Area and co-area formulas. Rectifiability.

[Fed69, 2.10.11] **Lemma.** If X is a complete separable metric space, Y is a metric space, $f : X \to Y$ is Lipschitz, $0 \le m < \infty$, $A \subseteq X$ is Borel, then

$$\int N(f|_A, y)\, \mathrm{d}\mathscr{H}^m(y) \le (\operatorname{Lip} f)^m \mathscr{H}^m(A)\,.$$

[Fed69, 2.10.25] **Theorem.** If X and Y are metric spaces, $f : X \to Y$ is Lipschitz, $A \subseteq X$, $0 \le k < \infty$, and $0 \le m < \infty$, then

$$\int^* \mathscr{H}^k(A \cap f^{-1}\{y\})\, \mathrm{d}\mathscr{H}^m(y) \le (\operatorname{Lip} f)^m \frac{\boldsymbol{\alpha}(k)\boldsymbol{\alpha}(m)}{\boldsymbol{\alpha}(m+k)} \mathscr{H}^{k+m}(A)\,,$$

provided either $\{y : \mathscr{H}^k(A \cap f^{-1}\{y\}) > 0\}$ is a union of countably many sets with finite $\mathscr{H}^m$ measure, or Y is boundedly compact.

[Fed69, 3.2.3] **Theorem.** Suppose $f : \mathbf{R}^m \to \mathbf{R}^n$, and $\operatorname{Lip}(f) < \infty$, and $m \le n$.

(a) If $A \subseteq \mathbf{R}^m$ is $\mathscr{L}^m$ measurable, then

$$\int_A J_m f\, \mathrm{d}\mathscr{L}^m = \int_{\mathbf{R}^n} N(f|_A, y)\, \mathrm{d}\mathscr{H}^m(y)\,.$$

(b) If $u : \mathbf{R}^m \to \mathbf{R}$ is $\mathscr{L}^m$ integrable, then

$$\int u(x) J_m f(x)\, \mathrm{d}\mathscr{L}^m(x) = \int_{\mathbf{R}^n} \sum_{x \in f^{-1}\{y\}} u(x)\, \mathrm{d}\mathscr{H}^m(y)\,. \tag{1}$$

[Fed69, 3.2.5] **Theorem.** Suppose $f : \mathbf{R}^m \to \mathbf{R}^n$, and $\operatorname{Lip}(f) < \infty$, and $m \le n$, and $A \subseteq \mathbf{R}^m$ is $\mathscr{L}^m$ measurable, and $g : \mathbf{R}^m \to \bar{\mathbf{R}}$. Then

$$\int_A g(f(x)) J_m f(x)\, \mathrm{d}\mathscr{L}^m(x) = \int_{\mathbf{R}^n} g(y) N(f|_A, y)\, \mathrm{d}\mathscr{H}^m(y)$$

given

(a) either g is $\mathscr{H}^m$ measurable

(b) or $N(f|_A, y) < \infty$ for $\mathscr{H}^m$ almost all $y \in \mathbf{R}^n$

(c) or $\mathbb{1}_A \cdot (g \circ f) \cdot J_m f$ is $\mathscr{L}^m$ measurable.

[Fed69, 3.2.11-12] **Theorem.** Suppose $f : \mathbf{R}^m \to \mathbf{R}^n$, and $\operatorname{Lip}(f) < \infty$, and $m > n$.

(a) If $A \subseteq \mathbf{R}^m$ is $\mathscr{L}^m$ measurable, then

$$\int_A J_n f\, \mathrm{d}\mathscr{L}^m = \int_{\mathbf{R}^n} \mathscr{H}^{m-n}(f^{-1}\{y\})\, \mathrm{d}\mathscr{L}^n(y)\,.$$

(b) If $u : \mathbf{R}^m \to \bar{\mathbf{R}}$ is $\mathscr{L}^m$ integrable, then

$$\int u(x) J_n f(x)\, \mathrm{d}\mathscr{L}^m(x) = \int_{\mathbf{R}^n} \int_{f^{-1}\{y\}} u(x)\, \mathrm{d}\mathscr{H}^{m-n}(x)\, \mathrm{d}\mathscr{L}^n(y)\,. \tag{2}$$

[Haj00, Theorem 11] **Theorem.** Let $f \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^m)$, $1 \le p < \infty$, $k = \min\{m, n\}$, and let $g : \mathbf{R}^n \to \mathbf{R}$ be either nonnegative measurable or measurable and such that $g \cdot J_k f \in L^1(\mathscr{L}^n)$. Then there ewists a representative of f such that both area (1) and co-area (2) formulas hold.

Remark. Formulas (1) and (2) still hold true given f is merely $\mathscr{L}^n$ approximately differentiable $\mathscr{L}^n$ almost everywhere and has the Lusin N property.

[Fed69, 3.2.14] **Definition.** Let $E \subseteq \mathbf{R}^n$, m be a positive integer, ϕ measures $\mathbf{R}^n$.

(a) E is *m rectifiable* if there exists $\varphi : \mathbf{R}^m \to \mathbf{R}^n$ with $\operatorname{Lip}(\varphi) < \infty$ and such that $E = \varphi[A]$ for some bounded set $A \subseteq \mathbf{R}^m$;

(b) E is *countably m rectifiable* if is a union of countably many m rectifiable sets;

(c) E is *countably (ϕ, m) rectifiable* if there exists a countably m rectifiable set $A \subseteq \mathbf{R}^n$ such that $\phi(E \sim A) = 0$;

(d) E is *(ϕ, m) rectifiable* if E is countably (ϕ, m) rectifiable and $\phi(E) < \infty$.

(e) E is *purely (ϕ, m) unrectifiable* if $\phi(E \cap \operatorname{im} \varphi) = 0$ for all $\varphi : \mathbf{R}^m \to \mathbf{R}^n$ with $\operatorname{Lip}(\varphi) < \infty$.

[Fed69, 3.2.29] **Theorem.** A set $W \in \mathbf{R}^n$ is countably $(\mathscr{H}^m, m)$ rectifiable *if and only if* there exists a countable family F of m dimensional submanifolds of $\mathbf{R}^n$ of class $\mathscr{C}^1$ such that $\mathscr{H}^m(W \sim \bigcup F) = 0$.

[Fed69, 3.2.18] **Lemma.** Assume $W \subseteq \mathbf{R}^n$ is $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable. Then for each $\lambda \in (1,\infty)$, there exist compact subsets $K_1, K_2, \ldots$ of $\mathbf{R}^m$ and maps $\psi_1, \psi_2, \ldots : \mathbf{R}^m \to \mathbf{R}^n$ such that

$$\{\psi_i[K_i] : i = 1,2,\ldots\} \text{ is disjointed}, \quad \mathscr{H}^m(W \sim \textstyle\bigcup_{i=1}^\infty \psi_i[K_i]) = 0,$$
$$\operatorname{Lip}(\psi_i) \le \lambda, \quad \psi_i|_{K_i} \text{ is injective}, \quad \operatorname{Lip}((\psi_i|_{K_i})^{-1}) \le \lambda,$$
$$\lambda^{-1}|v| \le |\operatorname{D}\psi_i(a)v| \le \lambda|v| \quad \text{for } a \in K_i,\ v \in \mathbf{R}^m.$$

[Fed69, 3.2.19] **Theorem.** Assume $W \subseteq \mathbf{R}^n$ is $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable. Then for $\mathscr{H}^m$ almost all $w \in W$

$$\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner W, w) = 1 \quad \text{and} \quad \operatorname{Tan}^m(\mathscr{H}^m \llcorner W, w) \in \mathbf{G}(n,m).$$

Moreover, if $f : W \to \mathbf{R}^\nu$ and $\operatorname{Lip}(f) < \infty$, then

$$(\mathscr{H}^m \llcorner W, m) \operatorname{ap} \operatorname{D} f(w) : \operatorname{Tan}^m(\mathscr{H}^m \llcorner W, w) \to \mathbf{R}^\nu$$

exists for $\mathscr{H}^m$ almost all $w \in W$.

[Fed69, 3.2.20] **Corollary.** Let $W \subseteq \mathbf{R}^n$ be $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable. Assume $m \le \nu$, and $f : W \to \mathbf{R}^\nu$, and $\operatorname{Lip}(f) < \infty$. Then

$$\int_W (g \circ f) J_m f \, \mathrm{d}\mathscr{H}^m = \int_{R^\nu} g(z) N(f,z) \, \mathrm{d}\mathscr{H}^m(z)$$

for any $g : \mathbf{R}^\nu \to \bar{\mathbf{R}}$.

[Mat75] **Theorem.** If $W \subseteq \mathbf{R}^n$ and $\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner W, w) = 1$ for $\mathscr{H}^m$ almost all $w \in W$, then W is countably $(\mathscr{H}^m, m)$ rectifiable.

[Pre87] **Theorem.** If μ is a Radon measure over $\mathbf{R}^n$ and $\boldsymbol{\Theta}^m(\mu, x) \in \mathbf{R}$ exists for μ almost all x, then $\mathbf{R}^n$ is countably (μ, m) rectifiable.

[Fed69, 3.2.22] **Theorem.** Let $m \ge \mu$, and $W \subseteq \mathbf{R}^n$ be $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable, and $Z \subseteq \mathbf{R}^\nu$ be $(\mathscr{H}^\mu, \mu)$ rectifiable and $\mathscr{H}^\mu$ measurable, and $f : W \to Z$, and $\operatorname{Lip}(f) < \infty$. For brevity let us write "ap" for "$(\mathscr{H}^m \llcorner W, m)$ ap".

(a) For $\mathscr{H}^m$ almost all $w \in W$, either $\operatorname{ap} J_\mu f(w) = 0$ or

$$\operatorname{im} \operatorname{ap} \operatorname{D} f(w) = \operatorname{Tan}^\mu(\mathscr{H}^\mu \llcorner Z, f(w)) \in \mathbf{G}(\nu, \mu).$$

(b) The levelset $f^{-1}\{z\}$ is $(\mathscr{H}^{m-\mu}, m-\mu)$ rectifiable and $\mathscr{H}^{m-\mu}$ measurable for $\mathscr{H}^\mu$ almost all $z \in Z$.

(c) For any $(\mathscr{H}^m \llcorner W)$ integrable function $g : W \to \bar{\mathbf{R}}$

$$\int_W g \cdot \operatorname{ap} J_\mu f \, \mathrm{d}\mathscr{H}^m = \int_Z \int_{f^{-1}\{z\}} g \, \mathrm{d}\mathscr{H}^{m-\mu} \, \mathrm{d}\mathscr{H}^\mu(z).$$

[Fed69, 3.2.23] **Theorem.** Assume $W \subseteq \mathbf{R}^n$ is m rectifiable and Borel, and $Z \subseteq \mathbf{R}^\nu$ is $(\mathscr{H}^\mu, \mu)$ rectifiable and Borel. Then $W \times Z \subseteq \mathbf{R}^n \times \mathbf{R}^\nu$ is $(\mathscr{H}^{m+\mu}, m+\mu)$ rectifiable and

$$\mathscr{H}^{m+\mu} \llcorner (W \times Z) = (\mathscr{H}^m \llcorner W) \times (\mathscr{H}^\mu \llcorner Z).$$

[Fed69, 3.2.24] **Beware,** there exist sets $W \subseteq \mathbf{R}^n$ and $Z \subseteq \mathbf{R}^\nu$ with $\mathscr{H}^m(W) = 0$ and $\mathscr{H}^\mu(Z) = 0$ but $\mathscr{H}^{m+\mu}(W \times Z) = \infty$. In particular, $\mathscr{H}^{m+\mu} \llcorner (W \times Z) \ne (\mathscr{H}^m \llcorner W) \times (\mathscr{H}^\mu \llcorner Z)$!

BV, Caccioppoli sets, and the Gauss-Green theorem. Let $U \subseteq \mathbf{R}^n$ be open.

[EG92, 5.1] **Definition.** A function $f \in L^1(U)$ has *bounded variation in U* if

$$\|\operatorname{D} f\|(U) = \sup\left\{ \int f \operatorname{div} \varphi \, \mathrm{d}\mathscr{L}^n : \varphi \in \mathscr{C}^1_c(U, \mathbf{R}^n),\ |\varphi| \le 1 \right\} < \infty.$$

We define

$$BV(U) = \{ f \in L^1(U) : \|\operatorname{D} f\|(U) < \infty \} \quad \text{and} \quad \|f\|_{BV(U)} = \|f\|_{L^1(U)} + \|\operatorname{D} f\|(U).$$

Definition. $f \in L^1(U)$ has *locally bounded variation in U* if $f \in BV(V)$ for all open sets $V \subseteq U$ such that $\operatorname{Clos} V \subseteq U$ is compact. We write $f \in BV_{\mathrm{loc}}(U)$.

Definition. An $\mathscr{L}^n$ measurable set $E \subseteq \mathbf{R}^n$ has *finite perimeter in* U if $\mathbb{1}_E \in BV(U)$.
Definition. E has *locally finite perimeter in* U if $\mathbb{1}_E \in BV_{\mathrm{loc}}(U)$.
Theorem. $f \in BV(U)$ *if and only if* there exists a Radon measure μ over $\mathbf{R}^n$ and a μ measurable function $\sigma : U \to \mathbf{R}^n$ satisfying $|\sigma(x)| = 1$ for μ almost all x and

$$\int_U f \operatorname{div} \varphi \, \mathrm{d}\mathscr{L}^n = -\int_U \varphi \bullet \sigma \, \mathrm{d}\mu \quad \text{for } \varphi \in \mathscr{C}_c^1(U, \mathbf{R}^n) .$$

Notation.

(a) If $f \in BV_{\mathrm{loc}}(U)$, then we write $\| \mathrm{D} f \| = \mu$ and ∇f for the density of the absolutely continuous part of the vector-valued Radon measure $\mu \llcorner \sigma$ with respect to the Lebesgue measure $\mathscr{L}^n$.

(b) If $E \subseteq \mathbf{R}^n$ has locally finite perimeter in U, then we write $\|\partial E\| = \| \mathrm{D} \mathbb{1}_E \|$ and $\nu_E = -\sigma$.

[Fed69, 4.5] **Remark.** We have $f \in BV_{\mathrm{loc}}(U)$ if and only if $\mathbf{E}^n \llcorner f \in \mathbf{N}_n^{\mathrm{loc}}(U)$, where $\mathbf{E}^n$ is the current naturally associated to the n-dimensional Euclidean space and $\mathbf{N}_n^{\mathrm{loc}}(U)$ denotes the vectorspace of locally normal currents in U; cf. [Fed69, 4.1.7].

[Fed69, 4.5.10] **Definition.** Let (Y, d) be a metric space, $f : \mathbf{R} \to Y$ be $\mathscr{L}^1$ measurable, $-\infty < a < b < \infty$. We define the *essential variation of* f *on* $[a, b]$, denoted $\operatorname{ess} \mathbf{V}_a^b f$, as the supremum of the set of numbers

$$\sum_{j=1}^{\nu} d(f(t_j), f(t_{j+1}))$$

corresponding to all finite sequences of points $t_1, t_2, \ldots, t_{\nu+1}$ of $\mathscr{L}^1$ approximate continuity of f with $a < t_1 \le t_2 \le \cdots \le t_{\nu+1} < b$.

[Fed69, 4.5.9(27)] **Definition.** For $i = 1, 2, \ldots, m$ and $z \in \mathbf{R}^{m-1}$ we define

$$\chi_{i,z} : \mathbf{R} \to \mathbf{R}^m , \qquad \chi_{i,z}(t) = (z_1, \ldots, z_{i-1}, t, z_i, \ldots, z_{m-1}) .$$

[Fed69, 4.5.10] **Lemma.** Assume $f : \mathbf{R}^m \to \mathbf{R}$ is $\mathscr{L}^m$ measurable and $m \ge 2$. Then $f \in BV_{\mathrm{loc}}(\mathbf{R}^m)$ if and only if

$$\int_K |f| \, \mathrm{d}\mathscr{L}^m < \infty \quad \text{whenever } K \subseteq \mathbf{R}^m \text{ is compact}$$
$$\text{and} \quad \int_{*Z} \operatorname{ess} \mathbf{V}_a^b (f \circ \chi_{i,z}) \, \mathrm{d}\mathscr{L}^{m-1}(z) < \infty$$

whenever $Z \subseteq \mathbf{R}^{m-1}$ is compact, $-\infty < a < b < \infty$, and $i \in \{1, 2, \ldots, m\}$.

[Fed69, 2.10.13] **Lemma.** Let Y be a metric space, $g : \mathbf{R} \to Y$ be continuous. Then

$$\operatorname{ess} \mathbf{V}_a^b g = \int N(g|_{[a,b]}, y) \, \mathrm{d}\mathscr{H}^1(y) \quad \text{whenever } -\infty < a < b < \infty .$$

[EG92, 5.1, Ex.1] **Remark.** We have $W_{\mathrm{loc}}^{1,1}(U) \subseteq BV_{\mathrm{loc}}(U)$. Moreover, for $f \in W_{\mathrm{loc}}^{1,1}(U)$ and any $A \subseteq U$

$$\| \mathrm{D} f \|(A) = \int_A |\operatorname{grad} f| \, \mathrm{d}\mathscr{L}^n \quad \text{and} \quad \nabla f = \operatorname{grad} f .$$

[EG92, 5.1, Ex.2] **Remark.** If $E \subseteq \mathbf{R}^n$ is open and the topological boundary $\operatorname{Bdry} E$ is a smooth hypersurface in $\mathbf{R}^n$ such that $\mathscr{H}^{n-1}(\operatorname{Bdry} E \cap K) < \infty$ for all compact $K \subseteq U$, then E has locally finite perimeter in U. Moreover, if $\mathscr{H}^{n-1}(\operatorname{Bdry} E) < \infty$, then

$$\|\partial E\| = \mathscr{H}^{n-1} \llcorner \operatorname{Bdry} E \quad \text{and} \quad \nu_E \text{ is the outer unit normal to } \operatorname{Bdry} E .$$

[EG92, 5.2.1] **Theorem.** If $f_i \in BV(U)$ and $f_i \to f$ in $L_{\mathrm{loc}}^1(U)$, then

$$\| \mathrm{D} f \|(U) \le \liminf_{i \to \infty} \| \mathrm{D} f_i \|(U) .$$

[EG92, 5.2.2] **Theorem.** Assume $f \in BV(U)$. Then there exist functions $f_i \in BV(U) \cap \mathscr{E}(U, \mathbf{R})$ such that

$$f_i \to f \quad \text{in } L^1(U) \quad \text{and} \quad \| \mathrm{D} f_i \|(U) \to \| \mathrm{D} f \|(U) \quad \text{as } i \to \infty$$
$$\text{and} \quad \mathscr{L}^n \llcorner \operatorname{grad} f_i \to \| \mathrm{D} f \| \llcorner \sigma \quad \text{weakly as vector-valued Radon measures} .$$

[EG92, 5.2.3] **Theorem.** Assume U is open and bounded in $\mathbf{R}^n$, $\operatorname{Bdry} U$ is a Lipschitz manifold, $f_i \in BV(U)$ satisfies $\sup\{\|f_i\|_{BV(U)} : i = 1, 2, \ldots\} < \infty$. Then there exists a subsequence f_{k_j} and a function $f \in BV(U)$ such that $f_{k_j} \to f$ in $L^1(U)$.

[EG92, 5.5] **Remark.** If $f : U \to \mathbf{R}$ is Lipschitsz, then the co-area formula gives

$$\int |\operatorname{grad} f| \, \mathrm{d}\mathscr{L}^n = \int \mathscr{H}^{n-1}(f^{-1}\{t\}) \, \mathrm{d}\mathscr{L}^1(t) \,.$$

Theorem. Let $f \in L^1(U)$ and define for $t \in \mathbf{R}$

$$E_t = \{x \in U : f(x) > t\} \,.$$

(a) If $f \in BV(U)$, then E_t has finite perimeter in U for $\mathscr{L}^1$ almost all t.

(b) If $f \in BV(U)$, then

$$\| \mathrm{D} f \|(U) = \int \| \partial E_t \|(U) \mathscr{L}^1(t) \,.$$

(c) If $\int \| \partial E_t \|(U) \mathscr{L}^1(t) < \infty$, then $f \in BV(U)$.

[EG92, 5.6.2] **Theorem.** Let E be bounded and of finite perimeter in $\mathbf{R}^n$. There exists $C = C(n) > 0$ such that

(a) $\mathscr{L}^n(E)^{1-1/n} \le C \| \partial E \|(\mathbf{R}^n)$,

(b) $\min\{\mathscr{L}^n(\mathbf{B}(x,r) \cap E), \mathscr{L}^n(\mathbf{B}(x,r) \sim E)\}^{1-1/n} \le C \| \partial E \|(\mathbf{U}(x,r))$ for $x \in \mathbf{R}^n$, $r \in (0,\infty)$.

[EG92, 5.7.1] **Definition.** Assume E has locally finite perimeter in $\mathbf{R}^n$ and $x \in \mathbf{R}^n$. We say that x belongs to the *reduced boundary* $\partial^* E$ of E if

(a) $\| \partial E \|(\mathbf{B}(x,r)) > 0$ for $r > 0$,

(b) $\lim_{r \downarrow 0} \| \partial E \|(\mathbf{B}(x,r))^{-1} \int_{\mathbf{B}(x,r)} \nu_E \, \mathrm{d} \| \partial E \| = \nu_E(x)$,

(c) $|\nu_E(x)| = 1$.

[EG92, 5.7.3] **Theorem.** Assume E has locally finite perimeter in $\mathbf{R}^n$.

(a) $\partial^* E$ is countably $(\mathscr{H}^{n-1}, n-1)$ rectifiable; cf. [Fed69, 4.2.16(2)].

(b) $\mathscr{H}^{n-1}(\partial^* E \cap K) < \infty$ for any compact set $K \subseteq \mathbf{R}^n$.

(c) $\nu_E(x) \in \operatorname{Nor}^{n-1}(\mathscr{H}^{n-1} \llcorner \partial^* E, x)$ for $\mathscr{H}^{n-1}$ almost all $x \in \partial^* E$.

(d) $\| \partial E \| = \mathscr{H}^{n-1} \llcorner \partial^* E$.

[EG92, 5.8] **Definition.** Assume E has locally finite perimeter in $\mathbf{R}^n$ and $x \in \mathbf{R}^n$. We say that x belongs to the *measure theoretic boundary* $\partial_* E$ of E if

$$\boldsymbol{\Theta}^{*n}(\mathscr{L}^n \llcorner E, x) > 0 \quad \text{and} \quad \boldsymbol{\Theta}^{*n}(\mathscr{L}^n \llcorner (\mathbf{R}^n \sim E), x) > 0 \,.$$

Lemma. $\partial^* E \subseteq \partial_* E$ and $\mathscr{H}^{n-1}(\partial_* E \sim \partial^* E) = 0$.

[Fed69, 4.5.6] **Theorem.** Assume E has locally finite perimeter in $\mathbf{R}^n$. Then

$$\int_E \operatorname{div} \varphi \, \mathrm{d}\mathscr{L}^n = \int_{\partial_* E} \varphi \bullet \nu_E \, \mathrm{d}\mathscr{H}^{n-1} \quad \text{for } \varphi \in \mathscr{C}^1_c(\mathbf{R}^n, \mathbf{R}^n) \,.$$

[EG92, 5.11] [Fed69, 4.5.11] **Theorem.** Let $E \subseteq \mathbf{R}^n$ be $\mathscr{L}^n$ measurable. Then E has locally finite perimeter in $\mathbf{R}^n$ *if and only if* $\mathscr{H}^{n-1}(\partial_* E \cap K) < \infty$ for all compact sets $K \subseteq \mathbf{R}^n$.

[EG92, 6.1.1] **Theorem.** Assume $f \in BV_{\mathrm{loc}}(\mathbf{R}^n)$. Then for $\mathscr{L}^n$ almost all $x \in \mathbf{R}^n$

$$\lim_{r \downarrow 0} \frac{1}{r} \left(\boldsymbol{\alpha}(n)^{-1} r^{-n} \int_{\mathbf{B}(x,r)} |f(y) - f(x) - \nabla f(x) \bullet (x-y)|^{n/(n-1)} \, \mathrm{d}\mathscr{L}^n \right)^{1-1/n} = 0 \,.$$

[EG92, 6.1.3] **Theorem.** Assume $f \in BV_{\mathrm{loc}}(\mathbf{R}^n)$. Then f is $(\mathscr{L}^n, n)$ approximately differentiable $\mathscr{L}^n$ almost everywhere. Moreover,

$$(\mathscr{L}^n, n) \operatorname{ap} \mathrm{D} f(x) u = \nabla f(x) \bullet u \quad \text{for } \mathscr{L}^n \text{ almost all } x \in \mathbf{R}^n \text{ and all } u \in \mathbf{R}^n \,.$$

Varifolds. Let $U \subseteq \mathbf{R}^n$ be open and $M \subseteq U$ be a smooth m dimensional submanifold (possibly open) such that the inclusion map $i : M \hookrightarrow \mathbf{R}^n$ is proper.

[All72, 2.5] **Definition.**

- *tangent vector fields:* $\mathscr{X}(M) = \{g \in \mathscr{C}^\infty_c(M, \mathbf{R}^n) : \forall x \in M \;\; g(x) \in \operatorname{Tan}(M,x)\}$;
- *normal vector fields:* $\mathscr{X}^\perp(M) = \{g \in \mathscr{C}^\infty_c(M, \mathbf{R}^n) : \forall x \in M \;\; g(x) \in \operatorname{Nor}(M,x)\}$;
- *tangent and normal parts of a vectorfield:* if $g \in \mathscr{C}^\infty_c(M, \mathbf{R}^n)$, then $\operatorname{Tan}(M,g) \in \mathscr{X}(M)$ and $\operatorname{Nor}(M,g) \in \mathscr{X}^\perp(M)$ are such that $g = \operatorname{Tan}(M,g) + \operatorname{Nor}(M,g)$;

- $\mathbf{G}_k(M) = \{(x,S) : x \in M\,,\ S \in \mathbf{G}(n,k)\,,\ S \subseteq \operatorname{Tan}(M,x)\}$;
- *the second fundamental form:* $\mathbf{b}(M,a) : \operatorname{Tan}(M,a) \times \operatorname{Tan}(M,a) \to \operatorname{Nor}(M,a)$ a symmetric bilinear mapping such that

$$\mathrm{D}g(a)w \bullet v = -\mathbf{b}(M,a)(v,w) \bullet g(a) \quad \text{for } v,w \in \operatorname{Tan}(M,a) \text{ and } g \in \mathscr{X}^{\perp}(M)\,;$$

- *the mean curvature vector:* $\mathbf{h}(M,a) \in \operatorname{Nor}(M,a)$ is characterized by

$$(\,\mathrm{D}g(a) \circ \operatorname{Tan}(M,a)_{\natural}) \bullet \operatorname{Tan}(M,a)_{\natural} = -g(a) \bullet \mathbf{h}(M,a) \quad \text{for } g \in \mathscr{X}^{\perp}(M)\,;$$

- for $(a,S) \in \mathbf{G}_k(M)$ the vector $\mathbf{h}(M,a,S) \in \operatorname{Nor}(M,a)$ is characterized by

$$(\,\mathrm{D}g(a) \circ \operatorname{Tan}(M,a)_{\natural}) \bullet S_{\natural} = -g(a) \bullet \mathbf{h}(M,a,S) \quad \text{for } g \in \mathscr{X}^{\perp}(M)\,.$$

[All72, 3.1] **Definition.** A Radon measure V over $\mathbf{G}_k(M)$ is called a k *dimensional varifold in* M. The weakly topologised space of k dimensional varifolds in M is denoted $\mathbf{V}_k(M)$.
For any $V \in \mathbf{V}_k(M)$ we define the *weight measure* $\|V\|$ over M by requiring

$$\|V\|(B) = V(\{(x,S) \in \mathbf{G}_k(M) : x \in B\}) \quad \text{for } B \subseteq M \text{ Borel}\,.$$

[All72, 3.2] **Definition.** If $F : M \to M'$ is a smooth map between smooth manifolds and $V \in \mathbf{V}_k(M)$, then we define $F_{\#}V \in \mathbf{V}_k(M')$ by

$$F_{\#}V(\alpha) = \int \alpha(F(x),\, \mathrm{D}F(x)[S]) \| {\textstyle\bigwedge_k} \mathrm{D}F(x) \circ S_{\natural} \| \,\mathrm{d}V(x,S) \quad \text{for } \alpha \in \mathscr{K}(\mathbf{G}_k(M'))\,.$$

Remark. Observe

$$\|\boldsymbol{\mu}_{r\,\#}V\| = r^k \boldsymbol{\mu}_{r\,\#}\|V\|\,.$$

[All72, 3.3] **Definition.** For $V \in \mathbf{V}_k(M)$ we define for $x \in M$ and $\beta \in \mathscr{K}(\mathbf{G}(n,k))$

$$V^{(x)}(\beta) = \lim_{r \downarrow 0} \|i_{\#}V\|(\mathbf{B}(x,r))^{-1} \int_{\mathbf{B}(x,r)\times\mathbf{G}(n,k)} \beta(S)\,\mathrm{d}(i_{\#}V)(y,S)\,.$$

[All72, 3.4] **Definition.** Let $V \in \mathbf{V}_k(M)$, $a \in M$, and $j : \operatorname{Tan}(M,a) \hookrightarrow \mathbf{R}^n$ be the inclusion map.

$$\operatorname{VarTan}(V,a) = \left\{ C \in \mathbf{V}_k(\operatorname{Tan}(M,a)) : j_{\#}C = \lim_{j\to\infty} (\boldsymbol{\mu}_{r_j} \circ \boldsymbol{\tau}_{-a} \circ i)_{\#}V \text{ for some } r_j \uparrow \infty \right\}\,.$$

[All72, 3.5] **Definition.** If $E \subseteq \mathbf{R}^n$ is countably $(\mathscr{H}^k, k)$ rectifiable and $\mathscr{H}^k(E \cap K) < \infty$ for $K \subseteq U$ compact, then define $\mathbf{v}(E) \in \mathbf{V}_k(U)$ by

$$\mathbf{v}(E)(\alpha) = \int_E \alpha(x, \operatorname{Tan}^k(\mathscr{H}^k \llcorner E, x))\,\mathrm{d}\mathscr{H}^k(x) \quad \text{for } \alpha \in \mathscr{K}(\mathbf{G}_k(U))\,.$$

Definition. We say that $V \in \mathbf{V}_k(M)$ is a *rectifiable varifold* if there exist countably $(\mathscr{H}^m, m)$ rectifiable sets $E_i \subseteq M$ and constants $c_i \in (0,\infty)$ such that

$$V = \sum_{i=1}^{\infty} c_i \mathbf{v}(E_i)\,.$$

If all c_i can be taken to be integers, then we say that V is an *integral varifold*.
The spaces of all k dimensional rectifiable and integral varifolds in M are denoted by

$$\mathbf{RV}_k(M) \quad \text{and} \quad \mathbf{IV}_k(M)\,.$$

Theorem. Let $V \in \mathbf{V}_k(M)$. Then $V \in \mathbf{RV}_k(M)$ *if and only if* for $\|V\|$ almost all a

$$\boldsymbol{\Theta}^m(i_{\#}\|V\|, a) \in (0,\infty) \quad \text{and} \quad V^{(a)}(\beta) = \beta(\operatorname{Tan}^k(i_{\#}\|V\|, a)) \quad \text{for } \beta \in \mathscr{K}(\mathbf{G}(n,k))\,.$$

Moreover, $V \in \mathbf{IV}_k(M)$ *if and only if* $V \in \mathbf{RV}_k(M)$ and $\boldsymbol{\Theta}^m(i_{\#}\|V\|, a)$ is a non-negative integer for $\|V\|$ almost all a.

[All72, 4.2] **Definition.** Let $V \in \mathbf{V}_k(M)$. Define $\delta V : \mathscr{X}(M) \to R$ the *first variation of* V by

$$\delta V(g) = \int (\mathrm{D}g(x) \circ S_\natural) \bullet S_\natural \,\mathrm{d}V(x,S) \quad \text{for } g \in \mathscr{X}(M)\,.$$

Definition. The *total variation measure* $\|\delta V\|$ is given by

$$\|\delta V\|(G) = \sup\{\delta V(g) : g \in \mathscr{X}(M)\,,\ \operatorname{spt} g \subseteq G\,,\ |g| \le 1\} \quad \text{for } G \subseteq M \text{ open}\,,$$
$$\|\delta V\|(A) = \inf\{\|\delta V\|(G) : A \subseteq G\,,\ G \subseteq M \text{ open}\} \quad \text{for arbitrary } A \subseteq M\,.$$

Definition. If $\delta V = 0$, we say that V is *stationary.* If $G \subseteq M$ is open and $\|\delta V\|(G) = 0$, we say that V is *stationary in* G.

[All72, 4.3] **Definition.** Assume $\|\delta V\|$ is a Radon measure. Then there exists a $\|\delta V\|$ measurable function $\boldsymbol{\eta}(V,\cdot)$ such that for $\|\delta V\|$ almost all x there holds $\boldsymbol{\eta}(V,x) \in \operatorname{Tan}(M,s)$ and

$$\delta V(g) = \int g(x) \bullet \boldsymbol{\eta}(V,x)\,\mathrm{d}\|\delta V\|(x) \quad \text{for } g \in \mathscr{X}(M)\,.$$

Setting $\mathbf{h}(V,x) = -\mathbf{D}(\|\delta V\|, \|V\|, x)\boldsymbol{\eta}(V,x)$ we obtain a $\|V\|$ measurable function such that

$$\delta V(g) = -\int g(x) \bullet \mathbf{h}(V,x)\,\mathrm{d}\|V\|(x) + \int g(x) \bullet \boldsymbol{\eta}(V,x)\,\mathrm{d}\|\delta V\|_{\mathrm{sing}}(x) \quad \text{for } g \in \mathscr{X}(M)\,,$$

where $\|\delta V\|_{\mathrm{sing}}$ denotes the singular part of $\|\delta V\|$ with respect to $\|V\|$.
We call $\mathbf{h}(V,x)$ the *generalized mean curvature vector of* V *at* x.

[All72, 4.4] **Remark.** If $V \in \mathbf{V}_k(M)$ and $g \in \mathscr{X}(U)$, then

$$\delta(i_\# V)(g) = \delta V(\operatorname{Tan}(M,g)) - \int \operatorname{Nor}(M,g)(x) \bullet \mathbf{h}(M,x,S)\,\mathrm{d}V(x,S)\,.$$

[All72, 4.5] **Lemma.** Let $W \subseteq U$ be open, $Y \subseteq \mathbf{R}^m$ be open, $\varphi : Y \to W$ and $\psi : W \to Y$ be smooth and such that $\psi \circ \varphi = \mathrm{id}_Y$ and $W \cap M = W \cap \operatorname{im}\varphi$, $V \in V_m(M)$. Then

$$\delta V(g) = \delta(\psi_\# V)(\|\textstyle\bigwedge_m \mathrm{D}\varphi\|\langle g \circ \varphi, \mathrm{D}\psi \circ \varphi\rangle) \quad \text{for } g \in \mathscr{X}(W \cap M)\,,$$
$$\int_Y \mathrm{D}\beta(y)v\,\mathrm{d}\|\psi_\# V\|(y) = \delta V\big((\|\textstyle\bigwedge_m \mathrm{D}\varphi\|^{-1}\beta \cdot \mathrm{D}\varphi(\cdot)v) \circ \psi\big) \quad \text{for } v \in R^m \text{ and } \beta \in \mathscr{D}(Y,\mathbf{R})\,.$$

[All72, 4.6] **Theorem.** Assume M is connected, $V \in \mathbf{V}_m(U)$, $\operatorname{spt}\|V\| \subseteq M$, $\|\delta V\|$ is a Radon measure, and

$$\delta V(g) = 0 \quad \text{for } g \in \mathscr{X}(M) \text{ with } \operatorname{Nor}(M,g) = 0\,.$$

Then there exists a constant $C > 0$ such that

$$V = C\mathbf{v}(M) \quad \text{and} \quad C = \|V\|(A)/\mathscr{H}^m(A) \quad \text{for any } A \subseteq M \text{ with } \mathscr{H}^m(A) \in (0,\infty)\,.$$

[All72, 4.7] **Example.** If $E \subseteq M$ is a set of locally finite perimeter in M, then $\mathbf{v}(E) \in \mathbf{V}_m(M)$ and

$$\delta\mathbf{v}(E)(g) = \int_{\partial_* E} g(x) \bullet \nu_E(x)\,\mathrm{d}\mathscr{H}^{m-1}(x) \quad \text{for } g \in \mathscr{X}(M)\,.$$

[All72, 4.8] **Example.** Let $0 < k < n$ and $T \in \mathbf{G}(n,k)$. Set $V(A) = \mathscr{H}^n(\{x : (x,T) \in A\})$ for $A \subseteq \mathbf{R}^n \times \mathbf{G}(n,k)$. Then

$$V \in \mathbf{V}_k(\mathbf{R}^n)\,, \quad \delta V = 0\,, \quad \|V\| = \mathscr{H}^n\,, \quad \boldsymbol{\Theta}^k(\|V\|,a) = 0 \quad \text{for } a \in \mathbf{R}^n\,.$$

[All72, 4.10] **Lemma.** Assume $r \in \mathbf{R}$, $V \in \mathbf{V}_k(U)$, $\|\delta V\|$ is a Radon measure, $f : W \to \mathbf{R}$ is continuous, $g \in \mathscr{X}(U)$, f is smooth in a neighborhood of $\operatorname{spt}\|V\| \cap f^{-1}\{r\} \cap \operatorname{spt} g$. Then

$$(\delta V \llcorner \{x : f(x) > r\})(g) = \delta\big(V \llcorner \{(x,S) : f(x) > r\}(g)\big)(g) + \lim_{h \downarrow 0} \frac{1}{h} \int_{\{(x,S) : r < f(x) \le r+h\}} S_\natural(g(x)) \bullet \operatorname{grad} f(x)\,\mathrm{d}V(x,S)\,.$$

Remark. Set $E_r = \{x \in U : f(x) > r\}$. In the language of [Men16b, §5] one could write

$$V\partial E_r(g) = \lim_{h \downarrow 0} \frac{1}{h} \int_{\{(x,S) : r < f(x) \le r+h\}} S_\natural(g(x)) \bullet \operatorname{grad} f(x)\,\mathrm{d}V(x,S)\,.$$

Theorem. Assume $V \in \mathbf{V}_k(U)$, $\|\delta V\|$ is a Radon measure, $-\infty \le a < b \le \infty$, $f : W \to \mathbf{R}$ is continuous and smooth in a neighborhood of $\operatorname{spt}\|V\| \cap f^{-1}(a,b)$. Then for $\mathscr{L}^1$ almost all $r \in (a,b)$ the measure $\|\delta(V \llcorner \{(x,S) : f(x) > r\})\|$ is a Radon measure and

$$\int_a^b \|\delta(V \llcorner \{(x,S) : f(x) > r\})\|(B)\,\mathrm{d}\mathscr{L}^1(r) \le \int_{B \cap f^{-1}(a,b) \times \mathbf{G}(n,k)} |S_\natural(\operatorname{grad} f(x))|\,\mathrm{d}V(x,S) + \int_a^b \|\delta V\|(B \cap \{x : f(x) > r\})\,\mathrm{d}\mathscr{L}^1(r)$$

for any Borel set $B \subseteq U$.

[All72, 4.12] **Remark.** Let $V \in \mathbf{V}_k(\mathbf{R}^n)$ and $r \in (0,\infty)$.

$$\|\delta(\boldsymbol{\mu}_{r\,\#} V)\| = r^{k-1} \boldsymbol{\mu}_{r\,\#} \|\delta V\| \,.$$

Remark. If $\boldsymbol{\Theta}^{k-1}(\|\delta V\|, a) = 0$, then all members of $\operatorname{VarTan}(V,a)$ are stationary.

[Men16b, 4.6] **Theorem.** Assume $U \subseteq \mathbf{R}^n$ is open, $V \in \mathbf{V}_k(U)$, $\|\delta V\|$ is Radon, $a \in U$, $s, r \in (0,\infty)$, $\mathbf{B}(a,r) \subseteq U$, $s \le r$. Then

$$r^{-k}\|V\|\,\mathbf{B}(a,r) - s^{-k}\|V\|\,\mathbf{B}(a,s) = \int_{(\mathbf{B}(a,r) \sim \mathbf{B}(a,s)) \times \mathbf{G}(n,k)} \frac{|P_\natural^\perp(x-a)|^2}{|x-a|^{k+2}}\,\mathrm{d}V(x,P) - \int_s^r \frac{1}{u^{k+1}} \int_{\mathbf{B}(a,u)} (x-a) \bullet \boldsymbol{\eta}(V,x)\,\mathrm{d}\|\delta V\|(x)\,\mathrm{d}\mathscr{L}^1(u)\,.$$

[All72, 5.1(3)] Suppose $M, R \in (0,\infty)$, $a \in U$, $\mathbf{B}(a,R) \subseteq U$, and $\|\delta V\|\,\mathbf{B}(a,r) \le M\|V\|\,\mathbf{B}(a,r)$ for all $r \in (0,R)$. Then the function

$$l_a(r) = r^{-k}\|V\|\,\mathbf{B}(a,r)\exp(Mr) \quad \text{for } r \in (0,R)$$

is non-decreasing.

[Sim83, 17.8] Suppose $R, p \in (0,\infty)$, $k \in \mathbb{N}$, $p > k$, $a \in U$, $\mathbf{B}(a,R) \subseteq U$, $V \in \mathbf{V}_k(U)$, V satisfies $H(p)$, and $\Gamma = (\int_{\mathbf{B}(a,R)} |\mathbf{h}(V,\cdot)|^p\,\mathrm{d}\|V\|)^{1/p}$. Then the function

$$u_a(r) = r^{-k}\|V\|\,\mathbf{B}(a,r) + \frac{\Gamma}{p-k} r^{1-k/p} \quad \text{for } r \in (0,R)$$

is non-decreasing.

[All72, 5.5(1)] Assume $V \in \mathbf{V}_k(U)$, $\|\delta V\|$ is Radon, $\boldsymbol{\Theta}^{*k}(\|V\|, x) > 0$ for $\|V\|$ almost all x. Then $V \in \mathbf{RV}_k(U)$.

[Men13] Assume $V \in \mathbf{IV}_k(U)$, $\|\delta V\|$ is Radon. Then there exists a countable collection $\mathcal{A}$ of k-dimensional submanifolds of $\mathbf{R}^n$ of class $\mathscr{C}^2$ such that

$$\|V\|(\mathbf{R}^n \sim \textstyle\bigcup \mathcal{A}) = 0 \quad \text{and} \quad \forall M \in \mathcal{A} \quad \mathbf{h}(M,x) = \mathbf{h}(V,x) \quad \text{for } \|V\| \text{ almost all } x \in M\,.$$

[All72, 5.6, 6.4] Assume that for $i \in \mathbb{N}$ we are given $M_i \in (0,\infty)$ and $G_i \subseteq U$ such that $\bigcup_i G_i = U$. Suppose $\vartheta : U \to (0,\infty)$ is continuous. Then

(a) $\left\{V \in \mathbf{RV}_k(U) : (\|V\| + \|\delta V\|)(G_i) \le M_i \text{ for } i \in \mathbb{N},\ \boldsymbol{\Theta}^k(\|V\|, x) \ge \vartheta(x)\right\}$ is compact.

(b) $\left\{V \in \mathbf{IV}_k(U) : (\|V\| + \|\delta V\|)(G_i) \le M_i \text{ for } i \in \mathbb{N}\right\}$ is compact.

Approximation of locally Lipschitz functions on varifolds. Let M be an m dimensional submanifold of class $\mathscr{C}^1$ of $\mathbf{R}^n$ and let $U \subseteq \mathbf{R}^n$ be open.

[Men16a, 3.1] **Theorem.** Suppose Y is a normed vectorspace, and $f : M \to Y$ is of class $\mathscr{C}^1$.

(a) If $\varrho(C,\delta)$ denotes the supremum of the set consisting of 0 and all numbers

$$|f(x) - f(a) - \langle \operatorname{Tan}(M,a)_\natural(x-a),\, \mathrm{D}f(a)\rangle| / |x-a|$$

corresponding to $\{x,a\} \subset C$ with $0 < |x-a| \le \delta$ whenever $C \subset M$ and $\delta > 0$, then $\varrho(C,\delta) \to 0$ as $\delta \to 0+$ whenever C is a compact subset of M.

(b) There exist an open subset V of $\mathbf{R}^n$ with $M \subset V$ and a function $g : V \to Y$ of class $\mathscr{C}^1$ with $g|M = f$ and

$$\mathrm{D}g(a) = \mathrm{D}f(a) \circ \operatorname{Tan}(M,a)_\natural \quad \text{for } a \in M.$$

[Men16a, 3.2] **Corollary.** There exists a function r of class $\mathscr{C}^1$ retracting some open subset of $\mathbf{R}^n$ onto M and satisfying

$$\mathrm{D}r(a) = \mathrm{Tan}(M, a)_\natural \quad \text{whenever } a \in M.$$

[Men16a, 3.3] **Lemma.** Suppose μ is a Radon measure over U, $h : U \to \mathbf{R}$ is of class $\mathscr{C}^1$, $A = \{x : h(x) \geq 0\}$, and $\varepsilon > 0$. Then there exists a *nonnegative* function $g : U \to \mathbf{R}$ of class $\mathscr{C}^1$ such that

$$\mu(A \sim \{x : h(x) = g(x)\}) \leq \varepsilon.$$

[Men16a, 3.4] **Lemma.** Suppose $A \subset U$, $f : U \to \mathbf{R}^l$ is of class $\mathscr{C}^1$, and $\varepsilon > 0$. Then there exist an open subset X of U and a function $g : \mathbf{R}^n \to \mathbf{R}^l$ of class $\mathscr{C}^1$ such that $A \subset X$, $f|X = g|X$, and

$$\mathrm{Lip}\, g \leq \varepsilon + \sup\{\mathrm{Lip}(f|A), \sup \| \mathrm{D}f \| [A]\}.$$

Moreover, if $l = 1$ and $f \geq 0$ then one may require $g \geq 0$.

[Men16a, 3.5] **Lemma.** Suppose $V \in \mathbf{RV}_m(U)$, and $\varepsilon > 0$.

(a) There exists an m dimensional submanifold M of class $\mathscr{C}^1$ of $\mathbf{R}^n$ with $\|V\|(U \sim M) \leq \varepsilon$.

(b) If Y is a finite dimensional normed vectorspace, f is a Y valued $\|V\|$ measurable function and A is set of points at which f is $(\|V\|, m)$ approximately differentiable, then there exists $g : U \to Y$ of class $\mathscr{C}^1$ such that

$$\|V\|(A \sim \{x : f(x) = g(x)\}) \leq \varepsilon.$$

[Men16a, 3.6] **Theorem.** Suppose $V \in \mathbf{RV}_m(U)$, C is a relatively closed subset of U, $f : U \to \mathbf{R}^l$ is locally Lipschitz, $\mathrm{spt}\, f \subset \mathrm{Int}\, C$, and $\varepsilon > 0$. Then there exists $g : U \to \mathbf{R}^l$ of class $\mathscr{C}^1$ satisfying

$$\mathrm{spt}\, g \subset C, \quad \mathrm{Lip}\, g \leq \varepsilon + \mathrm{Lip}\, f, \quad \|V\|(U \sim \{x : f(x) = g(x)\}) \leq \varepsilon.$$

Moreover, if $l = 1$ and $f \geq 0$ then one may require $g \geq 0$.

[Men16a, 3.7] **Corollary.** Suppose $V \in \mathbf{RV}_m(U)$, K is a compact subset of U, and $f : U \to \mathbf{R}^l$ is a Lipschitz function with $\mathrm{spt}\, f \subset \mathrm{Int}\, K$. Then there exists a sequence $f_i \in \mathscr{D}(U, \mathbf{R}^l)$ satisfying

$$f_i(x) \to f(x) \quad \text{uniformly for } x \in \mathrm{spt}\, \|V\| \text{ as } i \to \infty,$$
$$\big\|(\|V\|, m)\, \mathrm{ap}\, \mathrm{D}(f_i - f)\big\| \to 0 \quad \text{in } \|V\| \text{ measure as } i \to \infty,$$
$$\mathrm{spt}\, f_i \subset K \quad \text{for } i \in \mathbb{N}, \qquad \limsup_{i \to \infty} \mathrm{Lip}\, f_i \leq \mathrm{Lip}\, f.$$

Moreover, if $l = 1$ and $f \geq 0$ one may require $f_i \geq 0$ for $i \in \mathbb{N}$.

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