Some notation

[id & cf] The *identity map* on X and the *characteristic function* of some $E \subseteq X$ shall be denoted by

$$\mathrm{id}_X \quad \text{and} \quad \mathbb{1}_E\,.$$

[Df & grad f] Let X, Y be Banach spaces and $U \subseteq X$ be open. For the space of k *times continuously differentiable functions* $f : U \to Y$ we write $\mathscr{C}^k(U,Y)$. The *differential* of f at $x \in U$ is denoted

$$\mathrm{D}f(x) \in \mathrm{Hom}(X,Y)\,.$$

In case $Y = \mathbf{R}$ and X is a Hilbert space, we also define the *gradient* of f at $x \in U$ by

$$\operatorname{grad} f(x) = \mathrm{D}f(x)^* 1 \in X\,.$$

[Fed69, 2.10.9] Let $f : X \to Y$. For $y \in Y$ we define the *multiplicity*

$$N(f,y) = \mathrm{cardinality}(f^{-1}\{y\})\,.$$

[Fed69, 4.2.8] Whenever X is a vectorspace and $r \in \mathbf{R}$ we define the *homothety*

$$\boldsymbol{\mu}_r(x) = rx \quad \text{for } x \in X\,.$$

[Fed69, 2.7.16] Whenever X is a vectorspace and $a \in X$ we define the *translation*

$$\boldsymbol{\tau}_a(x) = x + a \quad \text{for } x \in X\,.$$

[Fed69, 2.5.13,14] Let X be a locally compact Hausdorff space. The space of all *continuous real valued functions on X with compact support* equipped with the supremum norm is denoted

$$\mathscr{K}(X)\,.$$

[Fed69, 4.1.1] Let X, Y be Banach spaces, $\dim X < \infty$, and $U \subseteq X$ be open. The space of all *smooth (infinitely differentiable) functions* $f : U \to Y$ is denoted

$$\mathscr{E}(U,Y)\,.$$

The space of all smooth functions $f : U \to Y$ with *compact support* is denoted

$$\mathscr{D}(U,Y)\,.$$

It is endowed with a *locally convex topology* as described in [Men16, Definition 2.13].

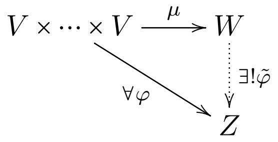
(Multi)linear algebra Let V, Z be vectorspaces.

[Fed69, 1.4.1] The vectorspace of all *k-linear anti-symmetric maps* $\varphi : V \times \cdots \times V \to Z$ shall be denoted by

$$\textstyle\bigwedge^k(V,Z)\,.$$

In case $Z = \mathbf{R}$, we write $\bigwedge^k V = \bigwedge^k(V,\mathbf{R})$.

[Fed69, 1.3.1] A vectorspace W together with $\mu \in \bigwedge^k(V, W)$ is the k^{th} *exterior power* of V if for any vectorspace Z and $\varphi \in \bigwedge^k(V, Z)$ there exists a unique linear map $\tilde{\varphi} \in \operatorname{Hom}(W, Z)$ such that $\varphi = \tilde{\varphi} \circ \mu$.



We shall write

$$W = \textstyle\bigwedge_k V \quad \text{and} \quad \mu(v_1, \dots, v_k) = v_1 \wedge \dots \wedge v_k\,.$$

We shall frequently identify $\varphi \in \bigwedge^k(V, Z)$ with $\tilde{\varphi} \in \operatorname{Hom}(\bigwedge_k V, Z)$.

[Fed69, 1.3.2] If $V = \operatorname{span}\{v_1, \dots, v_m\}$, then

$$\textstyle\bigwedge_k V = \operatorname{span}\big\{v_{\lambda(1)} \wedge \dots \wedge v_{\lambda(k)} : \lambda \in \Lambda(m,k)\big\} = \operatorname{span}\big\{v_\lambda : \lambda \in \Lambda(m,k)\big\}\,,$$

where $\Lambda(m, k) = \{\lambda : \{1, 2, \dots, k\} \to \{1, 2, \dots, m\} : \lambda \text{ is increasing}\}$.

[Fed69, 1.3.1] If $f \in \operatorname{Hom}(V, Z)$, then $\bigwedge_k f \in \operatorname{Hom}(\bigwedge_k V, \bigwedge_k Z)$ is characterised by

$$\textstyle\bigwedge_k f(v_1 \wedge \dots \wedge v_k) = f(v_1) \wedge \dots \wedge f(v_k) \quad \text{for } v_1, \dots, v_k \in V\,.$$

[Fed69, 1.3.4] If $f \in \operatorname{Hom}(V, V)$ and $\dim V = k < \infty$, then $\bigwedge_k V \simeq \mathbf{R}$. We define the *determinant* $\det f \in \mathbf{R}$ of f by requiring

$$\textstyle\bigwedge_k f(v_1 \wedge \dots \wedge v_k) = (\det f) v_1 \wedge \dots \wedge v_k\,,$$

whenever $v_1, \dots, v_k$ is a basis of V.

[Fed69, 1.4.5] If $f \in \operatorname{Hom}(V, V)$ and $\dim V = k < \infty$ and $v_1, \dots, v_k$ is basis of V and $\omega_1, \dots, \omega_k$ is the dual basis of $\bigwedge^1 V = \operatorname{Hom}(V, \mathbf{R})$, then we define the *trace* of f, denoted $\operatorname{tr} f$, by setting

$$\operatorname{tr} f = \sum_{i=1}^{k} \omega_i(f(v_i)) \in \mathbf{R}\,.$$

[Fed69, 1.7.5] If V is equipped with a scalar product (denoted by $\bullet$) and $\{v_1, \dots, v_m\}$ is an orthonormal basis of V, then $\bigwedge_k V$ is also equipped with a scalar product such that $\{v_\lambda : \lambda \in \Lambda(m, k)\}$ is orthonormal. In particular,

$$\operatorname{tr}(\textstyle\bigwedge_k f) = \displaystyle\sum_{\lambda \in \Lambda(m,k)} \textstyle\bigwedge_k f(v_\lambda) \bullet v_\lambda\,.$$

[Fed69, 1.7.2] *Orthogonal injections* are maps $f : X \to Y$ between inner product spaces such that $f(x) \bullet f(y) = x \bullet y$ whenever $x, y \in X$. We set

$$\mathbf{O}(n, m) = \{j \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n) : \forall x, y \in \mathbf{R}^m \;\; j(x) \bullet j(y) = x \bullet y\}\,.$$

[Fed69, 1.7.4] *Orthogonal projections* are maps $f : Y \to X$ between finite dimensional inner product spaces, such that $f^* : \bigwedge^1 X \to \bigwedge^1 Y$ is an orthogonal injection. We set

$$\mathbf{O}^*(n, m) = \{j^* : j \in \mathbf{O}(m, n)\}\,.$$

In case $n = m$ we write

$$\mathbf{O}(n) = \mathbf{O}^*(n, n) = \mathbf{O}(n, n)\,.$$

[Fed69, 1.7.4] If V, Z are equipped with scalar products and $f \in \operatorname{Hom}(V,Z)$, then the *adjoint map* $f^* \in \operatorname{Hom}(Z,V)$ is defined by the identity $f(v) \bullet z = v \bullet f^*(z)$ for $v \in V$ and $z \in Z$. We define the *(Hilbert-Schmidt) scalar product* and *norm* in $\operatorname{Hom}(V,Z)$ by setting for $f, g \in \operatorname{Hom}(V,Z)$

$$f \bullet g = \operatorname{tr}(f^* \circ g) \quad \text{and} \quad |f| = (f \bullet f)^{1/2} .$$

[Fed69, 1.7.6] If $f : X \to Y$ is an orthogonal injection [projection], then so is $\bigwedge_k f : \bigwedge_k X \to \bigwedge_k Y$.

[Fed69, 1.7.6] If V, Z are equipped with norms, then the *operator norm* of $f \in \operatorname{Hom}(V,Z)$ is

$$\|f\| = \sup\{|f(v)| : v \in V ,\, |v| \le 1\} .$$

[Fed69, 1.4.5] If $f \in \operatorname{Hom}(V,V)$ and $\dim V = m$ and $t \in \mathbf{R}$, then

$$\det(\mathrm{id}_V + tf) = \sum_{k=0}^{m} t^m \operatorname{tr}(\textstyle\bigwedge_k f) .$$

[Fed69, 1.6.1] The *Grassmannian* of k dimensional vector subspaces of $\mathbf{R}^n$ is defined to be the set

$$\mathbf{G}(n,k) = \left\{\xi \in \textstyle\bigwedge_k \mathbf{R}^n : \xi \text{ is simple}\right\} / \sim ,$$

where $\xi \sim \eta$ if and only if $\xi = c\eta$ for some $c \in \mathbf{R}$.

- **Exercise.** Let $\Psi : \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n$ be the diagonal map, i.e, $\Psi(x) = (x,x)$ for $x \in \mathbf{R}^n$ and let $p \in \mathbf{O}^*(n,m)$, $q \in \mathbf{O}^*(n,n-k)$ be fixed and such that $q \circ p^* = 0$. For $(g,h) \in \mathbf{O}(k) \times \mathbf{O}(n-k)$ we define $\varphi_{g,h} \in \mathbf{O}(n)$ to be the composition

$$\mathbf{R}^n \xrightarrow{\Psi} \mathbf{R}^n \times \mathbf{R}^n \xrightarrow{p \times q} \mathbf{R}^k \times \mathbf{R}^{n-k} \xrightarrow{g \times h} \mathbf{R}^k \times \mathbf{R}^{n-k} \xrightarrow{\simeq} \mathbf{R}^n .$$

Next, we define the right action of $(g,h) \in \mathbf{O}(k) \times \mathbf{O}(n-k)$ on $f \in \mathbf{O}(n)$ by

$$f \cdot (g,h) = f \circ \varphi_{g,h} .$$

Show that under this action $\mathbf{G}(n,k)$ is homeomorphic with the quotient space, i.e.,

$$\mathbf{G}(n,k) \simeq \mathbf{O}(n) / \mathbf{O}(k) \times \mathbf{O}(n-k) .$$

- **Exercise.** Consider the map

$$\pi : \{\xi \in \textstyle\bigwedge_k \mathbf{R}^n : \xi \text{ is simple}\} \to \mathbf{2}^{\mathbf{R}^n} , \quad \pi(\xi) = \{v \in \mathbf{R}^n : \xi \wedge v = 0\} .$$

Show that there exists a bijection $j : \operatorname{im} \pi \to \mathbf{G}(n,k)$.

Remark. The Hodge star (cf. [Fed69, 1.7.8]) operator $\star : \bigwedge_k \mathbf{R}^n \to \bigwedge_{n-k} \mathbf{R}^n$ gives rise to orthogonal complements under π, i.e.,

$$\pi(\xi)^\perp = \pi(\star \xi) .$$

- **Exercise.** Prove that $\mathbf{G}(n,m)$ is a smooth compact manifold of dimension $m(n-m)$; cf. [Fed69, 3.2.28(2)(4)].
Actually, $\mathbf{G}(n,m)$ can be isometrically embedded into the vectorspace $\bigodot_2 \bigwedge_m \mathbf{R}^n$.

[All72, 2.3] With $S \in \mathbf{G}(n,m)$ we associate the *orthogonal projection* $S_\natural \in \operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n)$ so that

$$S_\natural^* = S_\natural\,, \quad S_\natural \circ S_\natural = S_\natural\,, \quad \operatorname{im}(S_\natural) = S\,.$$

- **Exercise.** If $f \in \operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n)$ and $S \in \mathbf{G}(n,k)$, then

$$\frac{d}{dt}\Big|_{t=0} \left\| \textstyle\bigwedge_k((\mathrm{id}_{\mathbf{R}^m} + tf)\circ S_\natural)\right\|^2 = \frac{d}{dt}\Big|_{t=0} \left| \textstyle\bigwedge_k((\mathrm{id}_{\mathbf{R}^m} + tf)\circ S_\natural)\right|^2 = 2f \bullet S_\natural\,.$$

[All72, 8.9(3)] If $S,T \in \mathbf{G}(n,m)$, then

$$\|S_\natural - T_\natural\| = \|S_\natural^\perp \circ T_\natural\| = \|T_\natural^\perp \circ S_\natural\| = \|S_\natural \circ T_\natural^\perp\| = \|T_\natural \circ S_\natural^\perp\| = \|S_\natural^\perp - T_\natural^\perp\|\,.$$

[All72, 2.3(4)] If $\omega \in \operatorname{Hom}(\mathbf{R}^n, R)$ and $v \in \mathbf{R}^n$, then $\omega \cdot v \in \operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n)$ is given by $(\omega\cdot v)(u) = \omega(u)v$ and for $S \in \mathbf{G}(n,k)$

$$(\omega \cdot v)\bullet S_\natural = \omega(S_\natural(v)) = \langle S_\natural v, \omega\rangle\,.$$

Measures and measurable sets

[Fed69, 2.1.2] We say that ϕ *measures* X, if $\phi : \mathbf{2}^X \to \{t \in \bar{\mathbf{R}} : 0 \le t \le \infty\}$ and

$$\phi(A) \le \sum_{B\in F} \phi(B) \quad \text{whenever } F \subseteq \mathbf{2}^X \text{ is countable and } A \subseteq \bigcup F\,.$$

$A \subseteq X$ is said to be ϕ *measurable* if

$$\forall T \subseteq X \quad \phi(T) = \phi(T \cap A) + \phi(T \sim A)\,.$$

[Fed69, 2.2.3] Let X be a topological space and ϕ measure X. We say that ϕ is *Borel regular* if all open sets in X are ϕ measurable and for each $A \subseteq X$ there exists a Borel set B such that

$$A \subseteq B \quad \text{and} \quad \phi(A) = \phi(B)\,.$$

[Fed69, 2.2.5] Let X be a locally compact Hausdorff topological space and ϕ measure X. We say that ϕ is a *Radon measure* if all open sets are ϕ measurable and

$$\begin{gathered}\phi(K) < \infty \quad \text{for } K \subseteq X \text{ compact}\,,\\ \phi(V) = \sup\{\phi(K) : K \subseteq V \text{ compact}\} \quad \text{for } V \subseteq X \text{ open}\,,\\ \phi(A) = \inf\{\phi(V) : A \subseteq V\,, V \subseteq X \text{ is open}\} \quad \text{for arbitrary } A \subseteq X\,.\end{gathered}$$

[Mat95, 14.15] For $r > 0$ let $L(r)$ be the set of all maps $f : \mathbf{R}^n \to [0,\infty)$ such that $\operatorname{spt}(f) \subseteq \mathbf{B}(0,r)$ and $\operatorname{Lip}(f) \le 1$. The space of all Radon measures over $\mathbf{R}^n$ equipped with the weak topology is a complete separable metric space. The metric is given by

$$d(\phi,\psi) = \sum_{i=1}^{\infty} 2^{-1}\min\{1, F_i(\phi,\psi)\}\,, \quad \text{where} \quad F_r(\phi,\psi) = \sup\left\{\left|\textstyle\int f\,\mathrm{d}\phi - \int f\,\mathrm{d}\psi\right| : f \in L(r)\right\}\,.$$

[All72, 2.6(2)] Let X be locally compact Hausdorff space. If G is a family of opens sets of X such that $\bigcup G = X$ and $B : G \to [0,\infty)$, then the set

$$\{\phi : \phi \text{ is a Radon measure over } X\,,\ \phi(U) \le B(U) \text{ for } U \in G\}$$

is (weakly) compact in the space of all Radon measures over X. If ϕ_i, ϕ are Radon measures and $\lim_{i\to\infty} \phi_i = \phi$, then

$$\phi(U) \le \liminf_{i\to\infty} \phi(U) \quad \text{for } U \subseteq X \text{ open},$$
$$\phi(K) \ge \limsup_{i\to\infty} \phi(K) \quad \text{for } K \subseteq X \text{ compact},$$
$$\phi(A) = \lim_{i\to\infty} \phi_i(A) \quad \text{if } \operatorname{Clos} A \text{ is compact and } \phi(\operatorname{Bdry} A) = 0.$$

[Fed69, 2.10.2] Let $\boldsymbol{\Gamma}$ be the Euler function; see [Fed69, 3.2.13]. Assume X is a metric space. For $m \in [0,\infty)$, $\delta > 0$, and any $A \subseteq X$ we set

$$\zeta^m(A) = \boldsymbol{\alpha}(m) 2^{-m} \operatorname{diam}(A)^m, \quad \text{where} \quad \boldsymbol{\alpha}(m) = \boldsymbol{\Gamma}(1/2)^m / \boldsymbol{\Gamma}((m+2)/2),$$
$$\mathscr{H}^m_\delta(A) = \inf \left\{ \sum_{S\in G} \zeta^m(S) : \begin{array}{l} G \text{ a countable family of subsets of } X \text{ with} \\ A \subseteq \bigcup G \text{ and } \forall S \in G \ \operatorname{diam}(S) \le \delta \end{array} \right\}.$$

The m dimensional *Hausdorff measure* $\mathscr{H}^m(A)$ of $A \subseteq X$ is

$$\mathscr{H}^m(A) = \sup_{\delta > 0} \mathscr{H}^m_\delta(A) = \lim_{\delta \downarrow 0} \mathscr{H}^m_\delta(A).$$

[Fed69, 2.10.33] *Isodiametric inequality*: If $\varnothing \ne S \subseteq \mathbf{R}^m$, then

$$\mathscr{L}^m(S) = \mathscr{H}^m(S) \le \boldsymbol{\alpha}(m) 2^{-m} \operatorname{diam}(S)^m = \zeta^m(S).$$

[Fed69, 4.1.4] *Constancy theorem for distributions*: If $U \subseteq \mathbf{R}^n$ is open, Y is a Banach space, $T \in \mathscr{D}'(U,Y)$, $A \subseteq U$ is connected, and

$$\operatorname{spt} \mathrm{D}_j T \subseteq U \sim A \quad \text{for } j = 1,2,\ldots,n,$$

then there exists a continuous linear function $\alpha : Y \to \mathbf{R}$ such that

$$T(f) = \int_U \alpha \circ f \, \mathrm{d}\mathscr{L}^n \quad \text{whenever } f \in \mathscr{D}(U,Y) \text{ and } \operatorname{spt} f \subseteq A.$$

Approximate limits

[Fed69, 2.9.12] Let $A \subseteq \mathbf{R}^m$, $f : A \to \mathbf{R}^n$, ϕ be a Radon measure over $\mathbf{R}^m$, $x \in \mathbf{R}^m$.

$$\phi \operatorname{ap}\lim_{z\to x} f(z) = y \quad \iff \quad \forall \varepsilon > 0 \quad \lim_{r\downarrow 0} \frac{\phi(\{z \in \mathbf{B}(x,r) : |f(z) - y| > \varepsilon\})}{\phi(\mathbf{B}(x,r))} = 0,$$
$$\phi \operatorname{ap}\limsup_{z\to x} f(z) = \inf \left\{ t \in \mathbf{R} : \lim_{r\downarrow 0} \frac{\phi(\{z \in \mathbf{B}(x,r) : f(z) > t\})}{\phi(\mathbf{B}(x,r))} = 0 \right\},$$
$$\phi \operatorname{ap}\liminf_{z\to x} f(z) = \sup \left\{ t \in \mathbf{R} : \lim_{r\downarrow 0} \frac{\phi(\{z \in \mathbf{B}(x,r) : f(z) < t\})}{\phi(\mathbf{B}(x,r))} = 0 \right\}.$$

Densities

[Fed69, 2.10.19] Let ϕ be a Borel regular measure over a metric space X, $m \in \mathbf{R}$, $m \ge 0$, $a \in X$. We define

$$\boldsymbol{\Theta}^{*m}(\phi,a) = \limsup_{r\downarrow 0} \boldsymbol{\alpha}(m)^{-1} r^{-m} \phi(\mathbf{B}(a,r)), \quad \boldsymbol{\Theta}^m_*(\phi,a) = \liminf_{r\downarrow 0} \boldsymbol{\alpha}(m)^{-1} r^{-m} \phi(\mathbf{B}(a,r)).$$

If $\boldsymbol{\Theta}^m_*(\phi,a) = \boldsymbol{\Theta}^{*m}(\phi,a)$, then we write $\boldsymbol{\Theta}^m(\phi,a)$ for the common value.

[Fed69, 2.10.19(1)] If $A \subseteq X$, $t > 0$, and $\mathbf{\Theta}^{*m}(\phi, x) < t$ for all $x \in A$, then

$$\phi(A) \le 2^m t \mathscr{H}^m(A)\,.$$

[Fed69, 2.10.19(3)] If $A \subseteq X$, $t > 0$, and $\mathbf{\Theta}^{*m}(\phi, x) > t$ for all $x \in A$, then for any open set $V \subseteq X$ such that $A \subseteq V$

$$\phi(V) \ge t \mathscr{H}^m(A)\,.$$

[Fed69, 2.10.19(4)] If $A \subseteq X$, $\phi(A) < \infty$, and A is ϕ measurable, then

$$\mathbf{\Theta}^m(\phi \llcorner A, x) = 0 \quad \text{for } \mathscr{H}^m \text{ almost all } x \in X \sim A\,.$$

[Fed69, 2.10.19(2)(5)] If $A \subseteq X$, then

$$2^{-m} \le \mathbf{\Theta}^{*m}(\mathscr{H}^m \llcorner A, x) \le 1 \quad \text{for } \mathscr{H}^m \text{ almost all } x \in A\,.$$

Tangent and normal vectors Let X be a normed vectorspace, ϕ a measure over X, $a \in X$, m a positive integer, $S \subseteq X$.

[Fed69, 3.1.21] *Tangent cone*:

$$\operatorname{Tan}(S,a) = \{ v \in X : \forall \varepsilon > 0 \ \exists x \in S \ \exists r > 0 \ |x - a| < \varepsilon \text{ and } |r(x-a) - v| < \varepsilon \}\,,$$
$$\operatorname{Tan}(S,a) \cap \{ v : |v| = 1 \} = \bigcap_{\varepsilon > 0} \operatorname{Clos}\{ (x-a)/|x-a| : a \ne x \in S \cap \mathbf{U}(a,\varepsilon) \}\,.$$

If the norm in X comes from a scalar product, define the *normal cone*

$$\operatorname{Nor}(S,a) = \{ v \in X : \forall \tau \in \operatorname{Tan}(S,a) \quad v \bullet \tau \le 0 \}\,.$$

[Fed69, 3.2.16] *Approximate tangent cone*:

$$\operatorname{Tan}^m(\phi, a) = \bigcap \{ \operatorname{Tan}(S,a) : S \subseteq X,\ \mathbf{\Theta}^m(\phi \llcorner X \sim S, a) = 0 \}\,.$$

If the norm in X comes from a scalar product, define the *approximate normal cone*

$$\operatorname{Nor}^m(\phi, a) = \{ v \in X : \forall \tau \in \operatorname{Tan}^m(\phi,a) \quad v \bullet \tau \le 0 \}\,.$$

For $a \in X$, $v \in X$, and $\varepsilon > 0$ define the cone

$$\mathbf{E}(a, v, \varepsilon) = \{ x \in X : \exists r > 0 \quad |r(x-a) - v| < \varepsilon \}\,.$$

If the norm in X comes from a scalar product, $v \in X$, and $0 < \varepsilon < |v|$, then $b \in \mathbf{E}(a,v,\varepsilon)$ if and only if

$$b \ne a \quad \text{and} \quad \frac{b-a}{|b-a|} \bullet \frac{v}{|v|} > \left(1 - \frac{\varepsilon^2}{|v|^2} \right)^{1/2}\,.$$

Observe

$$v \in \operatorname{Tan}^m(\phi, a) \quad \iff \quad \forall \varepsilon > 0 \ \ \mathbf{\Theta}^{*m}(\phi \llcorner \mathbf{E}(a,v,\varepsilon), a) > 0\,.$$

Approximate differentiation Let X, Y be normed vectorspaces, ϕ be a measure over X, $A \subseteq X$, $f : A \to Y$, $a \in X$, m be a positive integer.

[Fed69, 3.2.16] We say that f is (ϕ, m) approximately differentiable at a if there exists an open neighbourhood U of a in X and a function $g : U \to Y$ such that

$$\mathrm{D}g(a) \text{ exists} \quad \text{and} \quad \boldsymbol{\Theta}^m(\phi \llcorner \{x \in A : f(x) \neq g(x)\}, a) = 0\,.$$

We then define

$$(\phi, m)\operatorname{ap}\mathrm{D}f(a) = \mathrm{D}g(a)|_{\operatorname{Tan}^m(\phi,a)} \in \operatorname{Hom}(\operatorname{Tan}^m(\phi, a), Y)\,.$$

Observe that $(\phi, m)\operatorname{ap}\mathrm{D}f(a)$ exists if and only if there exist $y \in Y$ and continuous $L \in \operatorname{Hom}(X, Y)$ such that for each $\varepsilon > 0$

$$\boldsymbol{\Theta}^m(\phi \llcorner X \sim \{x : |f(x) - y - L(x - a)| \le \varepsilon |x - a|\}, a) = 0\,.$$

Jacobians Assume $A \subseteq \mathbf{R}^m$ and $f : A \to \mathbf{R}^n$.

[Fed69, 3.2.1] If $a \in A$ and $\mathrm{D}f(a) \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n)$ exists, then the *k-dimensional Jacobian* $J_k f(a) \in \mathbf{R}$ of f at a is defined by

$$J_k f(a) = \|{\textstyle\bigwedge_k} \mathrm{D}f(a)\|\,.$$

In case $k = \min\{m, n\}$, we have

$$J_k f(a) = |{\textstyle\bigwedge_k} \mathrm{D}f(a)| = \operatorname{tr}\big({\textstyle\bigwedge_k}(\mathrm{D}f(a)^* \circ \mathrm{D}f(a))\big)^{1/2} = \operatorname{tr}\big({\textstyle\bigwedge_k}(\mathrm{D}f(a) \circ \mathrm{D}f(a)^*)\big)^{1/2}\,.$$

In particular, if $k = m \le n$, then

$$J_k f(a) = \det(\mathrm{D}f(a)^* \circ \mathrm{D}f(a))^{1/2}$$

and if $k = n \le m$, then

$$J_k f(a) = \det(\mathrm{D}f(a) \circ \mathrm{D}f(a)^*)^{1/2}\,.$$

If ϕ measures $\mathbf{R}^m$, m is a positive integer, $a \in \mathbf{R}^m$, and $(\phi, m)\operatorname{ap}\mathrm{D}f(a) \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n)$ exists, then the (ϕ, m) *approximate k-dimensional Jacobian* $(\phi, m)\operatorname{ap} J_k f(a) \in \mathbf{R}$ of f at a is defined by

$$(\phi, m)\operatorname{ap} J_k f(a) = \|{\textstyle\bigwedge_k}(\phi, m)\operatorname{ap}\mathrm{D}f(a)\|\,.$$

Lebesgue integral Assume ϕ measures X.

[Fed69, 2.4.1] We say that u is a ϕ *step function* if u is ϕ measurable, $\operatorname{im}(u)$ is a countable subset of $\mathbf{R}$, and

$$\sum_{y \in \operatorname{im}(u)} y\, \phi(u^{-1}\{y\}) \in \bar{\mathbf{R}}\,.$$

[Fed69, 2.4.2] Let $f : X \to \bar{\mathbf{R}}$. Set

$$\int^* f \,\mathrm{d}\phi = \inf\left\{ \sum_{y \in \operatorname{im}(u)} y\, \phi(u^{-1}\{y\}) : \begin{array}{l} u \text{ is a } \phi \text{ step function and} \\ u(x) \ge f(x) \text{ for } \phi \text{ almost all } x \end{array} \right\},$$

$$\int_* f \,\mathrm{d}\phi = \sup\left\{ \sum_{y \in \operatorname{im}(u)} y\, \phi(u^{-1}\{y\}) : \begin{array}{l} u \text{ is a } \phi \text{ step function and} \\ u(x) \le f(x) \text{ for } \phi \text{ almost all } x \end{array} \right\}.$$

We say that f is ϕ *integrable* if $\int_* f \,\mathrm{d}\phi = \int^* f \,\mathrm{d}\phi$ and then we write $\int f \,\mathrm{d}\phi$ for the common value. We say that f is ϕ *summable* if $|\int f \,\mathrm{d}\phi| < \infty$.

[Fed69, 2.9.1] If ϕ, ψ are Radon measures over $\mathbf{R}^n$ and $x \in \mathbf{R}^n$, we define

$$\mathbf{D}(\phi, \psi, x) = \lim_{r \downarrow 0} \phi(\mathbf{B}(x,r))/\psi(\mathbf{B}(x,r)) .$$

[Fed69, 2.9.5] $0 \le \mathbf{D}(\phi, \psi, x) < \infty$ for ψ almost all x.

[Fed69, 2.9.7] If $A \subseteq \mathbf{R}^n$ is ψ measurable, then

$$\int_A \mathbf{D}(\phi, \psi, x) \, \mathrm{d}\psi(x) \le \phi(A) ,$$

with equality if and only if ϕ is absolutely continuous with respect to ψ.

[Fed69, 2.9.19] If $\infty \le a < b \le \infty$ and $f : (a,b) \to \mathbf{R}$ is monotone (or, more generally, a function of bounded variation), then f is differentiable at $\mathscr{L}^1$ almost all $t \in (a,b)$ and

$$\left| \int_a^b f' \, \mathrm{d}\mathscr{L}^1 \right| \le |f(b) - f(a)| .$$

[Fed69, 2.5.12] **Theorem.** Let X be a locally compact separable metric space, E a separable normed vectorspace, $T : \mathscr{K}(X,E) \to \mathbf{R}$ be linear and such that

$$\sup\{T(\omega) : \omega \in \mathscr{K}(X,E) , \ \operatorname{spt}\omega \subseteq K , \ |\omega| \le 1\} < \infty \quad \text{whenever } K \subseteq X \text{ is compact} .$$

Define

$$\phi(U) = \sup\{T(\omega) : \omega \in \mathscr{K}(X,E) , \ |\omega| \le 1 , \ \operatorname{spt}\omega \subseteq U\} \quad \text{whenever } U \subseteq X \text{ is open} ,$$
$$\phi(A) = \inf\{\phi(U) : A \subseteq U , \ U \subseteq X \text{ is open}\} \quad \text{for arbitrary } A \subseteq X .$$

Then ϕ is a Radon measure over X and there exists a ϕ measurable map $k : X \to E^*$ such that $\|k(x)\| = 1$ for ϕ almost all x and

$$T(\omega) = \int \langle \omega(x), k(x) \rangle \, \mathrm{d}\phi(x) \quad \text{for } \omega \in \mathscr{K}(X,E) .$$

See also: [Sim83, 4.1]

Area and co-area formulas. Rectifiability.

[Fed69, 3.2.3] **Theorem.** Suppose $f : \mathbf{R}^m \to \mathbf{R}^n$, and $\operatorname{Lip}(f) < \infty$, and $m \le n$.

(a) If $A \subseteq \mathbf{R}^m$ is $\mathscr{L}^m$ measurable, then

$$\int_A J_m f \, \mathrm{d}\mathscr{L}^m = \int_{\mathbf{R}^n} N(f|_A, y) \, \mathrm{d}\mathscr{H}^m(y) .$$

(b) If $u : \mathbf{R}^m \to \mathbf{R}$ is $\mathscr{L}^m$ integrable, then

$$\int u(x) J_m f(x) \, \mathrm{d}\mathscr{L}^m(x) = \int_{\mathbf{R}^n} \sum_{x \in f^{-1}\{y\}} u(x) \, \mathrm{d}\mathscr{H}^m(y) .$$

[Fed69, 3.2.5] **Theorem.** Suppose $f : \mathbf{R}^m \to \mathbf{R}^n$, and $\operatorname{Lip}(f) < \infty$, and $m \le n$, and $A \subseteq \mathbf{R}^m$ is $\mathscr{L}^m$ measurable, and $g : \mathbf{R}^m \to \bar{\mathbf{R}}$. Then

$$\int_A g(f(x)) J_m f(x) \, \mathrm{d}\mathscr{L}^m(x) = \int_{\mathbf{R}^n} g(y) N(f|_A, y) \, \mathrm{d}\mathscr{H}^m(y)$$

given

(a) either g is $\mathscr{H}^m$ measurable

(b) or $N(f|_A, y) < \infty$ for $\mathscr{H}^m$ almost all $y \in \mathbf{R}^n$

(c) or $\mathbb{1}_A \cdot (g \circ f) \cdot J_m f$ is $\mathscr{L}^m$ measurable.

[Fed69, 3.2.11-12] **Theorem.** Suppose $f : \mathbf{R}^m \to \mathbf{R}^n$, and $\operatorname{Lip}(f) < \infty$, and $m > n$.

(a) If $A \subseteq \mathbf{R}^m$ is $\mathscr{L}^m$ measurable, then

$$\int_A J_n f \, \mathrm{d}\mathscr{L}^m = \int_{\mathbf{R}^n} \mathscr{H}^{m-n}(f^{-1}\{y\}) \, \mathrm{d}\mathscr{L}^n(y) .$$

(b) If $u : \mathbf{R}^m \to \bar{\mathbf{R}}$ is $\mathscr{L}^m$ integrable, then

$$\int u(x) J_n f(x) \, \mathrm{d}\mathscr{L}^m(x) = \int_{\mathbf{R}^n} \int_{f^{-1}\{y\}} u(x) \, \mathrm{d}\mathscr{H}^{m-n}(x) \, \mathrm{d}\mathscr{L}^n(y) .$$

[Fed69, 3.2.14] **Definition.** Let $E \subseteq \mathbf{R}^n$, m be a positive integer, ϕ measures $\mathbf{R}^n$.

(a) E is *m rectifiable* if there exists $\varphi : \mathbf{R}^m \to \mathbf{R}^n$ with $\operatorname{Lip}(\varphi) < \infty$ and such that $E = \varphi[A]$ for some bounded set $A \subseteq \mathbf{R}^m$;

(b) E is *countably m rectifiable* if is a union of countably many m rectifiable sets;

(c) E is *countably (ϕ, m) rectifiable* if there exists a countably m rectifiable set $A \subseteq \mathbf{R}^n$ such that $\phi(E \sim A) = 0$;

(d) E is *(ϕ, m) rectifiable* if E is countably (ϕ, m) rectifiable and $\phi(E) < \infty$.

(e) E is *purely (ϕ, m) unrectifiable* if $\phi(E \cap \operatorname{im} \varphi) = 0$ for all $\varphi : \mathbf{R}^m \to \mathbf{R}^n$ with $\operatorname{Lip}(\varphi) < \infty$.

[Fed69, 3.2.29] **Theorem.** A set $W \in \mathbf{R}^n$ is countably $(\mathscr{H}^m, m)$ rectifiable *if and only if* there exists a countable family F of m dimensional submanifolds of $\mathbf{R}^n$ of class $\mathscr{C}^1$ such that $\mathscr{H}^m(W \sim \bigcup F) = 0$.

[Fed69, 3.2.18] **Lemma.** Assume $W \subseteq \mathbf{R}^n$ is $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable. Then for each $\lambda \in (1, \infty)$, there exist compact subsets $K_1, K_2, \ldots$ of $\mathbf{R}^m$ and maps $\psi_1, \psi_2, \ldots : \mathbf{R}^m \to \mathbf{R}^n$ such that

$$\{\psi_i[K_i] : i = 1, 2, \ldots\} \text{ is disjointed}, \quad \mathscr{H}^m(W \sim \textstyle\bigcup_{i=1}^\infty \psi_i[K_i]) = 0 ,$$

$$\operatorname{Lip}(\psi_i) \le \lambda , \quad \psi_i|_{K_i} \text{ is injective}, \quad \operatorname{Lip}((\psi_i|_{K_i})^{-1}) \le \lambda ,$$

$$\lambda^{-1}|v| \le |\mathrm{D}\psi_i(a) v| \le \lambda |v| \quad \text{for } a \in K_i , \ v \in \mathbf{R}^m .$$

[Fed69, 3.2.19] **Theorem.** Assume $W \subseteq \mathbf{R}^n$ is $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable. Then for $\mathscr{H}^m$ almost all $w \in W$

$$\mathbf{\Theta}^m(\mathscr{H}^m \llcorner W, w) = 1 \quad \text{and} \quad \operatorname{Tan}^m(\mathscr{H}^m \llcorner W, w) \in \mathbf{G}(n, m)\,.$$

Moreover, if $f : W \to \mathbf{R}^\nu$ and $\operatorname{Lip}(f) < \infty$, then

$$(\mathscr{H}^m \llcorner W, m) \operatorname{ap} \mathrm{D} f(w) : \operatorname{Tan}^m(\mathscr{H}^m \llcorner W, w) \to \mathbf{R}^\nu$$

exists for $\mathscr{H}^m$ almost all $w \in W$.

[Fed69, 3.2.20] **Corollary.** Let $W \subseteq \mathbf{R}^n$ be $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable. Assume $m \le \nu$, and $f : W \to \mathbf{R}^\nu$, and $\operatorname{Lip}(f) < \infty$. Then

$$\int_W (g \circ f) J_m f \, \mathrm{d}\mathscr{H}^m = \int_{R^\nu} g(z) N(f, z) \, \mathrm{d}\mathscr{H}^m(z)$$

for any $g : \mathbf{R}^\nu \to \bar{\mathbf{R}}$.

[Mat75, Pre87] **Theorem.** If $W \subseteq \mathbf{R}^n$ and $\mathbf{\Theta}^m(\mathscr{H}^m \llcorner W, w) = 1$ for $\mathscr{H}^m$ almost all $w \in W$, then W is countably $(\mathscr{H}^m, m)$ rectifiable.

[Fed69, 3.2.22] **Theorem.** Let $m \ge \mu$, and $W \subseteq \mathbf{R}^n$ be $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable, and $Z \subseteq \mathbf{R}^\nu$ be $(\mathscr{H}^\mu, \mu)$ rectifiable and $\mathscr{H}^\mu$ measurable, and $f : W \to Z$, and $\operatorname{Lip}(f) < \infty$. For brevity let us write "ap" for "$(\mathscr{H}^m \llcorner W, m)$ ap".

(a) For $\mathscr{H}^m$ almost all $w \in W$, either $\operatorname{ap} J_\mu f(w) = 0$ or

$$\operatorname{im} \operatorname{ap} \mathrm{D} f(w) = \operatorname{Tan}^\mu(\mathscr{H}^\mu \llcorner Z, f(w)) \in \mathbf{G}(\nu, \mu)\,.$$

(b) The levelset $f^{-1}\{z\}$ is $(\mathscr{H}^{m-\mu}, m-\mu)$ rectifiable and $\mathscr{H}^{m-\mu}$ measurable for $\mathscr{H}^\mu$ almost all $z \in Z$.

(c) For any $(\mathscr{H}^m \llcorner W)$ integrable function $g : W \to \bar{\mathbf{R}}$

$$\int_W g \cdot \operatorname{ap} J_\mu f \, \mathrm{d}\mathscr{H}^m = \int_Z \int_{f^{-1}\{z\}} g \, \mathrm{d}\mathscr{H}^{m-\mu} \, \mathrm{d}\mathscr{H}^\mu(z)\,.$$

[Fed69, 3.2.23] **Theorem.** Assume $W \subseteq \mathbf{R}^n$ is m rectifiable and Borel, and $Z \subseteq \mathbf{R}^\nu$ is $(\mathscr{H}^\mu, \mu)$ rectifiable and Borel. Then $W \times Z \subseteq \mathbf{R}^n \times \mathbf{R}^\nu$ is $(\mathscr{H}^{m+\mu}, m + \mu)$ rectifiable and

$$\mathscr{H}^{m+\mu} \llcorner (W \times Z) = (\mathscr{H}^m \llcorner W) \times (\mathscr{H}^\mu \llcorner Z)\,.$$

[Fed69, 3.2.24] **Beware,** there exist sets $W \subseteq \mathbf{R}^n$ and $Z \subseteq \mathbf{R}^\nu$ with $\mathscr{H}^m(W) = 0$ and $\mathscr{H}^\mu(Z) = 0$ but $\mathscr{H}^{m+\mu}(W \times Z) = \infty$. In particular, $\mathscr{H}^{m+\mu} \llcorner (W \times Z) \ne (\mathscr{H}^m \llcorner W) \times (\mathscr{H}^\mu \llcorner Z)$!

BV, Caccioppoli sets, and the Gauss-Green theorem. Let $U \subseteq \mathbf{R}^n$ be open.

[EG92, 5.1] **Definition.** A function $f \in L^1(U)$ has *bounded variation in* U if

$$\|\mathrm{D}f\|(U) = \sup\left\{ \int f \operatorname{div}\varphi \, \mathrm{d}\mathscr{L}^n : \varphi \in \mathscr{C}^1_c(U, \mathbf{R}^n)\,,\ |\varphi| \le 1 \right\} < \infty\,.$$

We define

$$BV(U) = \{ f \in L^1(U) : \|\mathrm{D}f\|(U) < \infty \} \quad \text{and} \quad \|f\|_{BV(U)} = \|f\|_{L^1(U)} + \|\mathrm{D}f\|(U)\,.$$

Definition. $f \in L^1(U)$ has *locally bounded variation in* U if $f \in BV(V)$ for all open sets $V \subseteq U$ such that $\operatorname{Clos} V \subseteq U$ is compact. We write $f \in BV_{\mathrm{loc}}(U)$.

Definition. An $\mathscr{L}^n$ measurable set $E \subseteq \mathbf{R}^n$ has *finite perimeter in* U if $\mathbb{1}_E \in BV(U)$.

Definition. E has *locally finite perimeter in* U if $\mathbb{1}_E \in BV_{\mathrm{loc}}(U)$.

Theorem. $f \in BV(U)$ *if and only if* there exists a Radon measure μ over $\mathbf{R}^n$ and a μ measurable function $\sigma : U \to \mathbf{R}^n$ satisfying $|\sigma(x)| = 1$ for μ almost all x and

$$\int_U f \operatorname{div}\varphi \, \mathrm{d}\mathscr{L}^n = -\int_U \varphi \bullet \sigma \, \mathrm{d}\mu \quad \text{for } \varphi \in \mathscr{C}^1_c(U, \mathbf{R}^n)\,.$$

Notation.

(a) If $f \in BV_{\mathrm{loc}}(U)$, then we write $\|\mathrm{D}f\| = \mu$ and ∇f for the density of the absolutely continuous part of the vector-valued Radon measure $\mu \llcorner \sigma$ with respect to the Lebesgue measure $\mathscr{L}^n$.

(b) If $E \subseteq \mathbf{R}^n$ has locally finite perimeter in U, then we write $\|\partial E\| = \|\mathrm{D}\mathbb{1}_E\|$ and $\nu_E = -\sigma$.

[EG92, 5.1, Ex.1] **Remark.** We have $W^{1,1}_{\mathrm{loc}}(U) \subseteq BV_{\mathrm{loc}}(U)$. Moreover, for $f \in W^{1,1}_{\mathrm{loc}}(U)$ and any $A \subseteq U$

$$\|\mathrm{D}f\|(A) = \int_A |\operatorname{grad} f| \, \mathrm{d}\mathscr{L}^n \quad \text{and} \quad \nabla f = \operatorname{grad} f\,.$$

[EG92, 5.1, Ex.2] **Remark.** If $E \subseteq \mathbf{R}^n$ is open and the topological boundary $\operatorname{Bdry} E$ is a smooth hypersurface in $\mathbf{R}^n$ such that $\mathscr{H}^{n-1}(\operatorname{Bdry} E \cap K) < \infty$ for all compact $K \subseteq U$, then E has locally finite perimeter in U. Moreover, if $\mathscr{H}^{n-1}(\operatorname{Bdry} E) < \infty$, then

$$\|\partial E\| = \mathscr{H}^{n-1} \llcorner \operatorname{Bdry} E \quad \text{and} \quad \nu_E \text{ is the outer unit normal to } \operatorname{Bdry} E\,.$$

[EG92, 5.2.1] **Theorem.** If $f_i \in BV(U)$ and $f_i \to f$ in $L^1_{\mathrm{loc}}(U)$, then

$$\|\mathrm{D}f\|(U) \le \liminf_{i \to \infty} \|\mathrm{D}f_i\|(U)\,.$$

[EG92, 5.2.2] **Theorem.** Assume $f \in BV(U)$. Then there exist functions $f_i \in BV(U) \cap \mathscr{E}(U, \mathbf{R})$ such that

$$f_i \to f \quad \text{in } L^1(U) \quad \text{and} \quad \|\mathrm{D}f_i\|(U) \to \|\mathrm{D}f\|(U) \quad \text{as } i \to \infty$$
$$\text{and} \quad \mathscr{L}^n \llcorner \operatorname{grad} f_i \to \|\mathrm{D}f\| \llcorner \sigma \quad \text{weakly as vector-valued Radon measures}\,.$$

[EG92, 5.2.3] **Theorem.** Assume U is open and bounded in $\mathbf{R}^n$, $\operatorname{Bdry} U$ is a Lipschitz manifold, $f_i \in BV(U)$ satisfies $\sup\{\|f_i\|_{BV(U)} : i = 1, 2, \ldots\} < \infty$. Then there exists a subsequence f_{k_j} and a function $f \in BV(U)$ such that $f_{k_j} \to f$ in $L^1(U)$.

[EG92, 5.5] **Remark.** If $f : U \to \mathbf{R}$ is Lipschitsz, then the co-area formula gives

$$\int |\operatorname{grad} f| \, \mathrm{d}\mathscr{L}^n = \int \mathscr{H}^{n-1}(f^{-1}\{t\}) \, \mathrm{d}\mathscr{L}^1(t) \,.$$

Theorem. Let $f \in L^1(U)$ and define for $t \in \mathbf{R}$

$$E_t = \{x \in U : f(x) > t\} \,.$$

(a) If $f \in BV(U)$, then E_t has finite perimeter in U for $\mathscr{L}^1$ almost all t.

(b) If $f \in BV(U)$, then

$$\|\mathrm{D}f\|(U) = \int \|\partial E_t\|(U) \mathscr{L}^1(t) \,.$$

(c) If $\int \|\partial E_t\|(U) \mathscr{L}^1(t) < \infty$, then $f \in BV(U)$.

[EG92, 5.6.2] **Theorem.** Let E be bounded and of finite perimeter in $\mathbf{R}^n$. There exists $C = C(n) > 0$ such that

(a) $\mathscr{L}^n(E)^{1-1/n} \le C\|\partial E\|(\mathbf{R}^n)$,

(b) $\min\{\mathscr{L}^n(\mathbf{B}(x,r) \cap E), \mathscr{L}^n(\mathbf{B}(x,r) \sim E)\}^{1-1/n} \le C\|\partial E\|(\mathbf{U}(x,r))$ for $x \in \mathbf{R}^n$, $r \in (0,\infty)$.

[EG92, 5.7.1] **Definition.** Assume E has locally finite perimeter in $\mathbf{R}^n$ and $x \in \mathbf{R}^n$. We say that x belongs to the *reduced boundary* $\partial^* E$ of E if

(a) $\|\partial E\|(\mathbf{B}(x,r)) > 0$ for $r > 0$,

(b) $\lim_{r\downarrow 0} \|\partial E\|(\mathbf{B}(x,r))^{-1} \int_{\mathbf{B}(x,r)} \nu_E \, \mathrm{d}\|\partial E\| = \nu_E(x)$,

(c) $|\nu_E(x)| = 1$.

[EG92, 5.7.3] **Theorem.** Assume E has locally finite perimeter in $\mathbf{R}^n$.

(a) $\partial^* E$ is countably $(\mathscr{H}^{n-1}, n-1)$ rectifiable.

(b) $\mathscr{H}^{n-1}(\partial^* E \cap K) < \infty$ for any compact set $K \subseteq \mathbf{R}^n$.

(c) $\nu_E(x) \in \operatorname{Nor}^{n-1}(\mathscr{H}^{n-1} \llcorner \partial^* E, x)$ for $\mathscr{H}^{n-1}$ almost all $x \in \partial^* E$.

(d) $\|\partial E\| = \mathscr{H}^{n-1} \llcorner \partial^* E$.

[EG92, 5.8] **Definition.** Assume E has locally finite perimeter in $\mathbf{R}^n$ and $x \in \mathbf{R}^n$. We say that x belongs to the *measure theoretic boundary* $\partial_* E$ of E if

$$\boldsymbol{\Theta}^{*n}(\mathscr{L}^n \llcorner E, x) > 0 \quad \text{and} \quad \boldsymbol{\Theta}^{*n}(\mathscr{L}^n \llcorner (\mathbf{R}^n \sim E), x) > 0 \,.$$

Lemma. $\partial^* E \subseteq \partial_* E$ and $\mathscr{H}^{n-1}(\partial_* E \sim \partial^* E) = 0$.

Theorem. Assume E has locally finite perimeter in $\mathbf{R}^n$. Then

$$\int_E \operatorname{div} \varphi \, \mathrm{d}\mathscr{L}^n = \int_{\partial_* E} \varphi \bullet \nu_E \, \mathrm{d}\mathscr{H}^{n-1} \quad \text{for } \varphi \in \mathscr{C}^1_c(\mathbf{R}^n, \mathbf{R}^n) \,.$$

[EG92, 5.11] **Theorem.** Let $E \subseteq \mathbf{R}^n$ be $\mathscr{L}^n$ measurable. Then E has locally finite perimeter in $\mathbf{R}^n$ *if and only if* $\mathscr{H}^{n-1}(\partial_* E \cap K) < \infty$ for all compact sets $K \subseteq \mathbf{R}^n$.

[EG92, 6.1.1] **Theorem.** Assume $f \in BV_{\mathrm{loc}}(\mathbf{R}^n)$. Then for $\mathscr{L}^n$ almost all $x \in \mathbf{R}^n$

$$\lim_{r\downarrow 0} \frac{1}{r} \left(\boldsymbol{\alpha}(n)^{-1} r^{-n} \int_{\mathbf{B}(x,r)} |f(y) - f(x) - \nabla f(x) \bullet (x-y)|^{n/(n-1)} \, \mathrm{d}\mathscr{L}^n(y) \right)^{1-1/n} = 0 \,.$$

[EG92, 6.1.3] **Theorem.** Assume $f \in BV_{\mathrm{loc}}(\mathbf{R}^n)$. Then f is $(\mathscr{L}^n, n)$ approximately differentiable $\mathscr{L}^n$ almost everywhere. Moreover,

$$(\mathscr{L}^n, n) \operatorname{ap} \mathrm{D} f(x) u = \nabla f(x) \bullet u \quad \text{for } \mathscr{L}^n \text{ almost all } x \in \mathbf{R}^n \text{ and all } u \in \mathbf{R}^n \,.$$

Varifolds – definitions. Let $U \subseteq \mathbf{R}^n$ be open and $M \subseteq U$ be a smooth m dimensional submanifold (possibly open) such that the inclusion map $i : M \hookrightarrow \mathbf{R}^n$ is proper.

[All72, 2.5] **Definition.**

- *tangent vector fields:* $\mathscr{X}(M) = \{g \in \mathscr{C}_c^\infty(M, \mathbf{R}^n) : \forall x \in M \ \ g(x) \in \operatorname{Tan}(M,x)\}$;
- *normal vector fields:* $\mathscr{X}^\perp(M) = \{g \in \mathscr{C}_c^\infty(M, \mathbf{R}^n) : \forall x \in M \ \ g(x) \in \operatorname{Nor}(M,x)\}$;
- *tangent and normal parts of a vectorfield:* if $g \in \mathscr{C}_c^\infty(M, \mathbf{R}^n)$, then $\operatorname{Tan}(M,g) \in \mathscr{X}(M)$ and $\operatorname{Nor}(M,g) \in \mathscr{X}^\perp(M)$ are such that $g = \operatorname{Tan}(M,g) + \operatorname{Nor}(M,g)$;
- $\mathbf{G}_k(M) = \{(x,S) : x \in M\,,\ S \in \mathbf{G}(n,k)\,,\ S \subseteq \operatorname{Tan}(M,x)\}$;
- *the second fundamental form:* $\mathbf{b}(M,a) : \operatorname{Tan}(M,a) \times \operatorname{Tan}(M,a) \to \operatorname{Nor}(M,a)$ a symmetric bilinear mapping such that
$$\mathrm{D}g(a)w \bullet v = -\mathbf{b}(M,a)(v,w) \bullet g(a) \quad \text{for } v,w \in \operatorname{Tan}(M,a) \text{ and } g \in \mathscr{X}^\perp(M)\,;$$
- *the mean curvature vector:* $\mathbf{h}(M,a) \in \operatorname{Nor}(M,a)$ is characterized by
$$(\mathrm{D}g(a) \circ \operatorname{Tan}(M,a)_\natural) \bullet \operatorname{Tan}(M,a)_\natural = -g(a) \bullet \mathbf{h}(M,a) \quad \text{for } g \in \mathscr{X}^\perp(M)\,;$$
- for $(a,S) \in \mathbf{G}_k(M)$ the vector $\mathbf{h}(M,a,S) \in \operatorname{Nor}(M,a)$ is characterized by
$$(\mathrm{D}g(a) \circ \operatorname{Tan}(M,a)_\natural) \bullet S_\natural = -g(a) \bullet \mathbf{h}(M,a,S) \quad \text{for } g \in \mathscr{X}^\perp(M)\,.$$

[All72, 3.1] **Definition.** A Radon measure V over $\mathbf{G}_k(M)$ is called a *k dimensional varifold in M*. The weakly topologised space of k dimensional varifolds in M is denoted $\mathbf{V}_k(M)$.
For any $V \in \mathbf{V}_k(M)$ we define the *weight measure* $\|V\|$ over M by requiring
$$\|V\|(B) = V(\{(x,S) \in \mathbf{G}_k(M) : x \in B\}) \quad \text{for } B \subseteq M \text{ Borel}\,.$$

[All72, 3.2] **Definition.** If $F : M \to M'$ is a smooth map between smooth manifolds and $V \in \mathbf{V}_k(M)$, then we define $F_\# V \in \mathbf{V}_k(M')$ by
$$F_\# V(\alpha) = \int \alpha(F(x), \mathrm{D}F(x)[S]) \|{\textstyle\bigwedge_k} \mathrm{D}F(x) \circ S_\natural\| \,\mathrm{d}V(x,S) \quad \text{for } \alpha \in \mathscr{K}(\mathbf{G}_k(M'))\,,$$
with the understanding that $\alpha(F(x), \mathrm{D}F(x)[S]) \|\bigwedge_k \mathrm{D}F(x) \circ S_\natural\|$ equals zero whenever $\bigwedge_k \mathrm{D}F(x) \circ S_\natural = 0$.
Remark. Observe
$$\|\boldsymbol{\mu}_{r\#} V\| = r^k \boldsymbol{\mu}_{r\#} \|V\|\,.$$

[All72, 3.3] **Definition.** (Varifold disintegration; cf. [AFP00, §2.5]) For $V \in \mathbf{V}_k(M)$ we define for $x \in M$ and $\beta \in \mathscr{K}(\mathbf{G}(n,k))$
$$V^{(x)}(\beta) = \lim_{r \downarrow 0} \|i_\# V\|(\mathbf{B}(x,r))^{-1} \int_{\mathbf{B}(x,r) \times \mathbf{G}(n,k)} \beta(S) \,\mathrm{d}(i_\# V)(y,S)\,.$$

[All72, 3.4] **Definition.** Let $V \in \mathbf{V}_k(M)$, $a \in M$, and $j : \operatorname{Tan}(M,a) \hookrightarrow \mathbf{R}^n$ be the inclusion map.
$$\operatorname{VarTan}(V,a) = \left\{ C \in \mathbf{V}_k(\operatorname{Tan}(M,a)) : j_\# C = \lim_{j \to \infty} (\boldsymbol{\mu}_{r_j} \circ \boldsymbol{\tau}_{-a} \circ i)_\# V \text{ for some } r_j \uparrow \infty \right\}.$$

[All72, 3.5] **Definition.** If $E \subseteq \mathbf{R}^n$ is countably $(\mathscr{H}^k, k)$ rectifiable and $\mathscr{H}^k(E \cap K) < \infty$ for $K \subseteq U$ compact, then define $\mathbf{v}_k(E) \in \mathbf{V}_k(U)$ by

$$\mathbf{v}_k(E)(\alpha) = \int_E \alpha(x, \operatorname{Tan}^k(\mathscr{H}^k \llcorner E, x)) \, \mathrm{d}\mathscr{H}^k(x) \quad \text{for } \alpha \in \mathscr{K}(\mathbf{G}_k(U)) .$$

Definition. We say that $V \in \mathbf{V}_k(M)$ is a *rectifiable varifold* if there exist countably $(\mathscr{H}^m, m)$ rectifiable sets $E_i \subseteq M$ and constants $c_i \in (0, \infty)$ such that

$$V = \sum_{i=1}^{\infty} c_i \mathbf{v}_k(E_i) .$$

If all c_i can be taken to be integers, then we say that V is an *integral varifold.*
The spaces of all k dimensional rectifiable and integral varifolds in M are denoted by

$$\mathbf{RV}_k(M) \quad \text{and} \quad \mathbf{IV}_k(M) .$$

Theorem. Let $V \in \mathbf{V}_k(M)$. Then $V \in \mathbf{RV}_k(M)$ *if and only if* for $\|V\|$ almost all a

$$\boldsymbol{\Theta}^m(i_\# \|V\|, a) \in (0, \infty) \quad \text{and} \quad V^{(a)}(\beta) = \beta(\operatorname{Tan}^k(i_\# \|V\|, a)) \quad \text{for } \beta \in \mathscr{K}(\mathbf{G}(n,k)) .$$

Moreover, $V \in \mathbf{IV}_k(M)$ *if and only if* $V \in \mathbf{RV}_k(M)$ and $\boldsymbol{\Theta}^m(i_\# \|V\|, a)$ is a non-negative integer for $\|V\|$ almost all a.

[All72, 4.2] **Definition.** Let $V \in \mathbf{V}_k(M)$. Define $\delta V : \mathscr{X}(M) \to R$ the *first variation of* V by

$$\delta V(g) = \int (\mathrm{D}g(x) \circ S_\natural) \bullet S_\natural \, \mathrm{d}V(x, S) \quad \text{for } g \in \mathscr{X}(M) .$$

Definition. The *total variation measure* $\|\delta V\|$ is given by

$$\|\delta V\|(G) = \sup \{ \delta V(g) : g \in \mathscr{X}(M) ,\ \operatorname{spt} g \subseteq G ,\ |g| \le 1 \} \quad \text{for } G \subseteq M \text{ open} ,$$
$$\|\delta V\|(A) = \inf \{ \|\delta V\|(G) : A \subseteq G ,\ G \subseteq M \text{ open} \} \quad \text{for arbitrary } A \subseteq M .$$

Definition. If $\delta V = 0$, we say that V is *stationary.* If $G \subseteq M$ is open and $\|\delta V\|(G) = 0$, we say that V is *stationary in* G.

[All72, 4.3] **Definition.** Assume $\|\delta V\|$ is a Radon measure. Then there exists a $\|\delta V\|$ measurable function $\boldsymbol{\eta}(V, \cdot)$ such that for $\|\delta V\|$ almost all x there holds $\boldsymbol{\eta}(V,x) \in \operatorname{Tan}(M, s)$ and

$$\delta V(g) = \int g(x) \bullet \boldsymbol{\eta}(V,x) \, \mathrm{d}\|\delta V\|(x) \quad \text{for } g \in \mathscr{X}(M) .$$

Setting $\mathbf{h}(V,x) = -\mathbf{D}(\|\delta V\|, \|V\|, x)\boldsymbol{\eta}(V,x)$ we obtain a $\|V\|$ measurable function such that

$$\delta V(g) = -\int g(x) \bullet \mathbf{h}(V,x) \, \mathrm{d}\|V\|(x) + \int g(x) \bullet \boldsymbol{\eta}(V,x) \, \mathrm{d}\|\delta V\|_{\mathrm{sing}}(x) \quad \text{for } g \in \mathscr{X}(M) ,$$

where $\|\delta V\|_{\mathrm{sing}}$ denotes the singular part of $\|\delta V\|$ with respect to $\|V\|$.
We call $\mathbf{h}(V,x)$ the *generalized mean curvature vector of* V *at* x.

Varifolds – examples and basic facts. Let $U \subseteq \mathbf{R}^n$ be open and $M \subseteq U$ be a smooth m dimensional submanifold (possibly open) such that the inclusion map $i : M \hookrightarrow \mathbf{R}^n$ is proper.

[All72, 4.4] **Remark.** If $V \in \mathbf{V}_k(M)$ and $g \in \mathscr{X}(U)$, then

$$\delta(i_\# V)(g) = \delta V(\operatorname{Tan}(M,g)) - \int \operatorname{Nor}(M,g)(x) \bullet \mathbf{h}(M,x,S)\,\mathrm{d}V(x,S)\,.$$

[All72, 4.5] **Lemma.** Let $W \subseteq U$ be open, $Y \subseteq \mathbf{R}^m$ be open, $\varphi : Y \to W$ and $\psi : W \to Y$ be smooth and such that $\psi \circ \varphi = \mathrm{id}_Y$ and $W \cap M = W \cap \operatorname{im}\varphi$, $V \in V_m(M)$. Then

$$\delta V(g) = \delta(\psi_\# V)(\|{\textstyle\bigwedge_m}\mathrm{D}\varphi\| \langle g \circ \varphi, \mathrm{D}\psi \circ \varphi \rangle) \quad \text{for } g \in \mathscr{X}(W \cap M)\,,$$

$$\int_Y \mathrm{D}\beta(y) v \,\mathrm{d}\|\psi_\# V\|(y) = \delta V\big((\|{\textstyle\bigwedge_m}\mathrm{D}\varphi\|^{-1}\beta \cdot \mathrm{D}\varphi(\cdot)v) \circ \psi\big) \quad \text{for } v \in R^m \text{ and } \beta \in \mathscr{D}(Y,\mathbf{R})\,.$$

[All72, 4.6] **Theorem.** Assume M is connected, $\dim M = m$, $V \in \mathbf{V}_m(U)$, $\operatorname{spt}\|V\| \subseteq M$, $\|\delta V\|$ is a Radon measure, and

$$\delta V(g) = 0 \quad \text{for } g \in \mathscr{X}(M) \text{ with } \operatorname{Nor}(M,g) = 0\,.$$

Then there exists a constant $C > 0$ such that

$$V = C\mathbf{v}_m(M) \quad \text{and} \quad C = \|V\|(A)/\mathscr{H}^m(A) \quad \text{for any } A \subseteq M \text{ with } \mathscr{H}^m(A) \in (0,\infty)\,.$$

[All72, 4.7] **Example.** If $E \subseteq M$ is a set of locally finite perimeter in M, then $\mathbf{v}_m(E) \in \mathbf{V}_m(M)$ and

$$\delta\mathbf{v}_m(E)(g) = \int_{\partial_* E} g(x) \bullet \nu_E(x)\,\mathrm{d}\mathscr{H}^{m-1}(x) \quad \text{for } g \in \mathscr{X}(M)\,.$$

[All72, 4.8] **Example.** Let $0 < k < n$ and $T \in \mathbf{G}(n,k)$. Set $V(A) = \mathscr{H}^n(\{x : (x,T) \in A\})$ for $A \subseteq \mathbf{R}^n \times \mathbf{G}(n,k)$. Then

$$V \in \mathbf{V}_k(\mathbf{R}^n)\,, \quad \delta V = 0\,, \quad \|V\| = \mathscr{H}^n\,, \quad \boldsymbol{\Theta}^k(\|V\|,a) = 0 \quad \text{for } a \in \mathbf{R}^n\,.$$

- **Exercise.** Let $0 < k < n$, and Σ be a smooth k-dimensional submanifold of $\mathbf{R}^n$ with smooth boundary, and $\theta : \Sigma \to (0,\infty)$ be of class $\mathscr{C}^1$. Define

$$V(\alpha) = \int \alpha(x, \operatorname{Tan}(\Sigma,x))\theta(x)\,\mathrm{d}\mathscr{H}^k(x) \quad \text{for } \alpha \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n,k))\,.$$

For $g \in \mathscr{X}(\mathbf{R}^n)$ we have

$$\delta V(g) = -\int_\Sigma g(x) \bullet (\mathbf{h}(\Sigma,x) + \operatorname{Tan}(\Sigma,x)_\natural\big(\operatorname{grad}(\log\circ\theta)(x)\big)\theta(x)\,\mathrm{d}\mathscr{H}^k(x) + \int_{\partial\Sigma} g(x) \bullet \nu_\Sigma(x)\theta(x)\,\mathrm{d}\mathscr{H}^{k-1}(x)\,,$$

where $\nu_\Sigma(x)$ is the unit normal vector to Σ at $x \in \partial\Sigma$.
In particular,

$$\|\delta V\|_{\mathrm{sing}} = \theta\mathscr{H}^k \llcorner \partial\Sigma\,, \qquad \boldsymbol{\eta}(V,x) = \nu_\Sigma(x) \quad \text{for } x \in \partial\Sigma\,,$$

$$\mathbf{h}(V,x) = \mathbf{h}(\Sigma,x) + \operatorname{Tan}(\Sigma,x)_\natural\big(\operatorname{grad}(\log\circ\theta)(x)\big) \quad \text{for } x \in \Sigma\,.$$

[All72, 4.10] **Lemma.** Assume $r \in \mathbf{R}$, $V \in \mathbf{V}_k(U)$, $\|\delta V\|$ is a Radon measure, $f : W \to \mathbf{R}$ is continuous, $g \in \mathscr{X}(U)$, f is smooth in a neighborhood of $\operatorname{spt}\|V\| \cap f^{-1}\{r\} \cap \operatorname{spt} g$. Then

$$(\delta V \llcorner \{x : f(x) > r\})(g) = \delta\big(V \llcorner \{(x,S) : f(x) > r\}(g)\big)(g) \\ + \lim_{h \downarrow 0} \frac{1}{h} \int_{\{(x,S): r < f(x) \le r+h\}} S_\natural(g(x)) \bullet \operatorname{grad} f(x) \,\mathrm{d}V(x,S)\,.$$

Remark. Set $E_r = \{x \in U : f(x) > r\}$. In the language of [Men16, §5] one could write

$$V\partial E_r(g) = \lim_{h \downarrow 0} \frac{1}{h} \int_{\{(x,S): r < f(x) \le r+h\}} S_\natural(g(x)) \bullet \operatorname{grad} f(x) \,\mathrm{d}V(x,S)\,.$$

Theorem. Assume $V \in \mathbf{V}_k(U)$, $\|\delta V\|$ is a Radon measure, $-\infty \le a < b \le \infty$, $f : W \to \mathbf{R}$ is continuous and smooth in a neighborhood of $\operatorname{spt}\|V\| \cap f^{-1}(a,b)$. Then for $\mathscr{L}^1$ almost all $r \in (a,b)$ the measure $\|\delta(V \llcorner \{(x,S) : f(x) > r\})\|$ is a Radon measure and

$$\int_a^b \|\delta(V \llcorner \{(x,S) : f(x) > r\})\|(B) \,\mathrm{d}\mathscr{L}^1(r) \\ \le \int_{B \cap f^{-1}(a,b) \times \mathbf{G}(n,k)} |S_\natural(\operatorname{grad} f(x))| \,\mathrm{d}V(x,S) + \int_a^b \|\delta V\|(B \cap \{x : f(x) > r\}) \,\mathrm{d}\mathscr{L}^1(r)$$

for any Borel set $B \subseteq U$.

[All72, 4.12] **Remark.** Let $V \in \mathbf{V}_k(\mathbf{R}^n)$ and $r \in (0,\infty)$.

$$\|\delta(\boldsymbol{\mu}_{r\#} V)\| = r^{k-1} \boldsymbol{\mu}_{r\#} \|\delta V\|\,.$$

Remark. If $\mathbf{\Theta}^{k-1}(\|\delta V\|, a) = 0$, then all members of $\operatorname{VarTan}(V,a)$ are stationary.

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