1 Basis for a tensor product

1.1 Definition. Given vectorspaces A_i for $i \in I$ we define the *product* of A_i to be a vectorspace X together with linear maps $p_i \in \operatorname{Hom}(X, A_i)$ for $i \in I$ satisfying the following universal property: whenever Z is a vectorspace and $f_i \in \operatorname{Hom}(Z, A_i)$ for $i \in I$, then there exists a *unique* linear map $g \in \operatorname{Hom}(Z, X)$ such that $f_i = p_i \circ g$ for each $i \in I$.

We write $X = \prod_{i \in I} A_i$. If $I = \{1, 2, \dots, k\}$, we write $A_1 \times \dots \times A_k$.

1.2 Definition. Given vectorspaces A_i for $i \in I$ we define the *direct sum* of A_i to be a vectorspace X together with linear maps $j_i \in \operatorname{Hom}(A_i, X)$ for $i \in I$ satisfying the following universal property: whenever Z is a vectorspace and $f_i \in \operatorname{Hom}(A_i, Z)$ for $i \in I$, then there exists a *unique* linear map $g \in \operatorname{Hom}(X, Z)$ such that $f_i = g \circ j_i$ for each $i \in I$.

We write $X = \bigoplus_{i \in I} A_i$.

1.3 Definition. Given vectorspaces $A_1, \dots, A_k$ we define the *tensor product* of A_i to be a vectorspace X together with a k-linear map $\mu : A_1 \times \dots \times A_k \to X$ satisfying the following universal property: whenever Z is a vectorspace and $f : A_1 \times \dots \times A_k \to X$ is k-linear, then there exists a *unique* linear map $g \in \operatorname{Hom}(X, Z)$ such that $f = g \circ \mu$.

We write $X = A_1 \otimes \dots \otimes A_k$.

1.4 Remark. If V is a vectorspace and $\{v_i : i \in I\}$ is its basis, then

$$V = \bigoplus_{i \in I} \operatorname{span}\{v_i\} = \bigoplus_{i \in I} \mathbf{R} \,.$$

1.5 Lemma. *Let A_i for $i \in I$ and B be vectorspace. Then*

$$\Big(\bigoplus_{i \in I} A_i \Big) \otimes B = \bigoplus_{i \in I} \big(A_i \otimes B \big) \,.$$

Proof. Let $j_i \in \operatorname{Hom}(A_i, \bigoplus_{i \in I} A_i)$ be the maps comming from the definition of $\bigoplus_{i \in I} A_i$. Let $a_i \in \operatorname{Hom}(A_i \otimes B, \bigoplus_{i \in I} A_i \otimes B)$ be the maps comming from the definition of $\bigoplus_{i \in I} A_i \otimes B$. Let $\mu_i : A_i \times B \to A_i \otimes B$ be the 2-linear map from the definition of $A_i \otimes B$ and let $\mu : (\bigoplus_{i \in I} A_i) \times B \to (\bigoplus_{i \in I} A_i) \otimes B$ be the 2-linear map from the definition of $(\bigoplus_{i \in I} A_i) \otimes B$.

We shall check that $(\bigoplus_{i \in I} A_i) \otimes B$ together with maps $j_i \otimes \operatorname{id}_B$ satisfy the definition of $\bigoplus_{i \in I} (A_i \otimes B)$.

Assume we are given a vectorspace Z and maps $k_i \in \operatorname{Hom}(A_i \otimes B, Z)$. To make use of the definition of $(\bigoplus_{i \in I} A_i) \otimes B$ we need to construct a 2-linear map $k : (\bigoplus_{i \in I} A_i) \times B \to Z$. Consider the 2-linear maps $k_i \circ \mu_i : A_i \times B \to Z$. These give rise to maps $m_i : A_i \to \operatorname{Hom}(B, Z)$ such that $m_i(x)(y) = k_i \circ \mu_i(x, y)$. From the definition of $\bigoplus_{i \in I} A_i$ we obtain a unique map $m : \bigoplus_{i \in I} A_i \to \operatorname{Hom}(B, Z)$ such that $m_i = m \circ j_i$. The map m gives rise to the 2-linear map $k : (\bigoplus_{i \in I} A_i) \times B \to Z$ satisfying $k(x, y) = m(x)(y)$. From the definition of $(\bigoplus_{i \in I} A_i) \otimes B$ we get a unique map $l \in \operatorname{Hom}((\bigoplus_{i \in I} A_i) \otimes B, Z)$ such that $k = l \circ \mu$. We have

$$\begin{aligned} k_i \circ \mu_i(x, y) &= m_i(x)(y) = m(j_i(x))(y) \\ &= k(j_i(x), y) = l \circ \mu \circ (j_i \times \operatorname{id}_B)(x, y) = l \circ (j_i \otimes \operatorname{id}_B) \circ \mu_i(x, y) \,; \end{aligned}$$

hence

$$k_i \circ \mu_i = l \circ (j_i \otimes \operatorname{id}_B) \circ \mu_i \,.$$

It follows now from the definition of $A_i \otimes B$ that $k_i = l \circ (j_i \otimes \operatorname{id}_B)$. □

1.6 Corollary. *Let A and B be vectorspaces with bases $\{a_i : i \in I\}$ and $\{b_j : j \in J\}$ respectively. Then*

$$A \otimes B = \operatorname{span}\{a_i \otimes b_j : i \in I\,,\ j \in J\}\,.$$

Proof. We have

$$\begin{aligned} A \otimes B &= \Big(\bigoplus_{i \in I} \operatorname{span}\{a_i\}\Big) \otimes \Big(\bigoplus_{j \in J} \operatorname{span}\{b_j\}\Big) \\ &= \bigoplus_{i \in I, j \in J} \big(\operatorname{span}\{a_i\} \otimes \operatorname{span}\{b_j\}\big) = \operatorname{span}\{a_i \otimes b_j : i \in I\,,\ j \in J\}\,. \end{aligned}$$ □

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