Some notation

1. [id & cf] The *identity map* on X and the *characteristic function* of some $E \subseteq X$ shall be denoted by

$$\mathrm{id}_X \quad \text{and} \quad \mathbb{1}_E\,.$$

2. [Df & grad f] Let X, Y be Banach spaces and $U \subseteq X$ be open. For the space of k *times continuously differentiable functions* $f : U \to Y$ we write $\mathscr{C}^k(U, Y)$. The *differential* of f at $x \in U$ is denoted

$$\mathrm{D}f(x) \in \mathrm{Hom}(X, Y)\,.$$

In case $Y = \mathbf{R}$ and X is a Hilbert space, we also define the *gradient* of f at $x \in U$ by

$$\operatorname{grad} f(x) = \mathrm{D}f(x)^* 1 \in X\,, \quad \text{where } \mathbf{R} = \operatorname{span}\{1\}\,.$$

3. [Fed69, 2.10.9] Let $f : X \to Y$. For $y \in Y$ we define the *multiplicity*

$$N(f, y) = \operatorname{cardinality}\bigl(f^{-1}\{y\}\bigr)\,.$$

4. [Fed69, 4.2.8] Whenever X is a vectorspace and $r \in \mathbf{R}$ we define the *homothety*

$$\boldsymbol{\mu}_r(x) = rx \quad \text{for } x \in X\,.$$

5. [Fed69, 2.7.16] Whenever X is a vectorspace and $a \in X$ we define the *translation*

$$\boldsymbol{\tau}_a(x) = x + a \quad \text{for } x \in X\,.$$

6. [Men16, 2.10] Let X be a locally compact Hausdorff space. The space of all *continuous real valued functions on X with compact support* endowed with locally convex topology is denoted

$$\mathscr{K}(X)\,.$$

7. [Men16, 2.13] Let X, Y be Banach spaces, $\dim X < \infty$, and $U \subseteq X$ be open. The space of all *smooth (infinitely differentiable) functions* $f : U \to Y$ is denoted

$$\mathscr{E}(U, Y)\,.$$

The space of all *smooth functions $f : U \to Y$ with compact support* endowed with locally convex topology is denoted

$$\mathscr{D}(U, Y)\,.$$

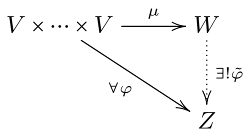
Multilinear algebra

Let V, Z be vectorspaces.

8. [Fed69, 1.4.1] The vectorspace of all *k-linear anti-symmetric maps* $\varphi : V \times \cdots \times V \to Z$ is denoted by

$$\textstyle\bigwedge^k(V, Z)\,.$$

9. [Fed69, 1.3.1] A vectorspace W together with $\mu \in \bigwedge^k(V, W)$ is the k^{th} *exterior power* of V if for any vectorspace Z and $\varphi \in \bigwedge^k(V, Z)$ there exists a unique linear map $\tilde{\varphi} \in \mathrm{Hom}(W, Z)$ such that $\varphi = \tilde{\varphi} \circ \mu$.



We shall write

$$W = \textstyle\bigwedge_k V \quad \text{and} \quad \mu(v_1, \ldots, v_k) = v_1 \wedge \cdots \wedge v_k\,.$$

We shall frequently identify $\varphi \in \bigwedge^k(V, Z)$ with $\tilde{\varphi} \in \mathrm{Hom}(\bigwedge_k V, Z)$.

10. [Fed69, 1.3.2] If $V = \operatorname{span}\{v_1, \ldots, v_m\}$, then

$$\textstyle\bigwedge_k V = \operatorname{span}\bigl\{v_{\lambda(1)} \wedge \cdots \wedge v_{\lambda(k)} : \lambda \in \Lambda(m, k)\bigr\} = \operatorname{span}\bigl\{v_\lambda : \lambda \in \Lambda(m, k)\bigr\}\,,$$

where

$$\Lambda(m, k) = \{\lambda : \{1, \ldots, k\} \to \{1, \ldots, m\} : \lambda \text{ is increasing}\}\,.$$

11. [Fed69, 1.3.1] If $f \in \mathrm{Hom}(V, Z)$, then $\bigwedge_k f \in \mathrm{Hom}(\bigwedge_k V, \bigwedge_k Z)$ is characterised by

$$\textstyle\bigwedge_k f(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k)$$

for $v_1, \ldots, v_k \in V$.

12. [Fed69, 1.3.4] If $f \in \mathrm{Hom}(V, V)$ and $\dim V = k < \infty$, then $\bigwedge_k V \simeq \mathbf{R}$. We define the *determinant* $\det f \in \mathbf{R}$ of f by requiring

$$\textstyle\bigwedge_k f(v_1 \wedge \cdots \wedge v_k) = (\det f) v_1 \wedge \cdots \wedge v_k\,,$$

whenever $v_1, \ldots, v_k$ is a basis of V.

13. [Fed69, 1.4.5] If $f \in \mathrm{Hom}(V, V)$ and $\dim V = k < \infty$ and $v_1, \ldots, v_k$ is basis of V and $\omega_1, \ldots, \omega_k$ is the dual basis of $\mathrm{Hom}(V, \mathbf{R})$, then we define the *trace* $\operatorname{tr} f \in \mathbf{R}$ of f by setting

$$\operatorname{tr} f = \sum_{i=1}^{k} \omega_i(f(v_i))\,.$$

14. [Fed69, 1.7.5] If V is equipped with a scalar product (denoted by $\bullet$) and $\{v_1, \ldots, v_m\}$ is an orthonormal basis of V, then $\bigwedge_k V$ is also equipped with a scalar product such that $\{v_\lambda : \lambda \in \Lambda(m, k)\}$ is orthonormal. In particular,

$$\operatorname{tr}(\textstyle\bigwedge_k f) = \displaystyle\sum_{\lambda \in \Lambda(m,k)} \textstyle\bigwedge_k f(v_\lambda) \bullet v_\lambda\,.$$

15. [Fed69, 1.7.4] If V, Z are equipped with scalar products and $f \in \mathrm{Hom}(V, Z)$, then the *adjoint map* $f^* \in \mathrm{Hom}(Z, V)$ is defined by the identity $f(v) \bullet z = v \bullet f^*(z)$ for $v \in V$ and $z \in Z$. We define the *(Hilbert-Schmidt) scalar product* and *norm* in $\mathrm{Hom}(V, Z)$ by setting for $f, g \in \mathrm{Hom}(V, Z)$

$$f \bullet g = \operatorname{tr}(f^* \circ g) \quad \text{and} \quad |f| = (f \bullet f)^{1/2}\,.$$

16. [Fed69, 1.7.6] If V, Z are equipped with norms, then the *operator norm* of $f \in \mathrm{Hom}(V, Z)$ is

$$\|f\| = \sup\{|f(v)| : v \in V\,,\ |v| \le 1\}\,.$$

17. [Fed69, 1.7.2] *Orthogonal injections*:

$$\mathbf{O}(n, m) = \bigl\{j \in \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n) : j^* \circ j = \mathrm{id}_{\mathbf{R}^m}\bigr\}\,.$$

18. [Fed69, 1.7.4] *Orthogonal projections*:

$$\mathbf{O}^*(n, m) = \{j^* : j \in \mathbf{O}(m, n)\}\,.$$

19. [Fed69, 1.4.5] If $f \in \mathrm{Hom}(V, V)$ and $\dim V = m$ and $t \in \mathbf{R}$, then

$$\det(\mathrm{id}_V + tf) = \sum_{k=0}^{m} t^m \operatorname{tr}(\textstyle\bigwedge_k f)\,.$$

20. [All72, 2.3] The *Grassmannian* of m dimensional vector subspaces of $\mathbf{R}^n$ is denoted by

$$\mathbf{G}(n, m)\,.$$

With $S \in \mathbf{G}(n, m)$ we associate the *orthogonal projection* $S_\natural \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ so that

$$S_\natural^* = S_\natural\,, \quad S_\natural \circ S_\natural = S_\natural\,, \quad \operatorname{im}(S_\natural) = S\,.$$

21. [Exercise] If $f \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ and $S \in \mathbf{G}(n,k)$, then

$$\frac{d}{dt}\Big|_{t=0} \|{\textstyle\bigwedge}_k((\mathrm{id}_{\mathbf{R}^m} + tf) \circ S_\natural)\|^2 = \frac{d}{dt}\Big|_{t=0} |{\textstyle\bigwedge}_k((\mathrm{id}_{\mathbf{R}^m} + tf) \circ S_\natural)|^2 = 2 f \bullet S_\natural .$$

22. [All72, 8.9(3)] If $S, T \in \mathbf{G}(n,m)$, then

$$\|S_\natural - T_\natural\| = \|S_\natural^\perp \circ T_\natural\| = \|T_\natural^\perp \circ S_\natural\| = \|S_\natural^\perp - T_\natural^\perp\| .$$

23. [All72, 2.3(4)] If $\omega \in \operatorname{Hom}(\mathbf{R}^n, R)$ and $v \in \mathbf{R}^n$, then $\omega \cdot v \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ is given by $(\omega \cdot v)(u) = \omega(u) v$ and for $S \in \mathbf{G}(n,k)$

$$(\omega \cdot v) \bullet S_\natural = \omega(S_\natural(v)) = \langle S_\natural v, \omega \rangle .$$

Measures and measurable sets

24. [Fed69, 2.1.2] We say that ϕ *measures* X, if $\phi : \mathbf{2}^X \to \{t \in \bar{\mathbf{R}} : 0 \le t \le \infty\}$ and

$$\phi(A) \le \sum_{B \in F} \phi(B)$$

whenever $F \subseteq \mathbf{2}^X$ is countable and $A \subseteq \bigcup F$.
$A \subseteq X$ is said to be ϕ *measurable* if

$$\forall T \subseteq X \quad \phi(T) = \phi(T \cap A) + \phi(T \sim A) .$$

25. [Fed69, 2.2.3] Let X be a topological space and ϕ measure X. We say that ϕ is *Borel regular* if all open sets in X are ϕ measurable and for each $A \subseteq X$ there exists a Borel set B such that

$$A \subseteq B \quad \text{and} \quad \phi(A) = \phi(B) .$$

26. [Fed69, 2.2.5] Let X be a locally compact Hausdorff topological space and ϕ measure X. We say that ϕ is a *Radon measure* if all open sets are ϕ measurable and

$$\begin{aligned} &\phi(K) < \infty \quad \text{for } K \subseteq X \text{ compact},\\ &\phi(V) = \sup\{\phi(K) : K \subseteq V \text{ compact}\} \quad \text{for } V \subseteq X \text{ open},\\ &\phi(A) = \inf\{\phi(V) : A \subseteq V, V \subseteq X \text{ is open}\} \quad \text{for arbitrary } A \subseteq X . \end{aligned}$$

27. [Men16, 2.11] The space of Daniell integrals on $\mathscr{K}(X)$ (cf. [Fed69, 2.5.6]) is denoted $\mathscr{K}(X)^*$ and coincides with the space of continuous linear functionals on $\mathscr{K}(X)$.

28. [Fed69, 2.5.5] If $\mu \in \mathscr{K}(X)^*$ and we set

$$\mu^+(f) = \sup\{\mu(k) : k \in \mathscr{K}(X),\ 0 \le k \le f\} \quad \text{and} \quad \mu^-(f) = -\inf\{\mu(k) : k \in \mathscr{K}(X),\ 0 \le k \le f\},$$

then μ^+ and μ^- are Radon measures. In particular,

$$\mathscr{M}(X) = \mathscr{K}(X)^* \cap \{\mu : \mu^- = 0\}$$

is the space of Radon measures over X.

29. [Fed69, 2.5.19] If $M : \mathscr{K}(X) \to [0,\infty)$, then

$$\mathscr{K}(X) \cap \{\mu : \mu^+ + \mu^- \le M\} \quad \text{is compact}.$$

30. [All72, 2.6(2)] Let X be locally compact Hausdorff space. If G is a family of opens sets of X such that $\bigcup G = X$ and $B : G \to [0,\infty)$, then the set

$$\{\phi \in \mathscr{M}(X) : \phi(U) \le B(U) \text{ for } U \in G\}$$

is (weakly) compact in $\mathscr{M}(X)$. If ϕ_i, ϕ are Radon measures and $\lim_{i\to\infty} \phi_i = \phi$, then

$$\begin{aligned} &\phi(U) \le \liminf_{i\to\infty} \phi(U) \quad \text{for } U \subseteq X \text{ open},\\ &\phi(K) \ge \limsup_{i\to\infty} \phi(K) \quad \text{for } K \subseteq X \text{ compact},\\ &\phi(A) = \lim_{i\to\infty} \phi_i(A) \quad \text{given } \operatorname{Clos} A \text{ is compact and } \phi(\operatorname{Bdry} A) = 0 . \end{aligned}$$

31. [Mat95, 14.15] For $r > 0$ let $L(r)$ be the set of all maps $f : \mathbf{R}^n \to [0,\infty)$ such that $\operatorname{spt}(f) \subseteq \mathbf{B}(0,r)$ and $\operatorname{Lip}(f) \le 1$. The space $\mathscr{M}(\mathbf{R}^n)$ of all Radon measures over $\mathbf{R}^n$ equipped with the weak topology is a complete separable metric space. The metric is given by

$$d(\phi,\psi) = \sum_{i=1}^\infty 2^{-1} \min\{1, F_i(\phi,\psi)\},$$

where $F_r(\phi,\psi) = \sup\{|\int f \,\mathrm{d}\phi - \int f \,\mathrm{d}\psi| : f \in L(r)\}$.

32. [Fed69, 2.10.2] Let $\mathbf{\Gamma}$ be the Euler function; see [Fed69, 3.2.13]. Assume X is a metric space. For $m \in [0,\infty)$, $\delta > 0$, and any $A \subseteq X$ we set

$$\boldsymbol{\alpha}(m) = \frac{\mathbf{\Gamma}(1/2)^m}{\mathbf{\Gamma}((m+2)/2)}, \quad \zeta^m(A) = \boldsymbol{\alpha}(m) 2^{-m} \operatorname{diam}(A)^m ,$$

$$\mathscr{H}^m_\delta(A) = \inf\left\{ \sum_{S \in G} \zeta^m(S) : \begin{array}{l} G \subseteq \mathbf{2}^X \text{ countable},\\ A \subseteq \bigcup G,\\ \forall S \in G \ \operatorname{diam}(S) \le \delta \end{array} \right\}.$$

The m dimensional *Hausdorff measure* $\mathscr{H}^m(A)$ of $A \subseteq X$ is

$$\mathscr{H}^m(A) = \sup_{\delta > 0} \mathscr{H}^m_\delta(A) = \lim_{\delta \downarrow 0} \mathscr{H}^m_\delta(A) .$$

33. [Fed69, 2.10.33] *Isodiametric inequality*: If $\varnothing \ne S \subseteq \mathbf{R}^m$, then

$$\mathscr{L}^m(S) = \mathscr{H}^m(S) \le \boldsymbol{\alpha}(m) 2^{-m} \operatorname{diam}(S)^m = \zeta^m(S) .$$

Approximate limits

34. [Fed69, 2.9.12] Let $A \subseteq \mathbf{R}^m$, $f : A \to \mathbf{R}^n$, ϕ be a Radon measure over $\mathbf{R}^m$, $x \in \mathbf{R}^m$.

$$\phi \operatorname{ap}\lim_{z\to x} f(z) = y \iff \forall \varepsilon > 0 \quad \lim_{r\downarrow 0} \frac{\phi(\{z \in \mathbf{B}(x,r) : |f(z) - y| > \varepsilon\})}{\phi(\mathbf{B}(x,r))} = 0,$$

$$\phi \operatorname{ap}\limsup_{z\to x} f(z) = \inf\left\{ t \in \mathbf{R} : \lim_{r\downarrow 0} \frac{\phi(\{z \in \mathbf{B}(x,r) : f(z) > t\})}{\phi(\mathbf{B}(x,r))} = 0 \right\},$$

$$\phi \operatorname{ap}\liminf_{z\to x} f(z) = \sup\left\{ t \in \mathbf{R} : \lim_{r\downarrow 0} \frac{\phi(\{z \in \mathbf{B}(x,r) : f(z) < t\})}{\phi(\mathbf{B}(x,r))} = 0 \right\}.$$

Densities

35. [Fed69, 2.10.19] Let ϕ be a Borel regular measure over a metric space X, $m \in \mathbf{R}$, $m \ge 0$, $a \in X$. We define

$$\mathbf{\Theta}^{*m}(\phi, a) = \limsup_{r\downarrow 0} \boldsymbol{\alpha}(m)^{-1} r^{-m} \phi(\mathbf{B}(a,r)),$$

$$\mathbf{\Theta}^m_*(\phi, a) = \liminf_{r\downarrow 0} \boldsymbol{\alpha}(m)^{-1} r^{-m} \phi(\mathbf{B}(a,r)).$$

If $\mathbf{\Theta}^m_*(\phi,a) = \mathbf{\Theta}^{*m}(\phi,a)$, then we write $\mathbf{\Theta}^m(\phi,a)$ for the common value.

36. [Fed69, 2.10.19(1)] If $A \subseteq X$, $t > 0$, and $\boldsymbol{\Theta}^{*m}(\phi, x) < t$ for all $x \in A$, then

$$\phi(A) \le 2^m t \mathscr{H}^m(A).$$

37. [Fed69, 2.10.19(3)] If $A \subseteq X$, $t > 0$, and $\boldsymbol{\Theta}^{*m}(\phi, x) > t$ for all $x \in A$, then for any open set $V \subseteq X$ such that $A \subseteq V$

$$\phi(V) \ge t \mathscr{H}^m(A).$$

38. [Fed69, 2.10.19(4)] If $A \subseteq X$, $\phi(A) < \infty$, and A is ϕ measurable, then

$$\boldsymbol{\Theta}^m(\phi \llcorner A, x) = 0 \quad \text{for } \mathscr{H}^m \text{ almost all } x \in X \sim A.$$

39. [Fed69, 2.10.19(2)(5)] If $A \subseteq X$, then

$$2^{-m} \le \boldsymbol{\Theta}^{*m}(\mathscr{H}^m \llcorner A, x) \le 1 \quad \text{for } \mathscr{H}^m \text{ almost all } x \in A.$$

Tangent and normal vectors

Let X be a normed vectorspace, ϕ a measure over X, $a \in X$, m a positive integer, $S \subseteq X$.

40. [Fed69, 3.1.21] *Tangent cone*:

$$\operatorname{Tan}(S,a) = \left\{ v \in X : \begin{array}{c} \forall \varepsilon > 0 \ \exists x \in S \ \exists r > 0 \\ |x - a| < \varepsilon \text{ and } |r(x-a) - v| < \varepsilon \end{array} \right\},$$

If the norm in X comes from a scalar product, define the *normal cone*

$$\operatorname{Nor}(S,a) = \{ v \in X : \forall \tau \in \operatorname{Tan}(S,a) \ \ v \bullet \tau \le 0 \}.$$

41. [Fed69, 3.2.16] *Approximate tangent cone*:

$$\operatorname{Tan}^m(\phi, a) = \bigcap \{ \operatorname{Tan}(S,a) : S \subseteq X,\ \boldsymbol{\Theta}^m(\phi \llcorner X \sim S, a) = 0 \}.$$

If the norm in X comes from a scalar product, define the *approximate normal cone*

$$\operatorname{Nor}^m(\phi, a) = \{ v \in X : \forall \tau \in \operatorname{Tan}^m(\phi,a) \ \ v \bullet \tau \le 0 \}.$$

For $a \in X$, $v \in X$, and $\varepsilon > 0$ define the cone

$$\mathbf{E}(a, v, \varepsilon) = \{ x \in X : \exists r > 0 \ \ |r(x-a) - v| < \varepsilon \}.$$

Observe

$$v \in \operatorname{Tan}^m(\phi, a) \iff \forall \varepsilon > 0 \ \ \boldsymbol{\Theta}^{*m}(\phi \llcorner \mathbf{E}(a,v,\varepsilon), a) > 0.$$

Approximate differentiation

Let X, Y be normed vectorspaces, ϕ be a measure over X, $A \subseteq X$, $f : A \to Y$, $a \in X$, m be a positive integer.

42. [Fed69, 3.2.16] We say that f is (ϕ, m) approximately differentiable at a if there exists an open neighbourhood U of a in X and a function $g : U \to Y$ such that

$$\mathrm{D}g(a) \text{ exists} \quad \text{and} \quad \boldsymbol{\Theta}^m(\phi \llcorner \{ x \in A : f(x) \ne g(x) \}, a) = 0.$$

We then define

$$(\phi, m) \operatorname{ap} \mathrm{D}f(a) = \mathrm{D}g(a)|_{\operatorname{Tan}^m(\phi,a)} \in \operatorname{Hom}(\operatorname{Tan}^m(\phi,a), Y).$$

Observe that $(\phi, m) \operatorname{ap} \mathrm{D}f(a)$ exists if and only if there exist $y \in Y$ and continuous $L \in \operatorname{Hom}(X,Y)$ such that for each $\varepsilon > 0$

$$\boldsymbol{\Theta}^m(\phi \llcorner X \sim \{ x : |f(x) - y - L(x-a)| \le \varepsilon |x - a| \}, a) = 0.$$

Jacobians

Assume $A \subseteq \mathbf{R}^m$ and $f : A \to \mathbf{R}^n$.

43. [Fed69, 3.2.1] If $a \in A$ and $\mathrm{D}f(a) \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n)$ exists, then the *k-dimensional Jacobian* $J_k f(a) \in \mathbf{R}$ of f at a is defined by

$$J_k f(a) = \| \textstyle\bigwedge_k \mathrm{D}f(a) \|.$$

In case $k = \min\{m, n\}$, we have

$$J_k f(a) = |\textstyle\bigwedge_k \mathrm{D}f(a)| = \operatorname{tr}\big(\textstyle\bigwedge_k (\mathrm{D}f(a)^* \circ \mathrm{D}f(a)) \big)^{1/2} = \operatorname{tr}\big(\textstyle\bigwedge_k (\mathrm{D}f(a) \circ \mathrm{D}f(a)^*) \big)^{1/2}.$$

In particular, if $k = m \le n$, then

$$J_k f(a) = \det(\mathrm{D}f(a)^* \circ \mathrm{D}f(a))^{1/2}$$

and if $k = n \le m$, then

$$J_k f(a) = \det(\mathrm{D}f(a) \circ \mathrm{D}f(a)^*)^{1/2}.$$

If ϕ measures $\mathbf{R}^m$, m is a positive integer, $a \in \mathbf{R}^m$, and $(\phi, m) \operatorname{ap} \mathrm{D}f(a) \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n)$ exists, then the (ϕ, m) *approximate k-dimensional Jacobian* $(\phi, m) \operatorname{ap} J_k f(a) \in \mathbf{R}$ of f at a is defined by

$$(\phi, m) \operatorname{ap} J_k f(a) = \| \textstyle\bigwedge_k (\phi, m) \operatorname{ap} \mathrm{D}f(a) \|.$$

Lebesgue integral

Assume ϕ measures X.

44. [Fed69, 2.4.1] We say that u is a *ϕ step function* if u is ϕ measurable, $\operatorname{im}(u)$ is a countable subset of $\mathbf{R}$, and

$$\sum_{y \in \operatorname{im}(u)} y\, \phi(u^{-1}\{y\}) \in \bar{\mathbf{R}}.$$

45. [Fed69, 2.4.2] Let $f : X \to \bar{\mathbf{R}}$. Set

$$\int^* f \, \mathrm{d}\phi = \inf_u \sum_{y \in \operatorname{im}(u)} y\, \phi(u^{-1}\{y\}),$$

where the infimum is taken with respect to all ϕ step functions u such that $u(x) \ge f(x)$ for ϕ almost all x. Similarly,

$$\int_* f \, \mathrm{d}\phi = \sup_u \sum_{y \in \operatorname{im}(u)} y\, \phi(u^{-1}\{y\}),$$

where the supremum is taken with respect to all ϕ step functions u such that $u(x) \le f(x)$ for ϕ almost all x.
We say that f is *ϕ integrable* if $\int_* f \, \mathrm{d}\phi = \int^* f \, \mathrm{d}\phi$ and then we write $\int f \, \mathrm{d}\phi$ for the common value. We say that f is *ϕ summable* if $|\int f \, \mathrm{d}\phi| < \infty$.

46. [Fed69, 2.9.1] If ϕ, ψ are Radon measures over $\mathbf{R}^n$ and $x \in \mathbf{R}^n$, we define

$$\mathbf{D}(\phi, \psi, x) = \lim_{r \downarrow 0} \phi(\mathbf{B}(x,r)) / \psi(\mathbf{B}(x,r)).$$

47. [Fed69, 2.9.5] $0 \le \mathbf{D}(\phi, \psi, x) < \infty$ for ψ almost all x.

48. [Fed69, 2.9.7] If $A \subseteq \mathbf{R}^n$ is ψ measurable, then

$$\int_A \mathbf{D}(\phi, \psi, x) \, \mathrm{d}\psi(x) \le \phi(A),$$

with equality if and only if ϕ is absolutely continuous with respect to ψ.

49. [Fed69, 2.9.19] If $\infty \le a < b \le \infty$ and $f : (a,b) \to \mathbf{R}$ is monotone, then f is differentiable at $\mathscr{L}^1$ almost all $t \in (a,b)$ and

$$\left| \int_a^b f' \, \mathrm{d}\mathscr{L}^1 \right| \le |f(b) - f(a)|.$$

50. [Fed69, 2.5.12] **Theorem.** Let X be a locally compact separable metric space, E a separable normed vectorspace, $T : \mathscr{K}(X, E) \to \mathbf{R}$ be linear and such that

$$\sup\{ T(\omega) : \omega \in \mathscr{K}(X,E),\ \operatorname{spt} \omega \subseteq K,\ |\omega| \le 1 \} < \infty$$

whenever $K \subseteq X$ is compact. Define

$$\phi(U) = \sup\{ T(\omega) : \omega \in \mathscr{K}(X,E),\ |\omega| \le 1,\ \operatorname{spt} \omega \subseteq U \}$$

whenever $U \subseteq X$ is open and

$$\phi(A) = \inf\{ \phi(U) : A \subseteq U,\ U \subseteq X \text{ is open} \}$$

for arbitrary $A \subseteq X$. Then ϕ is a Radon measure over X and there exists a ϕ measurable map $k : X \to E^*$ such that $\|k(x)\| = 1$ for ϕ almost all x and

$$T(\omega) = \int \langle \omega(x), k(x) \rangle \, \mathrm{d}\phi(x) \quad \text{for } \omega \in \mathscr{K}(X,E).$$

See also: [Sim83, 4.1]

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