Portland lectures on GMT

Def. $\Lambda^k(V, Z)$ - antisymmetric k-linear maps $V \times \ldots \times V \to Z$

Def. (W, μ) is called the k-th exterior power of V if $\mu \in \Lambda^k(V, W)$ and $\forall Z$ a vectorspace $\forall \varphi \in \Lambda^k(V, Z)$ $\exists!\, \tilde{\varphi} \in \mathrm{Hom}(W, Z) = \Lambda^1(W, Z)$

[Fed 69 Chapter 1]

$$\varphi = \tilde{\varphi} \circ \mu$$

$$\begin{tikzcd} V \times \ldots \times V \arrow[r, "\mu"] \arrow[dr, "k\text{-lin}"', "\forall \varphi"] & W \arrow[d, dashed, "\exists! \tilde{\varphi}"] \\ & Z \end{tikzcd}$$

Notation: $W = \Lambda_k V$

$$\mu(v_1, \ldots, v_k) = v_1 \wedge \ldots \wedge v_k$$

In other words

$\forall Z$ a vectorspace

$$\Lambda^k(V, Z) \underset{\text{bijection}}{\simeq} \mathrm{Hom}(\Lambda_k V, Z)$$

Def $f \in \mathrm{Hom}(V, Z)$ gives rise to

$$\Lambda_k f \in \mathrm{Hom}(\Lambda_k V, \Lambda_k Z)$$

s.t. $\Lambda_k f(v_1 \wedge \ldots \wedge v_k) = f(v_1) \wedge \ldots \wedge f(v_k)$

Proof.

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\mu_V"] \arrow[d, "f \times \ldots \times f"] & \Lambda_k V \arrow[d, dashed, "\exists! \Lambda_k f"] \\ Z \times \cdots \times Z \arrow[r, "\mu_Z"] & \Lambda_k Z \end{tikzcd}$$

Note $f \in \mathrm{Hom}(V, Z)$

$g \in \mathrm{Hom}(Z, W)$

Then $\Lambda_k(g \circ f) = \Lambda_k g \circ \Lambda_k f$

Def. $f, g \in \mathrm{Hom}(V, Z)$, V, Z finite dim. inner product spaces

$$f \bullet g = \mathrm{trace}(f^* \circ g)$$

$$|f|^2 = f \bullet f$$

[Hilbert-Schmidt norm]

Def. $f \in \mathrm{Hom}(V, Z)$, V, Z - normed spaces

$$\|f\| = \sup\{ |f(v)| : v \in V, |v| \le 1 \}$$

[OPERATOR NORM]

Def. [1.7.5] V an inner product space, $\dim V = m < \infty$

$(v_1, \ldots, v_m)$ an o.n.b. of V

Then $\Lambda_k V$ is an inner product space such that

$\{ v_{\lambda(1)} \wedge \ldots \wedge v_{\lambda(k)} : \lambda \in \Lambda(n, k) \}$ is o.n.b. of $\Lambda_k V$,

where $\Lambda(n, k) = \{ \lambda : \{1 \ldots k\} \to \{1 \ldots n\} : \lambda \text{ increasing} \}$

Remark (Exercise) In case $f \in \mathrm{Hom}(V, Z)$ and $\dim V = 1$ or $\dim Z = 1$ then $|f| = \|f\|$.

Portland lectures on GMT

9-12.07.2018

Def [3.2.1] $A \subseteq \mathbb{R}^m$, $a \in A$, $f : A \longrightarrow \mathbb{R}^n$, $Df(a) \in \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ exists

$$J_k f(a) = \| \Lambda_k Df(a) \|$$

[k-dimensional Jacobian of f at a]

Remark

(a) $k = \min\{n, m\} \Rightarrow \| \Lambda_k Df(a) \| = | \Lambda_k Df(a) |$

$$= \operatorname{trace}\left(\Lambda_k (Df(a)^* \circ Df(a)) \right)$$

$$= \operatorname{trace}\left(\Lambda_k (Df(a) \circ Df(a)^*) \right)$$

$\Downarrow$ $\dim \Lambda_k \mathbb{R}^m = 1$ OR $\dim \Lambda_k \mathbb{R}^n = 1$

Def. $g \in \operatorname{Hom}(V, V)$, $\dim V < \infty$, $(v_1, \dots, v_m)$ a basis of V

Then $\det g \in \mathbb{R}$ is the number characterised by

$$\Lambda_m g\, (v_1 \wedge \dots \wedge v_m) = (\det g)\, (v_1 \wedge \dots \wedge v_m)$$

noting $\Lambda_m g \in \operatorname{Hom}(\underbrace{\Lambda_m V}_{\simeq \mathbb{R}}, \underbrace{\Lambda_m V}_{\simeq \mathbb{R}})$

Remark

(b) $k = m \leq n \Rightarrow J_k f(a) = \det\left(Df(a)^* \circ Df(a) \right)^{1/2}$

$k = n \leq m \Rightarrow J_k f(a) = \det\left(Df(a) \circ Df(a)^* \right)^{1/2}$

Def. [1.7.4] $\mathbf{O}^*(n, m) = \{ p \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m) : p \circ p^* = \mathrm{id}_{\mathbb{R}^m} \}$ $\boxed{m \leq n}$

$= \{ p \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m) : \forall v, w \in \mathbb{R}^m \;\; p^*(v) \bullet p^*(w) = v \bullet w \}$

$= \{ p : p^* \text{ is an isometric embedding } \mathbb{R}^m \hookrightarrow \mathbb{R}^n \}$

Def. [Alm 00] $P \in G(n, k) \Rightarrow P_\natural \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is characterised by

$$P_\natural \circ P_\natural = P_\natural, \quad P_\natural^* = P_\natural, \quad \operatorname{im} P_\natural = P$$

Remark (Exercise)

(c) If $P, Q \in G(n, k)$, $p, q \in \mathbf{O}^*(n, k)$ are such that

$P_\natural = p^* \circ p$, $Q_\natural = q^* \circ q$, $\operatorname{im}(Df(a) \circ P_\natural) = Q$

then

$$\| \Lambda_k (Df(a) \circ P_\natural) \| = \left| \det \left(\overbrace{q \circ Df(a) \circ p^*}^{\in \operatorname{Hom}(\mathbb{R}^k, \mathbb{R}^k)} \right) \right|$$

! $p \in \mathbf{O}^*(n, k)$

$\Downarrow$

$\Lambda_j p \in \mathbf{O}^*(\Lambda_j \mathbb{R}^n, \Lambda_j \mathbb{R}^k)$ $\forall j \in \{0, 1, \dots, k\}$

Portland lectures on GMT 9-12. 07. 2018

Def. $N(f,y) = \mathcal{H}^0(f^{-1}\{y\})$

The Area formula

1. Lemma [Fed69, 2.10.11]

If $A \subseteq \mathbb{R}^m$ $\mathcal{H}^m$ measurable, $0 \le m \le n$ integers
$f : \mathbb{R}^m \to \mathbb{R}^n$, $\operatorname{Lip} f < \infty$,

then
$$\int_{\mathbb{R}^n} N(f|A, y)\, d\mathcal{H}^m(y) \le (\operatorname{Lip} f)^m \mathcal{H}^m(A)$$

Proof. Consider Borel partitions H_j of A such that

- $\lim_{j\to\infty} \sup\{\operatorname{diam} S : S \in H_j\} = 0$
- each element of H_j is a union of elements of H_{j+1}

Then
$$\int_{\mathbb{R}^n} N(f|A, y)\, d\mathcal{H}^m(y) \overset{\text{monotone convergence}}{=} \lim_{j\to\infty} \int_{\mathbb{R}^n} \sum_{S\in H_j} \mathbb{1}_{f[S]}(y)\, d\mathcal{H}^m(y)$$

Note: $\mathbb{1}_{f[S]}(y) = 1 \Leftrightarrow f^{-1}\{y\} \cap S \neq \emptyset$

$$= \lim_{j\to\infty} \sum_{S\in H_j} \mathcal{H}^m(f[S]) \le (\operatorname{Lip} f)^m \lim_{j\to\infty} \sum_{S\in H_j} \mathcal{H}^m(S)$$
$$= (\operatorname{Lip} f)^m \mathcal{H}^m(A).$$

$\square$

2. Lemma [Fed69, 3.2.2]

If $f : \mathbb{R}^m \to \mathbb{R}^n$, $m \le n$, f continuous, $\lambda > 1$

then there exists a countable family $\mathcal{G}$ of Borel sets s.t.

(a) $\{x \in \mathbb{R}^m : Df(x) \text{ is a monomorphism}\} \subseteq \bigcup \mathcal{G}$

(b) $\forall E \in \mathcal{G}$ $\exists s \in GL(m, \mathbb{R})$

- $f|_E$ is injective
- $\operatorname{Lip}(f|_E \circ s^{-1}) \le \lambda$ and $\operatorname{Lip}(s \circ (f|_E)^{-1}) \le \lambda$
- $\lambda^{-1}|s(v)| \le |Df(x)v| \le \lambda |s(v)|$ $\forall x \in E$ $\forall v \in \mathbb{R}^m$
- $\lambda^{-m}|\det s| \le J_m f(x) \le \lambda^m |\det s|$ $\forall x \in E$

Proof of lemma

Let $\varepsilon > 0$ be such that $\frac{1}{\lambda} + \varepsilon < 1 < \lambda - \varepsilon$

Let $S \subseteq GL(n, \mathbb{R})$ be countable and dense

For $i \in \mathbb{N}_+$ and $s \in S$ set

$$Z(s,i) = \{ a \in \mathbb{R}^m : \left(\tfrac{1}{\lambda} + \varepsilon\right)|s(v)| \le |Df(a)v| \le (\lambda - \varepsilon)|s(v)| \ \forall v \in \mathbb{R}^m$$

and

$$|f(b) - f(a) - Df(a)(b-a)| \le \varepsilon |s(b-a)| \ \forall b \in \mathbb{B}(a, 1/i) \}$$

Let $\mathcal{Z}(s,i)$ be a countable covering of $Z(s,i)$ by sets of diameter $\le 1/i$

Define $\mathcal{G} = \bigcup \{ \mathcal{Z}(s,i) : s \in S, \ i \in \mathbb{N}_+ \}$

Claim (Exercise) $\mathcal{G}$ satisfies (a) and (b) $\square$

Remark If $Df(a)$ is a monomorphism,

then $\exists g \in GL(m, \mathbb{R})$, $\exists h \in \mathbf{O}^*(n, m)$

$$Df(a) = h^* \circ g$$

3. Theorem (Area formula) [Fed 69, 3.2.3]

$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $m \le n$, $\operatorname{Lip} f < \infty$

1) If $A \subseteq \mathbb{R}^m$ is $\mathcal{L}^m$-measurable, then

$$\int_A J_m f \, d\mathcal{L}^m = \int_{\mathbb{R}^n} N(f|A, y) \, d\mathcal{H}^m(y)$$

2) If $u : \mathbb{R}^m \longrightarrow \mathbb{R}$ is $\mathcal{L}^m$-integrable, then

$$\int_{\mathbb{R}^m} u(x) \cdot J_m f(x) \, d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \sum_{x \in f^{-1}\{y\}} u(x) \, d\mathcal{H}^m(y)$$

Proof of theorem

Note that 2) follows from 1) by standard techniques.

$\boxed{\text{Step 1}}$ Assume $\mathcal{L}^m(A) = 0$.

Then $\int_A J_m f \, d\mathcal{L}^m = 0$

and by 1.

$$0 \leq \int_{\mathbb{R}^n} N(f|_A, y)\, d\mathcal{H}^m(y) \leq (\operatorname{Lip} f)^m \mathcal{H}^m(A) = 0.$$

$\boxed{\text{Step 2}}$ Assume $A \subseteq \{x : Df(x) \text{ is mono}\}$

Using Step 1 and Rademacher we assume $Df(x)$ exists for all $x \in A$

Employ 2. with some $\lambda > 1$ to get a Borel partition H of A s.t. 2 (a)(b) are satisfied.

Then $\forall B \in H \quad \exists s \in GL(m, \mathbb{R})$

- $\lambda^{-m} |\det s| \, \mathcal{L}^m(B) \leq \int_B J_m f \, d\mathcal{L}^m \leq \lambda^m |\det s| \, \mathcal{L}^m(B)$
- $\lambda^{-m} \mathcal{H}^m(s[B]) \leq \mathcal{H}^m(f[B]) \leq \lambda^m \mathcal{H}^m(s[B])$

because $f[B] = (f|_B \circ s^{-1})[s[B]]$

Since $\mathcal{H}^m(s[B]) = \mathcal{L}^m(s[B]) = |\det s| \, \mathcal{L}^m(B)$

we have

$$\lambda^{-2m} \mathcal{H}^m(f[B]) \leq \int_B J_m f \, d\mathcal{L}^m \leq \lambda^{2m} \mathcal{H}^m(f[B])$$

Summing over all $B \in H$ we get

$$\lambda^{-2m} \int_{\mathbb{R}^n} N(f|_A, y)\, d\mathcal{H}^m(y) \leq \int_A J_m f \, d\mathcal{L}^m \leq \lambda^{2m} \int_{\mathbb{R}^n} N(f|_A, y)\, d\mathcal{H}^m$$

We obtain the claim by passing to the limit $\lambda \downarrow 1$.

$\boxed{\text{Step 3}}$ Assume $A \subseteq \{x : J_m f(x) = 0\}$

Define
$$g : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m, \quad g(x) = (f(x), \varepsilon x) \qquad , \qquad p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \quad p(y, z) = y$$

Note
- $f = p \circ g$ and $Dg(x)$ is mono $\forall x \in \mathbb{R}^m$
- $\|Dg(x)\| \le \operatorname{Lip} f + \varepsilon$
- g is injective (Federer says "univalent")
- $J_m g \le \varepsilon(\operatorname{Lip} f + \varepsilon)^{m-1}$ because $|Dg(x)v| = \varepsilon v$ whenever $v \in \ker Df(x)$

Apply Step 2 to the map g to get

$$\mathcal{H}^m(f[A]) = \mathcal{H}^m(p \circ g[A]) \le \mathcal{H}^m(g[A])$$

$$= \int_A J_m g \, d\mathcal{L}^m \le \mathcal{L}^m(A) \cdot \varepsilon(\operatorname{Lip} f + \varepsilon)^{m-1}.$$

Letting $\varepsilon \downarrow 0$ we get $\mathcal{H}^m(f[A]) = 0$;

hence, $\int_{\mathbb{R}^n} N(f|A, y)\, d\mathcal{H}^m(y) = \int_{f[A]} N(f|A, y)\, d\mathcal{H}^m(y) = 0$ $\square$

4. Lemma [Fed 69, 3.2.9]

$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ continuous, $m \ge n$

There exists a countable family of Borel sets $\mathcal{G}$ s.t.

(a) $\{x \in \mathbb{R}^m : \operatorname{im} Df(x) = \mathbb{R}^n\} \subseteq \bigcup \mathcal{G}$

(b) $\forall E \in \mathcal{G}$ $\exists p \in \mathbf{O}^*(m, m-n)$ $\exists v : \mathbb{R}^n \times \mathbb{R}^{m-n} \longrightarrow \mathbb{R}^m$

$\operatorname{Lip} v < \infty$, $v(u(x)) = x$ for $x \in E$

$u(x) = (f(x), p(x))$, $u : \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^{m-n}$

Proof For $\lambda \in \Lambda(m, m-n)$ define

$p_\lambda(x_1 \dots x_m) = (x_{\lambda(1)}, \dots, x_{\lambda(m-n)})$, $p_\lambda \in \mathbf{O}^*(m, m-n)$

$u_\lambda(x) = (f(x), p_\lambda(x))$, $u_\lambda : \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^{m-n}$

$A_\lambda = \{x \in \mathbb{R}^m : Du_\lambda(x) \text{ is mono}\}$

Observe that
- $\ker Du_\lambda(x) = \ker Df(x) \cap \ker p_\lambda$
- $\{x \in \mathbb{R}^m : \operatorname{im} Df(x) = \mathbb{R}^n\} = \bigcup\{A_\lambda : \lambda \in \Lambda(m, m-n)\}$

Apply 2. to u_λ to find $\mathcal{G}_\lambda$ a covering of A_λ s.t.

$\forall E \in \mathcal{G}_\lambda$ $u_\lambda|_E$ is injective and bi-Lipschitz (invertible!)

Employ Kirszbraun [Fed69, 2.10.43]

to get $v : \mathbb{R}^n \times \mathbb{R}^{m-n} \longrightarrow \mathbb{R}^m$ s.t.

$$v|_{u_\lambda[E]} = \left(u_\lambda|_E\right)^{-1}$$

$\square$

5. Lemma [Fed69. 3.2.10]

Let $f, m, n, \mathcal{G}, E \in \mathcal{G}, p, v, u$ be as in 4.

Then (a) $\displaystyle\int_B J_n f \, d\mathcal{L}^m = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(B \cap f^{-1}\{y\}) \, d\mathcal{L}^n(y)$

for any $B \subseteq E$ $\mathcal{L}^m$-measurable

(b) $E \cap f^{-1}\{y\} = v\left[\{y\} \times p[E \cap f^{-1}\{y\}]\right]$ for $y \in \mathbb{R}^n$

Proof. First we show (b).

Set $v_y : \mathbb{R}^{m-n} \longrightarrow \mathbb{R}^m$, $v_y(z) = v(y, z)$. for $y \in \mathbb{R}^n$.

Since $v \circ u|_E = id_E$ we see that $v_y|_{p[E \cap f^{-1}\{y\}]}$ is injective

and $v_y\left[p[E \cap f^{-1}\{y\}]\right] = E \cap f^{-1}\{y\}$.

Simply because if $z \in E \cap f^{-1}\{y\}$, then

$f(z) = y$ and $p(z) \in p[E \cap f^{-1}\{y\}]$

so $v_y(p(z)) = v(u(z)) = z$;

on the other hand if $x \in p[E \cap f^{-1}\{y\}]$,

then there exists $z \in E \cap f^{-1}\{y\}$ s.t. $p(z) = x$

and $v_y(x) = v(u(z)) = z$.

Since $Du(x)^{-1} = Dv(u(x))$ we obtain

the commutative diagram:

$$\begin{tikzcd}
\mathbb{R}^{m-n} \arrow[r, hook, "j"] \arrow[d, "L_x"] & \mathbb{R}^n \times \mathbb{R}^{m-n} \arrow[r, two heads, "q"] \arrow[d, bend right, "Dv(u(x))"'] & \mathbb{R}^n \\
\operatorname{im} L_x \arrow[r, hook, "g"] & \mathbb{R}^m \arrow[u, bend right, "Du(x)"'] \arrow[ur, "Df(x)"] & W \arrow[l, hook', "h"'] \arrow[u]
\end{tikzcd}$$

$$\mathbb{R}^m \cong \operatorname{im} L_x \overset{\perp}{\oplus} W$$

$$\operatorname{im} L_x \perp W$$

where
$$q(y,z) = y$$
$$L_x = Dv_{f(x)}(p(x)) \Rightarrow g \circ L_x = Dv(f(x), p(x)) \circ j = Dv(u(x)) \circ j$$
$$\operatorname{im} L_x = Du(x)^{-1}\left[\{0\} \times \mathbb{R}^{m-n}\right] = \ker Df(x)$$
$$W = (\ker Df(x))^{\perp}$$
$$q \circ Du(x)\eta = q(Df(x)\eta, p(\eta)) = Df(x)\eta \quad \forall \eta \in \mathbb{R}^m$$

Hence,
$$Du(x) = q^* \circ q \circ Du(x) + j \circ j^* \circ Du(x)$$
$$= q^* \circ Df(x) + j \circ L_x^{-1} \circ g^*$$

Choose an o.n.b. of $\mathbb{R}^m$: $\eta_1, \dots, \eta_m$ s.t.
$$\ker Df(x) = \operatorname{span}\{\eta_1, \dots, \eta_{m-n}\}$$
We get:
$$\|\Lambda_m Du(x)\| = |Du(x)\eta_1 \wedge \dots \wedge Du(x)\eta_m|$$
$$= |L_x^{-1}\eta_1 \wedge \dots \wedge L_x^{-1}\eta_{m-n}| \cdot |Df(x)\eta_{m-n+1} \wedge \dots \wedge Df(x)\eta_m|$$
$$= \|\Lambda_n Df(x)\| \cdot \|\Lambda_{m-n} L_x^{-1}\| = \frac{\|\Lambda_n Df(x)\|}{\|\Lambda_{m-n} L_x\|},$$
$$\Rightarrow J_n f(x) = J_{m-n} V_{f(x)}(p(x)) \cdot J_m u(x)$$

Employing the Area Formula 3. we obtain
$$\int_B J_n f \, d\mathcal{L}^m = \int_B J_{m-n} V_{f(x)}(p(x)) \, J_m u(x) \, d\mathcal{L}^m(x)$$
$$\overset{\text{Area}}{=} \int_{u[B]} J_{m-n} V_y(z) \, d(\mathcal{L}^n \times \mathcal{L}^{m-n})(y,z)$$
$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n} \int_{p[B \cap f^{-1}\{y\}]} J_{m-n} V_y(z) \, d\mathcal{L}^{m-n}(z) \, \mathcal{L}^n(y)$$

$$\underset{(6)}{\overset{\text{Area}}{=}} \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(B \cap f^{-1}\{y\})\, d\mathcal{L}^n(y) \qquad \square$$

6. Lemma [Fed 69, 2.10.25] or [AFP00, 2.95]

$m, n, k \in \mathbb{N}$, $m \geq n+k$, $n \geq 1$

$A \subseteq \mathbb{R}^m$ is $\mathcal{H}^{n+k}$ measurable, $\mathcal{H}^{n+k}(A) < \infty$

$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $\operatorname{Lip} f < \infty$

Then
$$\int_{\mathbb{R}^n} \mathcal{H}^k(A \cap f^{-1}\{y\})\, d\mathcal{L}^n(y) \leq \frac{\alpha(k)\alpha(n)}{\alpha(n+k)} (\operatorname{Lip} f)^n \mathcal{H}^{n+k}(A)$$

Proof. Let $\delta > 0$ and $\mathcal{F}$ be a family of closed subsets of $\mathbb{R}^m$ s.t.

$A \subseteq \bigcup \mathcal{F}$, $\forall F \in \mathcal{F}$ $\operatorname{diam} F < \delta$,

$$\sum_{F \in \mathcal{F}} \zeta^{n+k}(F) \leq \mathcal{H}^{n+k}(A) + \delta \qquad \boxed{\zeta^l(F) = \frac{\alpha(l)(\operatorname{diam} F)^l}{2^l}}$$

Set
$$u = \sum_{F \in \mathcal{F}} \zeta^k(F)\, \mathbb{1}_{f[F]}$$

Observe
$$f^{-1}\{y\} \cap A \subseteq \bigcup \{F \in \mathcal{F} : F \cap f^{-1}\{y\} \neq \emptyset\}$$

Hence,
$$u(y) \geq \mathcal{H}^k_\delta(A \cap f^{-1}\{y\}) \qquad \forall y \in \mathbb{R}^n$$

Moreover,
$$\int u \, d\mathcal{L}^n = \sum_{F \in \mathcal{F}} \zeta^k(F)\, \mathcal{L}^n(f[F])$$

(isodiametric inequality [2.10.33]) $\to$
$$\leq \sum_{F \in \mathcal{F}} \alpha(k) \left(\frac{\operatorname{diam} F}{2}\right)^k \alpha(n) \left(\frac{\operatorname{diam} f[F]}{2}\right)^n$$
$$\leq \frac{\alpha(n)\alpha(k)}{\alpha(n+k)} (\operatorname{Lip} f)^n \sum_{F \in \mathcal{F}} \zeta^{n+k}(F)$$
$$\leq \frac{\alpha(n)\alpha(k)}{\alpha(n+k)} (\operatorname{Lip} f)^n \left(\mathcal{H}^{n+k}(A) + \delta\right),$$

Thus,
$$\int^* \mathcal{H}^k_\delta(A \cap f^{-1}\{y\})\, d\mathcal{L}^n(y) \leq \frac{\alpha(k)\alpha(n)}{\alpha(k+n)} (\operatorname{Lip} f)^n \left(\mathcal{H}^{n+k}(A) + \delta\right).$$

Two issues: (a) can we replace ζ^* with ζ ?
(b) can we pass to the limit $\delta \downarrow 0$ on the LHS?

[2.10.26] $\Rightarrow$ $y \longmapsto \mathcal{H}^k_\delta(A \cap f^{-1}\{y\})$ is $\mathcal{L}^n$-measurable for all $\delta > 0$ given A is compact

$\Rightarrow$ answer to (a): YES if A is compact.

Approximating A with compact sets (up to $\mathcal{H}^{n+k}$ null set) and using monotone convergence together with the observation that

$$\mathcal{H}^{n+k}(A) = 0 \quad \Rightarrow \quad \mathcal{H}^k(A \cap f^{-1}\{y\}) = 0 \text{ for } \mathcal{L}^n\text{a.e. } y$$

gives the claim.

$\square$

7. Theorem [Fed69, 3.2.11, 3.2.12]

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n, \quad \operatorname{Lip} f < \infty, \quad m \geq n$$

(a) If $A \subseteq \mathbb{R}^m$ is $\mathcal{L}^m$-measurable, then

$$\int_A J_n f \, d\mathcal{L}^m = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(f^{-1}\{y\} \cap A) \, d\mathcal{L}^n(y)$$

(b) If $g : \mathbb{R}^m \longrightarrow \overline{\mathbb{R}}$ is $\mathcal{L}^m$-integrable, then

$$\int_{\mathbb{R}^m} g \cdot J_n f \, d\mathcal{L}^m = \int_{\mathbb{R}^n} \int_{f^{-1}\{y\}} g(x) \, d\mathcal{H}^{m-n}(x) \, d\mathcal{L}^n(y).$$

Proof. Clearly (b) follows from (a) by std. techniques.

$\boxed{\text{Step 1}}$ Assume $\mathcal{L}^m(A) = 0$:
Then LHS $= 0$ and RHS $= 0$ by 6.

$\boxed{\text{Step 2}}$ Assume $A \subseteq \{x \in \mathbb{R}^m : \operatorname{im} Df(x) = \mathbb{R}^n\}$.
Conclusion follows from 5.

$\boxed{\text{Step 3}}$ Assume $A \subseteq \{x \in \mathbb{R}^m : J_n f(x) = 0\}$
$= \{x \in \mathbb{R}^m : \dim \operatorname{im} Df(x) < n\}$

Then LHS $= 0$.

Let $\varepsilon > 0$. Define

$$g : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad g(x,z) = f(x) + \varepsilon z$$

$$p : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad p(x,z) = z$$

For $x \in A$, $z \in \mathbb{R}^n$:

$$Dg(x,z)(v,w) = Df(x)v + \varepsilon w$$

$$\operatorname{im} Dg(x,z) = \mathbb{R}^n, \quad \|Dg(x,z)\| \le \operatorname{Lip} f + \varepsilon,$$

$$J_n g(x,z) \le \varepsilon (\operatorname{Lip} f + \varepsilon)^{n-1}$$

$\leftarrow$ because $\dim \operatorname{im} Df(x) \le n-1$

Hence,

$$\varepsilon(\operatorname{Lip} f + \varepsilon)^{n-1} \geq \int_{A \times B(0,1)} J_n g \, d(\mathcal{L}^m \times \mathcal{L}^n)$$

(co-area step 2) $\rightarrow$

$$= \int_{\mathbb{R}^n} \mathcal{H}^m(\underbrace{A \times B(0,1)}_{= Q} \cap g^{-1}\{y\}) \, d\mathcal{L}^n(y)$$

(by 6. with p, $\operatorname{Lip} p = 1$) $\rightarrow$

$$\geq C(m,n) \int_{\mathbb{R}^n} \int_{B(0,1)} \mathcal{H}^{m-n}(Q \cap g^{-1}\{y\} \cap p^{-1}\{w\}) \, d\mathcal{L}^n(w) \, d\mathcal{L}^n(y)$$

$(x,z) \in Q \cap g^{-1}\{y\} \cap p^{-1}\{w\}$
$\Updownarrow$
$z = w$
$g(x,z) = y = f(x) + \varepsilon w$
$\Updownarrow$
$f(x) = y - \varepsilon w$

$$= C(m,n) \int_{\mathbb{R}^n} \int_{B(0,1)} \mathcal{H}^{m-n}(A \cap f^{-1}\{y - \varepsilon w\}) \, d\mathcal{L}^n(w) \, d\mathcal{L}^n(y)$$

Fubini + change of variables

$$= \tilde{C}(m,n) \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(A \cap f^{-1}\{z\}) \, d\mathcal{L}^n(z)$$

Letting $\varepsilon \downarrow 0$ we see that RHS $= 0$.

$\square$

8. Def. [Fed 69, 3.2.14]

$E \subseteq \mathbb{R}^n$ is called <u>countably $(\mathcal{H}^m, m)$-rectifiable</u> if there exist Lipschitz maps $f_1, f_2, \ldots : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$\mathcal{H}^m\left(E \sim \bigcup_{i=1}^{\infty} \operatorname{im} f_i\right) = 0.$$

If, additionally, $\mathcal{H}^m(E) < \infty$, then E is called $(\mathcal{H}^m, m)$-rectifiable.

9. Lemma [Fed69, 3.2.18]

If $W \subseteq \mathbb{R}^n$ is $(\mathcal{H}^m, m)$ rectifiable and $\mathcal{H}^m$-measurable
$\lambda \in (1, \infty)$

then $\exists\ K_1, K_2, \ldots \subseteq \mathbb{R}^m$ compact
$\exists\ \psi_1, \psi_2, \ldots : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$$\psi_i[K_i] \cap \psi_j[K_j] = \emptyset \qquad \forall i \neq j$$

$$\psi_i[K_i] \subseteq W \qquad \forall i$$

$$\mathcal{H}^m\Big(W \sim \bigcup_{i=1}^{\infty} \psi_i[K_i]\Big) = 0$$

$\operatorname{Lip} \psi_i \leq \lambda$, $\psi_i|_{K_i}$ is injective,

$$\operatorname{Lip}\left(\psi_i|_{K_i}\right)^{-1} \leq \lambda$$

$$\lambda^{-1}|v| \leq D\psi_i(a)v \leq \lambda|v| \qquad \forall a \in K_i\ \forall v \in \mathbb{R}^m$$

Remark. This provides kind of an "atlas" for W

Remark. Using Rademacher [3.1.6] and Whitney [3.1.14] one can ask ψ_i to be of class $\mathcal{C}^1$.

10. Def. Let ϕ be a Borel regular measure over X ← normed vectorspace, $m \geq 0$, $a \in X$

[2.10.19]

(a) $$\Theta^{*m}(\phi, a) = \limsup_{r \downarrow 0} \frac{\phi\, \mathbb{B}(a, r)}{\alpha(m) r^m}$$

$$\Theta^{m}_{*}(\phi, a) = \liminf_{r \downarrow 0} \frac{\phi\, \mathbb{B}(a, r)}{\alpha(m) r^m}$$

If $\Theta^{*m}(\phi, a) = \Theta^{m}_{*}(\phi, a)$, then $\Theta^{m}(\phi, a) = \Theta^{*m}(\phi, a)$

[3.2.16]

(b) $a \in X$, $v \in X$, $\varepsilon > 0$

$$\mathbb{E}(a, v, \varepsilon) = \{x \in \mathbb{R}^N : \exists r > 0\ |r(x-a) - v| < \varepsilon\}$$

$$\operatorname{Tan}^m(\phi, a) = \{v \in X : \forall \varepsilon > 0\ \Theta^{*m}(\phi \llcorner \mathbb{E}(a, v, \varepsilon), a) > 0\}$$

[3.2.16] (C) $A \subseteq X$, $f : A \to Y$, $m \in \mathbb{N}_+$

f is (ϕ, m) approximately differentiable at $a \in X$ if there exists an open set $U \subseteq X$, $a \in U$ and a function $g : U \to Y$ s.t.

$Dg(a)$ exists and $\Theta^m(\phi \llcorner \{x \in A : f(x) \neq g(x)\}, a) = 0$

We define

$$(\phi, m)\,\mathrm{ap}\,Df(a) = Dg(a)\big|\mathrm{Tan}^m(\phi, a)$$

and

$$(\phi, m)\,\mathrm{ap}\,J_k f(a) = \|\Lambda_k\,(\phi, m)\,\mathrm{ap}\,Df(a)\|$$
$$= \|\Lambda_k\,Dg(a) \circ \mathrm{Tan}^m(\phi, a)_\natural\|.$$

11. Lemma [Fed 69, 3.2.19]

If $W \subseteq \mathbb{R}^n$ is $(\mathcal{H}^m, m)$-rectifiable and $\mathcal{H}^m$-measurable,

then (A) $\Theta^m(\mathcal{H}^m \llcorner W, w) = 1$

(B) $\mathrm{Tan}^m(\mathcal{H}^m \llcorner W, w) \in G(n, m)$

for $\mathcal{H}^m$ almost all $w \in W$

Moreover, if $f : W \to \mathbb{R}^\nu$, and $\mathrm{Lip} f < \infty$,

(C) then $(\mathcal{H}^m \llcorner W, m)\,\mathrm{ap}\,Df(w)$ exists for $\mathcal{H}^m$ almost all $w \in W$.

Remark. If W is $\mathcal{H}^m$-measurable and (A) holds, then W is countably $(\mathcal{H}^m, m)$ rectifiable

see: Mattila, Trans. AMS 205, 1975

- see also: Preiss, Ann. of Math. 125, 1987

- If W is $\mathcal{H}^m$-measurable and (B) holds, then W is countably $(\mathcal{H}^m, m)$ rectifiable.

see [Fed 69, 3.3.17].

12. Theorem [3.2.20]

$m \le \nu$, $W \subseteq \mathbb{R}^n$ is $(\mathcal{H}^m, m)$ rectifiable and $\mathcal{H}^m$-measurable
$f : W \to \mathbb{R}^\nu$, $\operatorname{Lip} f < \infty$

Then
$$\int_W f \circ g \ \operatorname{ap} J_m f \, d\mathcal{H}^m = \int_{\mathbb{R}^\nu} g(z)\, N(f,z)\, d\mathcal{H}^m(z)$$
whenever $g : \mathbb{R}^\nu \to \overline{\mathbb{R}}$

Here: $\operatorname{ap} J_m f = (\mathcal{H}^m \llcorner W, m) \operatorname{ap} J_m f$

13. Theorem [3.2.22]

$W \subseteq \mathbb{R}^n$ $(\mathcal{H}^m, m)$ rect, $\mathcal{H}^m$-meas,
$Z \subseteq \mathbb{R}^\nu$ $(\mathcal{H}^\mu, \mu)$ rect, $\mathcal{H}^\mu$-meas. $m \ge \mu$

$f : W \to Z$, $\operatorname{Lip} f < \infty$

Then
$$\int_W g \ \operatorname{ap} J_\mu f \, d\mathcal{H}^m = \int_Z \int_{f^{-1}\{z\}} g \, d\mathcal{H}^{m-\mu} \, d\mathcal{H}^\mu(z)$$
whenever $g : W \to \overline{\mathbb{R}}$ is $(\mathcal{H}^m \llcorner W)$-integrable.

Proofs: Use 9. to find "atlases" for W and Z and apply area and co-area in each "map" seperately (but be careful with null sets).

Remark. The area and co-area formulas (3. & 7.) hold (in certain sense) assuming merely that $f : \mathbb{R}^m \to \mathbb{R}^n$ is $(\mathcal{L}^m, m)$ ap. differentiable.

See: Hajłasz, Proc. Anal. Geom, 2000.

Remark. There exist $W \subseteq \mathbb{R}^n$, $Z \subseteq \mathbb{R}^\nu$ with $\mathcal{H}^m(W) = 0 = \mathcal{H}^\mu(Z)$
[3.2.24] Such that $\mathcal{H}^{m+\mu}(W \times Z) = \infty$.

In particular, $\mathcal{H}^{m+\mu} \llcorner (W \times Z) \neq (\mathcal{H}^m \llcorner W) \times (\mathcal{H}^\mu \llcorner Z)$.

Digression: $f \in BV_{loc}(U) \iff V = f \cdot \mathcal{L}^n \in \mathbb{V}_n(U)$
($U \subseteq \mathbb{R}^n$ open)

is such that $\|\delta V\|(K) < \infty$ whenever $K \subseteq U$ compact

$\iff T = \mathbf{E}^n \llcorner f \in \mathbf{N}_n^{loc}(U)$

see [Fed 69, 4.5.9]

Def. We say that an $\mathcal{L}^n$-measurable set $E \subseteq U$ has *locally finite perimeter in U* if $\mathbb{1}_E \in BV_{loc}(U)$.
(a.k.a. Caccioppoli set)

Therefore, I shall first talk about varifolds and then, if time allows, about Caccioppoli sets.

---

14. Def. M smooth (open) submanifold of $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$

$i : M \hookrightarrow U$ proper

$G_k(M) = \{(x,T) : x \in M,\ T \subseteq \operatorname{Tan}(M,x),\ T \in G(n,k)\}$

Varifolds $\longrightarrow$ $\mathbb{V}_k(M) =$ all positive Radon measures over $G_k(M)$ endowed with weak* topology

X locally compact Hausdorff space

$\rightsquigarrow \mathcal{K}(X) =$ space of continuous functions $X \to \mathbb{R}$ with compact support endowed with locally convex topology

! $\|\cdot\|_\infty$ norm on $\mathcal{K}(X)$ is not complete

see: Menne, Indiana Univ. Math. J. 65, 2016, §2.

$\mathbb{V}_k(M) \subseteq \mathcal{K}(G_k(M))^* \subseteq \mathbb{R}^{\mathcal{K}(G_k(M))}$

↑ infinite product with Tikhonov topology (Tychonoff) (Тихонов)

[see Fed 69, 2.5.19]

Weight measure $\longrightarrow$ $V \in \mathbb{V}_k(M) \rightsquigarrow \|V\|$ the Radon measure over M given by

$$\|V\|(A) = V(\{(x,S) \in G_k(M) : x \in A\})$$

Remark X - loc. cpt. Hausdorff space

[All 72, 2.6(2)] (a) If $M : \mathcal{K}(X) \to [0,\infty)$, then
[2.5.19] $\{\mu \in \mathcal{K}(X)^* ; \ \mu^+ + \mu^- \leq M\}$ is compact

$\mu, \mu_1, \mu_2, \mu_3, \ldots$ Radon measures over X
and $\lim_{i\to\infty} \mu_i = \mu$

(c) $\liminf_{i\to\infty} \mu_i(U) \geq \mu(U)$ whenever $U \subseteq X$ is open

$\limsup_{i\to\infty} \mu_i(K) \leq \mu(K)$ whenever $K \subseteq X$ is compact

(b) $\lim_{i\to\infty} \mu_i(f) = \mu(f)$ for any $f \in \mathcal{K}(X)$

(d) $\lim_{i\to\infty} \mu_i(A) = \mu(A)$ whenever $A \subseteq X$
Clos A is compact
$\mu(\operatorname{Bdry} A) = 0$.

Exercise μ Radon over X
$f : X \to \mathbb{R}$ is proper
Then $\mu(f^{-1}\{t\}) > 0$ for at most countably many $t \in \mathbb{R}$.

Def. Let $E \subseteq M$ be countably $(\mathcal{H}^k, k)$-rectifiable, $\mathcal{H}^k$-measurable,

[All 72 3.5] and such that $\mathcal{H}^k(E \cap K) < \infty$ for $K \subseteq U$ compact.

We define

$$\mathbb{V}_k(E) \in \mathbb{V}_k(M) \quad \text{by}$$

$$\mathbb{V}_k(E)(\alpha) = \int_E \alpha\left(x, \operatorname{Tan}^k(\mathcal{H}^k \llcorner E, x)\right) d\mathcal{H}^k(x) \quad \text{for } \alpha \in \mathcal{K}(G_k(M))$$

Def (Rectifiable and integral varifolds)

$$V \in R\mathbb{V}_k(M) \iff V \in \mathbb{V}_k(M) \text{ and}$$
$$\exists\, E_1, E_2, \dots \subseteq M \text{ as above}$$
$$\exists\, c_1, c_2, \dots \in (0, \infty)$$
$$V = \sum_{i=1}^{\infty} c_i \, \mathbb{V}_k(E_i)$$

$$V \in I\mathbb{V}_k(M) \iff V \in R\mathbb{V}_k(M) \text{ and}$$
$$\exists\, E_1, E_2, \dots \subseteq M$$
$$\exists\, c_1, c_2, \dots \in \mathbb{N}$$
$$V = \sum_{i=1}^{\infty} c_i \, \mathbb{V}_k(E_i)$$

Note

Rectifiability of V is something more than saying $\|V\|(M \sim E) = 0$ for some rect. set E. One also needs that $T = \operatorname{Tan}(E, x)$ for ~~\|V\|~~ V almost all (x, T).

Jump to 15.

16. Lemma (Definition) [All 72, 3.3]

$V \in \mathbb{V}_k(M)$, $x \in M$, $\beta \in \mathcal{K}(G(n,k))$

$$V^{(x)}(\beta) = \lim_{r \downarrow 0} \fint_{\mathbb{B}(x,r) \times G(n,k)} \beta(S) \, d(i_* V)(y, S)$$

Measure disintegration see also [AFP00, 2.5] Ambrosio, Fusco, Pallara

See also Fed 69, 2.5.20

Then $V^{(\cdot)}$ is $\|V\|$-measurable function with values in Radon measures over $G(n,k)$ and for $\|V\|$ almost all $x \in M$

$$V^{(x)}\left(G(n,k) \sim \{S : S \subseteq \operatorname{Tan}(M, x)\}\right) = 0$$

$$V^{(x)}(G(n,k)) = 1$$

$$\int \alpha(x, S) \, d(i_* V)(x, S) = \int V^{(x)}(\alpha_x) \, d\|V\|(x) \quad \forall \alpha \in \mathcal{K}(G_k(U))$$

where $\alpha_x(S) = \alpha(x, S)$.

Jump to 17.

Proof • For each $\beta \in \mathcal{K}(G(n,k))$ consider the Radon measure over M given by

$$\mu_\beta(\gamma) = \int \gamma(x)\,\beta(S)\, d(i_\# V)(x,S)$$

• Use the theory of symmetric derivation [Fed 69, 2.9. –] to obtain a $\|V\|$ measurable function $V_\beta : M \longrightarrow \mathbb{R}$ given by $V_\beta(x) = V^{(x)}(\beta)$ and such that

$$\int \gamma(x)\, V_\beta(x)\, d\|V\|(x) = \int \gamma(x)\, \beta(S)\, d(i_\# V)(x,S)$$
$$\forall \gamma \in \mathcal{K}(M)$$

• Choose a countable dense set D in $\mathcal{K}(G(n,k))$

• Approximate any $\alpha \in \mathcal{K}(G_k(U))$ by functions of the form $\sum_i \beta_i(S)\, \gamma_i(x)$ for some $\beta_i \in D$, $\gamma_i \in \mathcal{K}(M)$

$\square$

15. Def. [All 72, 3.2] $F : M \longrightarrow M'$ smooth, $V \in \mathbb{V}_k(M)$

<u>Then</u> $F_\# V$ is defined by

$$F_\# V(\alpha) = \int \alpha\big(F(x), DF(x)[S]\big)\, \|\Lambda_k DF(x) \circ S_\natural\|\, dV(x,S)$$
$$\forall \alpha \in \mathcal{K}(G_k(M'))$$

Remark. If F is proper, then $F_\# V \in \mathbb{V}_k(M')$

Remark If F is injective on E and $E \subseteq M$ is as in def. of $\mathbb{V}_k(E)$, then $F_\# \mathbb{V}_k(E) = \mathbb{V}_k(F[E])$, by the area formula.

Jump to 16.

17. Def [All 72, 3.4]

$V \in \mathbb{V}_k(M)$, $j : \operatorname{Tan}(M,a) \hookrightarrow \mathbb{R}^n$

$$\operatorname{VarTan}(V,a) = \Big\{ C \in \mathbb{V}_k(\operatorname{Tan}(M,a)) : j_\# C = \lim_{l\to\infty} (\mu_{r_l} \circ \tau_{-a} \circ i)_\# V \text{ for some } r_l \uparrow \infty \Big\}$$

$\mu_r(x) = rx$
$\tau_a(x) = x + a$
$i : M \hookrightarrow \mathbb{R}^n$

Lemma [All72, 3.4(1)] If $\Theta^{*k}(\|V\|, a) < \infty$, then $\operatorname{VarTan}(V,a)$ is compact and non-empty.

Exercise $V_k(\mathbb{R}^n)$ is a complete separable metric space

Proof of lemma

Let $f : (0,\infty) \to V_k(\mathbb{R}^n)$, $f(r) = (\mu_r \circ \tau_{-a} \circ i)_\# V$

Observe that $j_\# \operatorname{VarTan}(V,a) = \{ C \in \mathbb{V}_k(\mathbb{R}^n) : \exists r_l \uparrow \infty \ f(r_l) \to C \}$

$$= \bigcap_{r>0} \operatorname{Clos} f[(r,\infty)] \quad \leftarrow \text{closed}$$

Moreover,

$$\|j_\# C\| \, \mathcal{U}(0,R) \le \lim_{l\to\infty} \|(\mu_{r_l} \circ \tau_{-a} \circ i)_\# V\| \, \mathcal{U}(0,R)$$

(*)

$$= \lim_{l\to\infty} r_l^{\,k} \, i_\# \|V\|(a, R/r_l) \le \alpha(k) \, \Theta^{*k}(i_\# \|V\|, a) R^k < \infty$$

Thus, $j_\# \operatorname{VarTan}(V,a)$ is a closed subset of a compact set.

[see the Remark in 14.] □

18. Lemma [All72, 3.4(2)] Assume $V \in \mathbb{V}_k(M)$, $a \in M$, $\Theta^k(\|i_\# V\|, a) \in \mathbb{R}$ exists, $C \in \operatorname{VarTan}(V,a)$

Then

(i) $\|C\|(\operatorname{Tan}(M,a) \cap \mathbb{B}(0,r)) = \Theta^k(\|i_\# V\|, a)\, \alpha(k)\, r^k$

(ii) $$\int_{\mathbf{B}(0,1)\times G(n,k)} \beta(S)\, d(j_\# C)(x,S) = \Theta^k(\|i_\# V\|, a)\, \alpha(k)\, V^{(a)}(\beta) \quad \forall \beta \in \mathcal{K}(G(n,k)).$$

Proof For (i) repeat ⊛ which gives equality for $\mathcal{L}^1$ almost all R.
For (ii) take $\gamma \in \mathcal{K}(\mathbb{R}^n)$ which approximates $\mathbb{1}_{\mathbb{B}(0,1)}$.
Then

$$\int \gamma(x)\beta(S)\, d(j_{\#}C)(x,S) = \lim_{l\to\infty} r_l^k \int \gamma(r_l(x-a))\,\beta(S)\, dV(x,S)$$

$$= \lim_{l\to\infty} r_l^k \int \gamma(r_l(x-a)) \int \beta(S)\, dV^{(x)}(S)\, d\|V\|(x)$$

$\boxed{\gamma \approx \mathbb{1}_{\mathbb{B}(0,1)}}$

$$\simeq \lim_{l\to\infty} \frac{1}{(1/r_l)^k} \int_{\mathbb{B}(a,1/r_l)} V^{(x)}(\beta)\, d\|V\|(x) = \Theta^k(\|i_{\#}V\|, a)\,\alpha(k)\, V^{(a)}(\beta)$$

□

19. The first variation

[All72, 4.1] Assume

$h : (-\varepsilon,\varepsilon)\times M \longrightarrow M$ smooth

$h_t(x) = h(t,x)$, $h_0 = id_M$, $\forall t\in(-\varepsilon,\varepsilon)$ h_t is a diffeomorphism

$C = \operatorname{Clos}\{x\in M : \exists t\in(-\varepsilon,\varepsilon)\ h_t(x)\neq x\}$ is compact

G open neighbourhood of C in M

$V\in \mathbb{V}_k(M)$, $\|V\|(G) < \infty$

$$\mathring{h}_t(x) = \lim_{u\downarrow 0} \frac{1}{u}\left(h_{t+u}(x) - h_t(x)\right) \Rightarrow \mathring{h}_t \in \mathcal{X}(M).$$

smooth compactly supported vectorfields on M see All72, 2.5

We want to compute

$$⊛ = \frac{d}{dt}\Big|_{t=0} \|h_{t\#}V\|(G) = \int \frac{d}{dt}\Big|_{t=0} \|\Lambda_k Dh_t(x)\circ S_\natural\|\, dV(x,S)$$

Write $h_t(x) = \underbrace{h_0(x)}_{=x} + \mathring{h}_0(x) + o(t)$

So

$$⊛ = \int \frac{d}{dt}\Big|_{t=0} \left|\Lambda_k\left(id_{\operatorname{Tan}(M,x)} + tD\mathring{h}_0(x) + o(t)\right)\circ S_\natural\right| dV(x,S)$$

$$\overset{[\text{Fed69, 1.4.5}]}{\underset{[\text{All72, 2.3}]}{=}} \int \left(D\mathring{h}_0(x)\circ S_\natural\right)\bullet S_\natural\, dV(x,S)$$

Definition [All72, 4.2]

$\delta V : \mathcal{X}(M) \longrightarrow \mathbb{R}$ first variation of V

$$\delta V(g) = \int \underbrace{\left(Dg(x)\circ S_\natural\right)\bullet S_\natural}_{\operatorname{div}_S g}\, dV(x,S) \quad \text{for } g\in\mathcal{X}(M).$$

$\operatorname{div}_S g = \sum_{i=1}^k (Dg(x)e_i)\bullet e_i$ if $e_1,\dots,e_k$ is an o.n.b. of S

Def. $\|\delta V\|$ is the Borel regular measure over M characterised by

$$\|\delta V\|(G) = \sup\{\delta V(g) : g \in \mathscr{X}(M),\ \operatorname{spt} g \subseteq G,\ |g| \le 1\}$$

whenever G is open in M

$$\|\delta V\|(A) = \inf\{\|\delta V\|(G) : A \subseteq G,\ G \subseteq M \text{ open}\}$$

for arbitrary $A \subseteq M$.

$\|\delta V\|$ is called the *total variation* of V

Def. We say that

- V is *stationary* if $\delta V = 0$,
- V is *stationary in G* if $\|\delta V\|(G) = 0$.

20. Theorem / Definition

For $a \in M$ there exists a bilinear symmetric form

$$\mathbf{b}(M,a) : \operatorname{Tan}(M,a) \times \operatorname{Tan}(M,a) \longrightarrow \operatorname{Nor}(M,a)$$

such that

$$\mathbf{b}(M,a)(v,w) \bullet g(a) = - v \bullet (Dg(a)w)$$

whenever $g \in \mathscr{X}^{\perp}(M)$, $v, w \in \operatorname{Tan}(M,a)$

$\mathbf{b}(M,a)$ is called the *second fundamental form of M at a*.

[$\mathscr{X}^{\perp}(M) = \{g \in \mathscr{C}_c^{\infty}(M, \mathbb{R}^n) : g(x) \in \operatorname{Nor}(M,x) \text{ for } x \in M\}$

Def. $\mathbf{h}(M,a,S)$ – the mean curvature vector of M at $(a,S) \in G_k(M)$ is characterised by

$$(Dg(a) \circ \operatorname{Tan}(M,a)_{\natural}) \bullet S_{\natural} = - g(a) \bullet \mathbf{h}(M,a,S)$$

whenever $g \in \mathscr{X}^{\perp}(M)$

Def. $\mathbf{h}(M,a) = \mathbf{h}(M,a,\operatorname{Tan}(M,a)) \in \operatorname{Nor}(M,a)$ $\leftarrow k = m$

Remark: If $e_1, \dots, e_k$ is an o.n.b. of $S \in G(n,k)$, then

$$\mathbf{h}(M,a,S) = \sum_{i=1}^{k} \mathbf{b}(M,a)(e_i,e_i).$$

21. Def [All72, 4.3] Assume $\|\delta V\|$ is a Radon measure, i.e., $\|\delta V\|(K) < \infty$ whenever $K \subseteq M$ is compact.
Using [Fed 69, 2.9 and 2.5.12] we obtain

- $\eta(V,x) \in \operatorname{Tan}(M,x)$ with $|\eta(V,x)| = 1$

such that for $g \in \mathscr{X}(M)$
$$\delta V(g) = \int \eta(V,x) \bullet g(x) \, d\|\delta V\|$$
and define

Radon-Nikodym derivative; see [2.9]

- $$\mathbf{h}(V,x) = -\mathbf{D}(\|\delta V\|, \|V\|, x)\, \eta(V,x)$$
- $$\|\delta V\|_{\text{sing}} = \|\delta V\| \llcorner \{x : \mathbf{D}(\|\delta V\|, \|V\|, x) = \infty\}.$$

Then

$$\boxed{\begin{gathered}\delta V(g) = -\int g(x) \bullet \mathbf{h}(V,x) \, d\|V\|(x) + \int g(x) \bullet \eta(V,x) \, d\|\delta V\|_{\text{sing}} \\ \text{whenever } g \in \mathbf{L}^1(\|\delta V\|, \mathbb{R}^n) \text{ satisfies } g(x) \in \operatorname{Tan}(M,x)\end{gathered}}$$

Remark. [All72, 4.4] $i : M \hookrightarrow U$, $V \in \mathbf{V}_k(M)$, $g \in \mathscr{X}(U)$
$$\delta(i_\# V)(g) = \delta V(\operatorname{Tan}(M,g)) - \int \operatorname{Nor}(M,g) \bullet \mathbf{h}(M,x,S) \, dV(x,S)$$

22. Example Assume
$\Sigma \subseteq \mathbb{R}^n$ is a smooth k-dim. submanifold with boundary
$\Theta : \Sigma \to (0,\infty) \in \mathscr{C}^1$
$V \in \mathbf{V}_k(\mathbb{R}^n)$, $V(\alpha) = \int_\Sigma \alpha(x, \operatorname{Tan}(\Sigma,x)) \, \Theta(x) \, d\mathscr{H}^k(x) \quad \forall \alpha \in \mathscr{K}(\mathbb{R}^n \times G(n,k))$
Let $g \in \mathscr{X}(\mathbb{R}^n)$. Then
$$g(x) = \operatorname{Tan}(\Sigma,g)(x) + \operatorname{Nor}(\Sigma,g)(x)$$

- $(\mathrm{D}\operatorname{Nor}(\Sigma,g))(x) \bullet \operatorname{Tan}(\Sigma,x)_\natural = -g(x) \bullet \mathbf{h}(\Sigma,x)$
- $(\mathrm{D}\operatorname{Tan}(\Sigma,g))(x) \bullet \operatorname{Tan}(\Sigma,x)_\natural \Theta(x) =$

$$= \mathrm{D}(\Theta \cdot \operatorname{Tan}(\Sigma,g))(x) \bullet \operatorname{Tan}(\Sigma,x)_\natural - (\mathrm{D}\Theta(x)\, g(x)) \bullet \operatorname{Tan}(\Sigma,x)_\natural$$
$$= \operatorname{div}_\Sigma(\Theta \cdot \operatorname{Tan}(\Sigma,g))(x) - \operatorname{Tan}(\Sigma,x)_\natural(\operatorname{grad}\Theta(x)) \bullet g(x)$$

$$\Rightarrow \quad \delta V(g) = -\int_\Sigma g(x) \bullet \Big(\mathbb{h}(\Sigma,x) + \operatorname{Tan}(\Sigma,x)_\natural \big(\operatorname{grad}(\log\circ\Theta)(x)\big) \Big)\, \Theta(x)\, d\mathcal{H}^k(x)$$
$$+ \int_{\partial\Sigma} g(x) \bullet \nu_\Sigma(x)\; \Theta(x)\, d\mathcal{H}^{k-1}(x)$$

Hence,
$$\|\delta V\|_{\text{sing}} = \Theta \cdot \mathcal{H}^{k-1} \llcorner \partial\Sigma$$
$$\eta(V,x) = \nu_\Sigma(x) \qquad \text{for } x \in \partial\Sigma$$
$$\mathbb{h}(V,x) = \mathbb{h}(\Sigma,x) + \operatorname{Tan}(\Sigma,x)_\natural\big(\operatorname{grad}(\log\circ\Theta)(x)\big)$$

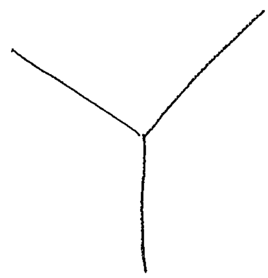
Example

$e_j = \exp(2\pi i\, j/3) \in \mathbb{C} \simeq \mathbb{R}^2$ for $j = 1,2,3$

$\Sigma = \{ t e_j : t \in [0,\infty),\ j = 1,2,3 \}$

$V = \mathbf{v}_1(\Sigma) \in \mathbb{V}_1(\mathbb{R}^2)$

$\Rightarrow \quad \delta V = 0$



Example If $\Sigma \subseteq \mathbb{R}^3$ is a minimal surface (i.e. $\mathbb{h}(\Sigma,x) = 0\ \forall x \in \Sigma$) then $\delta(\mathbf{v}_2(\Sigma)) = 0$.

23. Remark [All 72, 4.11, 4.12]

(a) if $V_i \xrightarrow[i\to\infty]{} V$ in $\mathbb{V}_k(M)$ and G is open in M then $\|\delta V\|(G) \le \liminf_{i\to\infty} \|\delta V_i\|(G)$ because sup of continuous functions is l.s.c.

(b) $V \in \mathbb{V}_k(\mathbb{R}^n)$, $r \in (0,\infty)$

Then
$$\|\mu_{r\,\#} V\| = r^k \mu_{r\,\#} \|V\|$$
$$\|\delta(\mu_{r\,\#} V)\| = r^{k-1} \mu_{r\,\#} \|\delta V\|$$

(c) If $\Theta^{k-1}(\|\delta V\|, a) = 0$ and $C \in \operatorname{VarTan}(V,a)$, then C is stationary, i.e., $\delta C = 0$

In particular, if $\Theta^k(\|V\|, a) \in (0,\infty)$ and $\mathbf{D}(\|\delta V\|, \|V\|, a) < \infty$, then $\Theta^k(\|\delta V\|, a) < \infty$

In particular, if $V \in \mathbf{RV}_k(\mathbb{R}^n)$, then (c) applies $\|V\|$ almost everywhere.

24. Lemma [All 72, 4.6 (1)] If $f \in \mathcal{D}(U, \mathbb{R})$, $V \in \mathbb{V}_k(U)$, $\operatorname{spt}\|V\| \subseteq f^{-1}\{0\}$, $\|\delta V\|$ is Radon,

then $S \subseteq \ker Df(x)$ for V almost all (x, S)

Proof Set $g(x) = f(x) \cdot \operatorname{grad} f(x)$ and note $g(x) = 0$ for $\|V\|$ a.a. x

Also $Dg(x) \bullet S_\natural = \left(Df(x) \cdot \operatorname{grad} f(x)\right) \bullet S_\natural = \left|S_\natural(\operatorname{grad} f(x))\right|^2$ for V almost all (x, S).

Thus,

$$0 = \int g(x) \bullet \eta(V, x)\, d\|\delta V\|(x) = \delta V(g) = \int \left|S_\natural(\operatorname{grad} f(x))\right|^2 dV(x, S)$$

$\operatorname{spt}\|\delta V\| \subseteq \operatorname{spt}\|V\| \subseteq f^{-1}\{0\}$

□

25. Corollary [4.6 (2)] $\{V \in \mathbb{V}_k(U) : \operatorname{spt}\|V\| \subseteq M,\ \|\delta V\| \text{ is Radon}\} \subseteq i_\# \mathbb{V}_k(M)$

which means that if $V \in$ LHS, then

$S \subseteq \operatorname{Tan}(M, x)$ for V almost all (x, S).

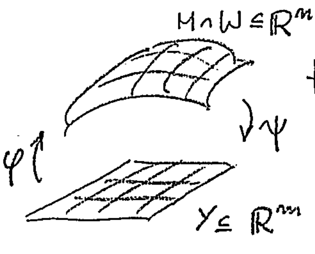
Proof. Represent $M \cap W = W \cap \bigcap_{i=1}^{n-m} f_i^{-1}\{0\}$

where $x \in W \subseteq M$, W is open, and $f_i \in \mathcal{D}(U, \mathbb{R})$.

Then use the previous lemma.

□

26. Theorem [4.6 (3)] If M is connected, $V \in \mathbb{V}_m(U)$, $\operatorname{spt}\|V\| \subseteq M$, $\|\delta V\|$ Radon, $\delta V(g) = 0$ for $g \in \mathscr{X}(U)$ with $\operatorname{Nor}(M, g) = 0$,



then $V = c\, \mathbb{V}_m(M)$, where $c = \dfrac{\|V\|(A)}{\mathcal{H}^m(A)}$ for any $A \subseteq M$ with $\mathcal{H}^m(A) > 0$.

"**Proof**".
- from 25. we know $V = i_\# \bar{V}$ for some $\bar{V} \in \mathbb{V}_m(M)$.
- using Remark in 21. we see $\delta \bar{V} = 0$.
- Set $T = \|\psi_\# \bar{V}\| \in \mathcal{D}'(Y, \mathbb{R})$
- Compute $D_j T = 0$ for $j = 1, 2, \dots, m$
- Apply constancy thm. [Fed 69, 4.1.4] to the distribution T to see that $\|\psi_\# \bar{V}\| = c\, \mathcal{L}^m$ for some $c \in \mathbb{R}$.

- Thus, $\|\tilde V\| \llcorner (M\cap W) = c\,\mathcal{H}^m \llcorner (M \cap W)$
- Since M is connected we get the conclusion. $\square$

27. Lemma [All 72, 4.10(1)]

If $V \in \mathbf{V}_k(U)$, $r \in \mathbb{R}$, $f : U \to \mathbb{R}$ continuous, $g \in \mathcal{X}(U)$, $\|\delta V\|$ is Radon, f is smooth in a neigh. of $\operatorname{spt}\|V\| \cap \operatorname{spt} g \cap f^{-1}\{r\}$

then

$$\delta V(\mathbb{1}_{\{f>r\}} \cdot g) = \delta\big(V \llcorner \{(x,S) : f(x) > r\}\big) + \lim_{h \downarrow 0} \frac{1}{h} \int_{\{(x,S) : r < f(x) \le r+h\}} S_\natural(g(x)) \bullet \operatorname{grad} f(x) \, dV(x,S)$$

Remark If $E_r = \{x \in U : f(x) > r\}$, then

$$((\delta V) \llcorner E_r)(g) - \delta(V \llcorner E_r \times G(n,k))(g) = V\,\partial E_r(g)$$

in the language of Menne, Indiana Univ. Math. J., 2016 §5

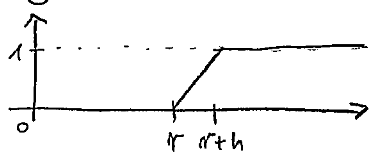
Proof of lemma

Let $\varphi \in \mathcal{D}(\mathbb{R},\mathbb{R})$ be s.t. f is smooth in a neigh. of $\operatorname{spt}\|V\| \cap f^{-1}[\operatorname{spt}\varphi'] \cap \operatorname{spt} g$.

Compute

$$D((\varphi\circ f)\cdot g)(x) \bullet S_\natural = (\varphi\circ f)(x)\, Dg(x) \bullet S_\natural + (\varphi'\circ f)(x) \operatorname{grad} f(x) \bullet S_\natural(g(x))$$

For $h > 0$ set $\gamma_h : \mathbb{R} \to \mathbb{R}$



Take $\varphi_j \in \mathcal{E}^\infty(\mathbb{R},\mathbb{R})$ s.t.

$$\|\varphi_j - \gamma_h\|_{L^\infty} \to 0$$

$$\varphi_j'(r) = 0, \quad \varphi_j'(r+h) = \frac{1}{h} \quad \text{for } j \in \mathbb{N}$$

$$\varphi_j'(s) \longrightarrow \gamma_h'(s) \quad \text{for } s \in \mathbb{R} \sim \{r, r+h\}$$

Use two definitions of δV and the dominated convergence theorem to obtain

$$\lim_{h\downarrow 0}\lim_{j\to\infty} \delta V((\varphi_j\circ f)g) = \lim_{h\downarrow 0}\lim_{j\to\infty}\int (\varphi_j\circ f)(x) g(x)\bullet \eta(V,x)\, d\|\delta V\|(x)$$

$$= \int \mathbb{1}_{\{f>r\}}(x)\, g(x)\bullet \eta(V,x)\, d\|\delta V\|(x) = \delta V(\mathbb{1}_{\{f>r\}}\, g) = \text{LHS}$$

$$\lim_{h\downarrow 0}\lim_{j\to\infty} \delta V((\varphi_j\circ f)g) = \lim_{h\downarrow 0}\lim_{j\to\infty}\int D((\varphi_j\circ f)g)(x)\bullet S_\natural\, dV(x,S)$$

$$= \int \mathbb{1}_{\{f>r\}}(x)\, Dg(x)\bullet S_\natural\, dV(x,S) + \lim_{h\downarrow 0}\frac{1}{h}\int_{\{r<f\le r+h\}\times G(n,k)} S_\natural(g(x))\bullet \operatorname{grad} f(x)\, dV(x,S)$$

$$= \text{RHS}$$

$\square$

28. Theorem [All72, 4.10(2)] V, f as before, $-\infty \le a < b \le \infty$, f is smooth in a neigh. of

$$\operatorname{spt}\|V\| \cap \{x : a < f(x) < b\}$$

Then for $\mathcal{L}^1$ almost all $r\in(a,b)$

(i) $\|\delta(V \llcorner \{(x,S) : f(x) > r\})\|$ is Radon

(ii)
$$\int_a^b \|\delta(V\llcorner\{(x,S) : f(x)>r\})\|(B)\, d\mathcal{L}^1(r)$$
$$\le \int_{B\cap\{a<f<b\}\times G(n,k)} |S_\natural(\operatorname{grad} f(x))|\, dV(x,S)$$
$$+ \int_a^b \|\delta V\|(B\cap\{f>r\})\, d\mathcal{L}^1(r)$$

whenever $B\subseteq U$ is Borel.

"Proof" Integrate 27. $\square$

Remark / Exercise: Let $A\subseteq\mathbb{R}^n$ be closed. Show that there exists a smooth ($\mathcal{C}^\infty$) non-negative function $f:\mathbb{R}^n\to\mathbb{R}$ such that $A=\{x : f(x)=0\}$.

29. Theorem If $r \in (0,\infty)$, $a \in \mathbb{R}^n$, $V \in \mathbf{V}_k(\mathbf{U}(a,r))$, $\varrho \in \mathcal{D}((0,r),\mathbb{R})$

[Menne 2016 Indiana U.M.J. 4.2]

[see also Sim 83 17.3]

then
$$-\int_0^r \varrho'(s)\, s^{-k}\, \|V\|\,\mathbf{B}(a,s)\, d\mathcal{L}^1(s)$$
$$= \int_{(\mathbf{U}(a,r)\sim\{a\})\times \mathbf{G}(n,k)} \varrho(|x-a|)\,|x-a|^{-k-2}\,|P_\natural^\perp(x-a)|^2\, dV(x,P)$$
$$- \delta V\left(x \longmapsto \left(\int_{|x-a|}^r s^{-k-1}\varrho(s)\, d\mathcal{L}^1(s)\right)(x-a)\right),$$

Proof. Assume $a=0$, $I=(-\infty,r)$, $J=(0,r)$.

If $\omega \in \mathcal{E}^\infty(I,\mathbb{R})$, $\sup \operatorname{spt}\omega < r$, $0 \notin \operatorname{spt}\omega'$,
$\theta : \mathbf{U}(0,r) \longrightarrow \mathbb{R}^n$, $\theta(x) = \omega(|x|)\,x$

then
$$D\theta(0) \bullet P_\natural = k\,\omega(0)$$
$$D\theta(x) \bullet P_\natural = |P_\natural x|^2 |x|^{-1}\omega'(|x|) + k\,\omega(|x|)$$
$$= -|P_\natural^\perp x|^2|x|^{-1}\omega'(|x|) + |x|\omega'(|x|) + k\,\omega(|x|)$$
for $x \in \mathbb{R}^n$ with $0<|x|<r$, $P \in \mathbf{G}(n,k)$

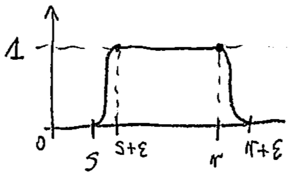
Define $\omega, \gamma \in \mathcal{E}^\infty(I,\mathbb{R})$
$$\omega(s) = \int_{\max\{s,0\}}^r \varrho(u)/u^{k+1}\, d\mathcal{L}^1(u), \quad \gamma(s) = s\,\omega'(s) + k\,\omega(s)$$

Then $\sup \operatorname{spt}\gamma \le \sup \operatorname{spt}\omega < r$, and $0 \notin \operatorname{spt}\omega'$, and
$$\omega'(s) = \frac{\varrho(s)}{s^{k+1}}, \quad \omega''(s) = \frac{-\varrho(s)(k+1)}{s^{k+2}} + \frac{\varrho'(s)}{s^{k+1}} \quad \text{for } s \in J$$
$$\gamma'(s) = s\,\omega''(s) + (k+1)\,\omega'(s) = s^{-k}\varrho'(s)$$

Hence,
$$\delta V(\theta) + \int_{\{(x,P):0<|x|<r\}} \varrho(|x|)|x|^{-k-2}\,|P_\natural^\perp x|^2\, dV(x,P) = \int \gamma(|x|)\, d\|V\|(x)$$
$$= -\int\int_{|x|}^r \gamma'(s)\, d\mathcal{L}^1(s)\, d\|V\|(x) = -\int_0^r \gamma'(s)\,\|V\|\,\mathbf{B}(a,s)\, d\mathcal{L}^1(s)$$
$$= -\int_0^r \varrho'(s)\, s^{-k}\,\|V\|\,\mathbf{B}(a,s)\, d\mathcal{L}^1(s)$$
□.

30. Corollary [Men 16, 4.5]

Assume $U \subseteq \mathbb{R}^n$ open, $V \in \mathbf{V}_k(U)$, $\|\delta V\|$ is Radon, $a \in U$, $s, r \in (0,\infty)$, $\mathbb{B}(a,r) \subseteq U$, $s \le r$.
Let $\varrho_\varepsilon \in \mathcal{D}((0, \operatorname{dist}(a, \mathbb{R}^n \sim U)), \mathbb{R})$ approximate the function $\mathbb{1}_{(s,r]} : \mathbb{R} \to \mathbb{R}$. Passing to the limit $\varepsilon \downarrow 0$ in 29. gives:



$$-\frac{\|V\|\,\mathbb{B}(a,s)}{s^k} + \frac{\|V\|\,\mathbb{B}(a,r)}{r^k} = \int_{(\mathbb{B}(a,r) \sim \mathbb{B}(a,s)) \times G(n,k)} \frac{|P_\natural^\perp(x-a)|^2}{|x-a|^{k+2}}\, dV(x,P)$$

$$- \int_{\mathbb{B}(a,r)} \eta(V,x) \bullet (x-a) \int_{\max\{|x-a|,s\}}^{r} \frac{d\mathcal{L}^1(u)}{u^{k+1}}\, d\|\delta V\|(x)$$

$$\left\{\begin{array}{l} x \in \mathbb{B}(a,r) \\ \max\{|x-a|, s\} \le u \le r \end{array}\right. \qquad \left\{\begin{array}{l} u \in (s,r) \\ |x-a| \le u \end{array}\right. \Rightarrow x \in \mathbb{B}(a,u)$$

Using Fubini in the last integral:

Monotonicity identity

$$\frac{\|V\|\,\mathbb{B}(a,r)}{r^k} - \frac{\|V\|\,\mathbb{B}(a,s)}{s^k} = \int_{\mathbb{B}(a,r) \sim \mathbb{B}(a,s)} \frac{|P_\natural^\perp(x-a)|^2}{|x-a|^{k+2}}\, dV(x,P)$$

$$- \int_s^r \frac{1}{u^{k+1}} \underbrace{\int_{\mathbb{B}(a,u)} (x-a) \bullet \eta(V,x)\, d\|\delta V\|(x)}_{= \delta V\left(x \mapsto \mathbb{1}_{\mathbb{B}(a,u)}(x) \cdot (x-a)\right)}\, d\mathcal{L}^1(u)$$

Remark If $\delta V = 0$, then

$$r \longmapsto \frac{\|V\|\,\mathbb{B}(a,r)}{r^k} \quad \text{is non-decreasing.}$$

31. Corollary [All72, 5.1(3)]

Suppose $\|\delta V\|\,\mathbb{B}(a,r) \le M\,\|V\|\,\mathbb{B}(a,r)$ for $0 < r < R$ where $M \in (0,\infty)$ is fixed.
Then $l_a(r) = \dfrac{\|V\|\,\mathbb{B}(a,r)}{r^k} \exp(M \cdot r)$ is non-decreasing on $(0,R)$.

In particular

$$\Theta^k(\|V\|, a) = \lim_{r \downarrow 0} l_a(r) \quad \text{exists.}$$

Proof. Let
$$f_a(r) = \frac{\|V\| \mathbb{B}(a,r)}{r^k}$$
$$g_a(r,s) = \int_s^r \frac{1}{u^{k+1}} \delta V\left(x \mapsto \mathbb{1}_{\mathbb{B}(a,u)}(x-a)\right) d\mathcal{L}^1(u)$$
$$h_a(r,s) = \int_{(\mathbb{B}(a,r) \sim \mathbb{B}(a,s)) \times G(n,k)} \frac{|P_\natural^\perp(x-a)|^2}{|x-a|^{k+2}} \, dV(x,P)$$

Monotonicity identity reads
$$f_a(s) + h_a(r,s) = f_a(r) + g_a(r,s) \qquad 0 < s \le r < R$$

Observe
$$\lim_{s \nearrow r} \frac{h_a(r,s)}{r-s} \geq 0 \qquad \text{for } \mathcal{L}^1 \text{ almost all } r \in (0,R)$$
$$\lim_{s \nearrow r} \frac{g_a(r,s)}{r-s} = \frac{1}{r^{k+1}} \delta V_x\left(\mathbb{1}_{\mathbb{B}(a,r)}(x-a)\right)$$

Thus,
$$f_a'(r) = \lim_{s \nearrow r} \frac{f_a(r) - f_a(s)}{r-s} = \lim_{s \nearrow r} \left(\frac{h_a(r,s)}{r-s} - \frac{g_a(r,s)}{r-s} \right) \geq - \lim_{s \nearrow r} \frac{g_a(r,s)}{r-s}$$
$$= -\frac{1}{r^k} \delta V_x\left(\mathbb{1}_{\mathbb{B}(a,r)} \frac{x-a}{r}\right) \geq - M f_a(r)$$
for $\mathcal{L}^1$ almost all $r \in (0,R)$

Therefore,
$$l_a'(r) = \exp(Mr)\left(f_a'(r) + M f_a(r)\right) \geq 0$$
$\Rightarrow$ l_a is non-decreasing.

$\square$

Remark. If $\|\delta V\|$ is a Radon measure, then for $\|V\|$ almost all $a \in U$
$$\mathbb{D}(\|\delta V\|, \|V\|, a) = \lim_{r \downarrow 0} \frac{\|\delta V\| \mathbb{B}(a,r)}{\|V\| \mathbb{B}(a,r)} \in (0,\infty); \quad \text{[see Fed 69 2.9.5]}$$
hence,
$$\exists R > 0 \quad \exists M > 0 \quad \forall r \in (0,r)$$
$$\|\delta V\| \mathbb{B}(a,r) \leq M \|V\| \mathbb{B}(a,r).$$

Thus,
- $\Theta^k(\|V\|, a) \in (0,\infty)$ for $\|V\|$ almost all $a \in U$
- $\Theta^k(\|V\|, \cdot)$ is a Borel function

Remark. The Preiss Rectifiability Theorem (Ann. of Math, 125, 1987) yields a countably $(\mathcal{H}^k, k)$ rectifiable set $E \subseteq U$ such that $\|V\|(U \sim E) = 0$.
However, this **does not** mean that $V \in \mathbf{RV}_k(U)$

32. Def. We say that $V \in \mathbf{V}_k(U)$ satisfies $H(p)$ if
(i) $p = 1$ and $\|\delta V\|$ is Radon
(ii) $p > 1$ and $\|\delta V\|$ is Radon, $\|\delta V\|_{sing} = 0$, $\mathbf{h}(V,\cdot) \in L^p(\|V\|, \mathbb{R}^n)$

33. Lemma [Sim 83, 17.8; All 72, 8.6] Assume $V \in \mathbf{V}_k(U)$ satisfies $H(p)$ for some $p > k$, $a \in U$, $R \in (0,\infty)$, $\mathbb{B}(a, 2R) \subseteq U$, $\Gamma = \left(\int_{\mathbb{B}(a,2R)} |\mathbf{h}(V,x)|^p \, d\|V\|(x) \right)^{1/p}$,

$$u_x(r) = \frac{\|V\| \mathbb{B}(x,r)}{r^k} + \frac{\Gamma}{p-k} r^{1-\frac{k}{p}} \quad \text{for } x \in \mathbb{B}(a,R), \ r \in (0,R).$$

Then u_x is non-decreasing for all $x \in \mathbb{B}(a,R)$.

Proof. Let $f_x(r)$, $g_x(r,s)$, $h_x(r,s)$ be as in the proof of 31.
Monotonicity identity gives

$$f_x(s) + h_x(r,s) = f_x(r) + g_x(r,s) \quad \text{for all } x \in \mathbb{B}(a,R) \text{ and } 0 < s \le r < R$$

As in 31. we have

$$f_x'(r) \geq -\lim_{s \uparrow r} \frac{g_x(r,s)}{r-s} = -\frac{1}{r^k} \int_{B(x,r)} \left(\frac{y-x}{r} \right) \bullet \mathbf{h}(V,y) \, d\|V\|(y)$$

$$\geq -\frac{1}{r^k} \left(\int_{\mathbb{B}(x,r)} |\mathbf{h}(V,y)|^p \, d\|V\|(y) \right)^{1/p} \left(\|V\| \mathbb{B}(x,r) \right)^{1-\frac{1}{p}}$$

$\mathbb{B}(x,r) \subseteq \mathbb{B}(a,2R)$

$$\geq -\Gamma \cdot f_x(r)^{1-\frac{1}{p}} \cdot \frac{1}{r^{k/p}}$$

Thus,

$$\left(f_x^{1/p} \right)'(r) = \frac{1}{p} f_x(r)^{\frac{1}{p}-1} f_x'(r) \Rightarrow \left(f_x^{1/p} \right)'(r) \geq -\frac{p}{r^{k/p}} \cdot \Gamma$$

$$\Rightarrow f_x^{1/p}(r) - f_x^{1/p}(s) \geq -\Gamma \cdot p \int_s^r \frac{d\mathcal{L}^1(t)}{t^{k/p}} = \frac{-\Gamma}{p-k} \left(r^{1-\frac{k}{p}} - s^{1-\frac{k}{p}} \right)$$

$$\Rightarrow u_x(r) \geq u_x(s)$$

$\square$

Corollary. For a fixed $r \in (0, R)$ the function
$$B(a,R) \ni x \longmapsto u_x(r) \text{ is u.s.c.}$$
← exercise

Hence,
$$B(a,R) \ni x \longmapsto \Theta^k(\|V\|, x) = \inf\{ u_x(r) : r \in (0,R) \} \text{ is u.s.c.}$$

Corollary. If, additionally, there exists $d \in (0,\infty)$ such that
$$\Theta^k(\|V\|, x) \geq d \quad \text{for } \|V\| \text{ almost all } x \in U$$
then
$$\Theta^k(\|V\|, x) \geq d \quad \text{for } \underline{\text{all}}\ x \in \operatorname{spt}\|V\|.$$

RECTIFIABILITY

34. Theorem [All 72, 5.5(1)] Assume
$V \in \mathbb{V}_k(U)$, $\|\delta V\|$ is Radon, $\Theta^{*k}(\|V\|, x) > 0$ for $\|V\|$ almost all x

Then $V \in R\mathbb{V}_k(U)$.

In particular, $T = \operatorname{Tan}^k(\|V\|, x)$ for V almost all (x,T).

35. Theorem [All 72, 5.6] [All 72, 6.4] Assume
$M_1, M_2, \ldots \in (0,\infty)$, $G_1, G_2, \ldots \subseteq U$, $\bigcup_{i=1}^{\infty} G_i = U$, $\Theta : U \to (0,\infty)$ is continuous

Then

(a) $\left\{ V \in R\mathbb{V}_k(U) : \begin{array}{l} (\|V\| + \|\delta V\|)(G_i) \leq M_i \text{ for } i = 1,2,\ldots \\ \Theta^k(\|V\|, x) \geq \Theta(x) \text{ for } \|V\| \text{ almost all } x \end{array} \right\}$ is compact.

(b) $\left\{ V \in I\mathbb{V}_k(U) : (\|V\| + \|\delta V\|)(G_i) \leq M_i \text{ for } i = 1,2,\ldots \right\}$ is compact.

36. Theorem [Menne, J. Geom. Anal. 2013] Assume $V \in I\mathbb{V}_k(U)$, $\|\delta V\|$ is Radon

Then $\exists\, \mathcal{G}$ – countable collection of $\mathcal{C}^2$-submanifolds of U

s.t. $\|V\|(U \sim \bigcup \mathcal{G}) = 0$

and $\forall M \in \mathcal{G}$ $\mathbf{h}(M,x) = \mathbf{h}(V,x)$ for $\|V\|$ almost all $x \in M$.

36. Examples

(a) $T \in G(n,k)$, $V \in \mathbb{V}_k(\mathbb{R}^n)$, $V(\alpha) = \int \alpha(x,T)\, d\mathcal{H}^n(x)$ for $\alpha \in \mathcal{K}(G_k(\mathbb{R}^n))$

Then
$$\delta V(g) = \left(\int Dg(x)\, d\mathcal{H}^n(x)\right) \bullet T_\natural = 0$$

so V is stationary ($\|\delta V\| = 0$ is Radon)

but $V \notin \mathbf{RV}_k(\mathbb{R}^n)$ unless $k = n$.

and $\Theta^k(\|V\|, x) = 0$ if $k < n$.

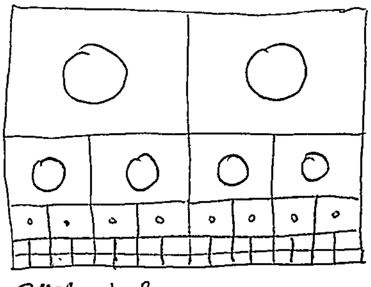
(b) $a \in \mathbb{R}^n$, $T \in G(n,k)$, $V \in \mathbb{V}_k(\mathbb{R}^n)$, $V(\alpha) = \alpha(a,T)$ for $\alpha \in \mathcal{K}(G_k(\mathbb{R}^n))$

Then $\delta V(g) = Dg(a) \bullet T$

and $\|\delta V\|(\{a\}) = \infty$ so $\|\delta V\|$ is not Radon.

(c) $m = n = k+1$, $\mathcal{F}$ = the Whitney decomposition of $\mathbb{R}^{n-1} \times (0,\infty)$

[Menne Adv. Calc. Var. 2009]



$\mathbb{R}^{n-1} \times \{0\}$

In each $Q \in \mathcal{F}$ of side length $2^{-\ell}$ place a sphere S_Q of radius $r_\ell < \frac{1}{2}2^{-\ell}$. Set $\Sigma = \bigcup\{S_Q : Q \in \mathcal{F}\}$ and $V = \mathbf{v}_k(\Sigma)$.

- $\|V\|([0,1]^n) = k\alpha(k) \sum_{\ell=1}^{\infty} (2^\ell r_\ell)^k$ $< \infty$?

If $r_\ell < \left(\frac{1}{4}\right)^\ell$, then $\|V\|$ is Radon and $V \in \mathbb{V}_k(\mathbb{R}^n)$.

- Let $p \in [1,\infty)$.

$$\int_{[0,1]^n} \mathbf{h}(V,\cdot)\, d\|V\| = k\alpha(k) \sum_{\ell=1}^{\infty} 2^{\ell k} \cdot r_\ell^k \cdot r_\ell^{-p} \qquad < \infty?$$

If $r_\ell \le \left(\frac{1}{4}\right)^{\frac{\ell k}{k-p}}$, then V satisfies $H(p)$.

Note. If $p \ge k$, then V cannot satisfy $H(p)$.

For $x \in \mathbb{R}^{n-1} \times \{0\} \subseteq \operatorname{spt}\|V\|$ we have $\Theta^k(\|V\|, x) = 0$

but there are points $x_\ell \in S_Q$, $\ell(Q) = 2^{-\ell}$

such that $x_\ell \longrightarrow x$ so $\Theta^k(\|V\|, \cdot)$ is not u.s.c.

$\Rightarrow \Theta^k(\|V\|, x_\ell) = 1$

37. Def. $f \in BV_{loc}(U) \iff f \in L^1(\mathcal{L}^n \llcorner U)$ and $\forall V \subset\subset U$ open
$$\sup\left\{ \int f \cdot \operatorname{div}\varphi \, d\mathcal{L}^n : \varphi \in \mathcal{X}(U), |\varphi| \le 1, \operatorname{spt}\varphi \subseteq V \right\}$$

$\iff$ distributional derivative of f is a Radon measure (vectorial) over U.

[Fed 69, 2.5.12] $\Rightarrow$ If $f \in BV_{loc}(U)$, then

$\exists\, \mu$ a Radon measure over U

$\exists\, \sigma : U \to \mathbb{R}^n$ μ-measurable with $|\sigma(x)| = 1$ for μ a.e. x

$$\int f \operatorname{div}\varphi \, d\mathcal{L}^n = \int \varphi \bullet \sigma \, d\mu \qquad \forall \varphi \in \mathcal{X}(U)$$

Def. $E \subseteq \mathbb{R}^n$ is of *locally finite perimeter in U* if $\mathbb{1}_E \in BV_{loc}(U)$

Def. $E \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable
$$\partial_* E = \left\{ x \in \mathbb{R}^n : \Theta^{*n}(\mathcal{L}^n \llcorner E, x) > 0 \;\&\; \Theta^{*n}(\mathcal{L}^n \llcorner \mathbb{R}^n \sim E, x) > 0 \right\}$$

Measure theoretic boundary

$$N(E,b) = \Big\{ u \in \mathbb{S}^{n-1} : \Theta^n\big(\mathcal{L}^n \llcorner \{x \in E : (x-b)\bullet u > 0\}, b\big) = 0$$
$$\& \; \Theta^n\big(\mathcal{L}^n \llcorner \{x \in \mathbb{R}^n \sim E : (x-b)\bullet u < 0\}, b\big) = 0 \Big\}$$

Measure theoretic normals at b

Theorem $E \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable

$\boxed{\text{Exercise: } \mathcal{H}^0(N(E,b)) \le 1}$

[EG 92, 5.11] 5.8

[Fed 69, 4.5.6] [4.5.11]

Then E is of loc. fin. perim in U if and only if
$$\mathcal{H}^{n-1}(\partial_* E \cap K) < \infty \quad \text{for any } K \subseteq U \text{ compact}$$

Moreover, if E is of loc. fin. perim. in U, then

- $\partial_* U$ is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable
- $N(E,b)$ has exactly one element for $\mathcal{H}^{n-1}$ almost all $b \in \partial_* E$ which we call $n_E(b)$
- $\displaystyle\int_E \operatorname{div}\varphi \, d\mathcal{L}^n = \int_{\partial_* E} \varphi \bullet n_E \, d\mathcal{H}^{n-1}$ (Gauss-Green thm.)

38. Example

(d) $E \subseteq M \subseteq U \subseteq \mathbb{R}^n$, E of loc. fin. perim in M, $\dim M = m$

Then $V = \mathbf{v}_m(E) \in \mathbf{V}_m(M)$

and $$\delta V(g) = \int_{\partial_* E} g \bullet n_E \, d\mathcal{H}^{m-1}$$

(e) If $f \in BV_{loc}(U)$ and $V = (\mathcal{L}^n \llcorner f) \times \mathrm{Dirac}(\mathbb{R}^n)$ and $f \geq 0$,

then $V \in \mathbf{V}_n(U)$ and $\|\delta V\|$ is Radon

because $$\delta V(g) = \int f \cdot \operatorname{div} g \, d\mathcal{L}^n$$

(f) If $V \in \mathbf{V}_n(U)$ and $\|\delta V\|$ is Radon,

then $\Theta^n(\|V\|, \cdot) \in BV_{loc}(U)$

because $\|V\| = \mathcal{L}^n \llcorner \Theta^n(\|V\|, \cdot)$

and $$\delta V(g) = \int \Theta^n(\|V\|, \cdot) \operatorname{div} g \, d\mathcal{L}^n .$$

[U. Menne, Proc. London. Math. Soc., 113, 2016, §3]

39. Theorem

[Menne, 3.1]

$m \leq n$, $M \subseteq \mathbb{R}^n$ m-dim submanifold

Y - normed vectorspace, $f : M \longrightarrow Y$ of class $\mathcal{C}^1$ rel. M

(1) If
$$\varrho(C, \delta) = \sup \{0\} \cup \left\{ |f(x) - f(a) - Df(a) \circ \mathrm{Tan}(M,a)_\natural (x-a)| \cdot \frac{1}{|x-a|} : x, a \in C,\ 0 < |x-a| \leq \delta \right\}$$

then $\varrho(C, \delta) \xrightarrow{\delta \downarrow 0} 0$

for any $C \subseteq M$ compact

(2) $\exists\, U \subseteq \mathbb{R}^n$ open $M \subseteq U$

$\exists\, g \in \mathcal{C}^1(U, Y)$ $g|_M = f$

$$Dg(a) = Df(a) \circ \mathrm{Tan}(M, a)_\natural \quad \text{for } a \in M$$

Proof (1) is immediate. (2) Apply the Whitney Extension Theorem [3.1.14] to polynomials $P_a(x) = f(a) + Df(a) \circ \mathrm{Tan}(M,a)_\natural (x-a)$ for $a \in M$, $x \in \mathbb{R}^n$

□

40. Corollary [Menne, 3.2] $\exists\, r : U \to M \quad r[U] = M, \quad r(x) = x$ for $x \in M$

$$Dr(a) = \operatorname{Tan}(M,a)_\natural \quad \text{for } a \in M$$

Proof Apply 39. to $f = \mathrm{id}_M : M \to \mathbb{R}^n$ to get $g : U \to \mathbb{R}^n$

Employ [Whitney 1957, p. 121] to get a retraction $h : U \to M \in \mathcal{C}^1$

set $r = h \circ g$. $\square$

41. Lemma [Menne, 3.3] $U \subseteq \mathbb{R}^n$ open, μ - Radon measure over U

$h : U \to \mathbb{R}, \in \mathcal{C}^1, \quad A = \{x : h(x) \geq 0\}, \quad \varepsilon > 0$

Then $\exists\, g : U \to \mathbb{R}_+ \in \mathcal{C}^1$

$$\mu\big(A \sim \{x : h(x) = g(x)\}\big) \leq \varepsilon$$

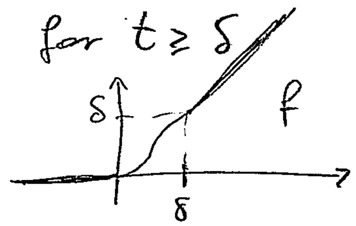
Proof In case A is compact:

find $\delta > 0$ s.t. $\mu(A \cap \{x : 0 < h(x) < \delta\}) \leq \varepsilon$

let $f : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^\infty$-smooth and s.t.

$f(t) = 0$ for $t \leq 0$, $f(t) = t$ for $t \geq \delta$

set $g = f \circ h$.



In general, use partition of unity. $\square$

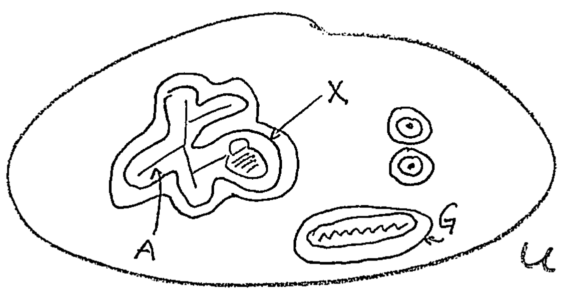
42. Lemma [Menne, 3.4] $l, n \in \mathbb{N}_+, \ U \subseteq \mathbb{R}^n$ open, $A \subseteq U$, $f : U \to \mathbb{R}^l \in \mathcal{C}^1, \varepsilon > 0$.

Then $\exists\, X \subseteq U$ open $\exists\, g : \mathbb{R}^n \to \mathbb{R}^l \in \mathcal{C}^1$

$$A \subseteq X, \quad f|X = g|X, \quad \operatorname{Lip} g \leq \varepsilon + \underbrace{\sup\{\operatorname{Lip}(f|A), \sup \|Df\|[A]\}}_{R}$$

Moreover, if $l = 1$ and $f \geq 0$, then $g \geq 0$.

Proof.



Find $G \subseteq U$ open s.t. $A \subseteq G$ and

$$\operatorname{Lip}(f|G) \leq R + \frac{\varepsilon}{2}$$

Use Kirszbraun Theorem [2.10.43] to obtain $g_0 : \mathbb{R}^n \to \mathbb{R}^l$

with $g_0|G = f|G$ and $\operatorname{Lip} g_0 = \operatorname{Lip}(f|G)$.

Use [Fed 69, 3.1.13] to construct a partition of unity $\{\varphi_i\}_{i=0}^{\infty}$ such that $\varphi_i \in \mathcal{D}(U,\mathbb{R})$, $\operatorname{spt}\varphi_i \subseteq U \sim A$ for $i=1,2,\dots$

$$\operatorname{spt}\varphi_0 \subseteq G, \quad \sum_{i=0}^{\infty} \varphi_i(x) = 1 \quad \text{for } x \in U$$

For each $i=1,2,\dots$ mollify g_0 to obtain $g_i \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^l)$ with

$$\operatorname{Lip} g_i \le \operatorname{Lip} g_0, \quad (\operatorname{Lip}\varphi_i)\|g_i - g_0\|_\infty \le \frac{1}{2}\frac{\varepsilon}{2^i}$$

If $l=1$ and $f \ge 0$, then $g_i \ge 0$.

Set $g = \sum_{j=0}^{\infty} \varphi_j g_j$ and $X = \operatorname{Int}(\varphi_0^{-1}\{1\})$.

Then $g \in \mathcal{C}^1$, $g|X = f|X = g_0|X$, and for $x,y \in U$

$$|g(x)-g(y)| = \Big|\sum_{j=0}^{\infty} \varphi_j(x)\big(g_j(x)-g_j(y)\big) + \big(\varphi_j(x)-\varphi_j(y)\big)\big(g_j(x)-g_0(y)\big)\Big|$$

$$\le (\operatorname{Lip} g_j)|x-y| + \frac{1}{2}\sum_{j=0}^{\infty}\frac{\varepsilon}{2^j} \le K + \varepsilon .$$

$\square$

43. Lemma [Menne, 3.5] $V \in \mathbf{RV}_m(U)$, $\varepsilon > 0$, $1 \le m \le n$, $U \subseteq \mathbb{R}^n$ open. Then

(1) $\exists\, M \subseteq U$ an m-dim. submanifold of U
$\|V\|(U \sim M) \le \varepsilon$

(2) If Y - finite dimensional normed vectorspace
$f : U \to Y$ is $\|V\|$ measurable
$A = \{x : (\|V\|, m)\operatorname{ap} Df(x) \text{ exists}\}$

then: $\exists\, g : U \to Y \in \mathcal{C}^1$
$\|V\|(A \sim \{x : f(x) = g(x)\}) \le \varepsilon$.

Proof. Exercise (Hint: Use Lusin approximation [3.1.16] and [3.2.29])

$\square$

44. Theorem [Menne, 3.6] $U \subseteq \mathbb{R}^n$ open, $V \in \mathbf{RV}_m(U)$, $C \subseteq U$ rel. closed

$f : U \to \mathbb{R}^l$ Lipschitz (locally)

$\operatorname{spt} f \subseteq \operatorname{Int} C$, $\varepsilon > 0$

Then $\exists\, g : U \to \mathbb{R}^l \in \mathscr{C}^1$, $\operatorname{spt} g \subseteq C$,

$$\operatorname{Lip} g \le \operatorname{Lip} f + \varepsilon$$

$$\|V\|(\{x \in U : f(x) \neq g(x)\}) \le \varepsilon$$

If $l = 1$, $f \ge 0$, then $g \ge 0$.

Proof. Let $X = \operatorname{Int} C$. Employ 43. to find $h : X \to \mathbb{R}^l \in \mathscr{C}^1$ and M an m-dim. submanifold of $\mathbb{R}^n$ s.t. $M \subseteq X$, $\|V\|(X \sim \{x \in M : f(x) = h(x)\}) \le \varepsilon$

$h \ge 0$ if $l = 0$ and $f \ge 0$.

By standard GMT:

$D(f|_M)(x) = D(h|_M)(x)$ for $\mathcal{H}^m$ almost all $x \in M$ satisfying $f(x) = h(x)$.

Set $B = \{x \in M : f(x) = h(x) \text{ and } D(f|_M)(x) = D(h|_M)(x)\}$

Then $\|V\|(X \sim B) < \varepsilon$

Use 40. to get a retraction

$r : G \to M \in \mathscr{C}^1$, $G \subseteq X$, G – open,

$Dr(a) = \operatorname{Tan}(M, a)_\natural$ for $a \in M$.

Then $\sup \|D(h \circ r)\|[B] \le \operatorname{Lip} f$

$\boxed{\operatorname{spt} f \subseteq \operatorname{Int} C}$

Apply 42. with

$(U \sim C) \cup G$, $(U \sim C) \cup B$, $((U \sim C) \times \{0\}) \cup (h \circ r)$

in place of

U, A, $f \in \mathscr{C}^1$

to get $g : U \to \mathbb{R} \in \mathscr{C}^1$ s.t.

$$g|_{U \sim X} = 0, \quad g|_B = f|_B, \quad \operatorname{Lip} g \le \operatorname{Lip} f + \varepsilon$$

and $g \ge 0$ if $l = 1$ and $f \ge 0$. $\square$

45. Corollary [Menne, 3.7] $1 \le m \le n$, $U \subseteq \mathbb{R}^n$ open, $V \in \mathbf{RV}_m(U)$, $K \subseteq U$ compact, $f : U \to \mathbb{R}^l$, $\operatorname{Lip} f < \infty$, $\operatorname{spt} f \subseteq \operatorname{Int} K$

Then $\exists\, f_i \in \mathcal{D}(U, \mathbb{R}^l)$

$f_i \rightrightarrows f$ on $\operatorname{spt} \|V\|$

$\| (\|V\|, m) \operatorname{ap} D (f_i - f) \| \longrightarrow 0$ in $\|V\|$ measure

$\operatorname{spt} f_i \subseteq K$,

$\limsup_{i \to \infty} (\operatorname{Lip} f_i) \le \operatorname{Lip} f$

If $l = 1$, $f \ge 0$, then $f_i \ge 0$.

Proof Exercise. (Hint: Apply 44. with $\varepsilon = \frac{1}{i}$
Then use Arzelà-Ascoli [2.10.21])

$\square$

h