



Geometric ellipticity

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$$\Phi : \{\text{geometric objects}\} \rightarrow \mathbf{R}$$

Goal: To study critical points of Φ .

Examples:

- ① The Plateau problem
- ② The isoperimetric problem

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where $P_F(T) \in \text{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ is such that

$$P_F(T) \circ P_F(T) = P_F(T), \quad \text{im } P_F(T) = T,$$

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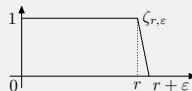
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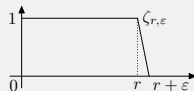
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Def. V is F -stationary iff. $\delta_F V \equiv 0$

Assume $\delta_F V = 0$, $\mathbf{B}(0, r + \varepsilon) \subseteq U$,
and $g(x) = \zeta_{r,\varepsilon}(|x|)x$.



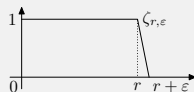
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Compute $\delta_F V(g)$ and pass to the limit $\varepsilon \downarrow 0$ to get

$$\frac{d}{dr} \left(\frac{\|V_F\| \mathbf{B}(0, r)}{r^k} \right) = \frac{d}{dr} \left(\int_{\mathbf{B}(0, r)} \frac{(\text{id}_{\mathbf{R}^n} - P_F(T))x \bullet x}{|x|^{k+2}} dV_F(x, T) \right).$$

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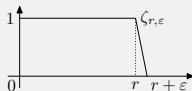
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A sufficient condition for LHS ≥ 0 would be

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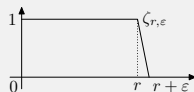
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Exercise. Let $Q \in \text{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ be a projection such that $Qx \bullet x \geq 0$ for all $x \in \mathbf{R}^n$. Show that $Q = Q^*$.

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More general result to be found in

W. Allard, *A characterization of the area integrand*, 1974.

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 $\varphi[M] \in \mathcal{A}$ whenever $M \in \mathcal{A}$ and $\varphi : U \rightarrow U$ is Lipschitz with
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If F is *elliptic* then

$$T = \text{Tan}(\Sigma, x) \text{ for } V \text{ a.a. } (x, T) \quad \implies \quad \Phi_F(V) = \Phi_F(\Sigma).$$

[E. R. Reifenberg F. Almgren, G. De Philippis, A. De Rosa, F. Ghiraldin, C. De Lellis, F. Maggi, Y. Fang, K., J. Harrison, H. Pugh, G. David ...]

Def. (S, D) is a *test pair* if D is a flat k -disc, $S \subseteq \mathbf{R}^n$ is (\mathcal{H}^k, k) -rectifiable and compact, $\partial D \subseteq S$ is not a Lipschitz retract of S .

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- $\text{UAE} \subseteq \mathbf{R}^{\text{G}(n,k)}$ is convex.
- If $k = n - 1$, then $F \in \text{UAE}$ iff. F comes from a uniformly convex norm.

- Q.** Are there any non-trivial elliptic integrands?
[e.g. a non-Euclidean norm ν on \mathbf{R}^n generates $\mathcal{H}_\nu^k = \Phi_F$,
with $F(T) = \alpha(k) / \mathcal{L}^k(\mathbf{B}^\nu(0,1) \cap T)$. Does $F \in \text{AE}$?]

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- Q. Is there a condition that can be easily checked for an integrand given by an explicit formula?

Desired property: $T = \text{Tan}(\text{spt} \|V\|, x)$ for V almost all (x, T) if $\delta_F V = 0$.

Def. $F \in \text{BC}$ iff. for all $V = (\mathcal{H}^k \llcorner T) \times \mu$

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Def. F satisfies the *atomic condition* ($F \in \text{AC}$) iff. given a probability measure μ over $\mathbf{G}(n, k)$ and setting

$$A_F(\mu) = \int P_F(T)^* d\mu(T) \in \text{Hom}(\mathbf{R}^n, \mathbf{R}^n),$$

there holds

$$\textcircled{1} \quad \dim \ker A_F(\mu) \leq n - k,$$

$$\textcircled{2} \quad \dim \ker A_F(\mu) = n - k \quad \implies \quad \mu = \text{Dirac}(T).$$

G. De Philippis, A. De Rosa, F. Ghiraldin, *Rectifiability of varifolds with locally bounded first variation with respect to anisotropic surface energies*, CPAM 2017

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Proof.

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hence, $S = T$ for μ almost all S , i.e., $\mu = \text{Dirac}(T)$.

Consequently, $A_F(\mu) = P_F(T)^*$ and $\dim \ker A_F(\mu) = n - k$.

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F satisfies the *scalar atomic condition* ($F \in \text{SAC}$) iff. 2 is satisfied with

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Two points S and T involved in checking condition 2.

Uniform version of SAC (a.k.a. **USAC**)

(corresponding to “uniform convexity” of \mathcal{G})

$$\exists \Gamma > 0 \quad \forall S, T \in \mathbf{G}(n, k) \quad P_F(S)^* \bullet Q_F(T) \geq \Gamma \|S_{\natural} - T_{\natural}\|^2$$

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See A. De Rosa and R. Tione, arXiv:2011.09922, 2020.

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Q. Is strict convexity of \mathcal{G} *necessary* for AC?

Thank you for listening.