8.9. (Exercise)
Let $S, T \in \mathbf{G}(n,k)$, $R \in \mathbf{G}(n,m)$, $\eta, \eta_1, \eta_2 \in \operatorname{Hom}(S, S^\perp)$ and $S_i = \{x + \eta_i(x) : x \in S\}$ for $i \in \{1,2\}$. Then

(1) $|S_\natural - T_\natural|^2 = 2 S_\natural \bullet T_\natural^\perp = 2 S_\natural^\perp \bullet T_\natural$,

(2) $|S_\natural \circ R_\natural|^2 = S_\natural \bullet R_\natural$,

(3) $\|S_\natural - T_\natural\| = \|S_\natural^\perp \circ T_\natural\| = \|S_\natural \circ T_\natural^\perp\|$,

(4) $2|T_\natural \bullet (\eta \circ S_\natural)|^2 \le |S_\natural - T_\natural|^2 |\eta|^2$,

(5) $\star\ \|S_{1\natural} - S_{2\natural}\| \le \|\eta_1 - \eta_2\|$,

$\star\ (1 - \|S_{1\natural} - S_\natural\|^2)\|\eta_1 - \eta_2\|^2 \le (1 + \|\eta_2\|^2)\|S_{1\natural} - S_{2\natural}\|^2$.

8.14. Lemma. (Exercise)
There exists $\Gamma \in (0,\infty)$ such that for all $T \in \mathbf{G}(n,k)$ and $l, \theta \in \operatorname{Hom}(T, T^\perp)$, defining $L \in \operatorname{Hom}(T, \mathbf{R}^n)$ by $L(x) = x + l(x)$, there holds

$$(\theta \circ T_\natural) \bullet (\operatorname{im} L)_\natural |\textstyle\bigwedge_k L| - l \bullet \theta \le \Gamma \|\theta\| \max\{\|l\|^3, \|l\|^{4k-1}\}.$$

8.15. Lemma. (Estimates for harmonic functions)
Let $T \in \mathbf{G}(n,k)$. There exists $\Gamma > 0$ such that for any *harmonic* function $h : T \cap \mathbf{U}(0,1) \to T^\perp$ and $y \in T \cap \mathbf{U}(0,\frac12)$ there holds

$$\max\{|h(0)|, \|\mathrm{D}h(0)\|\} \le \Gamma \left(\int_{T \cap \mathbf{U}(0,1)} |h|^2 \,\mathrm{d}\mathscr{H}^k \right)^{1/2},$$

$$\left|h(y) - h(0) - \mathrm{D}h(0)y\right| \le \Gamma \left(\int_{T \cap \mathbf{U}(0,1)} |h|^2 \,\mathrm{d}\mathscr{H}^k \right)^{1/2} |y|^2.$$

Proof of 8.16.
We assume 8.16 is false. Then we are given $\varepsilon \in (0,1)$ and, for each $i \in \mathbb{N}$, we choose $\theta \in (0, \frac{1}{16})$, $\Delta_i \in (0, \frac{1}{14})$, and $M_i \in (1,\infty)$ so that

(10) $$2^{k+1} \Gamma_{8.15}^2 (k\boldsymbol{\alpha}(k)/(k+4))\theta^2 \le \theta^{2(1-k/p)} 7^{-k-4},$$

(11) $$\lim_{i\to\infty} \Delta_i = 0, \quad M_i \ge \Gamma_{8.15} 7^{(k+2)/2},$$

(12) $$\frac{\left((2-\varepsilon)\boldsymbol{\alpha}(k)7^k\right)^{1/q-1/2}}{\theta^{(k+2)/2}\Delta_i} \le M_i (7\theta)^{1-k/p}.$$

Using homotheties, scaling, and rotations we obtain sequences V_i, a_i, r_i, U_i, μ_i, α_i such that

(13) $V_i \in \mathbf{V}_k(\mathbf{R}^n)$, $a_i \in \operatorname{spt}\|V_i\|$, $(U_i)_\natural a_i = 0$,

(14) $U_i \in \mathbf{G}(n,k)$, $\|(U_i)_\natural - T_\natural\| \le \Delta_i$,

(15) $\mu_i = \left(r^{-k-2} \int_{\mathbf{C}(U_i,0,7)} \operatorname{dist}(x,T)^2 \,\mathrm{d}\|V_i\|(x) \right)^{1/2} \le \Delta_i$,

(16) $\alpha_i = \left(\int_{\mathbf{C}(U_i,0,7)} |\mathbf{h}(V_i,\cdot)|^p \,\mathrm{d}\|V_i\| \right)^{1/p} \in [0,\infty)$,

(17) $\varepsilon \le s^{-k}\|V_i\|\,\mathbf{B}(x,s) \le (2-\varepsilon)\boldsymbol{\alpha}(k)$ for $s \in (0,7)$ and $\|V_i\|$ almost all $x \in \mathbf{C}(U_i,0,7)$,

(18) $1 \le \boldsymbol{\Theta}^k(\|V_i\|,x) \le 1 + \Delta_i$ for $\|V_i\|$ almost all $x \in \mathbf{C}(U_i,0,7)$,

and whenever $\tilde T \in \mathbf{G}(n,k)$ and $\tilde A$ is an affine k-plane parallel to $\tilde T$, then

(20) either $\|T_\natural - \tilde T_\natural\| > M_i \mu_i$,

(21) or $\left((7\theta)^{-k-2} \int_{\mathbf{C}(U_i,0,7\theta)} \operatorname{dist}(x,\tilde A)^2 \,\mathrm{d}\|V_i\|(x) \right)^{1/2} > \theta^{1-k/p} \max\{\mu_i, M_i 7^{1-k/p}\alpha_i\}$.

We set

$$\beta_i = \left(\int_{\mathbf{C}(U_i,0,6)} \|S_\natural - T_\natural\|^2 \,\mathrm{d}V(x,S) \right)^{1/2},$$

$$\nu_i = \sup\{ (7 - |(U_i)_\natural x|)^k |T_\natural^\perp x| : x \in \operatorname{spt}\|V_i\| \cap \mathbf{C}(U_i,0,7) \}.$$

We observe that

(22,23) $$\lim_{i\to\infty} a_i = 0, \quad \lim_{i\to\infty} \mu_i = 0, \quad \text{and} \quad \lim_{i\to\infty} \nu_i = 0.$$

W.l.o.g. $\operatorname{spt}\|V_i\| \cap \mathbf{C}(T,0,5) \subseteq \mathbf{U}(a_i,7)$. Next, we show that

(25) $$\left(\int_{\mathbf{U}(a_i,7)} |\mathbf{h}(V_i,\cdot)|^2 \,\mathrm{d}\|V_i\| \right)^{1/2} \le \Delta_i \mu_i.$$

We apply 8.12 to each V_i with $\mu = \sqrt{2}$ so to obtain maps $f_i : T \to T^\perp$ with $\operatorname{Lip} f_i \le 1$ and, setting,

$$F_i(x) = x + f_i(x), \quad C = \mathbf{C}(T,0,1), \quad D = T \cap \mathbf{U}(0,1),$$
$$X_i = C \cap \operatorname{im} F_i, \quad Y_i = \{ y \in D : \boldsymbol{\Theta}^k(\|V_i\|,x) \ge 1 \},$$

we get

(30,33) $$\|V_i\|(C \sim X_i) + \mathscr{H}^k(D \sim Y_i) \le P_{8.12}((\Delta_i\mu_i)^2 + \beta_i) \overset{8.13}{\lesssim} \mu_i^2.$$

Next, using (33) and 8.9, we prove that $\{\mu_i^{-1} f_i : i \in \mathbb{N}\}$ is bounded in $W^{1,2}$ so, up to a subsequence, it converges in L^2 to a $W^{1,2}$ function $h : D \to T^\perp$. Using 8.9 and 8.14 we compare $\mu_i^{-1} \int_D \mathrm{D}f_i(x) \bullet \mathrm{D}\varphi(x) \,\mathrm{d}\mathscr{H}^k(x)$ with $\delta V_i(\varphi \circ T_\natural)$ and show, employing (25), that h is harmonic.

Next, we set

$$L_i(y) = y + \mu_i \mathrm{D}h(0)y, \quad K_i(x) = L_i \circ T_\natural x + \mu_i h(0),$$
$$\tilde T_i = \operatorname{im} L_i, \quad \tilde A_i = \operatorname{im} K_i.$$

We observe that (20) is not satisfied, i.e.,

(39) $$\|(\tilde T_i)_\natural - T_\natural\| \overset{8.9(5)}{\le} \mu_i \|\mathrm{D}h(0)\| \overset{8.15}{\le} M_i \mu_i.$$

We show that

(40) $$\operatorname{dist}(x,\tilde A_i) \overset{8.15}{\le} \operatorname{dist}(x,T) + C\mu_i \quad \text{for } x \in C \cap \operatorname{spt}\|V_i\|,$$

(41) $$\operatorname{dist}(F_i(y),\tilde A_i) \overset{8.15}{\le} |f_i(y) - \mu_i h(y)| + C\mu_i|y|^2 \quad \text{for } x \in X_i,\ y = T_\natural x,\ |y| < \tfrac12,$$

Using (41), (40), (33), the area formula [Fed69, 3.2.20], and (10) we derive for all large $i \in \mathbb{N}$

$$\frac{1}{(7\theta)^{k+2}\mu_i^2} \int_{\mathbf{C}(U_i,0,7)} \operatorname{dist}(x,\tilde A_i)^2 \,\mathrm{d}\|V_i\|(x) < \left(\theta^{1-k/p}\right)^2.$$

Hence, (21) is not satisfied for large $i \in \mathbb{N}$ and, recalling (39), we get a contradiction.

8.17. Corollary. For all $\varepsilon \in (0,1)$ there exists $\gamma \in (0,1)$ and $N \in (1,\infty)$ such that for all V, a, r, T, U, A, μ, α satisfying (1)–(6) of 8.16 with γ in place of Δ assuming $r^{1-k/p}\alpha \le \gamma$ there holds

(1) $\operatorname{spt}\|V\| \cap U_\natural^{-1}\{a\} = \{a\}$,

(2) $S(a) = \operatorname{Tan}(\operatorname{spt}\|V\|, a) \in \mathbf{G}(n,k)$ and

$$\|T_\natural - S(a)_\natural\| \le N \max\{\mu, r^{1-k/p}\alpha\},$$

(3) for $s \in (0,r)$ there exists $T_s \in \mathbf{G}(n,k)$ and an affine k-plane A_s parallel to T_s such that

$$\|T_\natural - S(a)_\natural\| \le N \max\{\mu, r^{1-k/p}\alpha\} \left(\frac{s}{r}\right)^{1-k/p},$$

$$\left(s^{-k-2} \int_{\mathbf{C}(U,a,s)} \operatorname{dist}(x, A_s)^2 \,\mathrm{d}\|V\|(x)\right)^{1/2} \le N \max\{\mu, r^{1-k/p}\alpha\} \left(\frac{s}{r}\right)^{1-k/p}.$$

8.18. Lemma. Let $U \subseteq \mathbf{R}^k$ be open, $B \subseteq \mathbf{R}^k$ be closed and such that $\operatorname{Tan}(B,b) = R^k$ for $b \in B \cap U$. Then $B \cap U$ is open in $\mathbf{R}^k$.

In particular $B \cap U$ is both closed and open relative to U, so $B \cap U = U$.

8.19. Theorem. For all $\varepsilon \in (0,1)$ there exist $\eta \in (0,1)$ and $C \in (1,\infty)$ such that for all V, r, T, μ, α if

(1) $V \in \mathbf{V}_k(\mathbf{R}^n)$, $0 \in \operatorname{spt}\|V\|$, $r \in (0,\infty)$, $T \in \mathbf{G}(n,k)$,

(2) $\mu = \left(r^{-k-2} \int_{\mathbf{C}(T,0,2r)} \operatorname{dist}(x,T)^2 \,\mathrm{d}\|V\|(x)\right)^{1/2} \le \eta$,

(3) $0 \le r^{p-k}\alpha \le \eta$, $\left(\int_{\mathbf{C}(T,0,2r)} |\mathbf{h}(V,\cdot)|^p \,\mathrm{d}\|V\|\right)^{1/p} \le \alpha$,

(4) $\varepsilon \le s^{-k}\|V\|\,\mathbf{B}(x,s) \le (2-\varepsilon)\boldsymbol{\alpha}(k)$ for $\|V\|$ almost all $x \in \mathbf{C}(T,0,2r)$ and $s \in (0,r)$,

(5) $1 \le \boldsymbol{\Theta}^k(\|V\|, x) \le 1+\eta$ for $\|V\|$ almost all $x \in \mathbf{C}(T,0,2r)$,

then there exist $f : T \to T^\perp$ and $F : T \to \mathbf{R}^n$ such that

(6) $F(y) = y + f(y)$ for $y \in T$,

(7) $\operatorname{spt}\|V\| \cap \mathbf{C}(T,0,r) = \operatorname{im} F \cap \mathbf{C}(T,0,r)$,

(8) f is differentiable,

(9) $\|\mathrm{D}f(y) - \mathrm{D}f(z)\| \le C \max\{\mu, r^{1-k/p}\alpha\} \left(\frac{|y-z|}{r}\right)^{1-k/p}$.

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Sławomir Kolasiński
Instytut Matematyki, Uniwersytet Warszawski
ul. Banacha 2, 02-097 Warszawa, Poland
s.kolasinski@mimuw.edu.pl