Menger curvature for set of arbitrary dimension.

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1. Introduction

For any three points in $\mathbb{R}^n$ we define their Menger curvature as the inverse of the radius of the smallest circle passing through all of them. One can use this notion to define various kinds of global curvatures for non-smooth 1-dimensional curves. It turns out that finiteness of these curvatures imply self-avoidance effects and certain regularity results, which are important in applications for modelling long, entangled physical objects like DNA molecules and protein chains. Integral Menger curvatures, defined as the integral of Menger curvature in some power over all triples of points of a given curve, became useful also for applying topological constraints in variational problems. This allowed to prove the existence of minimizers of some constrained variational problems in a given isotopy class. Mathematically the deepest result so far, is a theorem by Léger [4] who proved that curves with finite integral Menger curvature are 1-rectifiable. This was a crucial step in the proof of Vitushkin's conjecture on removability of singularities of analytic functions.

Intensive research is being done on generalizations of Menger curvature for sets of higher dimension. It occurs that one cannot define k-dimensional Menger curvature using the radius of a circumsphere of $(k+2)$-points. This "obvious" generalization fails because of examples (see [5, Appendix B]) of very smooth manifolds for which this kind of curvature would be unbounded.

Strzelecki and von der Mosel [5] suggested a different notion of integral Menger-type curvature for surfaces in $\mathbb{R}^3$ and proved that finiteness of their functional implies Hölder regularity of the normal vector. They also applied their own results to prove existence of area minimizing surfaces in a given isotopy class under the constraint of bounded curvature. Our work is focused on generalizing these results to sets of arbitrary dimension and codimension. In [2] we introduce the following

Definition 1 *Let $T = \triangle(x_0,\dots,x_{m+1}) \subseteq \mathbb{R}^n$ be some $(m+1)$-dimensional simplex in $\mathbb{R}^n$, i.e. the convex hull of the set $\{x_0,\dots,x_{m+1}\} \subseteq \mathbb{R}^n$. The discrete curvature of T is*

$$\mathcal{K}(T) := \frac{\mathscr{H}^{m+1}(T)}{\operatorname{diam}(T)^{m+2}}.$$

Definition 2 *For $\Sigma \subseteq \mathbb{R}^n$ a $\mathscr{H}^m$-measurable set we define the p-energy functional as*

$$\mathcal{E}_p(\Sigma) := \int_{\Sigma^{m+2}} \mathcal{K}(\triangle(x_0,\dots,x_{m+1}))^p \, d\mu(x,\dots,x_{m+1}).$$

where $\mu = \mathscr{H}^m \otimes \dots \otimes \mathscr{H}^m$ is the product of $(m+2)$ copies of m-dimensional Hausdorff measure.

We show that for certain class of compact sets $\Sigma \subseteq \mathbb{R}^n$, finiteness of $\mathcal{E}_p(\Sigma)$ for $p > m(m+2)$ implies that Σ is a closed, $C^{1,\theta}$-manifold.

2. Fine sets

Some m-dimensional sets might have finite energy simply because they behave like lower dimensional sets. To avoid this effect we need to restrict the class of sets we want to examine.

Fix some compact set $\Sigma \subseteq \mathbb{R}^n$. We use the symbol $G(n,m)$ for the Grassmannian of m-dimensional subspaces in $\mathbb{R}^n$.

Definition 3 *For $x \in \Sigma$ and $r > 0$ we define the m-dimensional β-numbers of Σ as*

$$\beta_m(x,r) := \inf\left\{ \sup_{z\in\Sigma\cap\mathbb{B}(x,r)} \frac{\operatorname{dist}(z, x+H)}{r} : H \in G(n,m) \right\}.$$

We will also need the following definitions introduced by David, Kenig and Toro in [1].

Definition 4 *For $x \in \Sigma$ and $r > 0$ we define the following number*

$$\theta_m(x,r) := \frac{1}{r}\inf_P \{ D_H(\Sigma \cap \overline{\mathbb{B}}(x,r), (x+P)\cap\overline{\mathbb{B}}(x,r))\},$$

where D_H denotes the Hausdorff distance and the infimum is taken over all $P \in G(n,m)$.

Definition 5 *We say that $\Sigma \in \mathcal{F}(m)$ is an m-fine set if*

1. ***Ahlfors regularity.*** *There exists a constant $A_\Sigma > 0$ such that for all $x \in \Sigma$ and for all $r < \operatorname{diam}(\Sigma)$ we have*
$$\mathscr{H}^m(\Sigma \cap \mathbb{B}(x,r)) \ge A_\Sigma r^m.$$
2. ***No holes.*** *There exist two constants $M_\Sigma \ge 2$ and $R_\Sigma > 0$ such that for all $x \in \Sigma$ and $r < R_\Sigma$ we have*
$$\theta_m(x,r) \le M_\Sigma \beta_m(x,r).$$

This class of sets is very wide. For example any image of a smooth, compact manifold under a bi-lipschitz mapping or any finite union of such images belong to $\mathcal{F}(m)$.

3. Regularity

In [3] we prove the following

Theorem 1 *Let $\Sigma \in A(m)$ be such that $\mathcal{E}_p(\Sigma) \le E < \infty$ for some $p > m(m+2)$. Then Σ is a closed, $C^{1,\theta}$ manifold, where $\theta = \frac{p-m(m+2)}{(m+1)(m(m+1)(m+2)+p)}$. Moreover, we can cover Σ by balls of radius $R_0 = R_0(E, A_\Sigma)$ in such a way that in each of these balls Σ is a graph of $C^{1,\theta}$ function. Furthermore, all these functions have a common Hölder constant C_0, which depends only on E and A_Σ.*

To prove this we first prove that finiteness of $\mathcal{E}_p(\Sigma)$ implies that $\beta_m(x,r) \lesssim r^\theta$. Then we could use the result of David, Kenig and Toro [1, Proposition 9.1] which gives $C^{1,\theta}$ regularity at once but we need to know that R_0 and C_0 depend only on E and A_Σ so we give an independent proof. Next, we need to drop the dependence on A_Σ, so we prove

Theorem 2 *Let Σ be as before, then there exists a constant $R_1 = R_1(E) > 0$ such that for all $\rho \le R_1$ and all $x \in \Sigma$ we have*

$$\mathscr{H}^m(\Sigma \cap \mathbb{B}(x,\rho)) \ge \left(\tfrac{15}{16}\right)^{\frac{m}{2}} \mathscr{H}^m(\mathbb{B}^m)\rho^m.$$

And then we deduce the following

Corollary 1 *The constants R_0 and C_0 from Theorem 1 depend only on the energy bound E and do not depend on A_Σ.*

We believe that this will become useful in the proof of

Hypothesis 1 *There exists a natural number $N(E)$ which depends only on E such that the set*

$$\mathscr{X} := \{\Sigma : \mathcal{E}_p(\Sigma) \le E,\ 0 \in \Sigma,\ \mathscr{H}^m(\Sigma) \le 1\}$$

contains at most $N(E)$ non-homeomorphic sets.

4. Sketch of the proof of Theorem 2

First we use the fact proved in Theorem 1, that Σ is a closed, $C^{1,\theta}$ manifold. Hence, it has the property, that at each point one cat touch it with a cone

$$\forall x \in \Sigma\ \exists r(x) > 0\ \exists H \in G(n,m)\ \Sigma \cap C(x, r(x), H) = \emptyset,$$

where $C(x,r,H) = \mathbb{B}(x,r) \cap \{y : \operatorname{dist}(y-x, H) \ge \frac{1}{4}|y-x|\}$. Moreover a standard result from differential topology ensures that all the spheres centered at H and contained in $C(x,r,H)$ are linked with Σ. Therefore, for $v \in H$ with $|v| < r\sqrt{15/16}$ we have

$$\Sigma \cap \mathbb{B}(x+v, \tfrac{1}{4}r) \cap (x + H^\perp) \neq \emptyset.$$

This shows that the projection $\pi_H(\Sigma \cap \mathbb{B}(x,r))$ contains the disc $\mathbb{B}(x, r\sqrt{15/16}) \cap H$ and we can estimate the measure $\mathscr{H}^m(\Sigma \cap \mathbb{B}(x,r))$ by the measure of the projection which is exactly $(15/16)^{m/2}\mathscr{H}^m(\mathbb{B})r^m$.

The above also holds if instead of a cone we only have a "conical cap" $C(x,r,H) \setminus \mathbb{B}(x, \frac{1}{2}r)$ which does not intersect Σ and with the property that appropriate spheres are linked. To finish the proof we need to show that there exists a lower bound on $r(x) \ge R_1(E)$ which depend only on the energy E.

The crucial observation is that small, roughly regular simplices are exactly the reason why $\mathcal{E}_p$ might become infinite. We describe an algorithm similar to that presented in [5], which allows us to find at each point $x \in \Sigma$ one, almost regular simplex T with x being one of its vertices. Then we conclude that there is a lot (in the sense of measure) of regular simplices near T. More precisely, we move each vertex of T inside some small ball centered at that vertex and we obtain a set $\mathcal{U} \subseteq \Sigma^{m+2}$ of positive measure of roughly regular simplices. Then we can estimate the energy $\mathcal{E}_p(\Sigma)$ from below by an integral over $\mathcal{U}$. Since for roughly regular simplices $\mathcal{K}(T)$ behaves like $\operatorname{diam}(T)^{-1}$, we obtain a lower bound $R_1 > 0$ on the diameter of T. Our construction also ensures the existence of a "conical cap" $C(x, R_1, H(x)) \setminus \mathbb{B}(x, \frac{1}{2}R_1)$ with the desired properties described above.

The algorithm

At any point $x_0 \in \Sigma$ we can touch Σ by the cone $C(x_0, \rho_0, H_0)$ which does not intersect Σ. We increase the radius ρ_0 until we hit Σ. We then choose a well spread m-tuple of points in $\Sigma \cap \mathbb{B}(x_0, \rho_0)$. We do that just by choosing m points $y_1, \dots, y_m$ on $\partial\mathbb{B}(x_0, \rho_0\sqrt{15/16})$ such that the vectors $(y_1 - x_0), \dots, (y_m - x_0)$ form an orthogonal basis of H_0. Then we use the fact that appropriate spheres centered at these points are linked with Σ and we find points $x_i \in \Sigma \cap \mathbb{B}(x_0, \rho_0)$ for $i = 1, 2, \dots, m$. The points $x_0, x_1, \dots, x_m$ span some m-plane P. Now, we stop and analyze the situation. There are two possibilities.

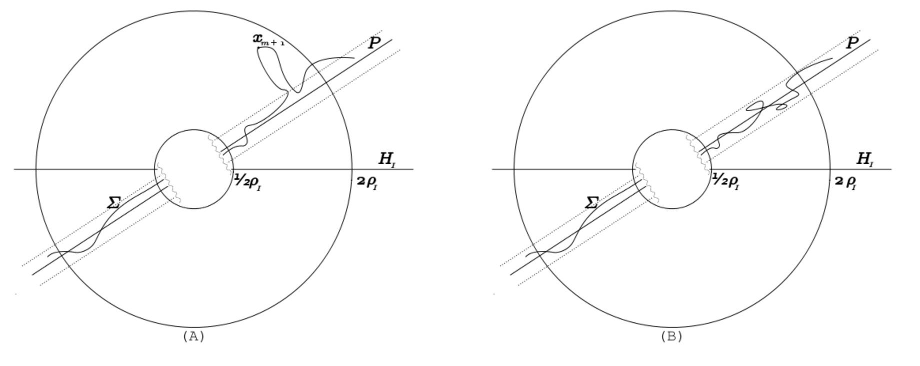

Figure 1: *The two possible configurations.*

Either we can find a point of Σ far from P at scale comparable to ρ_0, or Σ is almost flat at scale ρ_0 which means that it is very close to P. In the first case we can stop, since we have found a good simplex. In the second case we need to continue because there is no chance of finding a roughly regular simplex in $\mathbb{B}(x_0, \rho_0)$. We set $H_1 := P$ and repeat the procedure but now we consider not the cone $C(x_0, \rho_1, H_1)$ but only the "conical cap" $C(x_0, \rho_1, H_1) \setminus \mathbb{B}(x_0, \frac{1}{2}\rho_0)$. From the fact that Σ is close to $H_1 = P$ at scale ρ_0 we deduce that our "conical cap" does not intersect Σ for $\rho_1 \le 2\rho_0$. We increase ρ_1 until we hit Σ and iterate the whole algorithm. Of course the algorithm has to end after a finite number of steps because ρ_i grows geometrically and Σ is compact.

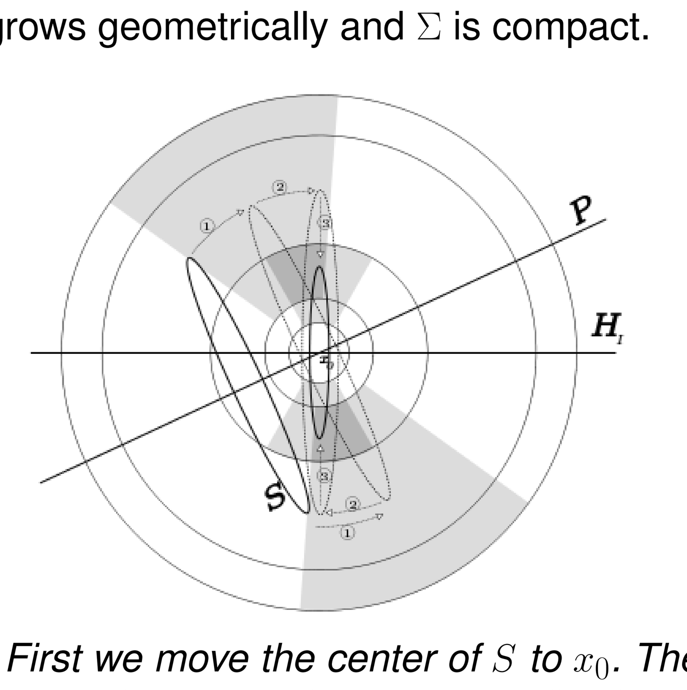

Figure 2: *First we move the center of S to x_0. Then we rotate S so that it is perpendicular to H_i. Finally we change the radius so that it is between $\frac{1}{2}\rho_{i-1}$ and ρ_i.*

In the course of the proof we build an increasing sequence of sets F_i made up from the conical caps $C(x_0, \rho_i, H_i) \setminus \mathbb{B}(x_0, \frac{1}{2}\rho_{i-1})$. For each i the set F_i does not intersect Σ, it contains the conical cap $C(x_0, \rho_i, H_i) \setminus \mathbb{B}(x_0, \frac{1}{2}\rho_{i-1})$. Using these properties of F_i we can construct an isotopy of any sphere centered at H and contained in F_i deforming it to some sphere contained in $F_1 = C(x_0, r_0, H_0)$. Since linking number is invariant under continuous deformations, we prove that spheres in F_i are linked with Σ and we obtain the thesis.

References

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