Some aspects of regularity theory for integral varifolds

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Definition (Almgren (1965))

Let $U \subseteq \mathbf{R}^n$ be open. An m-dimensional varifold in U is a Radon measure V on $U \times \mathbf{G}(n,m)$. The set of all m-varifolds in U with the topology of weak convergence is denoted by $\mathbf{V}_m(U)$.

Example

If M is an m-dimensional smooth submanifold of U, then the associated varifold $\mathbf{v}(M) \in \mathbf{V}_m(U)$ is defined by

$$\mathbf{v}(M)(f) = \int_M f(x, \operatorname{Tan}(M,x)) \, \mathrm{d}\mathcal{H}^m(x)$$

for $f \in \mathscr{C}_c^0(U \times \mathbf{G}(n,m))$.

A set $E \subseteq U$ is called $(\mathcal{H}^m, m)$ RECTIFIABLE if there exists a collection $\mathcal{C}$ of $\mathscr{C}^1$ submanifolds of U such that

$$\mathcal{H}^m(E \sim \bigcup \mathcal{C}) = 0 \quad \text{and} \quad \mathcal{H}^m(E) < \infty.$$

Definition

If $E \subseteq U$ is $(\mathcal{H}^m, m)$ rectifiable, then we define $\mathbf{v}(E) \in \mathbf{V}_m(U)$

$$\mathbf{v}(E)(f) = \int_E f(x, \operatorname{Tan}^m(\mathcal{H}^m \llcorner E, x)) \, \mathrm{d}\mathcal{H}^m(x) \, .$$

for $f \in \mathscr{C}_c^0(U \times \mathbf{G}(n, m))$.

Definition

A varifold V is called **rectifiable** if it is of the form

$$V = \sum_{i=1}^{\infty} \alpha_i \mathbf{v}(E_i),$$

where E_i are $(\mathcal{H}^m, m)$ rectifiable and $0 < \alpha_i < \infty$. In this case we write $V \in \mathbf{RV}_m(U)$.

Definition

If α_i are all *integer* numbers then V is called **integral** and we write $V \in \mathbf{IV}_m(U)$.

Remark

The numbers α_i are called **multiplicities** or **densities**.

Remark

$\mathbf{v}(E)$ is a *unit density integral varifold* for any $(\mathcal{H}^m, m)$ rectifiable set E.

- **Push forward**: $\phi \in \mathscr{C}^1_c(U, V)$ and $B \subseteq V \times \mathbf{G}(n,m)$,

  $$\phi_\# V(B) = \int_{\{(x,S):(\phi(x), D\phi(x)[S])\in B\}} |\textstyle\bigwedge_m (D\phi(x)\circ S_\natural)| \,\mathrm{d}V(x,S)\,.$$

- **Weight measure**: $p : U \times \mathbf{G}(n,m) \to U$ and $A \subseteq U$,

  $$\|V\|(A) = p_\sharp V(A) = V(A \times \mathbf{G}(n,m))\,.$$

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- **First and second variation**:

$$\delta V(g) = \frac{d}{dt} \|\phi_{t\#} V\| \Big|_{t=0} \,, \qquad \delta^2 V(g) = \frac{d^2}{dt^2} \|\phi_{t\#} V\| \Big|_{t=0} \,,$$

where

- $g \in \mathscr{C}_c^\infty(U, \mathbf{R}^n)$ is a *vector field*,
- ϕ_t is the *flow of diffeomorphisms* generated by g and
- $\operatorname{spt} g \subseteq G$ for some open set $G \subseteq U$ with $\|V\|(G) < \infty$.

- If $g \in \mathscr{C}_c^\infty(U, \mathbf{R}^n)$, then

$$\delta V(g) = \int Dg(x) \bullet S_\natural \, \mathrm{d}V(x,S) = \int \operatorname{div}_S g(x) \, \mathrm{d}V(x,S) \, .$$

- Total variation: largest Borel regular measure $\|\delta V\|$ on U such that whenever $G \subseteq U$ is open

$$\|\delta V\|(G) = \sup\{\delta V(g) : \operatorname{spt}(g) \subseteq G \text{ and } |g| \leq 1\}$$

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- If $\|\delta V\|$ is Radon, then

$$\delta V(g) = -\int g(x) \bullet \mathbf{h}(V,x) \, \mathrm{d}\|V\|(x) + \int g(x) \bullet \boldsymbol{\eta}(V,x) \, \mathrm{d}\|\delta V\|_{\mathrm{sing}}(x) \, ,$$

where

  - $\boldsymbol{\eta}(V,\cdot) : \mathbf{R}^n \to \mathbf{S}^{n-1}$ "unit normal at the boundary",
  - $\mathbf{h}(V,\cdot) : \mathbf{R}^n \to \mathbf{R}^n$ "mean curvature vector",
  - $\|\delta V\|_{\mathrm{sing}}$ "boundary measure".

Definition

V is called **stationary** if $\delta V = 0$, i.e.,

$$\|\delta V\|_{\text{sing}} = 0 \quad \text{and} \quad \mathbf{h}(V, \cdot) \equiv 0\,.$$

Example

- If $E \in \mathbf{G}(n, m)$, then $\mathbf{v}(E)$ is stationary.
- If $E \subseteq \mathbf{R}^3$ is a minimal surface (e.g. a *catenoid*), then $\mathbf{v}(E)$ is stationary.
- If $\alpha, \beta \in (0, \infty)$ and $V_1, V_2 \in \mathbf{V}_m(U)$ are stationary, then $\alpha V_1 + \beta V_2$ is stationary as well.

Theorem (Almgren (1965))

Let $G_1, G_2, \ldots$ be open subsets of $\mathbf{R}^n$ such that $\bigcup_i G_i = \mathbf{R}^n$ and $M_1, M_2, \ldots$ be real numbers. Then

$$\left\{V \in \mathbf{IV}_m(\mathbf{R}^n) : (\|V\| + \|\delta V\|)(G_i) \leq M_i\right\}$$

is compact in $\mathbf{IV}_m(\mathbf{R}^n)$ with respect to weak convergence.

Remark

The important thing is that the limit varifolds are still <u>integral</u>.

Sobolev maps $\sim$ limits of smooth maps

integral varifolds $\sim$ limits of smooth manifolds

varifolds $\approx$ currents without orientation

Theorem (Allard, Ann. of Math. (1972))

If $V \in \mathbf{V}_m(U)$, $m < p < \infty$,

$$\|\delta V\|_{sing} = 0 \quad \textit{and} \quad \mathbf{h}(V, \cdot) \in L^p_{loc}(\|V\|, \mathbf{R}^n),$$

then there exists an open set $G \subseteq U$ *such that*

- $\operatorname{reg} V := \operatorname{spt} \|V\| \cap G$ *is dense in* $\operatorname{spt} \|V\| \cap U$ *and*
- $\operatorname{reg} V$ *is an embedded* $\mathscr{C}^{1,\alpha}$ *submanifold of* $\mathbf{R}^n$, *where* $\alpha = 1 - m/p$.

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Example

- $C \subseteq \mathbf{R}$ a "thick" Cantor set, i.e., having $\mathcal{L}^1(C) > 0$,
- $f : \mathscr{C}^\infty(\mathbf{R})$ such that $C = \{x : f(x) = 0\}$ ("a smooth distance function"),
- $V = \mathbf{v}(\operatorname{graph} f) + \mathbf{v}(\mathbf{R} \times \{0\})$.

Then $\mathbf{h}(V, \cdot) \in L^\infty_{\text{loc}}$ but $\operatorname{sing} V = C$; hence, $\mathcal{H}^1(\operatorname{sing} V) > 0$.

Let $T \in \mathbf{G}(n,m)$, $a \in \mathbf{R}^n$, $0 < r < \infty$. Define

$$\operatorname{tilt}_2(a,r,T) = \left(r^{-m} \int_{\mathbf{B}(a,r)\times\mathbf{G}(n,m)} \|S_\natural - T_\natural\|^2 \, \mathrm{d}V(z,S) \right)^{1/2},$$

$$\operatorname{height}_2(a,r,T) = \left(r^{-m} \int_{\mathbf{B}(a,r)} |T_\natural^\perp z|^2 \, \mathrm{d}\|V\|(z) \right)^{1/2}.$$

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Remark

If $V = \mathbf{v}(\operatorname{graph} u)$ for some $u \in \mathscr{C}^1(T, T^\perp)$, $u(0) = 0$, $Du(0) = 0$, then

$$\begin{aligned} \operatorname{tilt}_2(0,r,T) &\sim \left(r^{-m} \int_{T \cap \mathbf{B}(0,r)} |Du(z)|^2 \, \mathrm{d}z \right)^{1/2}, \\ \operatorname{height}_2(0,r,T) &\sim \left(r^{-m} \int_{T \cap \mathbf{B}(0,r)} |u(z)|^2 \, \mathrm{d}z \right)^{1/2}. \end{aligned}$$

Suppose $V \in \mathbf{IV}_m(\mathbf{R}^n)$, $\|\delta V\|$ is a Radon measure,

$$\mathbf{h}(V,\cdot) \in L^p_{\mathrm{loc}} \quad \text{and} \quad \|\delta V\|_{\mathrm{sing}} = 0\,.$$

For which $0 < \alpha \leq 1$ (depending on m and p)

$$\limsup_{r \downarrow 0} r^{-\alpha}\, \mathrm{tilt}_2(a, r, T) < \infty$$

for V almost all (a, T)?

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for V almost all (a,T)?

Related PDE question:

Suppose $u \in \mathbf{W}^{1,1}(\Omega)$ and $\Delta u = f \in L^p$. For which $0 < \alpha \leq 1$

$$\limsup_{r \downarrow 0} r^{-\alpha} \Big(r^{-m} \int_{\mathbf{B}(x,r)} |Du(z) - Du(x)|^2 \,\mathrm{d}z \Big)^{1/2} < \infty$$

for $\mathcal{L}^m$ almost all $x \in \Omega$?

$V \in \mathbf{IV}_m(\mathbf{R}^n)$, $\mathbf{h}(V, \cdot) \in L^p_{\mathrm{loc}}$, $\|\delta V\|_{\mathrm{sing}} = 0$. For which $0 < \alpha \leq 1$

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(Schätzle, Ann. Sc. Norm. Super. Pisa Cl. Sci. (2004))

$$p > m\,, p \geq 2 \quad \Rightarrow \alpha = 1\,,$$

(Menne, J. Geom. Anal. (2013), Arch. Ration. Mech. Anal. (2012))

$$m > 2 \quad \Rightarrow \alpha = \min\left\{1\,, \tfrac{mp}{2(m-p)}\right\} \quad \left[\text{borderline } p = \frac{2m}{m+2}\right]\,,$$

$$m = 2\,, p > 1 \quad \Rightarrow \alpha = 1\,,$$

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$$m = 1\,, p \geq 1 \quad \Rightarrow \alpha = 1\,.$$

$V \in \mathbf{IV}_m(\mathbf{R}^n)$, $\mathbf{h}(V,\cdot) \in L^p_{\mathrm{loc}}$, $\|\delta V\|_{\mathrm{sing}} = 0$. For which $0 < \alpha \leq 1$

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$$m = 1\,, p \geq 1 \quad \Rightarrow \alpha = 1\,.$$

Theorem (K. & Menne (2015); arXiv:1501.07037)

Suppose $V \in \mathbf{IV}_2(\mathbf{R}^n)$ *(*$m = 2$*) and* $\|\delta V\|$ *is a Radon measure (*$p = 1$*). Then*

$$\lim_{r \downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a, r, T) = 0$$

for V *almost all* (a, T)*.*

Moreover, for any modulus of continuity $\omega : \mathbf{R} \to \mathbf{R}_+$*, there exists* $V \in \mathbf{IV}_2(\mathbf{R}^3)$ *such that*

$$\lim_{r \downarrow 0} r^{-1} \log(1/r)^{-1/2} \omega(r)^{-1} \operatorname{tilt}_2(a, r, T) > 0$$

for (a, T) *in a set of positive* V *measure.*

Proposition

Suppose $a \in \mathbf{R}^2$ ($m = 2$), $r > 0$, and $u \in \mathbf{W}_0^{1,1}(\mathbf{U}(a, r))$ is such that $\Delta u = f \in L^1$ ($p = 1$). Then for $\mathcal{L}^2$ almost all $b \in \mathbf{U}(a, r)$ there holds

$$\limsup_{s \downarrow 0} s^{-2} \| Du(\cdot) - Du(b) \|_{(2,\infty);\mathbf{U}(b,s)} < \infty .$$

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Remark

If $|Du|$ is additionally bounded then one obtains

$$\limsup_{s \downarrow 0} s^{-2} \log(1/r)^{-1/2} \|Du(\cdot) - Du(b)\|_{2;\mathbf{U}(b,s)} < \infty$$

for $\mathcal{L}^2$ almost all $b \in \mathbf{U}(a, r)$.

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for $\mathcal{L}^2$ almost all $b \in \mathbf{U}(a, r)$.

Recall: $\boxed{|Du(\cdot) - Du(b)| \quad \sim \quad \|S_\natural - T_\natural\| \leq 1}$

Assume $U \subseteq \mathbf{R}^2$ is open, and $\mathcal{L}^2(U) = 1$, and $f \in L^\infty(U)$, and $\|f\|_{\infty;U} = 1$. Set $A = \|f\|_{(2,\infty);U}$. Then

$$\begin{aligned}
\|f\|_{2;U}^2 &= \int_{U\cap\{x\,:\,|f(x)|\le A\}} |f|^2 \,\mathrm{d}\mathcal{L}^2 + 2\int_A^1 t\mathcal{L}^2(U\cap\{x : |f(x)| > t\})\,\mathrm{d}\mathcal{L}^1 t \\
&\le A + 2A\int_A^1 t^{-1}\,\mathrm{d}\mathcal{L}^1 t \\
&= A\,(1 + 2\ln(1/A)) = \|f\|_{(2,\infty);U}\big(1 + 2\ln(\|f\|_{(2,\infty);U}^{-1})\big)\,.
\end{aligned}$$

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Corollary

Whenever $x \in U$ *and*

$$\limsup_{r\to 0+} r^{-2}\|f\|_{(2,\infty);\mathbf{B}(x,r)} < \infty,$$

then

$$\limsup_{r\to 0+} r^{-2}\log(1/r)^{-1/2}\|f\|_{2;\mathbf{B}(x,r)} < \infty.$$

Proposition

Suppose $a \in \mathbf{R}^2$, $r > 0$, and $u \in \mathbf{W}_0^{1,\infty}(\mathbf{U}(a, r))$ is such that $\Delta u = f \in L^1$. Then for $\mathcal{L}^2$ almost all $b \in \mathbf{U}(a, r)$ there holds

$$\limsup_{s \downarrow 0} s^{-2} \log(1/r)^{-1/2} \| Du(\cdot) - Du(b) \|_{2;\mathbf{U}(b,s)} < \infty$$

Setup: $u \in \mathbf{W}_0^{1,\infty}(\mathbf{R}^2)$ and $\Delta u = f \in L^1$ and $a \in \mathbf{R}^2$, $r > 0$.
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1. **Derive a coercive estimate** (following Brakke):

$$4r^{-2} \int_{\mathbf{B}(a,r/2)} |Du|^2 \, \mathrm{d}\mathcal{L}^2$$
$$\leq C\lambda\big(r^{-2}\|f\|_{1;\mathbf{B}(a,r)} \|u\|_{r^{-2}\Phi;\mathbf{B}(a,r)}\big) + Cr^{-4}\|u\|^2_{2;\mathbf{B}(a,r)},$$

where

$$\Phi(t) = \exp(t^2) - 1, \quad \lambda(t) = t \log(1 + 1/t)^{1/2},$$

$$\|u\|_{\Phi,\mathbf{B}(a,r)} = \inf\left\{\gamma \in \mathbf{R}_+ : \int_{\mathbf{B}(a,r)} \Phi(\gamma^{-1}|u|) \, \mathrm{d}\mathcal{L}^2 \leq 1\right\}.$$

This is done by:

- taking a cut-off function $\varphi \in \mathscr{C}_0^\infty$,
- using φu as a test function,
- truncating at a specific height,
- and computing.

Setup: $u \in \mathbf{W}_0^{1,\infty}(\mathbf{R}^2)$ and $\Delta u = f \in L^1$ and $a \in \mathbf{R}^2$, $r > 0$.
Goal: $\limsup_{s \downarrow 0} s^{-2} \log(1/r)^{-1/2} \|Du(\cdot) - Du(b)\|_{2;\mathbf{U}(b,s)} < \infty$.

1. DERIVE A COERCIVE ESTIMATE:

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2. Use Poincaré inequality:
For $q > 2 = m$ and any $\mathcal{L}^2$ measurable set $A \subseteq \mathbf{U}(a,r)$

$$\Big\|u(\cdot) - \fint_A u \, \mathrm{d}\mathcal{L}^2\Big\|_{q;\mathbf{U}(a,r)} \leq Cq^{1/2} \frac{r^2}{\mathcal{L}^2(A)} r^{2/q} \|Du\|_{2;\mathbf{U}(a,r)}.$$

to derive the INTERPOLATION INEQUALITIES:

$$r^{-2/q}\|u\|_{q;\mathbf{U}(a,r)} \leq C\big(q^{1/2}\|Du\|_{2;\mathbf{U}(a,r)} + r^{-1}\|u\|_{2;A}\big)$$
$$\|u\|_{r^{-2}\Phi;\mathbf{U}(a,r)} \leq C\big(\|Du\|_{2;\mathbf{U}(a,r)} + r^{-1}\|u\|_{2;A}\big).$$

provided $\mathcal{L}^2(A) > \frac{1}{2}\mathcal{L}^2(\mathbf{U}(a,r))$.

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2. Apply interpolation inequalities:
For $q > 2$ and $A \subseteq \mathbf{U}(a, r)$ with $\mathcal{L}^2(A) > \frac{1}{2}\mathcal{L}^2(\mathbf{U}(a, r))$

$$r^{-2/q}\|u\|_{q;\mathbf{U}(a,r)} \leq C\big(q^{1/2}\|Du\|_{2;\mathbf{U}(a,r)} + r^{-1}\|u\|_{2;A}\big)$$
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$$\|u\|_{r^{-2}\Phi;\mathbf{U}(a,r)} \leq C\big(\|Du\|_{2;\mathbf{U}(a,r)} + r^{-1}\|u\|_{2;A}\big).$$

3. Use $\mathscr{C}^2$ rectifiability of u:
There exist $\mathscr{C}^2$ functions v_i such that if
$A_i = \{x : u(x) = v_i(x)\}$, then $\mathcal{L}^2\big(\mathbf{R}^2 \sim \bigcup_{i \in \mathbb{N}} A_i\big) = 0$.

Setup: $u \in \mathbf{W}_0^{1,\infty}(\mathbf{R}^2)$ and $\Delta u = f \in L^1$ and $a \in \mathbf{R}^2$, $r > 0$.
Goal: $\limsup_{s \downarrow 0} s^{-2} \log(1/r)^{-1/2} \|Du(\cdot) - Du(b)\|_{2;\mathbf{U}(b,s)} < \infty$.

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2. APPLY INTERPOLATION INEQUALITIES:
For $q > 2$ and $A \subseteq \mathbf{U}(a,r)$ with $\mathcal{L}^2(A) > \frac{1}{2}\mathcal{L}^2(\mathbf{U}(a,r))$

$$r^{-2/q}\|u\|_{q;\mathbf{U}(a,r)} \leq C\big(q^{1/2}\|Du\|_{2;\mathbf{U}(a,r)} + r^{-1}\|v_i\|_{2;A_i}\big)$$
$$\|u\|_{r^{-2}\Phi;\mathbf{U}(a,r)} \leq C\big(\|Du\|_{2;\mathbf{U}(a,r)} + r^{-1}\|v_i\|_{2;A_i}\big).$$

3. USE $\mathscr{C}^2$ RECTIFIABILITY OF u:
There exist $\mathscr{C}^2$ functions v_i such that if
$A_i = \{x : u(x) = v_i(x)\}$, then $\mathcal{L}^2\big(\mathbf{R}^2 \sim \bigcup_{i \in \mathbb{N}} A_i\big) = 0$.

Theorem (K. & Menne (2015); arXiv:1501.07037)

Suppose $V \in \mathbf{IV}_2(\mathbf{R}^n)$ *and* $\|\delta V\|$ *is a Radon measure. Then*

$$\lim_{r \downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a, r, T) = 0$$

for V *almost all* (a, T).

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V, \cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n, 2)$.

Goal: $\lim_{r \downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a, r, T) < \infty$.

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V, \cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n, 2)$.

Goal: $\lim_{r \downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a, r, T) < \infty$.

1. **Derive a coercive estimate** (following Brakke):

$$\operatorname{tilt}_2(a, r/2)^2 \le C r^{-2} \|\delta V\| \, \mathbf{B}(a, 4r)^2 + C\lambda\big(\operatorname{height}_\Phi(G, a, r) r^{-2} \|\delta V\| \, \mathbf{B}(a, 4r)\big) + C r^{-2} \operatorname{height}_2(G, a, r)^2 ,$$

where

$$\Phi(t) = \exp(t^2) - 1 , \quad \lambda(t) = t \log(1 + 1/t)^{1/2} ,$$

$$G = \{ z \in \operatorname{spt} \|V\| : \|V\| \, \mathbf{B}(z, s) \ge \delta s^2 \} ,$$

$$\operatorname{height}_\Phi(G, a, r) = \inf \{ \gamma \in \mathbf{R}_+ : r^{-2} \textstyle\int_{G \cap \mathbf{B}(a, 4r)} \Phi(\gamma^{-1} |T_\natural^\perp z|) \, \mathrm{d}\|V\|(z) \le 1 \} ,$$

$$\operatorname{tilt}_2(a, r) = \Big(r^{-m} \textstyle\int_{\mathbf{B}(a, r) \times \mathbf{G}(n, m)} \|S_\natural - T_\natural\|^2 \, \mathrm{d}V(z, S) \Big)^{1/2} ,$$

$$\operatorname{height}_2(G, a, r) = \Big(r^{-m} \textstyle\int_{G \cap \mathbf{B}(a, r)} |T_\natural^\perp z|^2 \, \mathrm{d}\|V\|(z) \Big)^{1/2} .$$

Recall the PDE case:

$$4 r^{-2} \textstyle\int_{\mathbf{B}(a, r/2)} |Du|^2 \, \mathrm{d}\mathcal{L}^2 \le C \lambda\big(r^{-2} \|f\|_{1; \mathbf{B}(a, r)} \|u\|_{r^{-2}\Phi; \mathbf{B}(a, r)} \big) + C r^{-4} \|u\|^2_{2; \mathbf{B}(a, r)} ;$$

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V,\cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n,2)$.

Goal: $\lim_{r \downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a,r,T) < \infty$.

1. **Derive a coercive estimate:**

$$\operatorname{tilt}_2(a,r/2)^2 \le Cr^{-2} \|\delta V\| \mathbf{B}(a,\, 4r)^2 + C\lambda\big(\operatorname{height}_\Phi(G,a,r) r^{-2} \|\delta V\| \mathbf{B}(a,\, 4r)\big) + Cr^{-2} \operatorname{height}_2(G,a,r)^2 \,,$$

2. **Use multivalued Lipschitz approximation:**

   There exists a set of *good points* $X \subseteq \mathbf{B}(a,\, r) \cap T$ and a 1-Lipschitz function $\tilde{f} : X \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ such that if

$$\eta := \|\delta V\| \mathbf{B}(a,\, r)^2 + r^2 \operatorname{tilt}_2(\mathbf{B}(a,\, r), r)^2 \,,$$

   then

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V,\cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n,2)$.

Goal: $\lim_{r\downarrow 0} r^{-1}\log(1/r)^{-1/2}\,\mathrm{tilt}_2(a,r,T) < \infty$.

1. **Derive a coercive estimate:**

$$\mathrm{tilt}_2(a,r/2)^2 \le Cr^{-2}\|\delta V\|\,\mathbf{B}(a,\,4r)^2 + C\lambda\big(\mathrm{height}_\Phi(G,a,r)r^{-2}\|\delta V\|\,\mathbf{B}(a,\,4r)\big) + Cr^{-2}\,\mathrm{height}_2(G,a,r)^2\,,$$

2. **Use multivalued Lipschitz approximation:** There exists a set of *good points* $X \subseteq \mathbf{B}(a,\,r)\cap T$ and a 1-Lipschitz function $\tilde{f} : X \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ such that if

$$\eta := \|\delta V\|\,\mathbf{B}(a,\,r)^2 + r^2\,\mathrm{tilt}_2(\mathbf{B}(a,\,r),r)^2\,,$$

then

- the Q-graph of $\tilde{f}$ is a subset of the support of V;

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V,\cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n,2)$.

Goal: $\lim_{r\downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a,r,T) < \infty$.

1. **Derive a coercive estimate**:

$$\operatorname{tilt}_2(a,r/2)^2 \le Cr^{-2}\|\delta V\|\,\mathbf{B}(a,\,4r)^2$$
$$+\,C\lambda\big(\operatorname{height}_\Phi(G,a,r)r^{-2}\|\delta V\|\,\mathbf{B}(a,\,4r)\big) + Cr^{-2}\operatorname{height}_2(G,a,r)^2\,,$$

2. **Use multivalued Lipschitz approximation**:
   There exists a set of *good points* $X \subseteq \mathbf{B}(a,\,r) \cap T$ and a 1-Lipschitz function $\tilde{f} : X \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ such that if

$$\eta := \|\delta V\|\,\mathbf{B}(a,\,r)^2 + r^2\operatorname{tilt}_2(\mathbf{B}(a,\,r),r)^2\,,$$

   then
   - the Q-graph of $\tilde{f}$ is a subset of the support of V;
   - the $\mathcal{L}^2$ measure of the *bad set* "downstairs", i.e., $\mathcal{L}^2(\mathbf{B}(a,\,r)\cap T\sim X)$ is controlled by good terms η;

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V,\cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n,2)$.

Goal: $\lim_{r\downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a,r,T) < \infty$.

1. **Derive a coercive estimate:**

$$\operatorname{tilt}_2(a,r/2)^2 \le Cr^{-2}\|\delta V\|\,\mathbf{B}(a,\,4r)^2$$
$$+\,C\lambda\big(\operatorname{height}_\Phi(G,a,r)r^{-2}\|\delta V\|\,\mathbf{B}(a,\,4r)\big) + Cr^{-2}\operatorname{height}_2(G,a,r)^2\,,$$

2. **Use multivalued Lipschitz approximation:**
There exists a set of *good points* $X \subseteq \mathbf{B}(a,\,r) \cap T$ and a 1-Lipschitz function $\tilde{f} : X \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ such that if

$$\eta := \|\delta V\|\,\mathbf{B}(a,\,r)^2 + r^2\operatorname{tilt}_2(\mathbf{B}(a,\,r),r)^2\,,$$

then

- the Q-graph of $\tilde{f}$ is a subset of the support of V;
- the $\mathcal{L}^2$ measure of the *bad set* “downstairs”, i.e., $\mathcal{L}^2(\mathbf{B}(a,\,r) \cap T \sim X)$ is controlled by good terms η;
- the $\|V\|$ measure of the *bad set* “upstairs”, i.e., $\|V\|(\mathbf{B}(a,\,r) \sim (X \times T^\perp))$ is controlled by η as well;

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V,\cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n,2)$.

Goal: $\lim_{r \downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a,r,T) < \infty$.

1. DERIVE A COERCIVE ESTIMATE:

$$\operatorname{tilt}_2(a,r/2)^2 \le Cr^{-2}\|\delta V\|\,\mathbf{B}(a,\,4r)^2$$
$$+\,C\lambda\big(\operatorname{height}_\Phi(G,a,r)r^{-2}\|\delta V\|\,\mathbf{B}(a,\,4r)\big) + Cr^{-2}\operatorname{height}_2(G,a,r)^2\,,$$

2. USE MULTIVALUED LIPSCHITZ APPROXIMATION:

There exists a set of *good points* $X \subseteq \mathbf{B}(a,\,r) \cap T$ and a 1-Lipschitz function $\tilde{f} : X \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ such that if

$$\eta := \|\delta V\|\,\mathbf{B}(a,\,r)^2 + r^2\operatorname{tilt}_2(\mathbf{B}(a,\,r),r)^2\,,$$

then

- the Q-graph of $\tilde{f}$ is a subset of the support of V;
- the $\mathcal{L}^2$ measure of the *bad set* “downstairs”, i.e., $\mathcal{L}^2(\mathbf{B}(a,\,r) \cap T \sim X)$ is controlled by good terms η;
- the $\|V\|$ measure of the *bad set* “upstairs”, i.e., $\|V\|(\mathbf{B}(a,\,r) \sim (X \times T^\perp))$ is controlled by η as well;
- the L^2 norm of $D\tilde{f}$ is also controlled by η;

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V,\cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n,2)$.

Goal: $\lim_{r \downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a,r,T) < \infty$.

1. **Derive a coercive estimate:**

$$\operatorname{tilt}_2(a,r/2)^2 \le Cr^{-2} \|\delta V\| \, \mathbf{B}(a,\, 4r)^2 + C\lambda\big(\operatorname{height}_\Phi(G,a,r) r^{-2} \|\delta V\| \, \mathbf{B}(a,\, 4r)\big) + Cr^{-2} \operatorname{height}_2(G,a,r)^2 ,$$

2. **Use multivalued Lipschitz approximation:**

There exists a set of *good points* $X \subseteq \mathbf{B}(a,\, r) \cap T$ and a 1-Lipschitz function $\tilde f : X \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ such that if

$$\eta := \|\delta V\| \, \mathbf{B}(a,\, r)^2 + r^2 \operatorname{tilt}_2(\mathbf{B}(a,\, r), r)^2 ,$$

then

- the Q-graph of $\tilde f$ is a subset of the support of V;
- the $\mathcal{L}^2$ measure of the *bad set* “downstairs”, i.e., $\mathcal{L}^2(\mathbf{B}(a,\, r) \cap T \sim X)$ is controlled by good terms η;
- the $\|V\|$ measure of the *bad set* “upstairs”, i.e., $\|V\|(\mathbf{B}(a,\, r) \sim (X \times T^\perp))$ is controlled by η as well;
- the L^2 norm of $D\tilde f$ is also controlled by η;
- the height excess on the set G is controlled by $\tilde f$ plus η.

  Recall: $G = \{ z \in \operatorname{spt} \|V\| : \|V\| \, \mathbf{B}(z,\, s) \ge \delta s^2 \}$

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V,\cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n,2)$.

Goal: $\lim_{r \downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a,r,T) < \infty$.

1. DERIVE A COERCIVE ESTIMATE:

$$\operatorname{tilt}_2(a,r/2)^2 \le Cr^{-2} \|\delta V\| \mathbf{B}(a,4r)^2 + C\lambda\big(\operatorname{height}_\Phi(G,a,r) r^{-2} \|\delta V\| \mathbf{B}(a,4r)\big) + Cr^{-2} \operatorname{height}_2(G,a,r)^2,$$

2. USE LIPSCHITZ APPROXIMATION:

Set $f(x) = \max\{|z| : z \in \tilde{f}(x)\}$.

Assume that $Q \in \mathbb{N}$, $a \in \operatorname{spt} \|V\|$, $T \in \mathbf{G}(n,2)$, $0 < r < \infty$. There exists $\Gamma > 1$ such that if

$$\|V\| \mathbf{B}(a,r) \approx Q\pi r^2,$$
$$\eta := \|\delta V\| \mathbf{B}(a,r)^2 + r^2 \operatorname{tilt}_2(\mathbf{B}(a,r),r)^2 \le \Gamma^{-1} r^2,$$

then there exists a Borel set $X \subseteq \mathbf{B}(a,r) \cap T$ and a function $f : X \to \mathbf{R}$, such that $\operatorname{Lip} f \le 1$ and

$$\mathcal{L}^2(\mathbf{B}(a,r) \cap T \sim X) + \|V\|(\mathbf{B}(a,r) \sim (X \times T^\perp)) \le \Gamma\eta$$
$$\operatorname{height}_2(G,a,r) \le \Gamma r(\|f\|_{2,X} + \eta)$$
$$\operatorname{height}_\Phi(G,a,r) \le \Gamma(\|f\|_{r^{-2}\Phi,X} + \eta^{1/2})$$
$$\|f\|_{2,X} \le \Gamma(\operatorname{height}_2(X \times T^\perp, r))$$
$$\|Df\|_{2,X} \le \Gamma\eta^{1/2}$$

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V,\cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n,2)$.

Goal: $\lim_{r\downarrow 0} r^{-1} \log(1/r)^{-1/2}\, \mathrm{tilt}_2(a,r,T) < \infty$.

1. **Derive a coercive estimate:**

$$\mathrm{tilt}_2(a,r/2)^2 \le C r^{-2} \|\delta V\|\, \mathbf{B}(a,\,4r)^2 + C\lambda\big(\mathrm{height}_\Phi(G,a,r) r^{-2} \|\delta V\|\, \mathbf{B}(a,\,4r)\big) + C r^{-2}\, \mathrm{height}_2(G,a,r)^2 ,$$

2. **Use Lipschitz approximation:**

$$\eta := \|\delta V\|\, \mathbf{B}(a,\,r)^2 + r^2\, \mathrm{tilt}_2(\mathbf{B}(a,\,r),r)^2 ,$$

$$\mathcal{L}^2(\mathbf{B}(a,\,r) \cap T \sim X) + \|V\|(\mathbf{B}(a,\,r) \sim (X \times T^\perp)) \le \Gamma \eta ,$$

$$\mathrm{height}_2(G,a,r) \le \Gamma r(\|f\|_{2,X} + \eta) ,$$

$$\mathrm{height}_\Phi(G,a,r) \le \Gamma(\|f\|_{r^{-2}\Phi,X} + \eta^{1/2}) ,$$

$$\|f\|_{2,X} \le \Gamma(\mathrm{height}_2(X \times T^\perp, r)) , \quad \|Df\|_{2,X} \le \Gamma \eta^{1/2} .$$

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V,\cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n,2)$.

Goal: $\lim_{r\downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a,r,T) < \infty$.

1. DERIVE A COERCIVE ESTIMATE:

$$\operatorname{tilt}_2(a,r/2)^2 \le Cr^{-2}\|\delta V\|\,\mathbf{B}(a,4r)^2 + C\lambda\big(\operatorname{height}_\Phi(G,a,r)r^{-2}\|\delta V\|\,\mathbf{B}(a,4r)\big) + Cr^{-2}\operatorname{height}_2(G,a,r)^2,$$

2. USE LIPSCHITZ APPROXIMATION:

$$\eta := \|\delta V\|\,\mathbf{B}(a,r)^2 + r^2 \operatorname{tilt}_2(\mathbf{B}(a,r),r)^2 ,$$
$$\mathcal{L}^2(\mathbf{B}(a,r)\cap T \sim X) + \|V\|(\mathbf{B}(a,r)\sim(X\times T^\perp)) \le \Gamma\eta ,$$
$$\operatorname{height}_2(G,a,r) \le \Gamma r(\|f\|_{2,X} + \eta) ,$$
$$\operatorname{height}_\Phi(G,a,r) \le \Gamma(\|f\|_{r^{-2}\Phi,X} + \eta^{1/2}) ,$$
$$\|f\|_{2,X} \le \Gamma(\operatorname{height}_2(X\times T^\perp,r)) , \quad \|Df\|_{2,X} \le \Gamma\eta^{1/2} .$$

3. APPLY INTERPOLATION INEQUALITIES:

$$r^{-2/q}\|f\|_{q;\mathbf{U}(a,r)} \le C\big(q^{1/2}\|Df\|_{2;\mathbf{U}(a,r)} + r^{-1}\|f\|_{2;A}\big)$$
$$\|f\|_{r^{-2}\Phi;\mathbf{U}(a,r)} \le C\big(\|Df\|_{2;\mathbf{U}(a,r)} + r^{-1}\|f\|_{2;A}\big).$$

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V,\cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n,2)$.

Goal: $\lim_{r\downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a,r,T) < \infty$.

1. **Derive a coercive estimate:**

$$\operatorname{tilt}_2(a,r/2)^2 \le Cr^{-2}\|\delta V\|\,\mathbf{B}(a,\,4r)^2 + C\lambda\big(\operatorname{height}_\Phi(G,a,r)r^{-2}\|\delta V\|\,\mathbf{B}(a,\,4r)\big) + Cr^{-2}\operatorname{height}_2(G,a,r)^2\,,$$

2. **Use Lipschitz approximation:**

$$\eta := \|\delta V\|\,\mathbf{B}(a,\,r)^2 + r^2\operatorname{tilt}_2(\mathbf{B}(a,\,r),r)^2\;,$$
$$\mathcal{L}^2(\mathbf{B}(a,\,r)\cap T\sim X) + \|V\|(\mathbf{B}(a,\,r)\sim(X\times T^\perp)) \le \Gamma\eta\;,$$
$$\operatorname{height}_2(G,a,r) \le \Gamma r(\|f\|_{2,X}+\eta)\;,$$
$$\operatorname{height}_\Phi(G,a,r) \le \Gamma(\|f\|_{r^{-2}\Phi,X}+\eta^{1/2})\;,$$
$$\|f\|_{2,X} \le \Gamma(\operatorname{height}_2(X\times T^\perp,r))\;,\quad \|Df\|_{2,X} \le \Gamma\eta^{1/2}\;.$$

3. **Apply interpolation inequalities:**

$$r^{-2/q}\|f\|_{q;\mathbf{U}(a,\,r)} \le C\big(q^{1/2}\|Df\|_{2;\mathbf{U}(a,\,r)} + r^{-1}\|f\|_{2;A}\big)$$
$$\|f\|_{r^{-2}\Phi;\mathbf{U}(a,\,r)} \le C\big(\|Df\|_{2;\mathbf{U}(a,\,r)} + r^{-1}\|f\|_{2;A}\big).$$

Setup: $V \in \mathbf{IV}_2(\mathbf{R}^n)$, $\mathbf{h}(V,\cdot) \in L^1_{\mathrm{loc}}$, $a \in \mathbf{R}^n$, $r > 0$, $T \in \mathbf{G}(n,2)$.

Goal: $\lim_{r \downarrow 0} r^{-1} \log(1/r)^{-1/2} \operatorname{tilt}_2(a,r,T) < \infty$.

1. DERIVE A COERCIVE ESTIMATE:

$$\operatorname{tilt}_2(a,r/2)^2 \le Cr^{-2}\|\delta V\|\,\mathbf{B}(a,4r)^2$$
$$+ C\lambda\big(\operatorname{height}_\Phi(G,a,r)r^{-2}\|\delta V\|\,\mathbf{B}(a,4r)\big) + Cr^{-2}\operatorname{height}_2(G,a,r)^2,$$

2. USE LIPSCHITZ APPROXIMATION:

$$\eta := \|\delta V\|\,\mathbf{B}(a,r)^2 + r^2 \operatorname{tilt}_2(\mathbf{B}(a,r),r)^2,$$
$$\mathcal{L}^2(\mathbf{B}(a,r) \cap T \sim X) + \|V\|(\mathbf{B}(a,r) \sim (X \times T^\perp)) \le \Gamma\eta,$$
$$\operatorname{height}_2(G,a,r) \le \Gamma r(\|f\|_{2,X} + \eta),$$
$$\operatorname{height}_\Phi(G,a,r) \le \Gamma(\|f\|_{r^{-2}\Phi,X} + \eta^{1/2}),$$
$$\|f\|_{2,X} \le \Gamma(\operatorname{height}_2(X \times T^\perp, r)), \quad \|Df\|_{2,X} \le \Gamma\eta^{1/2}.$$

3. APPLY INTERPOLATION INEQUALITIES:

$$r^{-2/q}\|f\|_{q;\mathbf{U}(a,r)} \le C\big(q^{1/2}\|Df\|_{2;\mathbf{U}(a,r)} + r^{-1}\|f\|_{2;A}\big)$$
$$\|f\|_{r^{-2}\Phi;\mathbf{U}(a,r)} \le C\big(\|Df\|_{2;\mathbf{U}(a,r)} + r^{-1}\|f\|_{2;A}\big).$$

4. USE $\mathscr{C}^2$ RECTIFIABILITY OF V: (MENNE, J. GEOM. ANAL. (2011))

Suppose $V \in \mathbf{IV}_m(U)$ and $\|\delta V\|$ is a Radon measure. Then there exists a countable collection $\mathcal{C}$ of m-dimensional submanifolds of U of class $\mathscr{C}^2$ such that $\|V\|(U \sim \bigcup \mathcal{C}) = 0$.

Thank you for your attention.