1. Show that there exists $C = C(m) > 1$ such that for all $P, Q \in \mathbf{G}(n,m)$
$$C^{-1}\|P_\natural - Q_\natural\|^2 \le 1 - \|\textstyle\bigwedge_m P_\natural \circ Q_\natural\| \le C\|P_\natural - Q_\natural\|^2 \,.$$
2. Let $P \in \mathbf{G}(n,m)$, and $\Sigma \subseteq \mathbf{R}^n$ be a compact subset of a graph of some $\mathscr{C}^1$ function $P \to P^\perp$. Prove that there exists $C = C(n,m) > 1$, such that
$$C^{-1}\int_\Sigma \|\operatorname{Tan}(\Sigma,x)_\natural - P_\natural\|^2 \,\mathrm{d}\mathscr{H}^m(x) \le \mathscr{H}^m(\Sigma) - \mathscr{H}^m(P_\natural[\Sigma]) \le C\int_\Sigma \|\operatorname{Tan}(\Sigma,x)_\natural - P_\natural\|^2 \,\mathrm{d}\mathscr{H}^m(x) \,.$$
Hint: Apply the area formula to $P_\natural$.
Remark: This shows that the *measure-excess* is comparable to the L^2-*tilt-excess.*
3. Let B be a Borel subset of a smooth closed m-dimensional submanifold Σ of $\mathbf{R}^n$ and $\phi : \mathbf{R}^n \to \mathbf{R}^N$ be an injective map of class $\mathscr{C}^1$. Using the area formula show that $\phi_\#(\mathbf{v}_m(B)) = \mathbf{v}_m(\phi[B])$.
4. Let $\Sigma \subseteq \mathbf{R}^n$ be $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable and $f : \Sigma \to \mathbf{R} \in L^1(\mathscr{H}^m \llcorner \Sigma)$. For $a \in \mathbf{R}^n$ and $r \in (0,\infty)$ define
$$\Phi_a(r) = \int_{\Sigma\cap\mathbf{B}(a,r)} f(x)\,\mathrm{d}\mathscr{H}^m(x)\,.$$
Prove that for any $a \in \mathbf{R}^n$ the function Φ_a is absolutely continuous and compute the derivative Φ_a'.
5. Let $u : \mathbf{R}^m \to \mathbf{R}^n$ be Lipschitz, $f \in L^1(\mathscr{L}^m)$. For $a \in \mathbf{R}^n$ and $r \in (0,\infty)$ define
$$\Phi_a(r) = \int_{u^{-1}[\mathbf{B}(a,r)]} f(x)\,\mathrm{d}\mathscr{H}^m(x)\,.$$
Prove that for any $a \in \mathbf{R}^n$ the function Φ_a is absolutely continuous and compute the derivative Φ_a'.
6. Show that there exists a $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable set $\Sigma \subseteq \mathbf{R}^n$ such that $\operatorname{Clos}\Sigma \sim \Sigma$ is purely $(\mathscr{H}^m, m)$ unrectifiable.
Hint. Consider an Alexander Horned Sphere type construction or read [Fed69, 4.2.25].
7. Assume $m \ge 2$. Let $f : \mathbf{R}^m \to \mathbf{R}^{n-m}$ be continuous. Define $F : \mathbf{R}^m \to \mathbf{R}^n$ by $F(x) = (x, f(x))$ for $x \in \mathbf{R}^m$ and let $\Sigma = \operatorname{im} F$ be the graph of f. Suppose
$$\mathscr{H}^m(\Sigma \cap K) < \infty \quad \text{whenever } K \subseteq \mathbf{R}^n \text{ is compact}\,.$$
Prove that $f \in BV_{\mathrm{loc}}(\mathbf{R}^m)^{n-m}$.
Does it follow that Σ is countably $(\mathscr{H}^m, m)$ rectifiable?
Remark. The example in [Fed69, 4.2.25] show that in case Σ is not a graph but only the *image* of a continuous function and $\mathscr{H}^m(\Sigma) < \infty$, then one *cannot* conclude that Σ is $(\mathscr{H}^m, m)$ rectifiable.
8. Let $0 < k \le m \le n$, $U \subseteq \mathbf{R}^n$ be open, and $M \subseteq U$ be a properly embedded smooth manifold of dimension m. Prove that $\mathbf{V}_k(M)$ is metrizable. Construct a metric.
9. Let $\Sigma \subseteq \mathbf{R}^n$ be $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable. Prove that for each $\varepsilon \in (0,1)$ there exists an (open) m-dimensional submanifold M of $\mathbf{R}^n$ of class $\mathscr{C}^1$ such that
$$\mathscr{H}^m(\Sigma \sim M) + \mathscr{H}^m(M \sim \Sigma) \le \varepsilon\,.$$
10. Let μ be a Radon measure over $\mathbf{R}^n$ and $a \in \mathbf{R}^n$. Prove that
$$\mu(\operatorname{Bdry}\mathbf{B}(a,r)) > 0 \quad \text{for at most countably many } r \in (0,\infty)\,.$$
In general, if $f : \mathbf{R}^n \to \mathbf{R}$ is proper, then
$$\mu(f^{-1}\{r\}) > 0 \quad \text{for at most countably many } r \in \mathbf{R}\,.$$

11. Let $T \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ be of rank k, i.e., $\bigwedge_k T \neq 0$ and $\bigwedge_{k+1} T = 0$. Show that

$$|\textstyle\bigwedge_k T| = \|\textstyle\bigwedge_k T\| \,.$$

12. Let $T \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$. Show that

$$2\operatorname{tr}(\textstyle\bigwedge_2 T) = (\operatorname{tr} T)^2 - \operatorname{tr}(T \circ T)\,,$$
$$\operatorname{tr}(\textstyle\bigwedge_2 (T + T^*)) = 2(\operatorname{tr} T)^2 - \operatorname{tr}(T \circ T) - \operatorname{tr}(T^* \circ T)\,.$$

Hint: [Fed69, 1.7.12] provides a possible solution.

13. Let $0 < k < n$, and Σ be a smooth k-dimensional submanifold of $\mathbf{R}^n$ with smooth boundary, and $\theta : \Sigma \to (0, \infty)$ be of class $\mathscr{C}^1$. Define

$$V(\alpha) = \int \alpha(x, \operatorname{Tan}(\Sigma, x))\theta(x)\, \mathrm{d}\mathscr{H}^k(x) \quad \text{for } \alpha \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n,k))\,.$$

Show that for $g \in \mathscr{X}(\mathbf{R}^n)$ we have

$$\delta V(g) = -\int_\Sigma g \bullet \Big(\mathbf{h}(\Sigma, \cdot) + \operatorname{Tan}(\Sigma, \cdot)_\natural\big(\operatorname{grad}(\log \circ \theta)\big)\Big)\theta\, \mathrm{d}\mathscr{H}^k + \int_{\operatorname{Bdry}\Sigma} \big(g \bullet \nu_\Sigma\big)\theta\, \mathrm{d}\mathscr{H}^{k-1}\,,$$

where ν_Σ is the function associating the unit normal vector with points of $\operatorname{Bdry}\Sigma$.
In particular,

$$\|\delta V\|_{\mathrm{sing}} = \theta\mathscr{H}^k \llcorner \operatorname{Bdry}\Sigma\,, \qquad \boldsymbol{\eta}(V, x) = \nu_\Sigma(x) \quad \text{for } x \in \operatorname{Bdry}\Sigma\,,$$
$$\mathbf{h}(V, x) = \mathbf{h}(\Sigma, x) + \operatorname{Tan}(\Sigma, x)_\natural\big(\operatorname{grad}(\log \circ \theta)(x)\big) \quad \text{for } x \in \Sigma\,.$$

Hint: The Stokes Theorem [Fed69, 4.1.31 pp. 391–392] might be useful.

14. Let $V \in \mathbf{V}_k(\mathbf{R}^n)$ and $r > 0$. Recall that $\boldsymbol{\mu}_r(x) = rx$. Prove that

$$\|\boldsymbol{\mu}_{r\#} V\| = r^k \boldsymbol{\mu}_{r\#}\|V\| \quad \text{and} \quad \|\delta(\boldsymbol{\mu}_{r\#} V)\| = r^{k-1}\boldsymbol{\mu}_{r\#}\|\delta V\|\,.$$

15. Let $\Sigma \subseteq \mathbf{R}^4 \simeq \mathbf{C}^2$ be an (affine) algebraic set of (complex) dimension 1, i.e., Σ is the zero-locus in $\mathbf{C}^2$ of some finite family of polynomials of type $\mathbf{C}^2 \to \mathbf{C}$. Show that $\delta \mathbf{v}_2(\Sigma) = 0$.

16. Show that there is a natural bijection between the set of m-dimensional varifolds in $\mathbf{R}^m$ with locally bounded first variation, i.e.,

$$\big\{V \in \mathbf{V}_m(\mathbf{R}^m) : \|\delta V\|(K) < \infty \text{ whenever } K \subseteq \mathbf{R}^n \text{ is compact}\big\}$$

and the set of non-negative real valued functions of locally bounded variation on $\mathbf{R}^m$.

17. Suppose X is locally compact Hausdorff space, the topology of X has a countable basis, and for each $r \in \mathbf{R}$ there is given a Radon measure μ_r over X in such a way that

$$\mu_r \le \mu_s \quad \text{whenever } -\infty < r \le s < \infty\,.$$

Prove that for almost all $r \in \mathbf{R}$ there exists a Radon measure $\mu'(r)$ over X such that

$$\mu'(r)(f) = \lim_{h \downarrow 0} h^{-1}\big(\mu_{r+h}(f) - \mu_r(f)\big)\,.$$

Remark. Reading [All72, 2.6(3)] and using [Fed69, 2.9.19] might help. One may also find the Stone-Weierstrass Theorem (e.g. [Rud76, 7.32]) useful.

18. Let C be the standard Cantor set in $\mathbf{R}$, and $f : \mathbf{R} \to \mathbf{R}$ be the associated function (i.e., $f(x) = \mathscr{H}^d(C \cap \{t : t \le x\})$ for $t \in \mathbf{R}$, where $d = \log 2/\log 3$), and V be the varifold in $\mathbf{R}^2 \simeq \mathbf{R} \times \mathbf{R}$ associated to the graph of a primitive function of f. Show that V is an integral varifold, and $\|\delta V\|$ is a Radon measure, and $\mathbf{h}(V, z) = 0$ for $\|V\|$ almost all z, and $\operatorname{spt}\|\delta V\|_{\mathrm{sing}}$ corresponds to C via the orthogonal projection onto the domain of f.

Definition For $1 \le p \le \infty$ we say that a varifold V *satisfies* $H(p)$ if

- in case $p = 1$, $\|\delta V\|$ is a Radon measure;
- in case $p > 1$, $\|\delta V\|$ is a Radon measure, the mean curvature vector $\mathbf{h}(V, \cdot)$ belongs to $L^p_{\mathrm{loc}}(\|V\|)$, and $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ (i.e. $\|\delta V\|_{\mathrm{sing}} = 0$).

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19. Let $V \in \mathbf{V}_m(\mathbf{R}^n)$ satisfy $H(m)$. Fix $0 < r < \infty$. Show that $\boldsymbol{\mu}_{r\#}V$ also satisfies $H(m)$. Moreover, if $m > 1$, then
$$\int_{\boldsymbol{\mu}_r[B]} |\mathbf{h}(\boldsymbol{\mu}_{r\#}V, z)|^m \,\mathrm{d}\|\boldsymbol{\mu}_{r\#}V\|(z) = \int_B |\mathbf{h}(V, z)|^m \,\mathrm{d}\|V\|(z)\,,$$
and, in case $m = 1$,
$$\|\delta(\boldsymbol{\mu}_{r\#}V)\|(\boldsymbol{\mu}_r[B]) = \|\delta V\|(B)\,,$$
whenever B is a Borel subset of $\mathbf{R}^n$.
20. Let $1 \le p < m < n$ and Z be an open subset of $\mathbf{R}^n$. Show that there exists a countable collection C of m-dimensional spheres in $\mathbf{R}^n$ such that $V = \sum_{M \in C} \mathbf{v}_m(C)$ satisfies $H(p)$ and $\operatorname{spt}\|V\| = \operatorname{Clos} Z$.

   Remark: In particular, it might be that $Z = \mathbf{R}^n$ which could not happen if $p \ge m$.
21. Let A be a closed subset of $\mathbf{R}^m$. Show that there exists a non-negative smooth (i.e. of class $\mathscr{C}^\infty$) function $f : \mathbf{R}^m \sim A \to \mathbf{R}$ such that, for some $C > 1$
$$C^{-1}\operatorname{dist}(x, A) \le f(x) \le C \operatorname{dist}(x, A) \quad \text{whenever } x \in \mathbf{R}^m\,.$$
   Prove that, in general, one cannot extend f to a $\mathscr{C}^1$ function on the whole of $\mathbf{R}^m$.

   Hint: Consider $m = 1$ and $\mathbf{R} \sim A = \bigcup\{(2^{-i}, 2^{-i+1}) : i \in \mathbb{N}\}$.

   Hint: Read [Ste70, VI,§2.1].

   Is it possible to construct a $\mathscr{C}^1$ function $f : \mathbf{R}^m \to \mathbf{R}$ satisfying, for some $C > 1$,
$$C^{-1}\operatorname{dist}(x, A)^2 \le f(x) \le C \operatorname{dist}(x, A)^2 \quad \text{whenever } x \in \mathbf{R}^m\,?$$
   Can one require f to be of class $\mathscr{C}^2$ in this case?
22. Let A be a closed subset of $\mathbf{R}^m$. Show that there exists a non-negative smooth (i.e. $\mathscr{C}^\infty$) function $f : \mathbf{R}^m \to \mathbf{R}$ such that $A = \{x : f(x) = 0\}$.

Definition An m-dimensional varifold V in $\mathbf{R}^n$ is called *singular at* $z \in \operatorname{spt}\|V\|$ if and only if there is no neighbourhood of z in which V corresponds to a positive multiple of an m-dimensional continuously differentiable submanifold.

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23. Suppose A is a closed subset of $\mathbf{R}^m$ with empty interior and positive $\mathscr{H}^m$ measure. Let $f : \mathbf{R}^m \to \mathbf{R}$ be a non-negative smooth function such that $A = \{x : f(x) = 0\}$. Define $M_1 = \mathbf{R}^m \times \{0\}$, and $M_2 = \operatorname{graph} f$, and $V = \mathbf{v}_m(M_1) + \mathbf{v}_m(M_2)$. Show that V is an integral varifold satisfying $H(\infty)$ which is singular at each point of $M_1 \cap M_2 \simeq A \times \{0\}$.
   In particular, the singular set of V has positive $\mathscr{H}^m$ measure.
24. Let $1 \le k < n$, and $f : \mathbf{R}^k \to \mathbf{R}^{n-k}$ be of class $\mathscr{C}^2$, and $\Sigma \subseteq \mathbf{R}^n$ be the graph of f. Define $p : \mathbf{R}^n \to \mathbf{R}^k$ and $q : \mathbf{R}^n \to \mathbf{R}^{n-k}$ by
$$p(x_1, \dots, x_n) = (x_1, \dots, x_k) \quad \text{and} \quad q(x_1, \dots, x_n) = (x_{k+1}, \dots, x_n)\,.$$
   Assume $f(0) = 0$ and $\mathrm{D}f(0) = 0$. Show that for $u, v \in \mathbf{R}^k \times \{0\}^{n-k}$
$$\mathbf{b}(\Sigma, 0)(u, v) = q^* \,\mathrm{D}^2 f(0)(p(u), p(v)) \quad \text{and} \quad \mathbf{h}(\Sigma, 0) = q^* \Delta f(0)\,.$$
25. Let V be associated with the unit sphere $\operatorname{Bdry}\mathbf{B}(0,1) \subseteq \mathbf{R}^n$. Compute δV.

26. Let M be a smooth m-dimensional submanifold of $\mathbf{R}^n$ and define $\tau : M \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ by $\tau(x) = \operatorname{Tan}(M, x)$ for $x \in M$. Prove that whenever $x \in M$ and $u, v \in \operatorname{Tan}(M, x)$, then

$$\mathbf{b}(M,x)(u,v) = \langle v, \mathrm{D}\tau(x)u \rangle = \mathrm{D}[y \mapsto \tau(y)v](x)u \,.$$

Hint. If $g \in \mathscr{X}^{\perp}(M)$, then $\langle u, \tau(x) \rangle \bullet g(x) = 0$ for all $x \in M$ and $u \in \mathbf{R}^n$.

27. Let V be associated with the following surface

$$\mathbf{R}^3 \cap \left\{ (x,y,z) : \cosh^2 z = x^2 + y^2 \right\} .$$

Compute δV.

28. Let Y be a Banach space. Prove that the image of the unique map

$$\mathscr{D}(\mathbf{R},\mathbf{R}) \otimes \cdots \otimes \mathscr{D}(\mathbf{R},\mathbf{R}) \otimes Y \to \mathscr{D}(\mathbf{R}^n, Y)$$

sending $\gamma_1 \otimes \cdots \otimes \gamma_n \otimes y$ to $(x_1, \ldots, x_n) \mapsto \gamma_1(x_1) \cdots \gamma_n(x_n) y$ is sequentially dense in its target.

Hint. Reading [Fed69, 1.1.3, 4.1.2, 4.1.3] might help.

29. Let $V \in \mathbf{G}(n,m)$, and $u \in V \sim \{0\}$ and let $(v_1, \ldots, v_m)$ be a basis of V. Then there exist $\alpha_1, \ldots, \alpha_m \in \mathbf{R}$ such that $u = \sum_{i=1}^m \alpha_i v_i$. Prove that

$$\alpha_i = \left(v_1 \wedge \cdots \wedge v_{i-1} \wedge u \wedge v_{i+1} \wedge \cdots \wedge v_m \right) \bullet \frac{v_1 \wedge \cdots \wedge v_m}{|v_1 \wedge \cdots \wedge v_m|^2} \,.$$

Remark: This is sometimes called the *Cramer's rule*; cf. [Lan87, VI, §4].

30. Let $S \in \mathbf{G}(n,k)$. Prove the following claims

$$\begin{gathered} S_\natural x \bullet S_\natural y = S_\natural x \bullet y \quad \text{and} \quad |S_\natural x|^2 = S_\natural x \bullet x \qquad \text{for } x, y \in \mathbf{R}^n \,, \\ \mathrm{id}_{\mathbf{R}^n} \bullet S_\natural = k \,, \\ (\omega v) \bullet S_\natural = \langle S_\natural v, \omega \rangle \qquad \text{for } \omega \in \operatorname{Hom}(\mathbf{R}^n, R) \text{ and } v \in \mathbf{R}^n \,, \\ f \bullet S_\natural = f^* \bullet S_\natural \qquad \text{for } f \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \,. \end{gathered}$$

Remark. If $\omega \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R})$ and $v \in \mathbf{R}^n$, then

$$\omega v \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \quad \text{is defined by} \quad (\omega v) w = \omega(w) v \,.$$

Remark. The scalar product on $\operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^m)$ is defined by

$$f \bullet g = \operatorname{tr}(f^* \circ g) \quad \text{for } f, g \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^m) \,.$$

31. Let $f \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ and $S \in \mathbf{G}(n,k)$. Show that

$$\frac{d}{dt}\Big|_{t=0} \left\| \textstyle\bigwedge_k \left((\mathrm{id}_{\mathbf{R}^m} + tf) \circ S_\natural \right) \right\|^2 = \frac{d}{dt}\Big|_{t=0} \left| \textstyle\bigwedge_k \left((\mathrm{id}_{\mathbf{R}^m} + tf) \circ S_\natural \right) \right|^2 = 2 f \bullet S_\natural \,.$$

Hint: Reading [Fed69, 1.4.5 and 1.7.6] might help.

32. Let $S, T \in \mathbf{G}(n,k)$. Prove that there exists a linear isometry $M \in \mathbf{O}(n)$ such that

$$M^{-1} \circ S_\natural \circ M = T_\natural \quad \text{and} \quad M^{-1} \circ S_\natural^\perp \circ M = T_\natural^\perp \,.$$

Deduce that $\|S_\natural \circ T_\natural^\perp\| = \|T_\natural \circ S_\natural^\perp\|$ and then prove that

$$\|S_\natural - T_\natural\| = \|S_\natural^\perp - T_\natural^\perp\| = \|T_\natural \circ S_\natural^\perp\| = \|T_\natural^\perp \circ S_\natural\| = \|S_\natural \circ T_\natural^\perp\| = \|S_\natural^\perp \circ T_\natural\| \,.$$

Hint. Read [All72, 8.9(3)].

33. Construct a closed k-dimensional submanifold Σ of $\mathbf{R}^n$ of class $\mathscr{C}^1$ such that for any k-dimensional submanifold Π of $\mathbf{R}^n$ of class $\mathscr{C}^2$ there holds $\mathscr{H}^k(\Sigma \cap \Pi) = 0$.

    Remark: This shows that there exist $\mathscr{C}^1$ manifolds which are *not* $\mathscr{C}^2$ rectifiable.

34. Let ω and η be two moduli of continuity (i.e. non-decreasing, strictly positive functions of type $(0,1) \to (0,\infty]$ with limit zero at zero) such that $\lim_{t\downarrow 0} \omega(t)/\eta(t) = 0$. Construct a submanifold of $\mathbf{R}^n$ of class $\mathscr{C}^{1,\eta}$ which is *not* $\mathscr{C}^{1,\omega}$ rectifiable.

    Hint: Read [Kah59].

35. For every positive integer i let $V_i = \mathbf{v}_m(M_i)$, where

$$M_i = \mathbf{R}^{m+1} \cap \left\{ z : \left| z - \frac{a}{i} \right| = \frac{1}{3i^{1+1/m}} \text{ for some } a \in \mathbf{Z}^{m+1} \right\},$$

and let $V = \lim V_i$. Show that V is, up to constant depending on m, the product of the Lebesgue measure over $\mathbf{R}^{m+1}$ with the $\mathbf{O}(m+1)$-invariant Radon probability measure over $\mathbf{G}(m+1,m)$; cf. [Fed69, 2.7.16(6)].

36. Recall that $\boldsymbol{\alpha}(m) = \mathbf{\Gamma}(1/2)^m / \mathbf{\Gamma}(m/2+1)$ for $m \in (0,\infty)$, where $\mathbf{\Gamma}(s) = \int_0^\infty \exp(-x) x^{s-1} \, \mathrm{d}\mathscr{L}^1(x)$ for $s \in (0,\infty)$; cf. [Fed69, 2.7.16, 3.2.13].
Let k be a positive integer, and $r \in (0,\infty)$, and $s \in (0,r)$, and $a \in \mathbf{R}^n$ be such that $|a| = r$. For $t \in (s-r, s+r)$ we define $\rho(s,t) \in (0,\infty)$ so that

$$\mathbf{B}(a,s) \cap \operatorname{Bdry} \mathbf{B}(0,t) = \mathbf{B}(ta/r, \rho(s,t)) \cap \operatorname{Bdry} \mathbf{B}(0,t).$$

Compute

$$\frac{\boldsymbol{\alpha}(k-1)}{\boldsymbol{\alpha}(k)} \lim_{s\downarrow 0} \int_{r-s}^{r+s} \frac{\rho(s,t)^{k-1}}{s^k} \, \mathrm{d}\mathscr{L}^1(t).$$

Remark: Compare with [All72, proof of 5.2(2)(f)].

37. Let $T \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ be an auto-morphism and let $(e_1, \dots, e_n)$ be an orthonormal basis of $\mathbf{R}^n$. Prove that

$$\left(T^{-1}\right)^* e_n \cdot \det T = *(Te_1 \wedge \cdots \wedge Te_{n-1}).$$

Hint: Consider the basis of $\mathbf{R}^n$ made of the vectors Te_i for $i = 1, 2, \dots, n$.

38. Let M be a closed m-dimensional oriented smooth submanifold of $\mathbf{R}^{m+1}$ with orientation form $\omega : M \to \bigwedge_m \mathbf{R}^n \cap \{\xi : |\xi| = 1\}$ and let $\psi : \mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$ be a diffeomorphism. For $p \in M$ let $\nu_M(p) = *\omega(p) \in \bigwedge_1 \mathbf{R}^{m+1}$ be the unit normal vector to M at p and let $\nu_{\psi[M]}(\psi(p))$ be the unit normal vector to $\psi[M]$ at $\psi(p)$. Prove that

$$\nu_{\psi[M]}(\psi(p)) = \left\langle \nu_M(p), \left(\mathrm{D}\psi(p)^* \right)^{-1} \right\rangle \cdot \frac{\det \mathrm{D}\psi(p)}{\left|\left\langle \omega(p), \bigwedge_m \mathrm{D}\psi(p) \right\rangle\right|}.$$

Remark: Compare with [SS81, last sentence on p. 743].

39. (*An extra exercise for those who mastered the use of wedge product and the Hodge star*) Let $p_0, p_1, \dots, p_{m+1} \in \mathbf{R}^n$ be points such that $(p_1 - p_0) \wedge \cdots \wedge (p_{m+1} - p_0) \neq 0$ and let $r > 0$ be the radius of the unique m-dimensional sphere passing through all the points $p_0, \dots, p_{m+1}$. Prove that

$$r = \frac{\left(|\xi(p_1 - p_0) \wedge \cdots \wedge \xi(p_{m+1} - p_0)|^2 - |(p_1 - p_0) \wedge \cdots \wedge (p_{m+1} - p_0)|^2 \right)^{1/2}}{2\left|(p_1 - p_0) \wedge \cdots \wedge (p_{m+1} - p_0)\right|},$$

where $\xi : \mathbf{R}^n \to \mathbf{R}^{n+1}$ is given by $\xi(x) = (x, |x|^2)$.

Let X be a normed vectorspace, ϕ a measure over X, $a \in X$, m a positive integer, $S \subseteq X$.

[Fed69, 3.1.21] *Tangent cone*:

$$\operatorname{Tan}(S,a) = \{v \in X : \forall \varepsilon > 0 \ \exists x \in S \ \exists r > 0 \ |x-a| < \varepsilon \text{ and } |r(x-a) - v| < \varepsilon\},$$

[Fed69, 3.2.16] *Approximate tangent cone*:

$$\operatorname{Tan}^m(\phi,a) = \bigcap\{\operatorname{Tan}(S,a) : S \subseteq X,\ \mathbf{\Theta}^m(\phi \llcorner X \sim S, a) = 0\}.$$

[Fed69, 3.2.14] *Rectifiable sets*: Let $E \subseteq \mathbf{R}^n$, m be a positive integer, ϕ measures $\mathbf{R}^n$.

(a) E is *m rectifiable* if there exists $\varphi : \mathbf{R}^m \to \mathbf{R}^n$ with $\operatorname{Lip}(\varphi) < \infty$ and such that $E = \varphi[A]$ for some bounded set $A \subseteq \mathbf{R}^m$;

(b) E is *countably m rectifiable* if is a union of countably many m rectifiable sets;

(c) E is *countably (ϕ,m) rectifiable* if there exists a countably m rectifiable set $A \subseteq \mathbf{R}^n$ such that $\phi(E \sim A) = 0$;

(d) E is *(ϕ,m) rectifiable* if E is countably (ϕ,m) rectifiable and $\phi(E) < \infty$.

(e) E is *purely (ϕ,m) unrectifiable* if $\phi(E \cap \operatorname{im}\varphi) = 0$ for all $\varphi : \mathbf{R}^m \to \mathbf{R}^n$ with $\operatorname{Lip}(\varphi) < \infty$.

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40. Show that

$$\operatorname{Tan}(S,a) \cap \{v : |v| = 1\} = \bigcap\big\{\operatorname{Clos}\{(x-a)/|x-a| : a \neq x \in S \cap \mathbf{U}(a,\varepsilon)\} : \varepsilon > 0\big\}.$$

41. For $a \in X$, $v \in X$, and $\varepsilon > 0$ define the cone

$$\mathbf{E}(a,v,\varepsilon) = \{x \in X : \exists r > 0 \ \ |r(x-a) - v| < \varepsilon\}.$$

If the norm in X comes from a scalar product, $v \in X$, and $0 < \varepsilon < |v|$, then

$$b \in \mathbf{E}(a,v,\varepsilon) \quad \iff \quad b \neq a \quad \text{and} \quad \frac{b-a}{|b-a|} \bullet \frac{v}{|v|} > \left(1 - \frac{\varepsilon^2}{|v|^2}\right)^{1/2}.$$

Show that

$$v \in \operatorname{Tan}^m(\phi,a) \quad \iff \quad \forall \varepsilon > 0 \ \ \mathbf{\Theta}^{*m}(\phi \llcorner \mathbf{E}(a,v,\varepsilon), a) > 0.$$

42. For $a \in \mathbf{R}^n$, $r \in (0,\infty]$, $s \in (0,1)$, $V \in \mathbf{G}(n,n-m)$ define (cf. [Fed69, 3.3.1])

$$X(a,r,V,s) = \big\{x \in \mathbf{R}^n : |V_\natural^\perp(x-a)| \le s|x-a| \text{ and } |x-a| < r\big\}.$$

Let ϕ be a radon measure over $\mathbf{R}^n$, $a \in \mathbf{R}^n$ be such that $\mathbf{\Theta}^{*m}(\phi,a) > 0$, and $T \in \mathbf{G}(n,m)$. Prove that

$$\operatorname{Tan}^m(\phi,a) = T \quad \iff \quad \forall s \in (0,1) \quad \mathbf{\Theta}^m(\phi \llcorner \mathbf{R}^n \sim X(a,\infty,T,s), a) = 0.$$

43. Let $A \subseteq \mathbf{R}^n$ be such that $\mathscr{H}^m(A) < \infty$. Show that there exist an $(\mathscr{H}^m,m)$ rectifiable set $A_1 \subseteq A$ and a purely $(\mathscr{H}^m,m)$ unrectifiable set $A_2 \subseteq A$ such that $A = A_1 \cup A_2$ and that this decomposition is unique up to a set of $\mathscr{H}^m$ measure zero.

44. Let $A \subseteq \mathbf{B}(0,1)$, $s \in (0,1)$, $p \in \mathbf{O}^*(n,m)$, $h \in \mathbf{R}$, $x,y \in A$ be such that

$$y \in A \cap X(x,\infty,\ker p,s),$$
$$|y-x| \ge \tfrac{3}{4}\sup\big\{|z-x| : z \in A \cap X(x,\infty,\ker p,s/4)\big\} = h,$$
$$C = p^{-1}\big[p[\mathbf{B}(x,sh/4)]\big].$$

Show that

$$A \cap C \subseteq X(x,2h,\ker p,s) \cup X(y,2h,\ker p,s).$$

Hint. Read [Fed69, 3.3.6].

45. Let $A \subseteq \mathbf{R}^n$, $V \in \mathbf{G}(n, n-m)$, $s \in (0,1)$, $r \in (0,\infty)$ be such that

$$\forall a \in A \quad A \cap X(a, r, V, s) = \varnothing .$$

Show that A is countably m rectifiable.

Hint. Read [Fed69, 3.3.5].

46. Let $A \subseteq \mathbf{R}^n$ be such that

$$\forall a \in A \; \exists V \in \mathbf{G}(n, n-m) \; \exists s \in (0,1) \; \exists r \in (0,\infty) \quad A \cap X(a, r, V, s) = \varnothing .$$

Show that A is countably m rectifiable.

Hint. The spaces $\mathbf{R}$ and $\mathbf{G}(n, n-m)$ are separable.

47. Let $A \subseteq \mathbf{R}^n$ be purely $(\mathscr{H}^m, m)$ unrectifiable. Show that for $\mathscr{H}^m$ almost all $a \in A$

$$\forall V \in \mathbf{G}(n, n-m) \; \forall s \in (0,1) \; \forall r \in (0,\infty) \quad A \cap X(a, r, V, s) \neq \varnothing .$$

48. Let $V \in \mathbf{G}(n, n-m)$, $A \subseteq \mathbf{R}^n$ be purely $(\mathscr{H}^m, m)$ unrectifiable. For each $r \in (0,1)$ let $f_r : A \to \mathbf{R}$ and $g_r : A \to \mathbf{R}$ be given by

$$f_r(a) = r^{-m} \mathscr{H}^m(A \cap X(a, r, V, s)) , \quad g_r(a) = r^{-m} \mathscr{H}^m(A \cap \mathbf{B}(a, r)) .$$

Prove that

$$\lim_{r \downarrow 0} \sup \operatorname{im} f_r = 0 \quad \Rightarrow \quad \lim_{r \downarrow 0} \sup \operatorname{im} g_r = 0 .$$

Hint. Use 44 and 47.

49. Let $A \subseteq \mathbf{R}^n$ be such that for $\mathscr{H}^m$ almost all $a \in A$ there exist $V \in \mathbf{G}(n, n-m)$ and $s \in (0,1)$ such that

$$\mathbf{\Theta}^m(\mathscr{H}^m \llcorner A \cap X(a, \infty, V, s), a) = 0 .$$

Prove that A is countably $(\mathscr{H}^m, m)$ rectifiable.

Remark. Compare [Fed69, 3.3.17].

50. Let A be such that $\operatorname{Tan}^m(\mathscr{H}^m \llcorner A, a) \in \mathbf{G}(n, m)$ for $\mathscr{H}^m$ almost all $a \in A$. Prove that A is countably $(\mathscr{H}^m, m)$ rectifiable.

51. Let ϕ be a Radon measure over $\mathbf{R}^n$ such that $0 < \mathbf{\Theta}^{*m}(\phi, a) < \infty$ and $\operatorname{Tan}^m(\phi, a) \in \mathbf{G}(n, m)$ for ϕ almost all a. Prove that $\mathbf{R}^n$ is countably (ϕ, m) rectifiable.

Hint. From [Fed69, 2.10.19, 2.10.6] it follows that ϕ and $\mathscr{H}^m$ are mutually absolutely continuous so setting

$$A = \{x : \mathbf{\Theta}^{*m}(\phi, x) > 0\} \quad \text{we have} \quad \phi = \mathbf{D}(\phi, \mathscr{H}^m \llcorner A, \cdot) \mathscr{H}^m \llcorner A .$$

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