \cap B(x, R)$ is a graph of some function $F_x \in C^{1,\alpha}(T_x\Sigma, T_x\Sigma^{\perp})$, where $\alpha = 1 - \frac{m(m+2)}{p}$. Moreover the radius R and the Hölder norm of DF_x depend only on E, m and p.

Conjectures

Definition

$$\tilde{K}(x) := \sup_{x_1,\dots,x_{m+1}} K(x, x_1, \dots, x_{m+1}),$$

$$\tilde{\mathscr{E}}_p := \int_\Sigma \tilde{K}(x)^p \, d\mathcal{H}^m_x .$$

Conjecture

If $\tilde{\mathscr{E}}_p(\Sigma) < \infty$ for some $p > m$, then Σ is locally a graph of a $W^{2,p}$ function.

Conjectures

Definition

$$\mathcal{A}_{p,E} := \{\Sigma \in \mathcal{F}(m) : \mathscr{E}_p(\Sigma) \le E\,, 0 \in \Sigma \text{ and } \mathcal{H}^m(\Sigma) \le 1\}.$$

Conjecture

Let $E > 0$ and $p > m(m+2)$. There exist a constant $N = N(E, m, p)$ and N sets $\Sigma_1, \ldots, \Sigma_N$ in $\mathcal{A}_{p,E}$ such that each other set $\Sigma \in \mathcal{A}_{p,E}$ is homeomorphic to one of the sets Σ_i for some i.

Thanks

Thank you for your attention

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but this is not the end.

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