Ellipticity in geometric variational problems

$\left.\begin{array}{l} 0 < k \le n \\ U \subseteq \mathbb{R}^n \text{ open} \end{array}\right\}$ fixed

(1)

We consider $\Phi : \{\text{geometric objects}\} \longrightarrow [0,\infty)$.

where geometric object $= \left\{\begin{array}{l} \text{a } k\text{-dim current in } U \\ \text{a } k\text{-dim varifold in } U \end{array}\right.$

Recall:

$\mathcal{D}^k(U) = \mathcal{D}(U, \Lambda^k \mathbb{R}^n)$ – the space of smooth compactly supported differential k-forms on U endowed with locally convex topology (non-metrizable TVS)

$\mathcal{D}_k(U) = \mathcal{D}'(U, \Lambda^k \mathbb{R}^n)$ – continuous linear functionals on $\mathcal{D}(U, \Lambda^k \mathbb{R}^n)$
$= k$-dim currents in U

$\Lambda^k \mathbb{R}^n$ – antisymmetric k-linear forms: $\underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{k \text{ times}} \longrightarrow \mathbb{R}$.

$\mathcal{R}_k(U) \subseteq \mathcal{D}_k(U)$ – rectifiable currents, i.e.,

$T \in \mathcal{R}_k(U) \iff$
- $\operatorname{spt} T \subseteq U$ is compact
- $\exists\, E \subseteq \operatorname{spt} T$ $\mathcal{H}^k$-measurable $(\mathcal{H}^k, k)$-rectifiable $(\mathcal{H}^k(E) < \infty)$

[F, 4.1.28 (4)]

$\exists\, \eta \in L^1(\mathcal{H}^k \llcorner E; \Lambda_k \mathbb{R}^n)$
for $\mathcal{H}^k$ almost all $x \in E$
$|\eta(x)| \in \mathbb{N}$, $\eta(x)$ is simple,
$\operatorname{Tan}^k(\mathcal{H}^k \llcorner E, x) = \{v \in \mathbb{R}^n : \eta(x) \wedge v = 0\}$

$$\boxed{\forall \phi \in \mathcal{D}^k(U) \quad T(\phi) = \int_E \langle \eta(x), \phi(x) \rangle \, d\mathcal{H}^k(x)}$$

Notation:

$\vec{T}(x) = \dfrac{\eta(x)}{|\eta(x)|}$ – orientation

$\|T\| = |\eta(x)| \cdot (\mathcal{H}^k \llcorner E)$ – multiplicity

$\mathbb{I}_k(U) \subseteq \mathcal{R}_k(U)$ – integral currents

$\mathbb{I}_k(U) = \bigcup \{\mathbb{I}_{k,K}(U) : K \subseteq U \text{ compact}\}$

$\mathbb{I}_{k,K}(U) = \{T \in \mathcal{R}_k(U) : \partial T \in \mathcal{R}_{k-1}(U),\ \operatorname{spt} T \subseteq K\}$

Recall. $T \in \mathcal{D}_k(U) \Rightarrow \partial T \in \mathcal{D}_{k-1}(U)$ s.t. $\partial T(\phi) = T(d\phi)$

Given $\Phi : U \times \Lambda_k \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

Parametric integrand

$\Phi(z, r\alpha) = r\, \Phi(z, \alpha)$ whenever $r > 0$

and $T \in \mathcal{R}_k(U)$ we define the action:

$$\langle \Phi, T \rangle = \int \Phi(z, \vec{T}(z)) \, d\|T\|(z)$$

Example: $\Phi(z, \alpha) = |\alpha| = \sqrt{\alpha \bullet \alpha}$ ← the area integrand (positive)

Ellipticity in geometric variational problems

Φ is called positive if $\Phi(z,\alpha) > 0$ whenever $\alpha \neq 0$.

If $a \in U$, then $\Phi_a : \Lambda_k \mathbb{R}^n \longrightarrow \mathbb{R}$, $\Phi_a(\alpha) = \Phi(a,\alpha)$,

Definition: Φ is called elliptic at $a \in U$ if

[F, 5.1.2]

$\exists c(a) > 0 \ \forall P \in G(n,k), \ \forall R \in \mathcal{R}_k(\mathbb{R}^n) \ \forall S \in \mathcal{R}_k(\mathbb{R}^n)$

$\operatorname{spt} S \subseteq P = \{v \in \mathbb{R}^n : v \wedge \gamma = 0\}$, $\vec{S}(z) = \gamma$ for $\|S\|$ almost all z.

$\partial S = \partial R$

$$\Rightarrow \int \Phi_a(\vec{R}(z))\, d\|R\|(z) - \int \Phi_a(\vec{S}(z))\, d\|S\|(z) \geq c(a)\big(\|R\|(U) - \|S\|(U)\big)$$

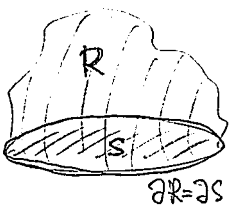
Remark: If $c(a) = 0$, then Φ is called **semielliptic** at a.

Φ is elliptic if $\inf\{c(a) : a \in K\} > 0$ for any $K \subseteq U$ compact.

Observe For R, S, γ as above we have

join

$$\partial(R-S) = 0 \quad \text{and} \quad R - S = \partial\big([\![0]\!] \Join (R-S)\big)$$



Hence, if $\chi \in \operatorname{Hom}(\Lambda_k \mathbb{R}^n, \mathbb{R}) \simeq \Lambda^k \mathbb{R}^n$ and $\phi : \mathbb{R}^n \longrightarrow \Lambda^k \mathbb{R}^n$

$\phi(x) = \chi$ ← constant

$\Rightarrow d\phi = 0$

then

$$\chi\Big(\int \vec{R}\, d\|R\| - \int \vec{S}\, d\|S\|\Big) = \int \chi \circ \vec{R}\, d\|R\| - \int \chi \circ \vec{S}\, d\|S\|$$
$$= (R-S)(\phi) = \big([\![0]\!] \Join (R-S)\big)(d\phi) = 0;$$

thus,

$$(*) \qquad \int \vec{R}\, d\|R\| = \int \vec{S}\, d\|S\| = \|S\|(U) \cdot \gamma$$

Let $F : \Lambda_k \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by
$c > 0$
$$F(\alpha) = \Phi_a(\alpha) - c\,|\alpha| \quad \Rightarrow \quad F(t\alpha) = t\,F(\alpha) \text{ for } t > 0$$

Note: F is convex $\iff F(\alpha + \beta) \leq F(\alpha) + F(\beta)$

Lemma: If F is convex, then Φ is elliptic at a and $c(a) \geq c$.

Proof. $\int \Phi_a \circ \vec{S}\, d\|S\| - c\,\|S\|(U) = F(\|S\|(U)\cdot\gamma)$

$$\overset{(*)}{=} F\Big(\int \vec{R}\, d\|R\|\Big) \underset{\text{(Jensen)}}{\leq} \int (F \circ \vec{R})\, d\|R\| = \int (\Phi_a \circ \vec{R})\, d\|R\| - c\,\|R\|(U)$$

$\square$

Ellipticity in geometric variational problems

<u>Remark</u>: If Φ_a is convex, then Φ is semielliptic at a.

<u>Def</u>. If $G: V \longrightarrow U \in \mathcal{C}^1$, then <u>pull-back</u>

$$G^{\#}\Phi(z,\alpha) = \Phi(G(z), \langle \alpha, \Lambda_k DG(z)\rangle)$$

[F, 5.1.4] <u>Thm</u>. If G is an immersion and Φ is elliptic, then $G^{\#}\Phi$ is elliptic

invariance under immersions

[F, 5.1.5] <u>Thm</u>. If Φ is positive and semielliptic, $K \subseteq U$ is compact, <u>then</u>

$$\mathcal{R}_{k,K}(U) \ni T \longmapsto \langle \Phi, T\rangle \in \mathbb{R}$$

is l.s.c. w.r.t. the flat norm on $\mathcal{R}_{k,K}(U)$ ("weak convergence").

<u>Flat norm</u>

$$\mathbb{F}_K(T) = \sup\left\{T(\phi): \begin{array}{l}\|\phi(x)\| \le 1 \\ \|d\phi(x)\| \le 1 \\ \text{for } x \in K\end{array}\right\}$$

[F., 4.1.12]

[F, 5.1.9] <u>Def</u>. $\tilde{\Phi}: U \times \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^n) \longrightarrow \mathbb{R}$

$$\tilde{\Phi}(z,\sigma) = \Phi\big(z, (\Lambda_k \sigma)(e_1 \wedge \dots \wedge e_k)\big)$$

<u>Note</u>. If $F: \mathbb{R}^k \to \mathbb{R}^n$ is injective of class $\mathcal{C}^1$, $K \subseteq \mathbb{R}^k$ compact, $F[K] \subseteq U$

<u>then</u> $$\langle \Phi, \underbrace{F_{\#}(\mathbb{E}^k \llcorner K)}_{\text{current cannonically associated to image } F \subseteq \mathbb{R}^n}\rangle = \langle F^{\#}\Phi, \mathbb{E}^k \llcorner K\rangle = \int_K \tilde{\Phi}(F(x), DF(x))\, d\mathcal{L}^k(x)$$

<u>Def</u>.

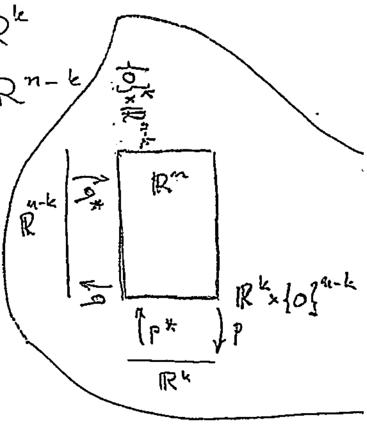
<u>Nonparametric integrand</u>

$$p(z_1 \dots z_m) = (z_1 \dots z_k) \quad : \mathbb{R}^n \longrightarrow \mathbb{R}^k$$

$$q(z_1 \dots z_m) = (z_{k+1} \dots z_m) \quad : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k}$$

$$\Phi^{\S}: U \times \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^{n-k}) \longrightarrow \mathbb{R}$$

$$\Phi^{\S}(z,\tau) = \tilde{\Phi}(z, p^* + q^* \circ \tau)$$



<u>Note</u> If $f: \mathbb{R}^k \longrightarrow \mathbb{R}^{n-k} \in \mathcal{C}^1$, $F = p^* + q^* \circ f$, $F[K] \subseteq U$

then

$$\langle \Phi, \underbrace{F_{\#}(\mathbb{E}^k \llcorner K)}_{\text{graph of } f \subseteq \mathbb{R}^n}\rangle = \int_K \Phi^{\S}(\underbrace{F(x)}_{\substack{\| \\ (x, f(x)) \in \mathbb{R}^n}}, Df(x))\, d\mathcal{L}^k(x)$$

Remark: If Φ is elliptic and of class $\mathcal{C}^2$, then for $w \in \mathcal{U}$, $\lambda \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^{n-k})$

[F, 5.1.10, 5.2.3, 5.2.17]

$$D^2 \Phi^{\S}_w(\lambda) : \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^{n-k}) \times \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^{n-k}) \longrightarrow \mathbb{R}$$

(2-linear symmetric form)

is strongly elliptic in the sense that it satisfies the nonparametric Legendre condition, i.e.,

$$\exists c > 0 \ \forall \xi \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}) \ \forall y \in \mathbb{R}^{n-k}$$

$\boxed{\xi y \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^{n-k}) \text{ of rank one}}$

$$D^2 \Phi^{\S}_w(\lambda)(\xi y, \xi y) \geq c\, |\xi|^2 \cdot |y|^2$$

Remark [F. 5.1.9] If $\Theta : \mathbb{R}^k \longrightarrow \mathbb{R}^{n-k}$ is of class $\mathcal{C}^1$,

$F = p^* + q^* \circ f : \mathbb{R}^k \longrightarrow \mathbb{R}^n \quad \Rightarrow F(x) = (x, f(x))$

$K \subseteq \mathbb{R}^k$ compact, $F[K] \subseteq \mathcal{U}$, $\operatorname{spt} \Theta \subseteq K$,

$h_t(z) = z + t(q^* \circ \Theta \circ p)(z)$ for $t \in (-\varepsilon, \varepsilon)$, $z \in \mathbb{R}^n$

then we have

$$\delta[F_\#(\mathbf{E}^k \llcorner K), \Phi, h] := \frac{d}{dt}\Big|_{t=0} \left\langle \Phi, h_{t\#} F_\#(\mathbf{E}^k \llcorner K) \right\rangle$$

$$= \int_K D\Phi^{\S}(F(x), Df(x))\,(q^* \circ \Theta(x), D\Theta(x))\, d\mathcal{L}^k(x)$$

Integrating by parts yields system of E-L equations in which second order derivatives of f are coupled with coefficients of $D^2 \Phi^{\S}_{F(x)}$.

$$\forall \nu \in \{1, \dots, n-k\}. \quad \sum_{i=1}^k \sum_{j=1}^k \sum_{\mu=1}^{n-k} D^2 \Phi^{\S}_{F(x)}(Df(x))(X_i v_\nu, X_j v_\mu) \cdot D^2 f_\mu(x)(e_i, e_j)$$

where $(e_1 \dots e_k)$ and $(X_1 \dots X_k)$ are dual o.n.b of $\mathbb{R}^k$ and $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$

and $(v_1 \dots v_{n-k})$ is an o.n.b. of $\mathbb{R}^{n-k}$

and $f_\mu(x) = f(x) \bullet v_\mu$.

Bottom line (Morel histori)

- Convex functions provide examples of elliptic integrands.
- Ellipticity of Φ implies ellipticity of E-L equations satisfied by Φ-stationary points which are locally graphs.

Ellipticity in geometric variational problems

$U \subseteq \mathbb{R}^n$ open
$0 < k \le n \in \mathbb{N}$

Recall. $\mathbb{V}_k(U)$ – space of Radon measures over $U \times G(n,k)$
$= k$-dim varifolds in U
$\simeq \underbrace{\mathcal{K}(U \times G(n,k))}^*$ see [F, 2.5.19]
↑ continuous compactly supported functions endowed with locally convex topology

Remark. There is no useful notion of a boundary for $V \in \mathbb{V}_k(U)$.

Given $\Phi : U \times G(n,k) \longrightarrow [0,\infty]$ we define its action on $V \in \mathbb{V}_k(U)$ by

$$\langle \Phi, V \rangle = \int \Phi(x,S)\, dV(x,S)$$

Def. $\Sigma \subseteq \mathbb{R}^n$ is called a k-set if $\mathcal{H}^k(\Sigma \cap K) < \infty$ for all $K \subseteq \mathbb{R}^n$ compact

Def. Σ is called a rectifiable k-set if Σ is $\mathcal{H}^k$-measurable and countably $(\mathcal{H}^k, k)$ rectifiable k-set.

Def If $\Sigma \subseteq U$ is a rect. k-set, then

$$\mathbb{V}_k(\Sigma)(\alpha) = \int_\Sigma \alpha(x, \mathrm{Tan}^k(\mathcal{H}^k \llcorner \Sigma, x))\, d\mathcal{H}^k(x) \quad \text{for } \alpha \in \mathcal{K}(U \times G(n,k))$$

and $\mathbb{V}_k(\Sigma) \in \mathbb{V}_k(U)$.

Notation: $\langle \Phi, \mathbb{V}_k(\Sigma) \rangle = \langle \Phi, \Sigma \rangle$

Def. If $a \in U$, then $\Phi^a(x,T) = \Phi(a,T) \quad \forall x \in U \ \forall T \in G(n,k)$

Def. • Φ is called Almgren uniformly elliptic at a if (AUE)
$\exists\, c(a) > 0 \ \forall \Sigma . \ \forall T \in G(n,k)$

- $D = T \cap \mathbb{B}(0,1)$, • $B = T \cap \partial \mathbb{B}(0,1)$
- Σ is a compact rect. k-set which cannot be deformed onto B by any map $f \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^n)$ such that $f(x) = x$ for $x \in B$, and $\mathcal{H}^k(\Sigma) > \mathcal{H}^k(D)$.

$$\langle \Phi^a, \Sigma \rangle - \langle \Phi^a, D \rangle > c(a)\left(\mathcal{H}^k(\Sigma) - \mathcal{H}^k(D)\right)$$

• Φ is called Almgren elliptic at a if $c(a) = 0$. (AE)

Def. $F \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R}^n)$ pull-back

$$\rightsquigarrow \quad F^*\Phi(x,T) = \Phi(F(x), DF(x)[T])\, \| \Lambda_k DF(x)|_T \|$$

Remark. (AUE) and (AE) are invariant under diffeomorphisms.

Remark. If Φ is (AUE), then one can find a minimizer of $\langle \Phi, \cdot \rangle$ in any class of (closed k-sets) competitors that is closed under taking Lip. deformations and local Hausdorff limits.
[Fang – K., 2018, Calc. Var. PDE]

Ellipticity in geometric variational problems

The problem: Given Φ how can we check whether it is (AUE) or (AE)?

Assume $\Psi : \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$ is convex and positive homogeneous

does it follow that $\Phi(x,T) = \Psi(T_\natural)$ is (AE)?

[[F, 3.2.29] : $G_0(n,k) = \{\xi \in \Lambda_k \mathbb{R}^n : \xi \text{ is simple and } |\xi| = 1\} \subseteq \Lambda_k \mathbb{R}^n$

$G(n,k) = T[G_0(n,k)] \subseteq \odot_2 \Lambda_k \mathbb{R}^n$

where $T(\xi) = \frac{1}{2}\xi^2$, $T : \Lambda_k \mathbb{R}^n \to \odot_2 \Lambda_k \mathbb{R}^n$.

The same question if $\Psi : \odot_2 \Lambda_k \mathbb{R}^n \to \mathbb{R}$ is convex.]

[Recall the computation from page 2]

Recall: If $F \in C^1(\mathbb{R}^n, \mathbb{R}^N)$, then

$$F_\# V(\alpha) = \int \alpha(F(x), DF(x)[T]) \, \| \Lambda_k DF(x) \circ T_\natural \| \, dV(x,T) \quad \text{for } \alpha \in \mathcal{K}(\mathbb{R}^N \times G(n,k))$$

Def If $g : U \to \mathbb{R}^n$ is smooth compactly supported vectorfield in U

$\varphi_t(x) = x + t g(x)$ for $t \in (-\varepsilon, \varepsilon)$, $V \in \mathbf{V}_k(U)$

then
$$\delta_\Phi V(g) = \frac{d}{dt}\Big|_{t=0} \langle \Phi, \varphi_{t\#} V \rangle$$

[DPDRG, 2017, CPAM]
$$= \int \langle g(x), D\Phi_T(x) \rangle + B_\Phi(x,T) \bullet Dg(x) \, dV(x,T)$$

where $\Phi_T : \mathbb{R}^n \to \mathbb{R}$, $\Phi_x : G(n,k) \to \mathbb{R}$, $\Phi_T(x) = \Phi_x(T) = \Phi(x,T)$

and $B_\Phi(x,T) \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is char. by

$$B_\Phi(x,T) \bullet L = \Phi(x,T) \cdot T_\natural \bullet L + \langle T^\perp \circ L \circ T + (T^\perp \circ L \circ T)^*, D\Phi_x(T) \rangle$$
$$\forall L \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$$

Def. Φ satisfies (AC) at $a \in U$ if

$\forall \mu \in$ Prob. measure over $G(n,k)$

$$A_a(\mu) = \int B_\Phi(a,T) \, d\mu(T) \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$$

a) $\dim \ker A_x(\mu) \leq n-k$

b) if $\dim \ker A_x(\mu) = n-k$, then $\mu = \mathrm{Dirac}(T_0)$ for some $T_0 \in G(n,k)$

Def. Φ satisfies (BC) at $a \in \mathcal{U}$ if
$\forall W \in \mathbb{V}_k(\mathbb{R}^n)$. $\forall \mu \in$ Prob. meas. over $G(n,k)$ $\forall T \in G(n,d)$
$W = (\mathcal{H}^k \llcorner T) \times \mu$
a) if $\delta_{\Phi^a} W = 0$, then $d \geq k$
b) if $k = d$ and $\delta_{\Phi^a} W = 0$, then $\mu = \text{Dirac}(T)$.

Lemma (De Rosa - K. - forthcomming)
Φ satisfies (AC) $\Longleftrightarrow$ Φ satisfies (BC).

Lemma (De Rosa - K.)
If Φ sat. (BC), then there exists a rectifiable minimizer of $\langle \Phi, \cdot \rangle$ in any class of k-sets closed under Lip. deform. and local Hausdorff convergence.

Theorem [DPDRG, 2017, CPAM]
If Φ satisfies (AC), $V \in \mathbb{V}_k(\mathcal{U})$, $\Theta^k_*(\|V\|, x) > 0$ for $\|V\|$ a.e. x,
at every $a \in \mathcal{U}$
and $\|\delta_\Phi V\|$ is a Radon measure
then V is a rectifiable varifold, i.e.,
$$V = \sum_{i=1}^{\infty} c_i \, \mathbf{v}_k(\Sigma_i) \quad \text{for some rectifiable } k\text{-sets } \Sigma_i \text{ and } c_i \in (0,\infty).$$

Moreover If Φ does not satisfy (AC) at $a \in \mathcal{U}$, then there exists $V \in \mathbb{V}_k(\mathbb{R}^n)$ such that $\|\delta_{\Phi^a} V\|$ is Radon but $V \llcorner \{(x,T) : \Theta^k_*(\|V\|,x) > 0\}$ is not rectifiable.

Theorem [DPDRG]
If $k = n-1$, then $\Phi : \mathcal{U} \times G(n,n-1) \to (0,\infty)$ satisfies (AC) at $a \in \mathcal{U}$
if and only if the function $G(\lambda \nu) = |\lambda| \cdot \Phi(a, \nu^\perp)$ for $\lambda \in \mathbb{R}, \nu \in \mathbb{S}^{n-1}$
$G : \mathbb{R}^n \to (0,\infty)$
is strictly convex.

Conjecture [De Rosa - K.]

If Φ satisfies (BC) at a, then Φ satisfies (AE) at a.

Idea of the proof:

Wlog $\Phi = \Phi^a$, i.e, $\Phi(x,T) = \Phi(a,T) \quad \forall x \in U \ \forall T \in G(n,k)$

Assume Φ satisfies (BC) but not (AE).

Choosing the basis of $\mathbb{R}^n$ appropriately we obtain

$$(*) \begin{cases} \exists\, S \subseteq \mathbb{R}^n \text{ a compact rect. } k\text{-set}, \ \mathcal{H}^k(S) > \mathcal{H}^k([0,1]^k) \\ \Sigma \text{ cannot be deformed onto } \partial[0,1]^n \cap (\mathbb{R}^k \times \{0\}^{n-k}) = B \end{cases}$$

and, setting $Q = [0,1]^n \cap (\mathbb{R}^k \times \{0\}^{n-k}) = [0,1]^k$,

$$\langle \Phi, S \rangle - \langle \Phi, Q \rangle \leq 0 \, ;$$

Since Φ satisfies (BC) we can find a minimizer of $\langle \Phi, \cdot \rangle$ in the class of all sets satisfying $(*)$ - denote it R.

If $\langle \Phi, R \rangle < \langle \Phi, S \rangle$, then we set $\Sigma = R$ $\boxed{\text{Note } \mathcal{H}^k(R) > \mathcal{H}^k(Q)}$

Otherwise $\langle \Phi, R \rangle = \langle \Phi, S \rangle = \langle \Phi, Q \rangle$ and we set $\Sigma = S$.

In any case, setting $V = \mathbf{v}_k(\Sigma)$, we have

$$\mathcal{H}^k(\Sigma) > \mathcal{H}^k(Q) \qquad \text{and} \qquad \delta_\Phi V(g) = 0$$
$$\langle \Phi, \Sigma \rangle \leq \langle \Phi, Q \rangle \qquad \text{for } g \in \mathcal{C}^1_c(\mathbb{R}^n, \mathbb{R}^n), \ \operatorname{spt} g \subseteq \mathbb{R}^n \sim B$$

BLOW - DOWN

We define

$$X_N = \sum_{\substack{z \in \mathbb{Z} \\ (\mu_{2^{-N}} \circ \tau_z)[Q] \subseteq Q}} (\mu_{2^{-N}} \circ \tau_z)_\# V \quad \text{and} \quad X = \lim_{N \to \infty} X_N \leftarrow \text{possibly a subsequence}$$

and $W = \sum_{z \in \mathbb{Z}} \tau_{z\,\#} X$

$\boxed{\text{Observe } \langle \Phi, X_N \rangle = \langle \Phi, \Sigma \rangle = \langle \Phi, V \rangle}$

Observe $\operatorname{spt} W = \mathbb{R}^k \times \{0\}^{n-k} =: T$, W is Radon, and $\tau_{x\,\#} W = W$ for $x \in \mathbb{R}^k \times \{0\}^{n-k}$ by construction.

Hence, $W = \Theta(\mathcal{H}^k \llcorner T) \times \mu$ where μ is some prob. meas. over $G(n,k)$ and $\Theta \in (0,\infty)$ is constant.

Actually, $\Theta = \dfrac{\mathcal{H}^k(\Sigma)}{\mathcal{H}^k(Q)} > 1$.

If we knew that $\delta_\Phi W = 0$, then from (BC) it would follow that $\mu = \mathrm{Dirac}(T)$ and

$$\langle \Phi, Q \rangle < \Theta \langle \Phi, Q \rangle = \langle \Phi, X \rangle$$
$$= \lim_{N\to\infty} \langle \Phi, X_N \rangle = \lim_{N\to\infty} \langle \Phi, \Sigma \rangle \le \langle \Phi, Q \rangle \quad ↯.$$

However, we do not know whether $\delta_\Phi W = 0$ (!) because $\delta_\Phi V(g) \neq 0$ for $g \in \mathcal{C}^1_c(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{spt} g \cap B \neq \emptyset$.

Moreover, if $A \in O(n)$ is a rotation or reflection then $\delta_\Phi (A_\# V)(g) \neq 0$ even when $\operatorname{spt} g \subseteq \mathbb{R}^n \setminus B$. due to anisotropy of Φ.