# Equivalence of ellipticity conditions for geometric variational problems

22. XI. 2018

(1)

Joint work with Antonio De Rosa

The problem:

$2 \leq d < n$ integers, $U \subseteq \mathbb{R}^n$ open

$\mathcal{C}$ a class of rel. closed, $(\mathcal{H}^d, d)$-rectifiable subsets of $U$

$F : \mathbb{R}^n \times G(n,d) \longrightarrow (0,\infty)$

$\Phi_F : \mathcal{C} \longrightarrow [0,\infty]$, $\Phi_F(X) = \int_X F(x, \mathrm{Tan}(X,x))\, d\mathcal{H}^d(x)$

(P) Find $X \in \mathcal{C}$ s.t. $\Phi_F(X) = \inf \{ \Phi_F(Y) : Y \in \mathcal{C} \}$

Strategy: $X \in \mathcal{C} \rightsquigarrow \mathbf{v}(X) \in C^0_c(\mathbb{R}^n \times G(n,d))^*$

$$\mathbf{v}(X)(\alpha) = \int_X \alpha(x, \mathrm{Tan}(X,x))\, d\mathcal{H}^d(x)$$

$X_i \in \mathcal{C}$ s.t. $\Phi_F(X_i) \xrightarrow[i\to\infty]{} \inf$

$\mathbf{v}(X_i) \xrightarrow{\ *\ } V$ up to subsequence (Banach Alaoglu)

$\Phi_F(X_i) = \mathbf{v}(X_i)(F) \longrightarrow V(F) =: \Phi_F(V)$ ← For free!

Q: 1) Is there an $(\mathcal{H}^d, d)$-rectifiable set $Z$ s.t. $V = \mathbf{v}(Z)$?

2) Is $Z \in \mathcal{C}$?

Answer: 1) Yes, given F is elliptic and $\mathcal{C}$ is closed under Lipschitz deformations $\mathbb{R}^n \to \mathbb{R}^n$ leaving $\mathbb{R}^n \setminus U$ fixed.

2) Yes, given $\mathcal{C}$ is closed under taking local Hausdorff limits.

[Fang & K., Calc. Var. PDE, 2018] or [Harrison & Pugh, Calc. Var. PDE, 2017]

<u>Def</u>. $(X, Q)$ is called a <u>test pair</u> if

- $Q$ is a $d$-dim. cube in $\mathbb{R}^n$
  $\Leftrightarrow \exists\, p : \mathbb{R}^d \to \mathbb{R}^n$ isometric embedding $Q = p[[0,1]^d]$
- $X$ is compact, $(\mathcal{H}^d, d)$-rectifiable $\subseteq \mathbb{R}^n$
- $\partial Q$ is not a retract of $X$.

<u>Def</u>. $F$ is called <u>Almgren elliptic</u> at $x \in \mathbb{R}^n$ <u>$(AE)_x$</u> if

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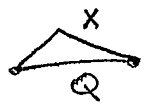
$$\Phi_{F^x}(X) > \Phi_{F^x}(Q)$$

whenever $(X, Q)$ is a test pair with $\mathcal{H}^d(X) > \mathcal{H}^d(Q)$.

<u>Def</u>. $F^x : \mathbb{R}^n \times G(n,d) \longrightarrow \mathbb{R}$, $F^x(y, T) = F(x, T)$.




<u>Def</u>. Assume $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, $V \in C^0_c(\mathbb{R}^n \times G(n,d))^*$,

$\varphi_t(x) = x + t g(x)$, $\varphi_t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\frac{d}{dt}\Big|_{t=0} \varphi_t(x) = g(x)$.

The <u>first variation of $V$ w.r.t. $F$</u>

$$\delta_F V(g) = \frac{d}{dt}\Big|_{t=0} \Phi_F(\varphi_{t\#} V),$$

where $$\varphi_{\#} V(\alpha) = \int \alpha(x, D\varphi(x)[T])\, J_d(\varphi|_{x+T})\, dV(x,T)$$

[if $\varphi$ is a diffeomorphism and $V = \mathbf{v}(X)$, then $\varphi_{\#} V = \mathbf{v}(\varphi[X])$]

<u>Def</u>. We say that $F$ satisfies <u>$(WBC)_x$</u> if
for any $W$ of the form $W = (\mathcal{H}^d \llcorner T) \times \mu$,
where $T \in G(n,d)$ and $\mu$ is a prob. measure on $G(n,d)$,
there holds:

$$\delta_{F^x} W = 0 \implies \mu = \mathrm{Dirac}(T).$$

If additionaly, for any $W$ of the form $W = (\mathcal{H}^d \llcorner T) \times \mu$,
where $T \in G(n,k)$ and $\mu$ is a prob. measure on $G(n,d)$,
there holds:

$$\delta_{F^x} W = 0 \implies k \geq d$$

then we say that $F$ satisfies <u>$(BC)_x$</u>.

**Remark** [CPAM 2018] : De Philippis, De Rosa, Ghiraldin define condition $(AC)_x$ which is sufficient and necessary for the implication:

$$\exists\, h \in L^1(\|V\|, \mathbb{R}^n) \qquad \delta_F V(g) = \int g(x) \cdot h(x)\, d\|V\|(x)$$

$$\Downarrow$$

$V = \Theta\, v(X)$ for some $\Theta \in L^1_{loc}(\mathcal{H}^d \llcorner X, \mathbb{R}_+)$ and some $(\mathcal{H}^d, d)$ rectifiable set $X \subseteq \mathbb{R}^n$.

**Theorem A:** De Rosa & K. $\quad (AC)_x \iff (BC)_x$

**Theorem B:** Answer to Q 1 & 2 is positive if $F \in (wBC)_x\ \forall x \in \mathbb{R}^n$

**Theorem C:** De Rosa & K. $\quad (wBC)_x \subseteq (AE)_x$

**Proof.** Assume $\exists\, F \in (wBC)_x \sim (AE)_x$.

Then $\exists\, (X, Q)$ a test pair with $\mathcal{H}^d(X) > \mathcal{H}^d(Q)$

and $\Phi_{F^x}(X) \le \Phi_{F^x}(Q)$.

Set $\mathcal{U} = \mathbb{R}^n \sim \partial Q$

$\mathcal{G} = \{ R \cap \mathcal{U} : (R, Q) \text{ is a test pair} \}$

Then $\mathcal{G}$ is closed under Lipschitz deformations and Hausdorff limits (given the limit is $(\mathcal{H}^d, d)$-rectifiable).

Since $F^x \in (wBC)_y$ for all $y \in \mathbb{R}^n$ we can apply Theorem B to find a minimiser $Y \in \mathcal{G}$ of $\Phi_{F^x}$.

Then either

$$\Phi_{F^x}(Y) < \Phi_{F^x}(Q) \quad \text{and we set } Z = Y$$

or

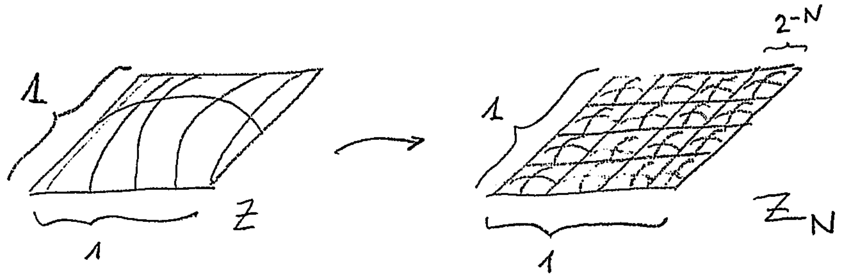
$$\Phi_{F^x}(Y) = \Phi_{F^x}(X) = \Phi_{F^x}(Q) \quad \text{and we set } Z = X.$$

In any case we get

- $(Z, Q)$ is a test pair
- $\mathcal{H}^d(Z) > \mathcal{H}^d(Q)$ ← key point to get a contradiction.
- $\Phi_{F^x}(Z) \le \Phi_{F^x}(Q)$

Wlog: $Q = [0,1]^d \times \{0\}^{n-d} \subseteq \mathbb{R}^n$

For $N \in \mathbb{N}$ define $Z_N = (2^{-N} Z) + \left( (2^{-N}\mathbb{Z}^d) \cap [0,1)^d \right) \times \{0\}^{n-d}$




Observe that

$$\forall N \in \mathbb{N} \quad \Phi_{F^x}(Z_N) = \Phi_{F^x}(Z) \quad \text{and} \quad \mathcal{H}^d(Z_N) = \mathcal{H}^d(Z).$$

Assume that $(Z_N, Q)$ is a test pair, i.e., that $\partial Q$ is not a retract of $Z_N$. $\Big\}$ !

(*) Then $Z_N$ is a minimizer of $\Phi_{F^x}$ in $\mathcal{C}$ $\quad \forall N \in \mathbb{N}$

Let $W_N = \mathbf{v}(Z_N)$ and $V_N = \sum_{v \in \mathbb{Z}^d \times \{0\}^{n-d}} \tau_{v \#} W_N$.

(*) $\Rightarrow$ $\delta_{F^x} W_N(g) = 0$
$\forall g \in C^1_c(\mathbb{R}^n \setminus \partial Q, \mathbb{R}^n)$

Up to a subsequence

$$W_N \longrightarrow W \quad \text{and} \quad V_N \longrightarrow V \quad \text{and} \quad V = \sum_{v \in \mathbb{Z}^d \times \{0\}^{n-d}} \tau_{v\#} W$$

<u>Observe</u>:

- $\operatorname{spt} \|V_N\| \subseteq T + \mathbb{B}(0, \varepsilon_N)$ with $\varepsilon_N = 2^{-N} \operatorname{diam} Z \xrightarrow[N \to \infty]{} 0$

  $\Rightarrow \operatorname{spt} V \subseteq T$ $\qquad T = \mathbb{R}^d \times \{0\}^{n-d}$

- $\forall v \in T \quad \tau_{v\#} V = V$

  $\Rightarrow V = \Theta(\mathcal{H}^d \llcorner T) \times \mu$ for some prob. meas. $\mu$ over $G(n,d)$.

  $\Rightarrow \delta_{F^x} V = 0$

Since $F \in (wBC)_x$ we get $\mu = \operatorname{Dirac}(T) \Rightarrow W = \Theta(\mathcal{H}^d \llcorner Q) \times \operatorname{Dirac}(T)$

Note that $\Theta = \dfrac{\mathcal{H}^d(Z)}{\mathcal{H}^d(Q)} > 1$. Hence,

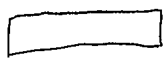
$$\Phi_{F^x}(Q) < \Theta\, \Phi_{F^x}(Q) = \Theta\, F(x,T)\, \mathcal{H}^d(Q) = W(F^x) = \Phi_{F^x}(W) = \Phi_{F^x}(Z)$$

$$\shortparallel$$

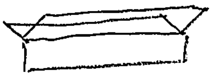
$$\Phi_{F^x}(Q) \quad \lightning \quad \square$$

To complete the proof we only need to show that $\partial Q$ is not a retract of $Z_N$.

**Example** The Adams surface



$M$ - a Möbius strip embedded in $\mathbb{R}^n$ so that $\partial M = (\partial [0,1]^2) \times \{0\}^{n-2}$,



$T$ - a triple Möbius strip embedded in $\mathbb{R}^n$ so that $\partial T = (\partial([-1,0] \times [0,1])) \times \{0\}^{n-2}$.

$$A = M \cup T \quad , \quad \partial A = (\partial([-1,1] \times [0,1])) \times \{0\}^{n-2}.$$

Then: $\partial M$ is not a retract of $M$

$\partial T$ is not a retract of $T$

but $\partial A$ is a retract of $A$ !

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**Fact** If $A \subseteq X$ are topological spaces and $(X, A)$ is a Borsuk pair (HEP, cofibration), then

a) if $A$ is contractible, then $X$ and $X/A$ are homotopy equivalent.

b) if $A$ is homeomorphic with $S^k$, then $A$ is a retract of $X$ if and only if $\exists f : X \to A$ s.t. $\deg(f|_A) = 1$.

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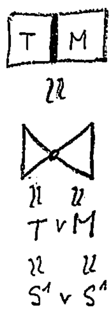
For the Adams surface: ($\approx$ means homotopy equivalence)




$M \approx S^1$, $T \approx S^1$, $\deg(\partial M \hookrightarrow M) = 2$,

$A \approx M \vee T \approx S^1 \vee S^1$, $\deg(\partial T \hookrightarrow T) = 3$,

Let $f : A \longrightarrow \partial A \simeq S^1$ be such that

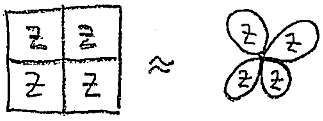
$$\deg(f|_M) = -1 \text{ and } \deg(f|_T) = 1.$$

Then $$\deg(f|_{\partial A}) = 3 - 2 = 1$$

This is possible because $\gcd(2,3) = 1$.

Assume $(Z, \partial Q)$ is a Borsuk pair and $N = 1$.

Then $\quad Z_1 \approx \bigvee_{i=1}^{2^d} Z$. $\leftarrow$ wedge sum of $2^d$ copies of <u>the same $Z$</u>




$\circledast$ We know that $\partial Q$ is not a retract of $Z$.

To get that $\partial Q$ is not a retract of $Z_1$, it suffices to show that for any $f, g : Z \to \partial Q$ there is a map $h : Z \to \partial Q$ s.t. $\deg(h|_{\partial Q}) = \gcd(\deg(f|_{\partial Q}), \deg(g|_{\partial Q}))$ because from $\circledast$ it follows that $\deg(h|_{\partial Q}) \neq 1$.

<u>Lemma</u> (A. Weber)

Assume $Z$ is a $d$-dimensional CW-complex
$\alpha \in \mathrm{Hom}(H_{d-1}(Z), \mathbb{Z})$
Then there exists $g : Z \to S^{d-1}$ s.t. $g_* = \alpha$.

Crucial that $Z$ is $d$-dim. and target of $g$ is $(d-1)$-dim.

<u>Proof</u>.

$$\begin{array}{ccccc} [Z, K(\mathbb{Z}, d-1)]_\sim & \xrightarrow{\simeq} & H^{d-1}(Z) & \xrightarrow{UCT} & \mathrm{Hom}(H_{d-1}(Z), \mathbb{Z}) \\ \Psi & & & & \Psi \\ \bar{g} & & \longmapsto & & \alpha \end{array}$$

where $K(\mathbb{Z}, d-1)$ is the Eilenberg – MacLane space.

The $d$-skeleton of $K(\mathbb{Z}, d-1)$ is $S^{d-1}$
(i.e. no cells in dimension $d$)

so $\bar{g}$ is homotopic to a map $g : Z \to S^{d-1} \subseteq K(Z, d-1)$.

$\square$

<u>Corollary</u> Assume $Z$ is a $d$-dim CW-complex,

$j : S^{d-1} \to Z$, $f, g : Z \to S^{d-1}$.

Then there exists $h : Z \to S^{d-1}$ s.t.

$$\deg(h \circ j) = \gcd(\deg(f \circ j), \deg(g \circ j))$$

<u>Proof</u> $\exists\, a, b \in \mathbb{Z}$ $\gcd(\deg(f \circ j), \deg(g \circ j)) = a \deg(f \circ j) + b \deg(g \circ j)$.

$\exists\, h : Z \to S^{d-1}$ $\quad h_* = a f_* + b g_*$

$\square$.

We still need to ensure that

- $(Z, \partial Q)$ is a Borsuk pair
- $Z$ is homotopy equivalent to a CW-complex

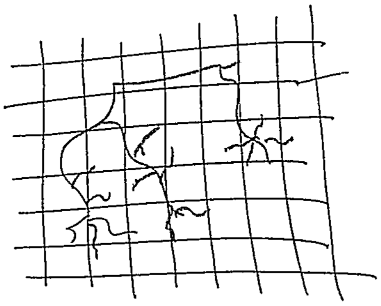
<u>Observation</u>: a) $\partial Q$ is a retract of $Z$

$\iff \exists \delta > 0 \quad \partial Q$ is a retract of $Z + \mathbb{B}(0,\delta)$.

b) If $\partial Q \subseteq \operatorname{Int} Z$, then $(Z, \partial Q)$ is a Borsuk pair.

To get CW-structure we use the deformation theorem.

- cover $Z$ with cubes of diameter $< \delta \leftarrow$ from b)
- deform $Z$ into the $d$-skeleton of these cubes



This gives:

i) $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ Lipschitz map

ii) $f(1,\cdot)[Z]$ is a cubical complex, <u>$\dim = d$</u>

iii) $f(1,\cdot)[Z]$ is a strong deformation retract of $f[I \times G]$

where $G \subseteq \mathbb{R}^n$ is open and $Z \subseteq G$.

<u>Remark</u> This is possible because

- $Z$ is compact
- $0 < \mathcal{H}^d(Z) < \infty$