New and old solutions to a generalised Plateau's problem

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Outline

$$D = \mathbf{R}^2 \cap \mathbf{B}(0,\ 1) \quad \text{a unit disc in } \mathbf{R}^2\,,$$

$$f : D \to \mathbf{R}^n\,, \qquad A(f) = \int_D Jf \, \mathrm{d}\mathcal{L}^2\,,$$

$$\Gamma \subseteq \mathbf{R}^n \quad \text{a closed Jordan curve}$$

Problem

Minimise $A(f)$ among all $f : D \to \mathbf{R}^n$, of suitable class, under the constraint

$$f[\partial D] = \Gamma\,.$$

- R. Garnier, *Ann. Sci. École Norm. Sup.*, 1928
- T. Radó, *Math. Z.*, 1930
- J. Douglas, *Trans. Amer. Math. Soc.*, 1931

$$B \subseteq \mathbf{R}^n \quad \text{a compact set}\,, \qquad G \quad \text{an abelian group}\,,$$
$$\check{\mathbf{H}}_k(B;G) \quad \text{the } k^{\text{th}} \text{ Čech homology group}\,,$$
$$\mathcal{H}^k(B) \quad \text{the } k \text{ dimensional Haudorff measure}\,,$$
$$\tau \in \check{\mathbf{H}}_{m-1}(B;G)$$

Problem

Minimise $\mathcal{H}^m(S)$ among compact sets $S \subseteq \mathbf{R}^n$ such that

$$i_*\tau = 0\,,$$

where $i_* : \check{\mathbf{H}}_{m-1}(B;G) \to \check{\mathbf{H}}_{m-1}(S \cup B;G)$ is induced by the inclusion map $i : B \hookrightarrow S \cup B$.

- E. R. Reifenberg and J. F. Adams, *Acta Math.*, 1960
- F. Almgren, *Ann. of Math.*, 1968

$$\mathbf{I}_k(\mathbf{R}^n) \quad k \text{ dimensional integral currents, i.e.,}$$

$$S \in \mathbf{I}_k(\mathbf{R}^n) \quad \Longleftrightarrow \quad S(\phi) = \int_E \langle \tau(x), \phi(x) \rangle \theta(x) \, \mathrm{d}\mathcal{H}^k(x) \,,$$

$$\mathrm{Mass}(S) = \int_E \theta(x) \, \mathrm{d}\mathcal{H}^k(x) \,, \quad \mathrm{Size}(S) = \mathcal{H}^k(\{x \in E : \theta(x) \neq 0\}) \,,$$

$$B \in \mathbf{I}_{m-1}(\mathbf{R}^n) \,, \partial B = 0$$

Problem (Mass minimisers)

Minimise $\mathrm{Mass}(S)$ *among* $S \in \mathbf{I}_m(\mathbf{R}^n)$ *such that* $\partial S = B$.

Problem (Size minimisers – unsolved)

Minimise $\mathrm{Size}(S)$ *among* $S \in \mathbf{I}_m(\mathbf{R}^n)$ *such that* $\partial S = B$.

- H. Federer and W. Fleming, *Ann. of Math.*, 1960

1. transversal intersections (parameterised solutions)
2. interior smoothness everywhere (mass minimisers)
3. choice of orientation (currents, homology)
4. choice of the coefficient group (currents, homology)
5. Reifenberg's solution only for compact groups

SEE:

- G. David, Should we solve Plateau's problem again?, in *Advances in analysis: the legacy of Elias M. Stein*, 108–145, Princeton Math. Ser., 50, Princeton Univ. Press, Princeton, NJ, 2014
- J. Harrison, H. Pugh, *Plateau's Problem: What's Next*, arXiv:1509.03797 [math.AP], 2015

$$B \subseteq \mathbf{R}^n \quad \text{a compact set}\,,$$
$$\mathcal{C} \subseteq \mathcal{C}_B = \{\gamma : \mathbf{S}^{n-m} \hookrightarrow \mathbf{R}^n \sim B : \gamma \text{ is a smooth embedding}\}$$
$$\text{if } \gamma \in \mathcal{C} \text{ and } \widetilde{\gamma} \in \mathcal{C}_B \text{ is smoothly isotopic to } \gamma, \text{ then } \widetilde{\gamma} \in \mathcal{C}\,,$$
$$\mathcal{F}(B,\mathcal{C}) = \left\{ K \subseteq \mathbf{R}^n : \begin{array}{l} K \text{ is compact } (\mathcal{H}^m, m) \text{ rectifiable} \\ K \cap \operatorname{im}\gamma \neq \varnothing \quad \text{for } \gamma \in \mathcal{C} \end{array} \right\}$$

Problem

Minimise $\mathcal{H}^m(S)$ *among* $S \in \mathcal{F}(B,\mathcal{C})$.

- J. Harrison and H. Pugh, *J. Geom. Anal.*, 2012
- C. De Lellis, F. Ghiraldin, F. Maggi, *J. Eur. Math. Soc. (JEMS)*, 2016
- G. De Philippis, A. De Rosa, F. Ghiraldin, *Adv. Math.*, 2016

$$B, K_0 \subseteq \mathbf{R}^n \quad \text{compact sets}\,,$$

$$\Sigma(B) = \left\{ \Phi(1,\cdot) : \begin{array}{c} \Phi : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n \text{ is continuous}\,, \\ \Phi(0,\cdot) = \mathrm{id}_{\mathbf{R}^n}\,, \quad \mathrm{Lip}\,\Phi(1,\cdot) < \infty\,, \\ \Phi(t,\cdot)[B] \subseteq B \text{ for } t \in [0,1] \end{array} \right\},$$

$$\mathcal{A}(B,K) = \big\{ \varphi[K_0] : \varphi \in \Sigma(B) \big\}$$

PROBLEM (PARTIALLY SOLVED)

Minimise $\mathcal{H}^m(S)$ among $S \in \mathcal{A}(B, K_0)$.

- G. David, *Princeton Math. Ser.*, 2014
- G. De Philippis, A. De Rosa, F. Ghiraldin, *Adv. Math.*, 2016
- C. De Lellis, F. Ghiraldin, F. Maggi, *J. Eur. Math. Soc. (JEMS)*, 2016

For a compact set $B \subseteq \mathbf{R}^n$ define axiomatically a class $\mathcal{P}(B)$ of competitors without referring to any particular notion of *boundary* or *spanning* and minimise $\mathcal{H}^m(S)$ among $S \in \mathcal{P}(B)$.

Usually the class $\mathcal{P}(B)$ is assumed to be closed under certain deformations (like sliding deformations) which are used in the proofs of existence of minimisers.

- C. De Lellis, F. Ghiraldin, F. Maggi, *J. Eur. Math. Soc. (JEMS)*, 2016
- G. De Philippis, A. De Rosa, F. Ghiraldin, *Adv. Math.*, 2016
- J. Harrison and H. Pugh, arXiv:1603.04492 [math.AP], 2016
- Y. Fang, arXiv:1310.4690 [math.CA], 2013

$$F : \mathbf{R}^n \times \mathbf{G}(n,m) \to (0,\infty) \quad \text{continuous}\,,$$
$$S = S_{\mathrm{reg}} \cup S_{\mathrm{irr}} \subseteq \mathbf{R}^n \quad \text{compact with } \mathcal{H}^m(S) < \infty\,,$$
$$\Phi_F(S) = \int_{S_{\mathrm{reg}}} F(x, \mathrm{Tan}(S_{\mathrm{reg}}, x))\, \mathrm{d}\mathcal{H}^m(x) + \mathcal{H}^m(S_{\mathrm{irr}})\,,$$
$$x \in \mathbf{R}^n \quad \Rightarrow \quad F^x(y,T) = F(x,T)\,.$$

DEFINITION (F. ALMGREN, *Ann. of Math.*, 1968)

We say that F is *elliptic* at $x \in \mathbf{R}^n$ if: there exists $C > 0$ such that given any disc $D \subseteq L$ lying in an affine m-plane $L \subseteq \mathbf{R}^n$ and a compact set $S \subseteq \mathbf{R}^n$ such that $\partial D \subseteq S$ is not a deformation retract of S, there holds

$$\Phi_{F^x}(S) - \Phi_{F^x}(D) \geq C\big(\mathcal{H}^m(S) - \mathcal{H}^m(D)\big)\,.$$

EXAMPLE

The AREA INTEGRAND: $F(x,T) = 1$ for $(x,T) \in \mathbf{R}^n \times \mathbf{G}(n,m)$. Then $\Phi_F(S) = \mathcal{H}^m(S)$.

Problem

Given an elliptic integrand F, and a compact set $B \subseteq \mathbf{R}^n$, and a good class of competitors $\mathcal{P}(B)$ "spanning B", minimise $\Phi_F(S)$ among $S \in \mathcal{P}(B)$.

- J. Harrison and H. Pugh, arXiv:1603.04492 [math.AP], 2016
- Y. Fang, arXiv:1310.4690 [math.CA], 2013
- C. De Lellis, A. De Rosa, F. Ghiraldin, arXiv:1602.08757 [math.AP], 2016

$$B \subseteq \mathbf{R}^n \text{ compact } (\mathcal{H}^m, m) \text{ rectifiable}, \quad U = \mathbf{R}^n \sim B,$$

$$\mathfrak{D}(U) = \left\{ \varphi \in \mathscr{C}^1(\mathbf{R}^n, \mathbf{R}^n) : \begin{array}{l} \exists a, r \quad \mathbf{B}(a, r) \subseteq U, \\ \varphi[\mathbf{B}(a, r)] \subseteq \mathbf{B}(a, r), \\ \varphi(x) = x \text{ for } x \in \mathbf{R}^n \sim \mathbf{B}(a, r) \end{array} \right\}$$

Theorem (to be optimised)

Assume F is an elliptic integrand and $\mathcal{P}(B)$ is a family of compact subsets of $\mathbf{R}^n$ such that

- *if $S \in \mathcal{P}(B)$ and $\varphi \in \mathfrak{D}(U)$, then $\varphi[S] \in \mathcal{P}(B)$;*
- *if $S_i \in \mathcal{P}(B)$ and $S_i \xrightarrow{\text{HD}} S$, then $S \in \mathcal{P}(B)$.*

Then there exists $S \in \mathcal{P}(B)$ such that

$$\Phi_F(S) \leq \Phi_F(R) \quad \textit{for } R \in \mathcal{P}(B).$$

Example

Let $\mathcal{P}(B)$ be the family of compact sets S which span some $\tau \in \check{\mathbf{H}}_{m-1}(B;G)$ in the sense of Reifenberg, i.e.,

$$i_*\tau = 0 \qquad \text{where } i : B \hookrightarrow S \cup B\,.$$

Then there exists a minimiser $R \in \mathcal{P}(B)$ of Φ_F.
Moreover, R is compact and $(\mathcal{H}^m, m)$ rectifiable.

1. Chose a minimising sequence $S_i \in \mathcal{P}(B)$ so that $\Phi_F(S_i) \to \inf\{\Phi_F(R) : R \in \mathcal{P}(B)\}$.
2. Define varifolds $V_i = \mathbf{v}(S_i)$ (Radon measures over $\mathbf{R}^n \times \mathbf{G}(n, m)$), so that

$$\Phi_F(S_i) = V_i(F) = \int F(x, T)\, \mathrm{d}V_i(x, T)\,.$$

3. Take the varifold limit $V_i \to V$ (weak limit of measures).
4. We get $V(F) = \inf\{\Phi_F(R) : R \in \mathcal{P}(B)\}$ for free!
5. Modify the sequence so that $S_i \xrightarrow{\text{HD}} S$ and $S \in \mathcal{P}(B)$ is such that $\mathcal{H}^m(S \sim \operatorname{spt} \|V\|) = 0$ (hair combing).
6. Show that $\operatorname{spt} \|V\|$ is $(\mathcal{H}^m, m)$ rectifiable and $V = \mathbf{v}(\operatorname{spt} \|V\|)$.
7. Then $V = \mathbf{v}(S)$ and $\Phi_F(S) = V(F) = \inf\{\Phi_F(R) : R \in \mathcal{P}(B)\}$.

$$\boldsymbol{\Theta}^m(\|V\|, x, r) = \frac{\|V\|\mathbf{B}(x, r)}{\boldsymbol{\alpha}(m)r^m}, \quad \boldsymbol{\Theta}^m(\|V\|, x) = \lim_{r \downarrow 0} \boldsymbol{\Theta}^m(\|V\|, x, r).$$

1. There exists $C_1, C_2 \in \mathbf{R}$ such that

   $$0 < C_1 \leq \boldsymbol{\Theta}^m(\|V\|, x, r) \leq C_2 < \infty$$

   for $x \in \operatorname{spt}\|V\| \cap U$ and $r \in (0, \operatorname{dist}(x, \mathbf{R}^n \sim U))$.
   Consequently

   $$\|V\| \llcorner U \approx \mathcal{H}^m \llcorner (\operatorname{spt}\|V\| \cap U).$$

2. Minimality of V then yields $(\mathcal{H}^m, m)$ rectifiability of $\operatorname{spt}\|V\|$.
3. Ellipticity of F gives then $\boldsymbol{\Theta}^m(\|V\|, x) = 1$ for $x \in U$ such that $\operatorname{Tan}(\operatorname{spt}\|V\|, x)$ is an m plane.
4. Area formula proves $V = \mathbf{v}(\operatorname{spt}\|V\|)$.

1. If $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ is of class $\mathscr{C}^1$, then $\varphi_\# : \mathbf{V}_m(\mathbf{R}^n) \to \mathbf{V}_m(\mathbf{R}^n)$ is continuous.
2. If $K \subseteq \mathbf{R}^n$ is compact, then the mass $\mathbf{M}(V) := V(\mathbf{R}^n)$ is continuous on $\{V \in \mathbf{V}_m(\mathbf{R}^n) : \operatorname{spt} \|V\| \subseteq K\}$.
3. Hence, $\|V\|\mathbf{B}(a, r) \approx \mathcal{H}^m(\phi[S_i \cap \mathbf{B}(a, r)])$ whenever $\{x : \varphi(x) \neq x\} \subseteq \mathbf{B}(a, r)$.
4. Smooth deformation of S_i onto m dimensional skeleton of a finite cubical complex.
5. Slicing theory for varifolds.

Definition (Almgren 1986)

Let $\mathcal{F}$ be a collection of n dimensional dyadic cubes in $\mathbf{R}^n$. We say that $\mathcal{F}$ is admissible if

- if $K, L \in \mathcal{F}$ and $K \neq L$, then $\operatorname{Int} K \cap \operatorname{Int} L = \varnothing$,
- if $K, L \in \mathcal{F}$ and $K \cap L \neq \varnothing$, then $\frac{1}{2} \leq \operatorname{side} L / \operatorname{side} K \leq 2$,
- if $K \in \mathcal{F}$, then $\partial K \subseteq \bigcup\{L \in \mathcal{F} : L \neq K\}$.

Example

1. Whitney cubes associated to an open set $U \subseteq \mathbf{R}^n$.
2. Tiling of $\mathbf{R}^n$ with isometric cubes.

Lemma

Let $Q = [-1,1]^n \subseteq \mathbf{R}^n$ *and* $a \in \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ *and* $\varepsilon > 0$. *There exists a* $\mathscr{C}^\infty$ *smooth map* $\varphi : \mathbf{R}^n \sim \{a\} \to \mathbf{R}^n$ *such that*

- $\varphi[Q \sim \{a\}] = \partial Q$,
- $\varphi(x) = x$ *whenever* $\operatorname{dist}(x, Q) \geq \varepsilon$,
- $\varphi[F] = F$ *for any face* F *of* Q *with* $\dim F < n$,
- $\|D\varphi(x)\| \leq C(n)|x-a|^{-1}$ *for* $x \in Q \sim \{a\}$.
- $|\varphi(x) - \varphi(y)| \leq (1+\varepsilon)|x-y|$ *for* $x, y \in \partial Q + \mathbf{B}(0, \varepsilon)$.
- φ *is smoothly isotopic to the identity on* $\mathbf{R}^n$.

Theorem

Assume $\mathcal{F}$ is admissible, and $\mathcal{A} \subseteq \mathcal{F}$ is finite, and $\Sigma \subseteq \mathbf{R}^n$ is compact, and $\mathcal{H}^m(\Sigma) < \infty$, and $\varepsilon > 0$. Set $G = \bigcup \mathcal{A} + \mathbf{U}(0, \varepsilon)$. There exists a $\mathscr{C}^\infty$ smooth map $f : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ such that

- $f(0, \cdot) = \mathrm{id}_{\mathbf{R}^n}$,
- $f(t, x) = x$ *for* $t \in [0,1]$ *and* $x \in \mathbf{R}^n \sim G$.
- $\operatorname{im} f(1, \cdot)$ *is a sum of* m *dimensional faces of cubes from* $\mathcal{A}$
- $\mathcal{H}^m(f(1,\cdot)[\Sigma \cap G]) < C(n,m)\mathcal{H}^m(\Sigma \cap G)$.
- *if* $\delta = \max\{\mathrm{side}\, Q : Q \in \mathcal{A}\}$ *and* Σ *is* $(\mathcal{H}^m, m)$ *rectifiable, then*

$$\mathcal{H}^{m+1}(f[[0,1] \times (\Sigma \cap G)]) < C(n,m)\delta\mathcal{H}^m(\Sigma \cap G)\,.$$

1. We can assume all S_i lie in a fixed compact set $K \subseteq \mathbf{R}^n$.
2. Take the Whitney cubical complex $\mathcal{W}$ associated with $\mathbf{R}^n \sim \operatorname{spt} \|V\|$.
3. For each $i \in \mathbb{N}$ apply the deformation theorem to S_i choosing only these cubes $Q \in \mathcal{W}$ for which $C_{\mathrm{dt}} \mathcal{H}^m(S_i \cap Q) < (\operatorname{side} Q)^{m-1}$.
4. We obtain a deformed competitors $\tilde{S}_i \in \mathcal{P}(B)$ which, far from $\operatorname{spt} \|V\|$, lie in the $(m-1)$ dimensional skeleton of a fixed cubical complex $\mathcal{W}$.
5. Choose a subsequence which converges in the Hausdorff metric on compact sets to some $S \subseteq \mathbf{R}^n$.
6. We obtain $\mathcal{H}^m(S \sim \operatorname{spt} \|V\|) = 0$.

Lemma (Almgren 1976)

Let $\rho : \mathbf{R}^n \to \mathbf{R}$ be proper of class $\mathscr{C}^1$ and $V \in \mathbf{V}_m(\mathbf{R}^n)$. For $\mathcal{L}^1$ almost all $t \in \mathbf{R}$ there exists a <u>*varifold slice*</u> *$\langle V, \rho, t\rangle \in \mathbf{V}_{m-1}(\mathbf{R}^n)$ characterised by*

$$\langle V, \rho, t\rangle(\beta) = \lim_{r \downarrow 0} \frac{\mu_\beta(\mathbf{B}(t, r))}{\mathcal{L}^1(\mathbf{B}(t, r))} \quad \textit{for } \beta \in \mathscr{C}^0_{\mathrm{c}}(\mathbf{R}^n \times \mathbf{G}(n, m-1)),$$

where

$$\mu_\beta(\varphi) = (V, \rho)(\beta \cdot (\varphi \circ \rho \circ \pi)),$$
$$(V, \rho)(\beta) = \int \beta(x, S \cap \ker D\rho(x)) \| D\rho(x) \circ S_\natural \| \, \mathrm{d}V(x, S)$$

for $\beta \in \mathscr{C}^0_{\mathrm{c}}(U \times \mathbf{G}(n, m-1))$ and $\varphi \in \mathscr{C}^0_{\mathrm{c}}(\mathbf{R})$.

Example

If $S \subseteq \mathbf{R}^n$ is $(\mathcal{H}^m, m)$ rectifiable and $\mathcal{H}^m$ measurable and $V = \mathbf{v}(S)$, then

$$\langle V, \rho, t \rangle = \mathbf{v}(S \cap \rho^{-1}[\{t\}]) \quad \text{for } \mathcal{L}^1 \text{ almost all } t\,.$$

For $t \in \mathbf{R}$ define $s : \mathbf{R} \to \mathbf{R}$ and $K_{\delta,t} : \mathbf{R}^n \to [0,1] \times \mathbf{R}^n$ by

$$s(r) = 0 \text{ if } r \leq 0\,, \quad s(r) = 1 \text{ if } r \geq 1\,, \quad s(r) = r \text{ if } r \in (0,1)\,,$$
$$K_{\delta,t}(x) = (s((t - \rho(x))/\delta), x)\,.$$

Lemma

Let $V \in \mathbf{RV}_m(\mathbf{R}^n)$. For $\mathcal{L}^1$ almost all $t \in \mathbf{R}$

$$\lim_{\delta \downarrow 0} K_{\delta,t\,\#} V = i_{0\,\#} V_0 + i_{1\,\#} V_1 + \mathbf{v}([0,1]) \times \langle V, \rho, t \rangle \in \mathbf{V}_m(\mathbf{R}^{n+1})\,,$$

where

$$V_0 = V \llcorner \{(x,S) \in \mathbf{R}^n \times \mathbf{G}(n,m) : \rho(x) \geq t\}\,,$$
$$V_1 = V \llcorner \{(x,S) \in \mathbf{R}^n \times \mathbf{G}(n,m) : \rho(x) < t\}\,,$$
$$i_t(x) = (t,x)\,.$$

1. There exists $C_1, C_2 \in \mathbf{R}$ such that
$$0 < C_1 \le \mathbf{\Theta}^m(\|V\|, x, r) \le C_2 < \infty$$
for $x \in \operatorname{spt}\|V\| \cap U$ and $r \in (0, \operatorname{dist}(x, \mathbf{R}^n \sim U))$.
2. Minimality of V then yields $(\mathcal{H}^m, m)$ rectifiability of $\operatorname{spt}\|V\|$.
3. Ellipticity of F gives then $\mathbf{\Theta}^m(\|V\|, x) = 1$ for $x \in U$ such that $\operatorname{Tan}(\operatorname{spt}\|V\|, x)$ is an m plane.
4. Area formula proves $V = \mathbf{v}(\operatorname{spt}\|V\|)$.

Lemma

Let $a \in \operatorname{spt}\|V\|$ *and* $\iota \in (0,\infty)$. *Set* $M_a(r) = \|V\|\mathbf{B}(a,r)$. *For all* $r \in (0, \operatorname{dist}(a, \mathbf{R}^n \sim U))$ *for which* $M_a'(r)$ *exists we have*

$$M_a(r) \le C M_a'(r)^{m/(m-1)} \quad \text{and} \quad \frac{M_a(r)}{r^m} \le \gamma + C\iota \frac{M_a'(r)}{r^{m-1}} ,$$

where $\gamma = \gamma(n,m,\iota)$ *and* $C = C(n,m,F)$

Corollary

For all $a \in \operatorname{spt}\|V\| \cap U$ *and all* $r \in (0, r_0)$*, where* $r_0 = \operatorname{dist}(a, \mathbf{R}^n \sim U)$*, we have*

$$m^{-m} C^{1-m} \le \boldsymbol{\Theta}^m(\|V\|, a, r) \le \max\{\tilde{C}, \boldsymbol{\Theta}^m(\|V\|, a, r_0)\} ,$$

where $\tilde{C} = \tilde{C}(n,m,F)$.

1. Choose $S \in \mathcal{P}(B)$ so that $\mathbf{v}(S)$ is weakly close to V.
2. Let $a \in \operatorname{spt} \mathbf{v}(S)$. Set $\rho(x) = |x - a|$ and $\Sigma = S \cap \rho^{-1}[\{r\}]$.
3. Set $\iota^{m-1} = C_{\mathrm{dt}} \mathcal{H}^{m-1}(\Sigma) \approx M_a'(r)$.
4. Cover Σ by dyadic cubes of side length ι.
5. The deformation theorem yields $f : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$.
6. Extend f to a homotopy that smashes everything to a point in the end.
7. Consider the weak limit

$$\lim_{\delta \downarrow 0} (f \circ K_{\delta,r})_\# \mathbf{v}(S) = f(0,\cdot)_\# \mathbf{v}(S \sim \mathbf{U}(a,r)) + f(1,\cdot)_\# \mathbf{v}(S \cap \mathbf{U}(a,r)) + \mathbf{v}(f_\#[[0,1] \times \Sigma]) \,.$$

8. Hence, for small $\delta > 0$ we get

$$M_a(r) \lesssim \mathcal{H}^m((f \circ K_{\delta,r})[S \cap \mathbf{U}(a,r)]) \lesssim \iota C_{\mathrm{dt}} \mathcal{H}^{m-1}(\Sigma) \,.$$

1. There exists $C_1, C_2 \in \mathbf{R}$ such that

$$0 < C_1 \leq \mathbf{\Theta}^m(\|V\|, x, r) \leq C_2 < \infty$$

   for $x \in \operatorname{spt}\|V\|$ and $r \in (0, \operatorname{dist}(x, \mathbf{R}^n \sim U))$.
2. Minimality of V then yields $(\mathcal{H}^m, m)$ rectifiability of $\operatorname{spt}\|V\|$.
3. Ellipticity of F gives then $\mathbf{\Theta}^m(\|V\|, x) = 1$ for $x \in U$ such that $\operatorname{Tan}(\operatorname{spt}\|V\|, x)$ is an m plane.
4. Area formula proves $V = \mathbf{v}(\operatorname{spt}\|V\|)$.

Lemma

Let $G \subseteq \mathbf{R}^n$ *be open and* $K \subseteq G$ *be purely* $(\mathcal{H}^m, m)$ *unrectifiable with* $\mathcal{H}^m(K) < \infty$. *Let* $f : \mathbf{R}^n \to \mathbf{R}^n$ *be of class* $\mathscr{C}^k$, *where* $k \geq n - m + 1$. *Suppose*

$$\dim \operatorname{im} \mathrm{D}f(x) \leq m \quad \text{for all } x \in G\,.$$

Then for every $\varepsilon \in \mathbf{R}$ *with* $0 < \varepsilon < \operatorname{dist}(K, \mathbf{R}^n \sim G)$ *there exists a map* $f_\varepsilon : \mathbf{R}^n \to \mathbf{R}^n$ *of class* $\mathscr{C}^k$ *satisfying*

$$f(x) = f_\varepsilon(x) \quad \text{for } x \in \mathbf{R}^n \sim G\,,$$
$$|f(x) - f_\varepsilon(x)| \leq \varepsilon \quad \text{and} \quad |\mathrm{D}f(x) - \mathrm{D}f_\varepsilon(x)| \leq \varepsilon \quad \text{for } x \in \mathbf{R}^n\,,$$
$$\mathcal{H}^m(f_\varepsilon[K]) \leq \varepsilon \mathcal{H}^m(K)\,.$$

1. Let $a \in R = \operatorname{spt}\|V\|$ be such that $\boldsymbol{\Theta}^{m*}(R_{\mathrm{irr}}, a) > 0$ and $\boldsymbol{\Theta}^{m}(R_{\mathrm{reg}}, a) = 0$.
2. Find $r > 0$ with $\boldsymbol{\Theta}^{m}(R_{\mathrm{reg}}, a, r) < \varepsilon \ll \boldsymbol{\Theta}^{m}(R_{\mathrm{irr}}, a, r)$.
3. Apply deformation theorem to $R \cap \mathbf{B}(a, r)$ to find $f : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$.
4. Find a $\mathscr{C}^1$ perturbation φ of $f(1, \cdot)$ such that $\mathcal{H}^m(\varphi[R_{\mathrm{irr}}]) \le \varepsilon \mathcal{H}^m(R_{\mathrm{irr}})$.
5. Then $\mathcal{H}^m(\varphi[R \cap \mathbf{B}(a, r)]) < \varepsilon C \mathcal{H}^m(R \cap \mathbf{B}(a, r))$.
6. Choose $S \in \mathcal{P}(B)$ so that $\mathbf{v}(S)$ is weakly close to V.
7. Then $\mathcal{H}^m(\varphi[S \cap \mathbf{B}(a, r)]) < \varepsilon C \mathcal{H}^m(S \cap \mathbf{B}(a, r))$.
8. Consequently,
   $\Phi_F(\varphi[S]) < V(F) = \inf\{\Phi_F(K) : K \in \mathcal{P}(B)\}$.
   A contradiction!

1. There exists $C_1, C_2 \in \mathbf{R}$ such that
$$0 < C_1 \le \boldsymbol{\Theta}^m(\|V\|, x, r) \le C_2 < \infty$$
for $x \in \operatorname{spt} \|V\|$ and $r \in (0, \operatorname{dist}(x, \mathbf{R}^n \sim U))$.
2. Minimality of V then yields $(\mathcal{H}^m, m)$ rectifiability of $\operatorname{spt} \|V\|$.
3. Ellipticity of F gives then $\boldsymbol{\Theta}^m(\|V\|, x) = 1$ for $x \in U$ such that $\operatorname{Tan}(\operatorname{spt} \|V\|, x)$ is an m plane.
4. Area formula proves $V = \mathbf{v}(\operatorname{spt} \|V\|)$.

1. Setup:

$$0 = a \in \operatorname{spt}\|V\| \cap U\,, \quad T = \operatorname{Tan}(\operatorname{spt}\|V\|, a) \in \mathbf{G}(n,m)\,,$$
$$r_i \downarrow 0\,, \quad \delta_i \downarrow 0\,, \quad 0 < \varepsilon_i \ll r_i^m\,,$$
$$K_i = \left\{ r_i x : |T_\natural x| \le 1,\ |T_\natural^\perp x| \le \delta_i \right\},$$
$$\xi_i \in \mathscr{C}^\infty(\mathbf{R}^n, \mathbf{R}^n) \quad \text{projects } K_i \text{ onto } T\,, \quad R_i = \xi_i[S_i]\,.$$

2. Minimality of V gives

$$\Phi_F(S_i) - \varepsilon_i \le \Phi_F(V) \le \Phi_F(R_i)$$
$$\le \Phi_F(S_i) - \Phi_F(S_i \cap K_i) + \Phi_F(R_i \cap K_i) + \Phi_F(S_i \div R_i)\,,$$

$\Theta^m(\|V\|, a) = 1$ for $\|V\|$ almost all a (idea of the proof)

1. Setup:

$$0 = a \in \operatorname{spt}\|V\| \cap U\,, \quad T = \operatorname{Tan}(\operatorname{spt}\|V\|, a) \in \mathbf{G}(n,m)\,,$$
$$r_i \downarrow 0\,, \quad \delta_i \downarrow 0\,, \quad 0 < \varepsilon_i \ll r_i^m\,,$$
$$K_i = \left\{ r_i x : |T_\natural x| \le 1,\ |T_\natural^\perp x| \le \delta_i \right\}\,,$$
$$\xi_i \in \mathscr{C}^\infty(\mathbf{R}^n, \mathbf{R}^n) \quad \text{projects } K_i \text{ onto } T\,, \quad R_i = \xi_i[S_i]\,.$$

2. Minimality of V gives

$$\frac{\Phi_F(S_i \cap K_i) - \Phi_F(R_i \cap K_i)}{r_i^m} \le \frac{\varepsilon_i + \Phi_F(S_i \div R_i)}{r_i^m} \to 0\,.$$

3. Since $D_i = R_i \cap K_i = T \cap \mathbf{B}(0, r_i)$ ellipticity of F yields

$$\Phi_{F^a}(S_i \cap K_i) - \Phi_{F^a}(R_i \cap K_i) \gtrsim \mathcal{H}^m(S_i \cap K_i) - \mathcal{H}^m(R_i \cap K_i)\,.$$

Hence,

$$\frac{\mathcal{H}^m(S_i \cap K_i)}{r_i^m} - \boldsymbol{\alpha}(m) = \frac{\mathcal{H}^m(S_i \cap K_i) - \mathcal{H}^m(D_i)}{r_i^m} \to 0\,.$$

1. There exists $C_1, C_2 \in \mathbf{R}$ such that
$$0 < C_1 \leq \boldsymbol{\Theta}^m(\|V\|, x, r) \leq C_2 < \infty$$
for $x \in \operatorname{spt} \|V\|$ and $r \in (0, \operatorname{dist}(x, \mathbf{R}^n \sim U))$.
2. Minimality of V then yields $(\mathcal{H}^m, m)$ rectifiability of $\operatorname{spt} \|V\|$.
3. Ellipticity of F gives then $\boldsymbol{\Theta}^m(\|V\|, x) = 1$ for $x \in U$ such that $\operatorname{Tan}(\operatorname{spt} \|V\|, x)$ is an m plane.
4. Area formula proves $V = \mathbf{v}(\operatorname{spt} \|V\|)$.

1. Consider $T \in \mathbf{G}(n,m)$ and sets K_i and R_i as before.
2. Define $Q_i = \boldsymbol{\mu}_{1/r_i}[R_i \cap K_i]$.
3. We know $\mathcal{H}^m(Q_i) \to \boldsymbol{\alpha}(m)$ as $i \to \infty$.
4. Area formula applied to $T_\natural$ gives

$$\boldsymbol{\alpha}(m) = \mathcal{H}^m(D) = \int_{Q_i} \|{\textstyle\bigwedge_m} T_\natural \circ \operatorname{Tan}(Q_i,x)_\natural\| \, \mathrm{d}\mathcal{H}^m(x) \to \boldsymbol{\alpha}(m) \,.$$

5. Note: $1 - \|{\textstyle\bigwedge_m} T_\natural \circ \operatorname{Tan}(Q_i,x)_\natural\| \approx \|T_\natural - \operatorname{Tan}(Q_i,x)_\natural\|^2$.
6. Hence,

$$V(\alpha) = \int_{\operatorname{spt}\|V\|} \alpha(x, \operatorname{Tan}(\operatorname{spt}\|V\|, x)) \, \mathrm{d}\mathcal{H}^m(x) \,,$$

whenever $\alpha \in \mathscr{C}^0_{\mathrm{c}}(U \times \mathbf{G}(n,m))$.

Q.E.D.