The Plateau problem – old and new

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Classical formulations of the problem

Reifenberg's (co)homological spanning

$B \subseteq \mathbf{R}^n$ a compact set, G an abelian group,
$\check{\mathbf{H}}_k(B;G)$ the k^{th} Čech homology group,
$\mathcal{H}^k(B)$ the k dimensional Haudorff measure,

Problem. *Let L be a subgroup of $\check{\mathbf{H}}_{m-1}(B;G)$. Minimize $\mathcal{H}^m(S)$ among compact subsets S of $\mathbf{R}^n$ such that the composition of maps*

$$L \hookrightarrow \check{\mathbf{H}}_{m-1}(B;G) \xrightarrow{i_*} \check{\mathbf{H}}_{m-1}(S \cup B;G)$$

equals zero, where i_* is induced by the inclusion map $i : B \hookrightarrow S \cup B$.

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$$\check{\mathbf{H}}^{m-1}(S \cup B;G) \xrightarrow{i^*} \check{\mathbf{H}}^{m-1}(B;G) \twoheadrightarrow \check{\mathbf{H}}^{m-1}(B;G)/L$$

equals zero, where i^* is induced by the inclusion map $i : B \hookrightarrow S \cup B$.

See:

- E. R. Reifenberg and J. F. Adams, *Acta Math.*, 1960
- F. Almgren, *Ann. of Math.*, 1968

Currents of Federer and Fleming

$$\mathbf{I}_k(\mathbf{R}^n) \quad k \text{ dimensional integral currents,}$$

$$S \in \mathbf{I}_k(\mathbf{R}^n) \quad \Rightarrow \quad S(\phi) = \int_E \langle \tau(x), \phi(x) \rangle \theta(x) \, \mathrm{d}\mathcal{H}^k(x) ,$$

$$\mathbf{Mass}(S) = \int_E \theta(x) \, \mathrm{d}\mathcal{H}^k(x) ,$$

$$\mathbf{Size}(S) = \mathcal{H}^k(\{x \in E : \theta(x) \neq 0\}) ,$$

$$B \in \mathbf{I}_{m-1}(\mathbf{R}^n) , \quad \partial B(\phi) = B(d\phi) = 0 .$$

Problem (*Mass minimizers*).
Minimize $\mathbf{Mass}(S)$ among $S \in \mathbf{I}_m(\mathbf{R}^n)$ satisfying

$$\partial S = B .$$

Problem (*Size minimizers* – unsolved).
Minimize $\mathbf{Size}(S)$ among $S \in \mathbf{I}_m(\mathbf{R}^n)$ satisfying

$$\partial S = B .$$

See:

- H. Federer and W. Fleming, *Ann. of Math.*, 1960

Drawbacks

Solutions of the Plateau's problem should model the behaviour of soap films!

- Mass minimizers do *not* minimize measure.
- Mass minimizers are smooth everywhere and cannot model soap films with singularities.
- Choice of orientation (currents, homology).
- Choice of the coefficient group restricts the set of competitors (currents, homology).
- Reifenberg's solution only for compact groups.

See:

- G. David, Should we solve Plateau's problem again?, in *Advances in analysis: the legacy of Elias M. Stein*, 108–145, Princeton Math. Ser., 50, Princeton Univ. Press, Princeton, NJ, 2014
- J. Harrison, H. Pugh, *Plateau's Problem: What's Next*, arXiv:1509.03797, 2015
- J. Harrison and H. Pugh. Plateau's problem. *In Open problems in mathematics*, 273–302. Springer, [Cham], 2016

New formulations of the problem

Spanning in terms of linking numbers

$$B \subseteq \mathbf{R}^n \quad \text{a compact set} ,$$

$$\mathcal{C}_B = \{\gamma : \mathbf{S}^{n-m} \hookrightarrow \mathbf{R}^n \sim B : \gamma \text{ a smooth embedding}\} ,$$

$$\mathcal{C} \subseteq \mathcal{C}_B ,$$

if $\gamma \in \mathcal{C}$ and $\tilde{\gamma} \in \mathcal{C}_B$ is smoothly isotopic to γ, then $\tilde{\gamma} \in \mathcal{C}$,

$$\mathcal{F}(B,\mathcal{C}) = \left\{ K \subseteq \mathbf{R}^n : \begin{array}{l} K \text{ compact } (\mathcal{H}^m, m) \text{ rectifiable} \\ K \cap \operatorname{im}\gamma \neq \varnothing \quad \text{for } \gamma \in \mathcal{C} \end{array} \right\} .$$

Example. If B is an orientable $(m-1)$ dimensional manifold, then we can take

$$\mathcal{C} = \{\gamma \in \mathcal{C}_B : \operatorname{lk}(\operatorname{im}\gamma, B) = 1\} .$$

Problem. *Minimize $\mathcal{H}^m(S)$ among $S \in \mathcal{F}(B,\mathcal{C})$.*

See:

- J. Harrison and H. Pugh, *J. Geom. Anal.*, 2012
- C. De Lellis, F. Ghiraldin, F. Maggi, *J. Eur. Math. Soc. (JEMS)*, 2017
- G. De Philippis, A. De Rosa, F. Ghiraldin, *Adv. Math.*, 2016

Sliding deformations and sliding minimizers

$$B, K_0 \subseteq \mathbf{R}^n \quad \text{compact sets} ,$$

$$\Sigma(B) = \left\{ \Phi(1,\cdot) : \begin{array}{c} \Phi : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n \text{ continuous} , \\ \Phi(0,\cdot) = \mathrm{id}_{\mathbf{R}^n} , \quad \operatorname{Lip}\Phi(1,\cdot) < \infty , \\ \Phi(t,\cdot)[B] \subseteq B \text{ for } t \in [0,1] \end{array} \right\} ,$$

$$\mathcal{A}(B,K) = \{\varphi[K_0] : \varphi \in \Sigma(B)\}$$

Problem (partially solved).
Minimize $\mathcal{H}^m(S)$ among $S \in \mathcal{A}(B,K_0)$.

Remark. One can find $K \subseteq \mathbf{R}^n$ satisfying

$$\inf\{\mathcal{H}^m(S) : S \in \mathcal{A}(B,K_0)\} = \mathcal{H}^m(K) = \inf\{\mathcal{H}^m(S) : S \in \mathcal{A}(B,K)\} ,$$

but it is *not known* whether $K \in \mathcal{A}(B,K_0)$.

See:

- G. David, *Princeton Math. Ser.*, 2014
- G. De Philippis, A. De Rosa, F. Ghiraldin, *Adv. Math.*, 2016
- C. De Lellis, F. Ghiraldin, F. Maggi, *J. Eur. Math. Soc. (JEMS)*, 2017

A unifying approach

Let $B \subseteq \mathbf{R}^n$ be compact.

- Define the class of competitors $\mathcal{P}(B)$ axiomatically without referring to any particular notion of *boundary* or *spanning*.
- Consider general elliptic functionals Φ instead of the Hausdorff measure $\mathcal{H}^m$.
- Minimize $\Phi(S)$ among $S \in \mathcal{P}(B)$.

Remark. Usually the class $\mathcal{P}(B)$ is assumed to be closed under certain deformations (like sliding deformations).

See:

- C. De Lellis, F. Ghiraldin, F. Maggi, *J. Eur. Math. Soc. (JEMS)*, 2017
- G. De Philippis, A. De Rosa, F. Ghiraldin, *Adv. Math.*, 2016
- C. De Lellis, A. De Rosa, F. Ghiraldin, arXiv:1602.08757, 2016
- J. Harrison and H. Pugh, arXiv:1603.04492, 2016
- Y. Fang. Existence of minimizers for the Reifenberg plateau problem. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5), 16(3):817–844, 2016

Elliptic integrands and a generalized Plateau problem

Elliptic integrands

$$F : \mathbf{R}^n \times \mathbf{G}(n,m) \to (0,\infty) \quad \text{continuous} ,$$

$$S = S_{\mathrm{reg}} \cup S_{\mathrm{irr}} \subseteq \mathbf{R}^n \quad \text{compact with } \mathcal{H}^m(S) < \infty ,$$

$$\Phi_F(S) = \int_{S_{\mathrm{reg}}} F(x, \operatorname{Tan}(S_{\mathrm{reg}}, x)) \, \mathrm{d}\mathcal{H}^m(x) + \mathcal{H}^m(S_{\mathrm{irr}}) ,$$

$$x \in \mathbf{R}^n \quad \Rightarrow \quad F^x(y,T) = F(x,T) .$$

Definition (F. Almgren, *Ann. of Math.*, 1968). We say that F is *elliptic* at $x \in \mathbf{R}^n$ if: there exists $C > 0$ such that given any disc $D \subseteq L$ lying in an affine m-plane $L \subseteq \mathbf{R}^n$ and a compact set $S \subseteq \mathbf{R}^n$ which cannot be deformed onto $\partial D \subseteq S$ with an *admissible deformation*, there holds

$$\Phi_{F^x}(S) - \Phi_{F^x}(D) \geqslant C\big(\mathcal{H}^m(S) - \mathcal{H}^m(D)\big) .$$

Definition. F is *bounded* if $\sup \operatorname{im} F / \inf \operatorname{im} F < \infty$.

Example. The *area integrand*: $F(x,T) = 1$ for $(x,T) \in \mathbf{R}^n \times \mathbf{G}(n,m)$. Then $\Phi_F(S) = \mathcal{H}^m(S)$.

An existence result; see [?]

Let $U \subseteq \mathbf{R}^n$ be open. Let $\mathfrak{D}(U)$ be the family of deformations which are compositions of a finite number of maps $f : \mathbf{R}^n \to \mathbf{R}^n$ such that for some ball $B \subseteq U$

$$\{x \in \mathbf{R}^n : f(x) \neq x\} \subseteq B \quad \text{and} \quad f[B] \subseteq B .$$

Let $\mathcal{C}$ be a family of closed subsets of $\mathbf{R}^n$ such that

- if $S \in \mathcal{C}$ and $f \in \mathfrak{D}(U)$, then $f[S] \in \mathcal{C}$;
- if $S_i \in \mathcal{C}$ and $S_i \xrightarrow{i \to \infty} S$ locally in U in Hausdorff distance, then $S \in \mathcal{C}$.

Theorem. *Let F be a bounded elliptic $\mathscr{C}^0$ integrand. Assume $\mu = \inf\{\Phi_F(T \cap U) : T \in \mathcal{C}\} \in (0,\infty)$. There exist $S \in \mathcal{C}$ and a sequence $S_i \in \mathcal{C}$ such that*

- *$S \cap U$ is $(\mathcal{H}^m, m)$ rectifiable;*
- *$\lim_{i\to\infty} \Phi_F(S_i \cap U) = \Phi_F(S \cap U) = \mu$;*
- *$\lim_{i\to\infty} \mathbf{v}(S_i \cap U) = \mathbf{v}(S \cap U)$ in $\mathbf{V}_m(U)$;*
- *$S_i \xrightarrow{i \to \infty} S$ locally in U in Hausdorff distance.*

An application

Corollary. *Suppose $B \subseteq \mathbf{R}^n$ is compact. Let $\mathcal{P}(B)$ be the family of compact sets $S \subseteq \mathbf{R}^n$ which span some subgroup $L \subseteq \check{\mathbf{H}}_{m-1}(B;G)$ (or $L \subseteq \check{\mathbf{H}}^{m-1}(B;G)$) in the sense of Reifenberg. Then there exists a minimizer $R \in \mathcal{P}(B)$ of Φ_F. Moreover, R is compact and $(\mathcal{H}^m, m)$ rectifiable.*

Remark. If we assumed that $\mathfrak{D}(U)$ contains only diffeomorphisms $\mathscr{C}^1$ isotopic to identity, then we could apply the theorem also in case $\mathcal{P}(B) = \mathcal{F}(B,\mathcal{C})$ (spanning defined in terms of linking numbers).