Imagine a two dimensional surface whose boundary is just a circle. It is intuitively obvious that among such surfaces the flat disc is the one with minimal area but a formal proof is not that simple. Moreover, we consider certain anisotropic spaces where the value of area depends on the slope of the surface. This means that the area of a flat disc varies as we rotate it about some axis. To make things even more interesting we place everything in an ambient space of arbitrary high dimension $n$ and we also consider “surfaces” of arbitrary dimension $k$. The boundary is always fixed and has to be a $(k-1)$-dimensional sphere. With all these generalisations the problem to describe minimisers of area becomes really hard (which fact makes mathematicians rather happy). For $k = 2$ and $n \geq 4$ the problem was solved only in 2012. For other choices of $k$ the problem is open.

Actually, there are even more nuances. Given a function (a norm) that computes an anisotropic length of a vector (i.e. rotations may change the length) one can construct many sensible and meaningful notions of $k$-dimensional area. My first goal is to describe those anisotropic notions of $k$-area for which the flat $k$-disc minimises $k$-area among $k$-surfaces having a fixed $(k-1)$-sphere as boundary. I shall call them elliptic. Understanding the deep nature of ellipticity is the main far-reaching objective of the project and may lead to answering other long-standing open questions some of which are outlined below.

Any function $F$ computing some type of (anisotropic) $k$-area of a surface gives rise (by a differentiation process) to the notion of mean $F$-curvature. Assume a $k$-surface $\Sigma$ lies inside a region $U$ and touches the boundary of $U$ at some point $p$. Intuitively, a notion of curvature describes how fast the surface bends so $\Sigma$ should have bigger curvature than the boundary of $U$ at $p$. This type of comparison is called the maximum principle and is known to hold for the classical notion of curvature. In the anisotropic case the maximum principle is proven in co-dimension one only, i.e., if $n - k = 1$. In higher co-dimension the problem is open – especially that the notion of mean $F$-curvature depends also on the dimension $k$ and it is not clear how to compare curvatures of objects of different dimensions.

If the mean $F$-curvature of a surface $\Sigma$ is globally bounded, then the surface should not bend too fast at any point. This possibly excludes long and very thin tentacles or at least should give a bound on their thickness. If this is the case, there is a radius $R > 0$ such that the intersection of $\Sigma$ with any ball centred at $\Sigma$ of radius $0 < r < R$ must have $k$-area proportional to $r^k$. This is true for the usual notion of $k$-area but is not known for other anisotropic $k$-areas. It is actually a very big question that holds off many other developments.

A bit more subtle open problem, is the very definition of the mean $F$-curvature in case $F$ is not differentiable. This is especially important since such $F$ are used to model, e.g., liquid crystals.

References