Ellipticity in geometric variational problems

23.X.2018

[sem. Top Alg]

- Thanks for the invitation to Marcin. Apologise for talking about trivial things!
- Brief history: July 2018 : e-mail to A. Weber

Result = the following lemma:

Lemma (due to Andrzej Weber)

Assume Z is a d-dim. CW-complex and $\alpha \in \operatorname{Hom}(H_{d-1}(Z), \mathbb{Z})$.
Then there is $g : Z \to \mathbb{S}^{d-1}$ s.t. $g_* = \alpha$.

Proof.

$$[Z, K(\mathbb{Z}, d-1)]_{\sim} \xrightarrow{\simeq} H^{d-1}(Z) \xrightarrow{UCT} \operatorname{Hom}(H_{d-1}(Z), \mathbb{Z})$$

$$\bar{g} \longmapsto \alpha$$

Since the d-dim skeleton of $K(\mathbb{Z}, d-1) = \mathbb{S}^{d-1}$
there is $g : Z \to \mathbb{S}^{d-1} \subseteq K(\mathbb{Z}, d-1)$ s.t. $\bar{g} \approx g$. □.

Corollary: Assume $j : \mathbb{S}^{d-1} \to Z$ is given.
Then
$$D(j) = \{ \deg(f \circ j) : f : Z \to \mathbb{S}^{d-1} \} = \{ m \cdot A : m \in \mathbb{Z} \}$$
where $A = \min(\{ |k| : k \in D \} \sim \{0\})$.

Proof Let $f, g : Z \to \mathbb{S}^{d-1}$. Then there are $a, b \in \mathbb{Z}$ s.t.
$$\gcd(\deg(f \circ j), \deg(g \circ j)) = a \cdot \deg(f \circ j) + b \cdot \deg(g \circ j)$$
From lemma there exists $h : Z \to \mathbb{S}^{d-1}$ s.t.
$$h_* = a f_* + b g_* \quad \text{and} \quad \deg(h \circ j) = \gcd(\deg(f \circ j), \deg(g \circ j))$$ □

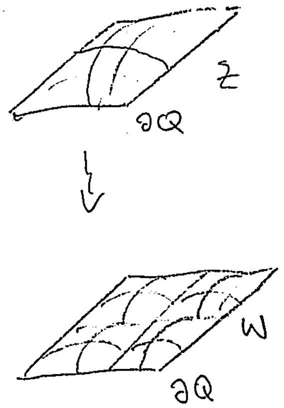
Corollary: Assume $W = \bigvee_{i=1}^{N} Z$ and
$$l : \mathbb{S}^{d-1} \longrightarrow \bigvee_{i=1}^{N} \mathbb{S}^{d-1} \xrightarrow{\bigvee_{i=1}^{N} j} W$$
Then
$$D(l) = D(j).$$

Proof: If $f : W \to \mathbb{S}^{d-1}$, then $\deg(f \circ l) = \sum_{i=1}^{N} \deg(f_i \circ j)$ □

23. X. 2018

(1/2)

Corollary: Assume $Z \subseteq \mathbb{R}^n$ is a d-dim. CW-complex, $Q \subseteq T \in G(n,d)$ is a unit d-dim. cube, $\partial Q \subseteq Z$, $\partial Q \hookrightarrow Z$ is a cofibration, $W \subseteq \mathbb{R}^n$ is composed of 2^d-copies of $\frac{1}{2} Z$ placed side-by-side so that $\partial Q \subseteq W$.



If ∂Q is not a retract of Z, then ∂Q is not a retract of W.

Proof: **Step 1:** Assume $\partial Q \subseteq X$ and $j : \partial Q \hookrightarrow X$ is a cofibration. Then

$$\partial Q \text{ is a retract of } X \iff 1 \in D(j).$$

$\Rightarrow$ trivial

$\Leftarrow$ Let $f : X \to \partial Q$ be s.t. $\deg(f \circ j) = 1$.

Let $h : I \times \partial Q \to \partial Q$ be s.t. $h(0,\cdot) = f \circ j$, $h(1,\cdot) = id_{\partial Q}$

Define $\bar{g} : (\{0\} \times X) \cup (I \times \partial Q) \to \partial Q$

$$\bar{g}(0,\cdot) = f, \quad \bar{g}(t,\cdot) = h(t,\cdot) \text{ for } t > 0.$$

Extend $\bar{g}$ to $g : I \times X \to \partial Q$.

Define $r : Z \to \partial Q$ by $r = g(1,\cdot)$. $\square$

Step 2: Assume $1 \notin D(j)$.

Using HEP we observe that W is htp. equiv. to $\bigvee_{i=1}^{2^d} Z$ and the inclusion $\partial Q \xrightarrow{k} W$ corresponds under this equivalence, to

$$\begin{tikzcd} l : \partial Q \arrow[r] & \bigvee_{i=1}^{2^d} \partial Q \arrow[r] & \bigvee_{i=1}^{2^d} Z \\ k : \partial Q \arrow[r, hook] & W \arrow[ur, leftrightarrow, "\approx"'] & \end{tikzcd}$$

From the last corollary we get

$$1 \notin D(j) = D(l) = D(k).$$

Hence, step 1 shows that ∂Q is not a retract of W. $\square$

Remark If M is a Möbius strip, T a triple Möbius strip, $M, T \subseteq \mathbb{R}^n$, $W = M \cup T$, then ∂W is a retract of W.

$M \cap T = \partial M \cap \partial T$ is a line segment

Ellipticity in geometric variational problems

Seminarium z topologii algebraicznej 23. X. 2018.

Joint work with Antonio De Rosa

The problem

$2 \le d < n$ integers, $U \subseteq \mathbb{R}^n$ open,

$\mathcal{G}$ a class of relatively closed subsets of U,

$\Phi : \mathcal{G} \to [0, \infty]$ a functional

(P) Find $X \in \mathcal{G}$ s.t. $\Phi(X) = \inf \{ \Phi(Y) : Y \in \mathcal{G} \}$

Def. $A \subseteq \mathbb{R}^n$, $\delta \in (0,1)$, $s \in [0, \infty)$.

$$\mathcal{H}^s_\delta(A) = \inf \Big\{ \sum_{P \in \mathcal{G}} \alpha(s) \cdot 2^{-s} \operatorname{diam}(P)^s : \mathcal{G} \subseteq 2^{\mathbb{R}^n} \text{ countable}, \ \forall P \in \mathcal{G} \ \operatorname{diam} P \le \delta, \ A \subseteq \bigcup \mathcal{G} \Big\}$$

$$\alpha(s) = \frac{\Gamma(1/2)^s}{\Gamma(\frac{s}{2}+1)}$$

$$\mathcal{H}^s(A) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A)$$

$A \subseteq \mathbb{R}^n$ is <u>$\mathcal{H}^s$-measurable</u> <u>iff</u> $\mathcal{H}^s(T) = \mathcal{H}^s(T \cap A) + \mathcal{H}^s(T \sim A)$ $\forall T \subseteq \mathbb{R}^n$.

Def $A \subseteq \mathbb{R}^n$ is <u>$(\mathcal{H}^d, d)$-rectifiable</u> if $\mathcal{H}^d(A) < \infty$ and there exists a countable family $\mathcal{F}$ of submanifolds of $\mathbb{R}^n$ of class 1 s.t. $\mathcal{H}^d(A \sim \bigcup \mathcal{F}) = 0$.

Assume $\mathcal{G}$ contains only d-dimensional submanifolds of U of class 1 and $\Phi_F : \mathcal{G} \to [0, \infty]$ is defined by

$$\Phi_F(X) = \int_X F(x, \operatorname{Tan}(X,x)) \, \boxed{dx}, \quad \text{where } F : \mathbb{R}^n \times G(n,d) \to (0,\infty)$$

dx: Riemannian measure on X. F is given, continuous integrand, bounded $0 < \sup F / \inf F < \infty$

Model case: $F \equiv 1$ Plateau problem

To solve (P) we take a minimizing seq., i.e., $X_i \in \mathcal{G}$ s.t. $\lim_{i \to \infty} \Phi_F(X_i) = \inf$.

We want to pass to the limit with $X_i \to$?

$\forall i \quad X_i \rightsquigarrow V_i \in C_c^0(\mathbb{R}^n \times G(n,d))^*$ ← i.e. a Radon measure. Hahn Banach

②

23. X. 2018

$\boxed{V_i = \mathbb{v}(X_i)}$ $\qquad V_i(\alpha) = \int_{X_i} \alpha(x, \operatorname{Tan}(X_i, x))\, dx$

Then $\quad \Phi_F(X_i) = V_i(F)$

(Banach Alaoglu) $\quad V_i \xrightarrow{*} V \in C_c^0(\mathbb{R}^n \times G(n,d))^*$ ← Varifolds

$$\inf = \lim_{i\to\infty} \Phi_F(X_i) = \lim_{i\to\infty} V_i(F) = V(F) =: \Phi_F(V)$$

__Question__: Is there a "regular enough" set $X \subseteq U$ such that $V(F) = \int_X F(x, \operatorname{Tan}(X,x))$ "dx", i.e., is $V = \mathbb{v}(X)$?

__Answer__: Yes, given F is __elliptic__ and $\mathcal{C}$ is closed under Lipschitz deformations leaving $\mathbb{R}^n \sim U$ fixed, i.e.,

$\forall Z \in \mathcal{C} \quad \forall f \in \operatorname{Lip}(\mathbb{R}^n, \mathbb{R}^n) \quad f(x) = x$ for $x \in \mathbb{R}^n \sim U$

$$\Downarrow$$

$$f[Z] \in \mathcal{C}$$

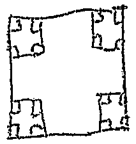
[see Feng, K., Calc. Var. PDE, 2018]

__Remark__ • If $\mathcal{C}$ is closed under taking Hausdorff limits, then the minimizer X is in $\mathcal{C}$.

• In general, X will not be a manifold but a d-__rectifiable set__, i.e.,

$$X \subseteq N \cup \bigcup_{i=1}^{\infty} M_i \quad, \text{where } M_i \subseteq U \text{ submanifolds of dim. } d \text{ of class } 1$$

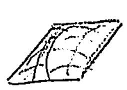
N - an $\mathcal{H}^d$-nullset.



$G(\frac{1}{4}) \times G(\frac{1}{4})$

__Def.__ (X, Q) is called a __test pair__ if

- Q is a d-dimensional cube in $\mathbb{R}^n$, ← $\exists\, p : \mathbb{R}^d \to \mathbb{R}^n$ isometric injection $Q = p[[0,1]^d]$
- X is a compact and d-rectifiable subset of $\mathbb{R}^n$,
- $\boxed{\partial Q \text{ is not a retract of } X.}$

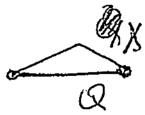


__Def__ $F^x(y, T) = F(x, T) \quad \forall y \in \mathbb{R}^n \ \forall T \in G(n,d)$

__Def.__ F is called __Almgren elliptic__ $(AE)_x$ if

(1968) $\quad \Phi_{F^x}(X) = \int_X F(x, \operatorname{Tan}(X, y))\, dy > \int_Q F(x, \operatorname{Tan}(Q, y))\, dy = \Phi_{F^x}(Q)$

whenever (X, Q) is a test pair with $\mathcal{H}^d(X) > \mathcal{H}^d(Q)$.



__Def__. Assume $g \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ a vectorfield, $V \in C_c^0(\mathbb{R}^n \times G(n,d))^*$,

$\varphi_t(x) = x + t \cdot g(x)$, $\quad \varphi_t : \mathbb{R}^n \to \mathbb{R}^n$, $\quad \frac{d}{dt}\big|_{t=0} \varphi_t(x) = g(x)$

We define $\qquad \delta_F V(g) = \frac{d}{dt}\Big|_{t=0} \Phi_F(\varphi_{t\#} V)$. ← first variation of V w.r.t. F

where $\varphi_{\#} V(\alpha) = \int \alpha\big(\varphi(x), D\varphi(x)[T]\big) \cdot J_d(\varphi|_{x+T})\, dV(x,T)$ ③

23.X.2018

φ diffeo, $V = \mathbf{v}(X)$ $\Rightarrow$ $\varphi_{\#} V = \mathbf{v}(\varphi[X])$

<u>Def</u>. We say that F satisfies <u>(wBC)</u> at $x \in \mathbb{R}^n$ if

[2018] for any W of the form $W = (\mathcal{H}^d \llcorner T) \times \mu$, where $T \in G(n,d)$ and μ is a prob. measure on $G(n,d)$, if $\delta_{F^x} W = 0$, then $\mu = \mathrm{Dirac}(T)$.

<u>Remark</u>. • De Philippis, De Rosa, Ghiraldin defined condition (AC) which is sufficient and necessary for the implication

[CPAM 2018]

! Jeśli V minimalizuje Φ_F, to $\delta_F V = 0$

$$\delta_F V(g) = \int g(x) \cdot h(x)\, d\|V\|(x) \quad \text{for some } h \in L^1(\|V\|, \mathbb{R}^n)$$

$$\Downarrow$$

V is naturally associated to a rectifiable set. $(V = \mathbf{v}(X))$

• (AC) $\Leftrightarrow$ (BC), (BC) $\Rightarrow$ (wBC) [De Rosa, K.]

In particular, if $\delta_F V = 0$, then V is rectifiable

<u>Theorem</u> (De Rosa, K.) $x \in \mathbb{R}^n$

$$(\mathrm{wBC})_x \Rightarrow (\mathrm{AE})_x$$

<u>Proof</u>. Assume not, i.e., $\exists F \in (\mathrm{wBC})_x \sim (\mathrm{AE})_x$.

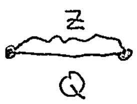
Then $\exists (X, Q)$ a test pair with $\mathcal{H}^d(X) > \mathcal{H}^d(Q)$ but $\Phi_{F^x}(X) \le \Phi_{F^x}(Q)$.

Employing the result of Feng & K. with $U = \mathbb{R}^n \sim \partial Q$ we find a minimiser Y of Φ_{F^x} in $\mathcal{G} = \{R \cap U : (R,Q) \text{ a test pair}\}$. $\boxed{U = \mathbb{R}^n \sim \partial Q}$

Then either $\Phi_{F^x}(Y) < \Phi_{F^x}(Q)$ and set $Z = Y$

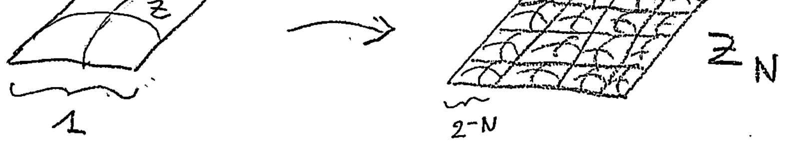
or $\Phi_{F^x}(Y) = \Phi_{F^x}(X) = \Phi_{F^x}(Q)$ and set $Z = X$

In any case we get $\mathcal{H}^d(Z) > \mathcal{H}^d(Q)$ and $\Phi_{F^x}(Z) \le \Phi_{F^x}(Q)$ and (Z, Q) is a test pair.



Wlog. $Q = [0,1]^d \times \{0\}^{n-d} \subseteq \mathbb{R}^n$

For $N \in \mathbb{N}$ define $Z_N = \mu_{2^{-N}}[Z] + \big((2^{-N}\mathbb{Z}^d) \cap [0,1)^d\big) \times \{0\}^{n-d}$



$\boxed{\mu_r(x) = r x}$

We call Z_N the 2^N-multiplication of Z.

Observe that $\Phi_{F^x}(Z_N) = \Phi_{F^x}(Z) \quad \forall N \in \mathbb{N}$. (**) (4)

23.X.2018

$\boxed{\text{Assume that } (Z_N, Q) \text{ is a test pair}}$!

Then Z_N is a minimizer of Φ_{F^x} in the class

$\mathcal{C} = \{R \cap U : (R, Q) \text{ is a test pair}\}$; hence,

if W_N is the associated varifold, then $\boxed{W_N = \mathbf{v}(Z_N)}$

(*) $\delta_{F^x} W_N(g) = 0$ for $g \in C^1_c(\underbrace{\mathbb{R}^n \sim \partial Q}_{U}, \mathbb{R}^n)$.

We take $V_N = \sum_{v \in \mathbb{Z}^d \times \{0\}^{n-d}} \tau_{v\#} W_N$ $\quad \boxed{\tau_v(x) = v + x}$

We pass to the limit $V_N \longrightarrow V$, $W_N \longrightarrow W$. $\boxed{\text{up to subsequence}}$

Note: $\forall v \in \underbrace{\mathbb{R}^d \times \{0\}^{n-d}}_{T}$ $\tau_{v\#} V = V \implies V = \Theta(\mathcal{H}^d \llcorner T) \times \mu$

! $\operatorname{spt}\|V_N\| \subseteq T + B(0, \varepsilon_N)$, where $\varepsilon_N \overset{\boxed{2^{-N}\operatorname{diam} Z}}{=} \to 0$.

$\overset{(*)}{\implies} \delta_{F^x} V = 0$, $\quad \Theta = \dfrac{\mathcal{H}^d(Z)}{\mathcal{H}^d(Q)} > 1$!

From $(wBC)_x$ we deduce $\mu = \text{Dirac}(T)$

and we get:

$$\Phi_{F^x}(Q) < \underbrace{\Theta\, \Phi_{F^x}(Q)}_{\Theta F(x,T)\mathcal{H}^d(Q) = F(x,T)\|W\|(\mathbb{R}^n)} = \Phi_{F^x}(W) \overset{(**)}{=} \Phi_{F^x}(Z) \le \Phi_{F^x}(Q)$$

$\lightning$ $\square$

We only need to show that (Z_N, Q) is a test pair,

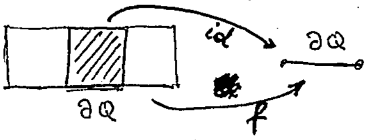
i.e., that ∂Q is not a retract of Z_N.

Assume:

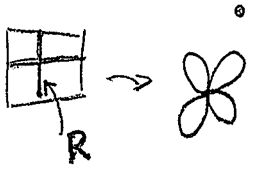
- $N = 1$: Z_N is a 2-multiplication of Z_{N-1} and we use induction
- $Z \subseteq \mathbb{R}^n$ is open and htp. equiv. to a d-dim. CW-complex
- (Z, R) is a Borsuk pair for any finite cubical complex $R \subseteq Z$. (HEP)
- the 2^{Nd} copies of $\mu_{2^{-N}}[Z \sim \partial Q]$ do not intersect inside Z_N
- $j : \partial Q \hookrightarrow Z_N$ is the inclusion map.

Then

- ∂Q is a retract of Z_N iff $\exists f : Z_N \to \partial Q$, $\deg(f \circ j) = 1$ by HEP



- Since $N = 1$, Z_N is homotopy equivalent



to $\bigvee_{i=1}^{2^d} Z$. Let $\psi : Z_N \xrightarrow[\text{HEP}]{\approx} Z_N/R \overset{\text{HEP}}{\approx} \bigvee_{i=1}^{2^d} Z$

- ∂Q is a retract of Z_N iff

$$\exists\ f : \bigvee_{i=1}^{2^d} Z \longrightarrow \partial Q \quad \deg(f\circ\psi\circ j) = 1$$

$$\Downarrow \qquad\qquad \|$$

$$f_1, \dots, f_{2^d} : Z \longrightarrow \partial Q$$

$$\deg(f\circ\psi\circ j) = \sum_{i=1}^{2^d} \deg(f_i\circ j) = 1$$

Assume ∂Q is a retract of Z_N.

We get $f_1 \dots f_{2^d}$ with $(f_i)_* : H_{d-1}(Z) \to H_{d-1}(\partial Q)$.

and $\sum_{i=1}^{2^d} (f_i)_* [\partial Q] = [\partial Q] \in H_{d-1}(\partial Q)$

We ask whether there exists a map $g : Z \longrightarrow \partial Q$

s.t. $g_* = \sum_{i=1}^{2^d} (f_i)_* : H_{d-1}(Z) \longrightarrow H_{d-1}(\partial Q)$.

If so, then $\deg(g\circ j) = 1$ and ∂Q is a retract of Z $\lightning$.

Lemma Assume Z is a d-dim CW-complex

$\forall\ \alpha \in \mathrm{Hom}(H_{d-1}(Z), \mathbb{Z})$

$\exists\ g : Z \longrightarrow S^{d-1}$. $\quad g_* = \alpha$

Proof [Due to A. Weber]

$$\begin{array}{ccccc} [Z, K(\mathbb{Z}, d-1)]_\sim & \xrightarrow{\cong} & H^{d-1}(Z) & \xrightarrow{\text{UCT}} & \mathrm{Hom}(H_{d-1}(Z), \mathbb{Z}) \\ \tilde g & \longmapsto & \beta & \longmapsto & \alpha \end{array}$$

We find $\tilde g : Z \longrightarrow K(\mathbb{Z}, d-1)$

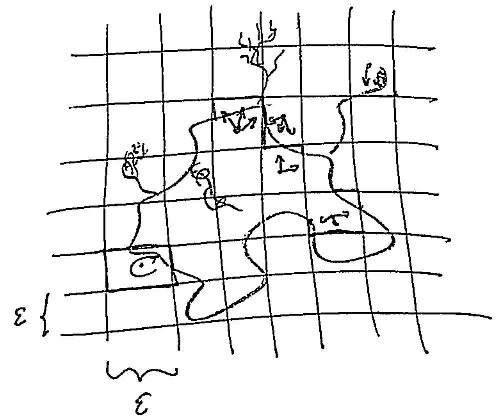
but the d-skeleton of $K(\mathbb{Z}, d-1)$ is S^{d-1}

and Z is d-dimensional.

Hence, $\tilde g$ is homotopic to a map $g : Z \longrightarrow S^{d-1}$. $\square$.

Q: How to pass from an arbitrary compact set Z with $0 < \mathcal{H}^d(Z) < \infty$ to a d-dim. CW-complex? 23.X.2018

Use a well established tool of GMT: the deformation thm.



$f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \in \mathrm{Lip}$

- $\exists\ C = C(n, d)$

$\mathcal{H}^d(f(1,\cdot)[Z]) \leq C\, \mathcal{H}^d(Z)$

$\mathcal{H}^{d+1}(f[I \times Z]) \leq C\, \varepsilon\, \mathcal{H}^d(Z)$

given Z is d-rectifiable.

- $f[I \times U] \subseteq Z + \mathbb{B}(0, \varepsilon)$

Features

- changes topology!
- $f(1,\cdot)[Z]$ is a d-dim cubical complex.
- $f(1,\cdot)[Z]$ is a strong deformation retract of $f[I \times Z]$

Even more: there exists an open set $U \subseteq \mathbb{R}^n$ s.t. $f(1,\cdot)[Z]$ is a strong def. retr. of $f[I \times U]$.

and $Z \subseteq U \Rightarrow \partial Q \subseteq U$

$\Rightarrow (U, \partial Q)$ is a Borsuk pair.

- one can arrange so that $f(t, x) = x$ for $x \in \partial Q$ $\forall t \in I$.

Remark: ∂Q is a retract of Z

$$\Updownarrow$$

$\exists\ \varepsilon > 0$ ∂Q is a retract of $Z + \mathbb{B}(0, \varepsilon)$.