We are given $\varepsilon \in (0,1)$ and we are looking for $\eta \in (0,\infty)$. We assume

$$n,k \in \mathbb{N}\,, \quad p \in [2,\infty)\,, \quad 1 \le k < p\,, \quad R,d \in (0,\infty)\,,$$
$$V \in \mathbf{V}_k(\mathbf{R}^n)\,, \quad a \in \operatorname{spt}\|V\|\,,$$
$$\boldsymbol{\Theta}^k(\|V\|,x) \ge d \quad \text{for } \|V\| \text{ almost all } x \in \mathbf{U}(a,R)\,,$$
$$\|V\|\,\mathbf{U}(a,R) \le (1+\eta) d\boldsymbol{\alpha}(k) R^k\,,$$
$$\left(\int_{\mathbf{U}(a,R)} |\mathbf{h}(V,\cdot)|^p \,\mathrm{d}\|V\|\right)^{1/p} \le \frac{\eta d^{1/p}}{R^{1-k/p}}\,,$$

We know that if $\eta > 0$ is small enough, then for $b \in \mathbf{U}(a,(1-\varepsilon)R) \cap \operatorname{spt}\|V\|$ and $0 < r < (1-\varepsilon)(R - |a-b|)$ there exists and affine k-plane $X(b,r)$ such that

$$(1-\varepsilon)d \le \frac{\|V\|\,\mathbf{B}(b,r)}{\boldsymbol{\alpha}(k) r^k} \le (1+\varepsilon)d$$
$$\text{and} \quad \text{H-dist}_{\mathbf{U}(b,r)}(\operatorname{spt}\|V\|, X(b,r)) \le \varepsilon r\,.$$

Exercise. Let $G \subseteq \mathbf{R}^n$ be open, μ be a Radon measure over G, $\varepsilon, \Delta \in (0,1)$, $R \in (0,\infty)$, $T \in \mathbf{G}(n,k)$, $p \in [1,\infty)$. Suppose

$$R^{-k} \int_G \left(\frac{\operatorname{dist}(x,T)}{R}\right)^p \mathrm{d}\mu(x) \le \varepsilon^p\,,$$
$$r^{-k}\mu\,\mathbf{B}(x,r) \ge \Delta \quad \text{for } \mu \text{ almost all } x,\ r \in (0,R)\,.$$

Prove that for μ almost all x satisfying $\operatorname{dist}(x, \mathbf{R}^n \sim G) > R$ there holds

$$\operatorname{dist}(x,T) \le 2\varepsilon R \Delta^{-1/p}\,.$$

8.13. Lemma. (Caccioppoli type inequality)
Suppose $U, G \subseteq \mathbf{R}^n$ are open, $\delta \in (0,\infty)$, $U + \mathbf{U}(0,\delta) \subseteq G$, $T \in \mathbf{G}(n,k)$, $V \in \mathbf{V}_k(G)$,

$$\alpha^2 = \int_G |\mathbf{h}(V,\cdot)|^2 \,\mathrm{d}\|V\| \in [0,\infty)\,,$$
$$\mu^2 = \int_G |T_\natural^\perp x|^2 \,\mathrm{d}\|V\|(x)\,,$$
$$\beta^2 = \int_{U\times\mathbf{G}(n,k)} \|S_\natural - T_\natural\|^2 \,\mathrm{d}V(x,S)\,.$$

Then

$$\beta^2 \le \max\{\delta^2\alpha^2,\, 9\delta^{-2}\mu^2\}\,.$$

Corollary. Taking $U = \mathbf{C}(T,a,r)$, $\delta = r$, $G = \mathbf{C}(T,a,2r)$, we obtain

$$\int_{\mathbf{C}(T,a,r)\times\mathbf{G}(n,k)} \|S_\natural - T_\natural\|^2 \,\mathrm{d}V(x,S) \le \max\left\{ r^2 \int_{\mathbf{C}(T,a,2r)} |\mathbf{h}(V,\cdot)|^2 \,\mathrm{d}\|V\|,\ 9\int_{\mathbf{C}(T,a,2r)} \left(\frac{|T_\natural^\perp x|}{r}\right)^2 \mathrm{d}\|V\|(x) \right\}$$

8.14. Lemma. (Exercise)
There exists $\Gamma \in (0,\infty)$ such that for all $T \in \mathbf{G}(n,k)$ and $l, \theta \in \operatorname{Hom}(T, T^\perp)$, defining $L \in \operatorname{Hom}(T,\mathbf{R}^n)$ by $L(x) = x + l(x)$, there holds

$$(\theta \circ T_\natural) \bullet (\operatorname{im} L)_\natural |{\textstyle\bigwedge_k} L| - l \bullet \theta \le \Gamma \|\theta\| \max\{\|l\|^3, \|l\|^{4k-1}\}\,.$$

8.15. Lemma. (Estimates for harmonic functions)
Let $T \in \mathbf{G}(n,k)$. There exists $\Gamma > 0$ such that for any *harmonic* function $h : T \cap \mathbf{U}(0,1) \to T^\perp$ and $y \in T \cap \mathbf{U}(0,\frac{1}{2})$ there holds

$$\max\{|h(0)|,\, \|\mathrm{D}h(0)\|\} \le \Gamma\left(\int_{T\cap\mathbf{U}(0,1)} |h|^2 \,\mathrm{d}\mathscr{H}^k\right)^{1/2},$$
$$\left|h(y) - h(0) - \mathrm{D}h(0)y\right| \le \Gamma\left(\int_{T\cap\mathbf{U}(0,1)} |h|^2 \,\mathrm{d}\mathscr{H}^k\right)^{1/2} |y|^2\,.$$

8.16. Theorem. (core of the proof)
For all $\varepsilon \in (0,1)$ there exist $\Delta \in (0,\frac{1}{14})$, $\theta \in (0,\frac{1}{16})$, $M \in (1,\infty)$ such that for all $V, a, r, T, U, A, \mu, \alpha$ if

(1) $V \in \mathbf{V}_k(\mathbf{R}^n)$, $a \in \operatorname{spt}\|V\|$, $r \in (0,\infty)$,

(2) $T, U \in \mathbf{G}(n,k)$, $\|T_\natural - U_\natural\| \le \Delta$,
A is a k-dimensional affine plane parallel to T,

(3) $\mu^2 = r^{-k-2}\int_{\mathbf{C}(U,a,r)} \operatorname{dist}(x,A)^2 \,\mathrm{d}\|V\|(x) \le \Delta^2$,

(4) $\alpha^p = \int_{\mathbf{C}(U,a,r)} |\mathbf{h}(V,\cdot)|^p \in [0,\infty)$,

(5) $\varepsilon \le s^{-k}\|V\|\,\mathbf{B}(x,s) \le (2-\varepsilon)\boldsymbol{\alpha}(k)$ for $s \in (0,r)$ and $\|V\|$ almost all $x \in \mathbf{C}(U,a,r)$,

(6) $1 \le \boldsymbol{\Theta}^k(\|V\|,x) \le 1 + \Delta$ for $\|V\|$ almost all $x \in \mathbf{C}(U,a,r)$,

then there exist $\tilde{T} \in \mathbf{G}(n,k)$ and $\tilde{A} \subseteq \mathbf{R}^n$ such that

(7) $\tilde{A}$ is an affine k-dimensional plane parallel to $\tilde{T}$,

(8) $\|T_\natural - \tilde{T}_\natural\| \le M\mu$,

(9) $\left((\theta r)^{-k-2}\int_{\mathbf{C}(U,a,\theta r)} \operatorname{dist}(x,\tilde{A})^2 \,\mathrm{d}\|V\|(x)\right)^{1/2} \le \theta^{1-k/p}\max\{\mu, M r^{1-k/p}\alpha\}$.

Corollary. Iterating 8.16 we obtain a convergent sequence $\{A_i : i \in \mathbb{N}\}$ of affine k-planes such that setting

$$\operatorname{height}_2(a,Z,s)^2 = r^{-k-2}\int_{\mathbf{C}(U,a,s)} \operatorname{dist}(x,Z)^2 \,\mathrm{d}\|V\|(x)\,,$$

we get

$$\operatorname{height}_2(a,A_j,\theta^j r) \le \theta^{j(1-k/p)}\max\{\operatorname{height}_2(a,A_0,r),\, M\alpha r^{1-k/p}\}\,.$$

In particular, if $0 < s < r$ and $j \in \mathbb{N}$ is such that $\theta^{j+1} \le s/r < \theta^j$, then

$$\operatorname{height}_2(a,A_j,s) \le \left(\frac{s}{r}\right)^{1-k/p}\max\{\operatorname{height}_2(a,A_0,r),\, M\alpha r^{1-k/p}\}\,.$$

The area formula

Assume $W \subseteq \mathbf{R}^n$ is $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable, $m \le \nu$, $f : W \to \mathbf{R}^\nu$, $\operatorname{Lip}(f) < \infty$. Then

$$\int_W (g \circ f) J_m f \, \mathrm{d}\mathscr{H}^m = \int_{R^\nu} g(z) N(f,z) \, \mathrm{d}\mathscr{H}^m(z)$$

for any $g : \mathbf{R}^\nu \to \bar{\mathbf{R}}$.

(see [Fed69, 3.2.20])

The co-area formula

Assume $m \ge \mu$, $W \subseteq \mathbf{R}^n$ is $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable, $Z \subseteq \mathbf{R}^\nu$ is $(\mathscr{H}^\mu, \mu)$ rectifiable and $\mathscr{H}^\mu$ measurable, $f : W \to Z$, $\operatorname{Lip}(f) < \infty$. Then (we write "ap" for "$(\mathscr{H}^m \llcorner W, m)$ ap")

(1) for $\mathscr{H}^m$ almost all $w \in W$, either $\operatorname{ap} J_\mu f(w) = 0$ or $\operatorname{im} \operatorname{ap} \mathrm{D} f(w) = \operatorname{Tan}^\mu(\mathscr{H}^\mu \llcorner Z, f(w)) \in \mathbf{G}(\nu, \mu)$;

(2) $f^{-1}\{z\}$ is $(\mathscr{H}^{m-\mu}, m-\mu)$ rectifiable and $\mathscr{H}^{m-\mu}$ measurable for $\mathscr{H}^\mu$ almost all $z \in Z$;

(3) for any $(\mathscr{H}^m \llcorner W)$ integrable function $g : W \to \bar{\mathbf{R}}$

$$\int_W g \cdot \operatorname{ap} J_\mu f \, \mathrm{d}\mathscr{H}^m = \int_Z \int_{f^{-1}\{z\}} g \, \mathrm{d}\mathscr{H}^{m-\mu} \, \mathrm{d}\mathscr{H}^\mu(z) \, .$$

(see [Fed69, 3.2.22])

References

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