New and old solutions to generalised Plateau's problems

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(joint **ongoing** work with Yangqin Fang)

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X Forum of partial differential equations

1. Classical formulations of the problem
2. New formulations of the problem
3. Elliptic integrands and generalised Plateau's problem
4. Almgren's methods rediscovered

Prove existence (and possibly also regularity) of soap films spanned on a given wire,
i.e.,
find a *surface* having minimal *area* among surfaces spanning a given *boundary*.

- Surface?
- Area?
- Spanning a boundary?

$$B \subseteq \mathbf{R}^n \quad \text{a compact set}\,, \qquad G \quad \text{an abelian group}\,,$$
$$\check{\mathbf{H}}_k(B;G) \quad \text{the } k^{\text{th}} \text{ Čech homology group}\,,$$
$$\mathcal{H}^k(B) \quad \text{the } k \text{ dimensional Haudorff measure}\,,$$
$$\tau \in \check{\mathbf{H}}_{m-1}(B;G)$$

Problem

Minimise $\mathcal{H}^m(S)$ among compact sets $S \subseteq \mathbf{R}^n$ such that

$$i_* \tau = 0\,,$$

where $i_* : \check{\mathbf{H}}_{m-1}(B;G) \to \check{\mathbf{H}}_{m-1}(S \cup B;G)$ is induced by the inclusion map $i : B \hookrightarrow S \cup B$.

- E. R. Reifenberg and J. F. Adams, *Acta Math.*, 1960
- F. Almgren, *Ann. of Math.*, 1968

$$\mathbf{I}_k(\mathbf{R}^n) \subseteq \mathscr{D}'(\mathbf{R}^n, \textstyle\bigwedge^k \mathbf{R}^n) \quad k \text{ dimensional integral currents,}$$

$$S \in \mathbf{I}_k(\mathbf{R}^n) \quad \Rightarrow \quad S(\phi) = \int_E \langle \tau(x), \phi(x) \rangle \theta(x) \, \mathrm{d}\mathcal{H}^k(x) \, ,$$

$$\mathrm{Mass}(S) = \int_E \theta(x) \, \mathrm{d}\mathcal{H}^k(x) \, , \quad \mathrm{Size}(S) = \mathcal{H}^k(\{x \in E : \theta(x) \neq 0\}) \, ,$$

$$B \in \mathbf{I}_{m-1}(\mathbf{R}^n) \, , \quad \partial B(\phi) = B(d\phi) = 0$$

Problem (Mass minimisers)

Minimise $\mathrm{Mass}(S)$ *among* $S \in \mathbf{I}_m(\mathbf{R}^n)$ *such that* $\partial S = B$.

Problem (Size minimisers – unsolved)

Minimise $\mathrm{Size}(S)$ *among* $S \in \mathbf{I}_m(\mathbf{R}^n)$ *such that* $\partial S = B$.

- H. Federer and W. Fleming, *Ann. of Math.*, 1960

Solutions of the Plateau's problem should model the behaviour of soap films!

1. Interior smoothness everywhere (mass minimisers).
2. Choice of orientation (currents, homology).
3. Choice of the coefficient group (currents, homology).
4. Reifenberg's solution only for compact groups.

SEE:

- G. David, Should we solve Plateau's problem again?, in *Advances in analysis: the legacy of Elias M. Stein*, 108–145, Princeton Math. Ser., 50, Princeton Univ. Press, Princeton, NJ, 2014
- J. Harrison, H. Pugh, *Plateau's Problem: What's Next*, arXiv:1509.03797, 2015

$$B \subseteq \mathbf{R}^n \quad \text{a compact set},$$

$$\mathcal{C} \subseteq \mathcal{C}_B = \{\gamma : \mathbf{S}^{n-m} \hookrightarrow \mathbf{R}^n \sim B : \gamma \text{ is a smooth embedding}\}$$

$$\text{if } \gamma \in \mathcal{C} \text{ and } \tilde{\gamma} \in \mathcal{C}_B \text{ is smoothly isotopic to } \gamma, \text{ then } \tilde{\gamma} \in \mathcal{C},$$

$$\mathcal{F}(B,\mathcal{C}) = \left\{ K \subseteq \mathbf{R}^n : \begin{array}{l} K \text{ is compact } (\mathcal{H}^m, m) \text{ rectifiable} \\ K \cap \operatorname{im}\gamma \neq \varnothing \quad \text{for } \gamma \in \mathcal{C} \end{array} \right\}$$

If B is an orientable $(m-1)$ dimensional manifold, then we can take

$$\mathcal{C} = \{\gamma \in \mathcal{C}_B : \operatorname{lk}(\operatorname{im}\gamma, B) = 1\}.$$

Problem

Minimise $\mathcal{H}^m(S)$ among $S \in \mathcal{F}(B,\mathcal{C})$.

- J. Harrison and H. Pugh, *J. Geom. Anal.*, 2012
- C. De Lellis, F. Ghiraldin, F. Maggi, *J. Eur. Math. Soc. (JEMS)*, 2016
- G. De Philippis, A. De Rosa, F. Ghiraldin, *Adv. Math.*, 2016

$$B, K_0 \subseteq \mathbf{R}^n \quad \text{compact sets}\,,$$

$$\Sigma(B) = \left\{ \Phi(1,\cdot) : \begin{array}{c} \Phi : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n \text{ is continuous}\,, \\ \Phi(0,\cdot) = \mathrm{id}_{\mathbf{R}^n}\,, \quad \mathrm{Lip}\,\Phi(1,\cdot) < \infty\,, \\ \Phi(t,\cdot)[B] \subseteq B \text{ for } t \in [0,1] \end{array} \right\}\,,$$

$$\mathcal{A}(B,K) = \{\varphi[K_0] : \varphi \in \Sigma(B)\}$$

Problem (partially solved)

Minimise $\mathcal{H}^m(S)$ among $S \in \mathcal{A}(B,K_0)$.

One can find $K \subseteq \mathbf{R}^n$ satisfying

$$\inf\{\mathcal{H}^m(S) : S \in \mathcal{A}(B,K_0)\} = \mathcal{H}^m(K) = \inf\{\mathcal{H}^m(S) : S \in \mathcal{A}(B,K)\}$$

but it is *not known* whether $K \in \mathcal{A}(B,K_0)$.

- G. David, *Princeton Math. Ser.*, 2014
- G. De Philippis, A. De Rosa, F. Ghiraldin, *Adv. Math.*, 2016
- C. De Lellis, F. Ghiraldin, F. Maggi, *J. Eur. Math. Soc. (JEMS)*, 2016

Let $B \subseteq \mathbf{R}^n$ be compact.

1. Define the class of competitors $\mathcal{P}(B)$ axiomatically without referring to any particular notion of *boundary* or *spanning*.
2. Consider general elliptic functionals Φ instead of the Hausdorff measure $\mathcal{H}^m$.
3. Minimise $\Phi(S)$ among $S \in \mathcal{P}(B)$.

Remark

Usually the class $\mathcal{P}(B)$ is assumed to be closed under certain deformations (like sliding deformations).

- C. De Lellis, F. Ghiraldin, F. Maggi, *J. Eur. Math. Soc. (JEMS)*, 2016
- G. De Philippis, A. De Rosa, F. Ghiraldin, *Adv. Math.*, 2016
- C. De Lellis, A. De Rosa, F. Ghiraldin, arXiv:1602.08757, 2016
- J. Harrison and H. Pugh, arXiv:1603.04492, 2016
- Y. Fang, arXiv:1310.4690, 2013

$$F : \mathbf{R}^n \times \mathbf{G}(n,m) \to (0,\infty) \quad \text{continuous}\,,$$
$$S = S_{\mathrm{reg}} \cup S_{\mathrm{irr}} \subseteq \mathbf{R}^n \quad \text{compact with } \mathcal{H}^m(S) < \infty\,,$$
$$\Phi_F(S) = \int_{S_{\mathrm{reg}}} F(x, \mathrm{Tan}(S_{\mathrm{reg}}, x))\, \mathrm{d}\mathcal{H}^m(x) + \mathcal{H}^m(S_{\mathrm{irr}})\,,$$
$$x \in \mathbf{R}^n \quad \Rightarrow \quad F^x(y,T) = F(x,T)\,.$$

Definition (F. Almgren, *Ann. of Math.*, 1968)

We say that F is <u>*elliptic*</u> at $x \in \mathbf{R}^n$ if: there exists $C > 0$ such that given any disc $D \subseteq L$ lying in an affine m-plane $L \subseteq \mathbf{R}^n$ and a compact set $S \subseteq \mathbf{R}^n$ which cannot be deformed onto $\partial D \subseteq S$ with an *admissible deformation*, there holds

$$\Phi_{F^x}(S) - \Phi_{F^x}(D) \geqslant C\big(\mathcal{H}^m(S) - \mathcal{H}^m(D)\big)\,.$$

Example

The area integrand: $F(x,T) = 1$ for $(x,T) \in \mathbf{R}^n \times \mathbf{G}(n,m)$. Then $\Phi_F(S) = \mathcal{H}^m(S)$.

- Consider $\mathbf{R}^n$ with a non-Euclidean norm $|||\cdot|||$.
- This gives rise to a non-Euclidean metric ρ on $\mathbf{R}^n$.
- This yields a non-standard Hausdorff measure $\mathcal{H}^m_\rho$.

Problem

Does there exist an integrand $F : \mathbf{R}^n \times \mathbf{G}(n,m) \to (0,\infty)$ *such that*

$$\mathcal{H}^m_\rho(A) = \Phi_F(A) = \int_A F(x, \mathrm{Tan}(A,x))\, \mathrm{d}\mathcal{H}^m(x)$$

for any $(\mathcal{H}^m, m)$ *rectifiable set* $A \subseteq \mathbf{R}^n$. *Is* F *elliptic?*

$$B \subseteq \mathbf{R}^n \text{ compact } (\mathcal{H}^m, m) \text{ rectifiable}, \quad U = \mathbf{R}^n \sim B,$$

$$\mathfrak{D}(U) = \left\{ \varphi \in \mathscr{C}^1(\mathbf{R}^n, \mathbf{R}^n) : \begin{array}{l} \exists a, r \quad \mathbf{B}(a, r) \subseteq U, \\ \varphi[\mathbf{B}(a, r)] \subseteq \mathbf{B}(a, r), \\ \{x : \varphi(x) \neq x\} \subseteq \mathbf{B}(a, r) \end{array} \right\}$$

THEOREM (TO BE OPTIMISED)

Assume F is an elliptic integrand and $\mathcal{P}(B)$ is a family of compact subsets of $\mathbf{R}^n$ such that

- *if $S \in \mathcal{P}(B)$ and $\varphi \in \mathfrak{D}(U)$, then $\varphi[S] \in \mathcal{P}(B)$;*
- *if $S_i \in \mathcal{P}(B)$ and $S_i \xrightarrow{\text{HD}} S$, then $S \in \mathcal{P}(B)$.*

Then there exists $S \in \mathcal{P}(B)$ such that

$$\Phi_F(S) \leqslant \Phi_F(R) \quad \textit{for } R \in \mathcal{P}(B).$$

Example

Let $\mathcal{P}(B)$ be the family of compact sets S which span some $\tau \in \check{\mathbf{H}}_{m-1}(B;G)$ in the sense of Reifenberg, i.e.,

$$i_*\tau = 0\,, \qquad \text{where } i : B \hookrightarrow S \cup B\,.$$

Then there exists a minimiser $R \in \mathcal{P}(B)$ of Φ_F. Moreover, R is compact and $(\mathcal{H}^m, m)$ rectifiable.

Remark

If we assumed that $\mathfrak{D}(U)$ contains only diffeomorphisms $\mathscr{C}^1$ isotopic to identity, then we could apply the theorem also in case $\mathcal{P}(B) = \mathcal{F}(B, \mathcal{C})$ (spanning defined in terms of linking numbers).

1. Chose a minimising sequence $S_i \in \mathcal{P}(B)$ so that $\Phi_F(S_i) \to \inf\{\Phi_F(R) : R \in \mathcal{P}(B)\}$.
2. Define varifolds $V_i = \mathbf{v}(S_i)$ (Radon measures over $\mathbf{R}^n \times \mathbf{G}(n,m)$), so that
$$\Phi_F(S_i) = V_i(F) = \int F(x,T)\,\mathrm{d}V_i(x,T)\,.$$
3. Take the varifold limit $V_i \to V$ (weak limit of measures).
4. We get for free!
$$V(F) = \inf\{\Phi_F(R) : R \in \mathcal{P}(B)\}\,.$$
5. Modify the sequence so that $S_i \xrightarrow{\text{HD}} S$ and $S \in \mathcal{P}(B)$ is such that $\mathcal{H}^m(S \sim \operatorname{spt}\|V\|) = 0$ (HAIR COMBING).
6. Show that $\operatorname{spt}\|V\|$ is $(\mathcal{H}^m, m)$ rectifiable and $V = \mathbf{v}(\operatorname{spt}\|V\|)$.
7. Then $V = \mathbf{v}(S)$; hence,
$$\Phi_F(S) = V(F) = \inf\{\Phi_F(R) : R \in \mathcal{P}(B)\}\,.$$

$$\boldsymbol{\Theta}^m(\|V\|, x, r) = \frac{\|V\|\mathbf{B}(x, r)}{\boldsymbol{\alpha}(m) r^m}, \quad \boldsymbol{\Theta}^m(\|V\|, x) = \lim_{r \downarrow 0} \boldsymbol{\Theta}^m(\|V\|, x, r).$$

1. There exists $C_1, C_2 \in \mathbf{R}$ such that

   $$0 < C_1 \leqslant \boldsymbol{\Theta}^m(\|V\|, x, r) \leqslant C_2 < \infty$$

   for $x \in \operatorname{spt}\|V\| \cap U$ and $r \in (0, \operatorname{dist}(x, \mathbf{R}^n \sim U))$. Consequently

   $$\|V\| \llcorner U \approx \mathcal{H}^m \llcorner (\operatorname{spt}\|V\| \cap U).$$

2. Minimality of V then yields $(\mathcal{H}^m, m)$ rectifiability of $\operatorname{spt}\|V\|$.
3. Ellipticity of F gives then $\boldsymbol{\Theta}^m(\|V\|, x) = 1$ for $x \in U$ such that $\operatorname{Tan}(\operatorname{spt}\|V\|, x)$ is an m plane.
4. Area formula proves $V = \mathbf{v}(\operatorname{spt}\|V\|)$.

1. If $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ is of class $\mathscr{C}^1$, then $\varphi_\# : \mathbf{V}_m(\mathbf{R}^n) \to \mathbf{V}_m(\mathbf{R}^n)$ is continuous.
2. If $K \subseteq \mathbf{R}^n$ is compact, then the mass

$$\mathbf{M}(V) := V(\mathbf{R}^n)$$

   is continuous on $\{V \in \mathbf{V}_m(\mathbf{R}^n) : \operatorname{spt} \|V\| \subseteq K\}$.
3. Hence,

$$\|V\|\mathbf{B}(a, r) \approx \mathcal{H}^m(\phi[S_i \cap \mathbf{B}(a, r)])$$

   whenever $\{x : \varphi(x) \neq x\} \subseteq \mathbf{B}(a, r) \supseteq \varphi[\mathbf{B}(a, r)]$.
4. Smooth deformation theorem:
   deforms a compact set $\Sigma \subseteq \mathbf{R}^n$ onto an m dimensional skeleton of a finite cubical complex covering Σ.
5. Slicing theory for varifolds.

THEOREM

Assume $\mathcal{F}$ is an admissible collection of cubes (e.g. Whitney decomposition of U or tiling of $\mathbf{R}^n$ with isometric cubes), and $\mathcal{A} \subseteq \mathcal{F}$ is finite, and $\Sigma \subseteq \mathbf{R}^n$ is compact, and $\mathcal{H}^m(\Sigma) < \infty$, and $\varepsilon > 0$. Set $G = \bigcup \mathcal{A} + \mathbf{U}(0, \varepsilon)$. There exists a $\mathscr{C}^\infty$ smooth map $f : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ such that

- $f(0,\cdot) = \mathrm{id}_{\mathbf{R}^n}$ *and* $f(t,x) = x$ *for* $t \in [0,1]$ *and* $x \in \mathbf{R}^n \sim G$
- $f(1,\cdot)[\Sigma \cap G]$ *is a sum of* m *dimensional faces of cubes from* $\mathcal{A}$
- $\mathcal{H}^m(f(1,\cdot)[\Sigma \cap G]) < C_{\mathrm{dt}}(n,m)\mathcal{H}^m(\Sigma \cap G)$
- *if* $\delta = \max\{\mathrm{side}\, Q : Q \in \mathcal{A}\}$ *and* Σ *is* $(\mathcal{H}^m, m)$ *rectifiable, then*

$$\mathcal{H}^{m+1}(f[[0,1] \times (\Sigma \cap G)]) < C_{\mathrm{dt}}(n,m)\delta\mathcal{H}^m(\Sigma \cap G)$$

Remark

Assume $V \in \mathbf{V}_m(\mathbf{R}^n)$ and $\rho : \mathbf{R}^n \to \mathbf{R}$ is a proper map of class $\mathscr{C}^1$. For $\mathcal{L}^1$ almost all $t \in \mathbf{R}$ there exists a varifold slice

$$\langle V, \rho, t \rangle \in \mathbf{V}_{m-1}(\mathbf{R}^n) .$$

If $S \subseteq \mathbf{R}^n$ is $(\mathcal{H}^m, m)$ rectifiable and $\mathcal{H}^m$ measurable and $V = \mathbf{v}(S)$, then

$$\langle V, \rho, t \rangle = \mathbf{v}(S \cap \rho^{-1}[\{t\}]) \quad \text{for } \mathcal{L}^1 \text{ almost all } t .$$

- Almgren, *Mem. Amer. Math. Soc.*, 1976.

$$\Theta^m(\|V\|, x, r) = \frac{\|V\|\mathbf{B}(x, r)}{\alpha(m) r^m}, \quad \Theta^m(\|V\|, x) = \lim_{r \downarrow 0} \Theta^m(\|V\|, x, r).$$

1. There exists $C_1, C_2 \in \mathbf{R}$ such that

$$0 < C_1 \leqslant \Theta^m(\|V\|, x, r) \leqslant C_2 < \infty$$

   for $x \in \operatorname{spt}\|V\| \cap U$ and $r \in (0, \operatorname{dist}(x, \mathbf{R}^n \sim U))$.
2. Minimality of V then yields $(\mathcal{H}^m, m)$ rectifiability of $\operatorname{spt}\|V\|$.
3. Ellipticity of F gives then $\Theta^m(\|V\|, x) = 1$ for $x \in U$ such that $\operatorname{Tan}(\operatorname{spt}\|V\|, x)$ is an m plane.
4. Area formula proves $V = \mathbf{v}(\operatorname{spt}\|V\|)$.

Lemma

Let $a \in \operatorname{spt}\|V\|$ *and* $\iota \in (0, \infty)$. *Set* $M_a(r) = \|V\|\mathbf{B}(a, r)$. *For all* $r \in (0, \operatorname{dist}(a, \mathbf{R}^n \sim U))$ *for which* $M_a'(r)$ *exists we have*

$$M_a(r) \leqslant C M_a'(r)^{m/(m-1)} \quad \text{and} \quad \frac{M_a(r)}{r^m} \leqslant \gamma + C\iota \frac{M_a'(r)}{r^{m-1}},$$

where $\gamma = \gamma(n, m, \iota)$ *and* $C = C(n, m, F)$

Corollary

For all $a \in \operatorname{spt}\|V\| \cap U$ *and all* $r \in (0, r_0)$, *where* $r_0 = \operatorname{dist}(a, \mathbf{R}^n \sim U)$, *we have*

$$m^{-m} C^{1-m} \leqslant \boldsymbol{\Theta}^m(\|V\|, a, r) \leqslant \max\{\tilde{C}, \boldsymbol{\Theta}^m(\|V\|, a, r_0)\},$$

where $\tilde{C} = \tilde{C}(n, m, F)$.

$\|V\|\mathbf{B}(a, r) =: M_a(r) \leqslant CM_a'(r)^{m/(m-1)}$ (idea of the proof)

1. Choose $S \in \mathcal{P}(B)$ so that $\mathbf{v}(S)$ is weakly close to V.
2. Let $a \in \operatorname{spt}\|V\|$ and $\Sigma = S \cap \partial\mathbf{B}(a, r)$.
3. Set $\varepsilon^{m-1} = C_{\mathrm{dt}}\mathcal{H}^{m-1}(\Sigma) \approx M_a'(r)$.
4. Cover Σ with a family $\mathcal{A}$ of dyadic cubes of side length ε.
5. The deformation theorem yields $f : [0, 1] \times \mathbf{R}^n \to \mathbf{R}^n$ with

$$\mathcal{H}^{m-1}(f(1, \cdot)[\Sigma]) < C_{\mathrm{dt}}\mathcal{H}^{m-1}(\Sigma) \leqslant \varepsilon^{m-1}$$
$$\mathcal{H}^{m}(f[[0, 1] \times \Sigma]) < \varepsilon C_{\mathrm{dt}}\mathcal{H}^{m-1}(\Sigma) = \varepsilon^{m}$$

   Thus, $f(1, \cdot)[\Sigma] \subseteq \bigcup \mathrm{skel}_{\text{m-2}}\, \mathcal{A}$.
6. Set
$$\tilde{f}(t, x) = \begin{cases} f(2t, x) & \text{if } 0 \leqslant t < 1/2 \\ (2 - 2t)f(1, x) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$
7. Define $K_\delta(x) = ((r - |x - a|)/\delta, x) : \mathbf{R}^n \to \mathbf{R} \times \mathbf{R}^n$.
8. Consider the weak limit
$$\lim_{\delta \downarrow 0} (f \circ K_\delta)_\# \mathbf{v}(S) = f(0, \cdot)_\# \mathbf{v}(S \sim \mathbf{U}(a, r))$$

$\|V\|\mathbf{B}(a,r) =: M_a(r) \leqslant C M_a'(r)^{m/(m-1)}$ (idea of the proof)

2. Let $a \in \operatorname{spt}\|V\|$ and $\Sigma = S \cap \partial\mathbf{B}(a,r)$.
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$$\lim_{\delta \downarrow 0}(f \circ K_\delta)_{\#}\mathbf{v}(S) = f(0,\cdot)_{\#}\mathbf{v}(S \sim \mathbf{U}(a,r))$$
$$+ f(1,\cdot)_{\#}\mathbf{v}(S \cap \mathbf{U}(a,r)) + \mathbf{v}(f_{\#}[[0,1] \times \Sigma]) .$$

$\|V\|\mathbf{B}(a,r) =: M_a(r) \leqslant C M_a'(r)^{m/(m-1)}$ (idea of the proof)

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9. Hence, for small $\delta > 0$ we get

$\|V\|\mathbf{B}(a,r) =: M_a(r) \leqslant CM_a'(r)^{m/(m-1)}$ (idea of the proof)

5. The deformation theorem yields $f : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ with

$$\mathcal{H}^{m-1}(f(1,\cdot)[\Sigma]) < C_{\mathrm{dt}}\mathcal{H}^{m-1}(\Sigma) \leqslant \varepsilon^{m-1}$$
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Thus, $f(1,\cdot)[\Sigma] \subseteq \bigcup \mathrm{skel}_{\mathrm{m\text{-}2}}\,\mathcal{A}$.

6. Set
$$\tilde f(t,x) = \begin{cases} f(2t,x) & \text{if } 0 \leqslant t < 1/2 \\ (2-2t)f(1,x) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$

7. Define $K_\delta(x) = ((r-|x-a|)/\delta, x) : \mathbf{R}^n \to \mathbf{R}\times\mathbf{R}^n$.

8. Consider the weak limit
$$\lim_{\delta\downarrow 0}(f\circ K_\delta)_\#\mathbf{v}(S) = f(0,\cdot)_\#\mathbf{v}(S\sim\mathbf{U}(a,r)) + f(1,\cdot)_\#\mathbf{v}(S\cap\mathbf{U}(a,r)) + \mathbf{v}(f_\#[[0,1]\times\Sigma]).$$

9. Hence, for small $\delta > 0$ we get
$$M_a(r) \lesssim \mathcal{H}^m((f\circ K_\delta)[S\cap\mathbf{U}(a,r)])$$

$$\mathcal{H}^{m-1}(f(1,\cdot)[\Sigma]) < C_{\mathrm{dt}}\mathcal{H}^{m-1}(\Sigma) \leqslant \varepsilon^{m-1}$$
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7. Define $K_\delta(x) = ((r - |x-a|)/\delta, x) : \mathbf{R}^n \to \mathbf{R} \times \mathbf{R}^n$.

8. Consider the weak limit
$$\lim_{\delta\downarrow 0}(f \circ K_\delta)_\#\mathbf{v}(S) = f(0,\cdot)_\#\mathbf{v}(S \sim \mathbf{U}(a, r)) + f(1,\cdot)_\#\mathbf{v}(S \cap \mathbf{U}(a, r)) + \mathbf{v}(f_\#[[0,1]\times\Sigma]) \,.$$

9. Hence, for small $\delta > 0$ we get
$$M_a(r) \lesssim \mathcal{H}^m((f\circ K_\delta)[S \cap \mathbf{U}(a, r)]) \lesssim \varepsilon C_{\mathrm{dt}}\mathcal{H}^{m-1}(\Sigma) = \varepsilon^m \approx M_a'(r)^{m/(m-1)} \,.$$

$\|V\|\mathbf{B}(a,r) =: M_a(r) \leqslant C M_a'(r)^{m/(m-1)}$ (idea of the proof)

$$\mathcal{H}^m(f[[0,1]\times\Sigma]) < \varepsilon C_{\mathrm{dt}} \mathcal{H}^{m-1}(\Sigma) = \varepsilon^m$$

Thus, $f(1,\cdot)[\Sigma] \subseteq \bigcup \mathrm{skel}_{\text{m-2}}\, \mathcal{A}$.

6. Set
$$\tilde{f}(t,x) = \begin{cases} f(2t,x) & \text{if } 0 \leqslant t < 1/2 \\ (2-2t)f(1,x) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$

7. Define $K_\delta(x) = ((r - |x-a|)/\delta, x) : \mathbf{R}^n \to \mathbf{R} \times \mathbf{R}^n$.

8. Consider the weak limit
$$\lim_{\delta\downarrow 0} (f \circ K_\delta)_\# \mathbf{v}(S) = f(0,\cdot)_\# \mathbf{v}(S \sim \mathbf{U}(a,r)) + f(1,\cdot)_\# \mathbf{v}(S \cap \mathbf{U}(a,r)) + \mathbf{v}(f_\#[[0,1]\times\Sigma])\,.$$

9. Hence, for small $\delta > 0$ we get
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$$\boldsymbol{\Theta}^m(\|V\|, x, r) = \frac{\|V\|\mathbf{B}(x, r)}{\boldsymbol{\alpha}(m) r^m}, \quad \boldsymbol{\Theta}^m(\|V\|, x) = \lim_{r \downarrow 0} \boldsymbol{\Theta}^m(\|V\|, x, r).$$

1. There exists $C_1, C_2 \in \mathbf{R}$ such that

$$0 < C_1 \leqslant \boldsymbol{\Theta}^m(\|V\|, x, r) \leqslant C_2 < \infty$$

for $x \in \operatorname{spt}\|V\|$ and $r \in (0, \operatorname{dist}(x, \mathbf{R}^n \sim U))$.

2. Minimality of V then yields $(\mathcal{H}^m, m)$ rectifiability of $\operatorname{spt}\|V\|$.
3. Ellipticity of F gives then $\boldsymbol{\Theta}^m(\|V\|, x) = 1$ for $x \in U$ such that $\operatorname{Tan}(\operatorname{spt}\|V\|, x)$ is an m plane.
4. Area formula proves $V = \mathbf{v}(\operatorname{spt}\|V\|)$.

Lemma

Let $G \subseteq \mathbf{R}^n$ be open and $K \subseteq G$ be purely $(\mathcal{H}^m, m)$ unrectifiable with $\mathcal{H}^m(K) < \infty$. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be of class $\mathscr{C}^k$, where $k \geqslant n - m + 1$. Suppose

$$\dim \operatorname{im} \mathrm{D}f(x) \leqslant m \quad \text{for all } x \in G\,.$$

Then for every $\varepsilon \in \mathbf{R}$ with $0 < \varepsilon < \operatorname{dist}(K, \mathbf{R}^n \sim G)$ there exists a map $f_\varepsilon : \mathbf{R}^n \to \mathbf{R}^n$ of class $\mathscr{C}^k$ satisfying

$$f(x) = f_\varepsilon(x) \quad \text{for } x \in \mathbf{R}^n \sim G\,,$$
$$|f(x) - f_\varepsilon(x)| \leqslant \varepsilon \quad \text{and} \quad |\mathrm{D}f(x) - \mathrm{D}f_\varepsilon(x)| \leqslant \varepsilon \quad \text{for } x \in \mathbf{R}^n\,,$$
$$\mathcal{H}^m(f_\varepsilon[K]) \leqslant \varepsilon \mathcal{H}^m(K)\,.$$

1. Let $a \in R = \operatorname{spt}\|V\|$ be such that $\boldsymbol{\Theta}^{m*}(R_{\mathrm{irr}}, a) > 0$ and $\boldsymbol{\Theta}^{m}(R_{\mathrm{reg}}, a) = 0$.
2. Find $r > 0$ with $\boldsymbol{\Theta}^{m}(R_{\mathrm{reg}}, a, r) < \varepsilon \ll \boldsymbol{\Theta}^{m}(R_{\mathrm{irr}}, a, r)$.
3. Apply deformation theorem to $R \cap \mathbf{B}(a, r)$ to find $f : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$.
4. Find a $\mathscr{C}^1$ perturbation φ of $f(1,\cdot)$ such that $\mathcal{H}^m(\varphi[R_{\mathrm{irr}}]) \leqslant \varepsilon \mathcal{H}^m(R_{\mathrm{irr}})$.
5. Then $\mathcal{H}^m(\varphi[R \cap \mathbf{B}(a, r)]) < \varepsilon C \mathcal{H}^m(R \cap \mathbf{B}(a, r))$.
6. Choose $S \in \mathcal{P}(B)$ so that $\mathbf{v}(S)$ is weakly close to V.
7. Then $\mathcal{H}^m(\varphi[S \cap \mathbf{B}(a, r)]) < \varepsilon C \mathcal{H}^m(S \cap \mathbf{B}(a, r))$.
8. Consequently,
   $\Phi_F(\varphi[S]) < V(F) = \inf\{\Phi_F(K) : K \in \mathcal{P}(B)\}$.
   A contradiction!

$$\boldsymbol{\Theta}^m(\|V\|, x, r) = \frac{\|V\|\mathbf{B}(x, r)}{\boldsymbol{\alpha}(m) r^m}, \quad \boldsymbol{\Theta}^m(\|V\|, x) = \lim_{r \downarrow 0} \boldsymbol{\Theta}^m(\|V\|, x, r).$$

1. There exists $C_1, C_2 \in \mathbf{R}$ such that
$$0 < C_1 \leqslant \boldsymbol{\Theta}^m(\|V\|, x, r) \leqslant C_2 < \infty$$
for $x \in \operatorname{spt}\|V\|$ and $r \in (0, \operatorname{dist}(x, \mathbf{R}^n \sim U))$.
2. Minimality of V then yields $(\mathcal{H}^m, m)$ rectifiability of $\operatorname{spt}\|V\|$.
3. Ellipticity of F gives then $\boldsymbol{\Theta}^m(\|V\|, x) = 1$ for $x \in U$ such that $\operatorname{Tan}(\operatorname{spt}\|V\|, x)$ is an m plane.
4. Area formula proves $V = \mathbf{v}(\operatorname{spt}\|V\|)$.

1. Setup:

$$0 = a \in \operatorname{spt}\|V\| \cap U\,, \quad T = \operatorname{Tan}(\operatorname{spt}\|V\|, a) \in \mathbf{G}(n, m)\,,$$
$$r_i \downarrow 0\,, \quad \delta_i \downarrow 0\,, \quad 0 < \varepsilon_i \ll r_i^m\,,$$
$$K_i = \left\{ r_i x : |T_\natural x| \leqslant 1,\ |T_\natural^\perp x| \leqslant \delta_i \right\},$$
$$\xi_i \in \mathscr{C}^\infty(\mathbf{R}^n, \mathbf{R}^n) \quad \text{projects } K_i \text{ onto } T\,, \quad R_i = \xi_i[S_i]\,.$$

2. Minimality of V gives

$$\begin{aligned}\Phi_F(S_i) - \varepsilon_i &\leqslant \Phi_F(V) \leqslant \Phi_F(R_i)\\ &\leqslant \Phi_F(S_i) - \Phi_F(S_i \cap K_i) + \Phi_F(R_i \cap K_i) + \Phi_F(S_i \div R_i)\,,\end{aligned}$$

1. Setup:

$$0 = a \in \operatorname{spt}\|V\| \cap U\,, \quad T = \operatorname{Tan}(\operatorname{spt}\|V\|, a) \in \mathbf{G}(n,m)\,,$$
$$r_i \downarrow 0\,, \quad \delta_i \downarrow 0\,, \quad 0 < \varepsilon_i \ll r_i^m\,,$$
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$$\xi_i \in \mathscr{C}^\infty(\mathbf{R}^n, \mathbf{R}^n) \quad \text{projects } K_i \text{ onto } T\,, \quad R_i = \xi_i[S_i]\,.$$

2. Minimality of V gives

$$\frac{\Phi_F(S_i \cap K_i) - \Phi_F(R_i \cap K_i)}{r_i^m} \leqslant \frac{\varepsilon_i + \Phi_F(S_i \div R_i)}{r_i^m} \to 0\,.$$

3. Since $D_i = R_i \cap K_i = T \cap \mathbf{B}(0, r_i)$ ellipticity of F yields

$$\Phi_{F^a}(S_i \cap K_i) - \Phi_{F^a}(R_i \cap K_i) \gtrsim \mathcal{H}^m(S_i \cap K_i) - \mathcal{H}^m(R_i \cap K_i)\,.$$

Hence,

$$\frac{\mathcal{H}^m(S_i \cap K_i)}{r_i^m} - \boldsymbol{\alpha}(m) = \frac{\mathcal{H}^m(S_i \cap K_i) - \mathcal{H}^m(D_i)}{r_i^m} \to 0\,.$$

$$\boldsymbol{\Theta}^m(\|V\|, x, r) = \frac{\|V\|\mathbf{B}(x, r)}{\boldsymbol{\alpha}(m) r^m}, \quad \boldsymbol{\Theta}^m(\|V\|, x) = \lim_{r \downarrow 0} \boldsymbol{\Theta}^m(\|V\|, x, r).$$

1. There exists $C_1, C_2 \in \mathbf{R}$ such that

$$0 < C_1 \leqslant \boldsymbol{\Theta}^m(\|V\|, x, r) \leqslant C_2 < \infty$$

   for $x \in \operatorname{spt}\|V\|$ and $r \in (0, \operatorname{dist}(x, \mathbf{R}^n \sim U))$.
2. Minimality of V then yields $(\mathcal{H}^m, m)$ rectifiability of $\operatorname{spt}\|V\|$.
3. Ellipticity of F gives then $\boldsymbol{\Theta}^m(\|V\|, x) = 1$ for $x \in U$ such that $\operatorname{Tan}(\operatorname{spt}\|V\|, x)$ is an m plane.
4. Area formula proves $V = \mathbf{v}(\operatorname{spt}\|V\|)$.

1. Consider $T \in \mathbf{G}(n,m)$ and sets K_i and R_i as before.
2. Define $Q_i = \boldsymbol{\mu}_{1/r_i}[R_i \cap K_i]$.
3. We know $\mathcal{H}^m(Q_i) \to \boldsymbol{\alpha}(m)$ as $i \to \infty$.
4. Area formula applied to $T_\natural$ gives

$$\boldsymbol{\alpha}(m) = \mathcal{H}^m(D) = \int_{Q_i} \|{\textstyle\bigwedge_m} T_\natural \circ \mathrm{Tan}(Q_i,x)_\natural\| \,\mathrm{d}\mathcal{H}^m(x) \to \boldsymbol{\alpha}(m)\,.$$

5. Note: $1 - \|{\textstyle\bigwedge_m} T_\natural \circ \mathrm{Tan}(Q_i,x)_\natural\| \approx \|T_\natural - \mathrm{Tan}(Q_i,x)_\natural\|^2$.
6. Hence,

$$V(\alpha) = \int_{\mathrm{spt}\,\|V\|} \alpha(x, \mathrm{Tan}(\mathrm{spt}\,\|V\|, x))\,\mathrm{d}\mathcal{H}^m(x)\,,$$

whenever $\alpha \in \mathscr{C}^0_{\mathrm{c}}(U \times \mathbf{G}(n,m))$.

Q.E.D.