We are given $\varepsilon \in (0,1)$ and we are looking for $\eta \in (0,\infty)$. We assume

$$n, k \in \mathbb{N}, \quad p \in [2,\infty), \quad 1 \le k < p, \quad R, d \in (0,\infty),$$
$$V \in \mathbf{V}_k(\mathbf{R}^n), \quad a \in \operatorname{spt}\|V\|,$$
$$\mathbf{\Theta}^k(\|V\|, x) \ge d \quad \text{for } \|V\| \text{ almost all } x \in \mathbf{U}(a,R),$$
$$\|V\|\,\mathbf{U}(a,R) \le (1+\eta) d \boldsymbol{\alpha}(k) R^k,$$
$$\left(\int_{\mathbf{U}(a,R)} |\mathbf{h}(V,\cdot)|^p \,\mathrm{d}\|V\| \right)^{1/p} \le \frac{\eta d^{1/p}}{R^{1-k/p}}.$$

On day 1 we have proved that, choosing $\eta > 0$ small enough, for any $b \in \mathbf{U}(a,(1-\varepsilon)R) \cap \operatorname{spt}\|V\|$ and $0 < r < (1-\varepsilon)(R - |a-b|)$ there exists an affine k-plane $X(b,r)$ such that

$$(1-\varepsilon) d \le \frac{\|V\|\,\mathbf{B}(b,r)}{\boldsymbol{\alpha}(k) r^k} \le (1+\varepsilon) d$$
$$\text{and} \quad \text{H-dist}_{\mathbf{U}(b,r)}(\operatorname{spt}\|V\|, X(b,r)) \le \varepsilon r.$$

Applying the Reifenberg Topological Disc Theorem (e.g. [DT12]) we conclude that $\operatorname{spt}\|V\| \cap \mathbf{U}(a,(1-\varepsilon)R)$ is a topological disc.

6.1. Lemma. Suppose

(1) $\nu \in \mathbb{N}_+$, $1 \le \mu < \infty$, $0 < \xi < 1$, $1 < M < \infty$, $0 < R < \infty$, $T \in \mathbf{G}(n,k)$, $Y \subseteq \mathbf{R}^n$, $V \in \mathbf{V}_k(\mathbf{R}^n)$,

(2) $\mathscr{H}^0(Y) \le \nu + 1$,

(3) $|y - z| \le \mu |T_\natural^\perp(y-z)|$ for $y, z \in Y$,

(4) $(M+1)\operatorname{diam} Y \le R$,

(5) $R\|\delta V\|\,\mathbf{B}(y,r) \le \xi \|V\|\,\mathbf{B}(y,r)$ for $y \in Y$ and $r \in (0,R)$,

(6) $\int_{\mathbf{B}(y,r)\times\mathbf{G}(n,k)} \|S_\natural - T_\natural\| \,\mathrm{d}V(x,S) \le \xi \|V\|\,\mathbf{B}(y,r)$ for $y \in Y$, $r \in (0,R)$.

Then there exist $V_1, V_2 \in \mathbf{V}_k(\mathbf{R}^n)$ and a partition $Y = Y_0 \cup Y_1 \cup Y_2$ of Y such that

(7) $V \ge V_1 + V_2$,

(8) $\mathscr{H}^0(Y_1) \le \nu$ and $\mathscr{H}^0(Y_2) \le \nu$,

(9) $(M \operatorname{diam} Y)\|\delta V_j\|\,\mathbf{B}(y,r)$
$\le 2M(\nu+1)(\exp\xi)(3\nu M\mu)^{k+1}\xi\|V_j\|\,\mathbf{B}(y,r)$
for $j \in \{1,2\}$, $y \in Y_j$, and $r \in (0, M\operatorname{diam} Y)$,

(10) $\int_{\mathbf{B}(y,r)\times\mathbf{G}(n,k)} \|S_\natural - T_\natural\| \,\mathrm{d}V_j(x,S)$
$\le M(\exp\xi)(3\nu M\mu)^k \xi \|V_j\|\,\mathbf{B}(y,r)$
for $j \in \{1,2\}$, $y \in Y_j$, and $r \in (0, M\operatorname{diam} Y)$,

(11) $\mathbf{\Theta}^k(\|V\|, y) = \mathbf{\Theta}^k(\|V_j\|, y)$ for $j \in \{1,2\}$ and $y \in Y_j$,

(12) $((1+1/M)^k + (\nu+1)/M)(\exp\xi)\frac{\|V\|(Y+\mathbf{B}(0,R))}{\boldsymbol{\alpha}(k)R^k}$
$\ge \sum_{y\in Y_0} \mathbf{\Theta}^k(\|V\|,y) + \sum_{j=1}^2 \frac{\|V_j\|(Y_j + \mathbf{B}(0, M\operatorname{diam} Y))}{(M\operatorname{diam} Y)^k}$.

6.2. Theorem. For all $\nu \in \mathbb{N}_+$, $\lambda \in (1,\infty)$, $\mu \in [1,\infty)$ there exists $\gamma \in (0,\infty)$ such that for all V, Y, R, T if

$$V \in \mathbf{V}_k(\mathbf{R}^n), \quad Y \subseteq \mathbf{R}^n, \quad R \in (0,\infty), \quad T \in \mathbf{G}(n,k),$$
$$\mathscr{H}^0(Y) \le \nu, \quad \operatorname{diam} Y \le \gamma R,$$
$$|y - z| \le \mu |T_\natural^\perp(y-z)| \text{ for } y, z \in Y,$$
$$R\|\delta V\|\,\mathbf{B}(y,r) \le \gamma \|V\|\,\mathbf{B}(y,r) \text{ for } y \in Y,\ r \in (0,R),$$
$$\int_{\mathbf{B}(y,r)\times\mathbf{G}(n,k)} \|S_\natural - T_\natural\| \,\mathrm{d}V(x,S) \le \gamma \|V\|\,\mathbf{B}(y,r)$$
$$\text{for } y \in Y,\ r \in (0,R),$$

then

$$\sum_{y\in Y} \mathbf{\Theta}^k(\|V\|, y) \le \frac{\|V\|(Y + \mathbf{B}(0,R))}{\boldsymbol{\alpha}(k) R^k}.$$

8.11. Lemma. (CONTROL OF HOLES) If

$$V \in \mathbf{V}_k(\mathbf{R}^n), \quad r \in (0,\infty), \quad \zeta \in (0,1), \quad T \in \mathbf{G}(n,k),$$
$$y \in T, \quad T_\natural^\perp[\operatorname{spt}\|V\|] \text{ is compact}, \quad \|V\|\,\mathbf{C}(T,y,\zeta r) = 0,$$

then

$$(\zeta(k+1) - k)\|V\|\,\mathbf{C}(T,y,r)$$
$$\le \int_{\mathbf{B}(y,r)\times\mathbf{G}(n,k)} \|S_\natural - T_\natural\|^2 \,\mathrm{d}V(x,S) + r\|\delta V\|\,\mathbf{C}(T,y,r).$$

8.12. Lemma. (LIPSCHITZ APPROXIMATION) For all $\varepsilon \in (0,1)$, $\mu \in (1,\infty)$ there exist $\xi, P \in (0,\infty)$ such that if

(1) $V \in \mathbf{V}_k(\mathbf{R}^n)$ and $T \in \mathbf{G}(n,k)$,

(2) $\mathbf{\Theta}^k(\|V\|, x) \ge 1$,
$\|V\|\,\mathbf{B}(x,1) \le (2-\varepsilon)\boldsymbol{\alpha}(k)$,
and $\|V\|\,\mathbf{B}(x,r) \ge \varepsilon r^k$
for $\|V\|$ almost all $x \in \mathbf{C}(T,0,5)$ and $r \in (0,1)$,

(3) $\alpha = \left(\int_{\mathbf{C}(T,0,5)} |\mathbf{h}(V,\cdot)|^2 \right)^{1/2} \in [0,\infty)$,
$\beta = \left(\int_{\mathbf{C}(T,0,5)\times\mathbf{G}(n,k)} \|S_\natural - T_\natural\|^2 \right)^{1/2}$,

(4) $\|V\|\,\mathbf{C}(T,0,1) > 0$,

(5) $\operatorname{dist}(x,T) \le \xi$ for $x \in \operatorname{spt}\|V\|$,

then there exists $f : T \to T^\perp$ and $F : T \to \mathbf{R}^n$ such that

(6) $F(x) = x + f(x)$ for $x \in T$,

(7) $\sup \operatorname{im}|f| \le \sup\{\operatorname{dist}(x,T) : x \in \operatorname{spt}\|V\|\}$,

(8) $|f(y) - f(z)| \le \frac{1}{\sqrt{\mu^2-1}}|y-z|$,

(9) $\|V\|(\mathbf{C}(T,0,1) \sim \operatorname{im} F)$
$+ \mathscr{H}^k(T \cap \mathbf{U}(0,1) \cap \{y : \mathbf{\Theta}^k(\|V\|, F(y)) < 1\})$
$\le P(\alpha^2 + \beta^2)$.

If there is more than one layer, one needs a more robust approximation. Let $Q \in \mathbb{N}$ and $\mathbf{Q}_Q$ denote the space of unordered tuples of Q points in $\mathbf{R}^n$ (it may be identified with the space of 0 dimensional integral currents in $\mathbf{R}^n$ with mass Q). Almgren [Alm00] developed a way to approximate integral currents with Lipschitz $\mathbf{Q}_Q$-valued functions (see also [DLS11]). Menne (see [Men10, 3.15] and [Men12, 5.7]) provided $\mathbf{Q}_Q$-valued approximation for integral varifolds.

References

[All72] William K. Allard. On the first variation of a varifold. *Ann. of Math. (2)*, 95:417–491, 1972.

[Alm00] Frederick J.jun. Almgren. *Almgren's big regularity paper. Q-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2. Edited by V. Scheffer and Jean E. Taylor.* Singapore: World Scientific, 2000.

[DLS11] Camillo De Lellis and Emanuele Nunzio Spadaro. Q-valued functions revisited. *Mem. Amer. Math. Soc.*, 211(991):vi+79, 2011. URL: `http://dx.doi.org/10.1090/S0065-9266-10-00607-1`, `doi:10.1090/S0065-9266-10-00607-1`.

[DT12] Guy David and Tatiana Toro. Reifenberg parameterizations for sets with holes. *Mem. Amer. Math. Soc.*, 215(1012):vi+102, 2012. URL: `http://dx.doi.org/10.1090/S0065-9266-2011-00629-5`, `doi:10.1090/S0065-9266-2011-00629-5`.

[Fed69] Herbert Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[Men10] Ulrich Menne. A Sobolev Poincaré type inequality for integral varifolds. *Calc. Var. Partial Differential Equations*, 38(3-4):369–408, 2010. URL: `http://dx.doi.org/10.1007/s00526-009-0291-9`, `doi:10.1007/s00526-009-0291-9`.

[Men12] Ulrich Menne. Decay estimates for the quadratic tilt-excess of integral varifolds. *Arch. Ration. Mech. Anal.*, 204(1):1–83, 2012. URL: `http://dx.doi.org/10.1007/s00205-011-0468-1`, `doi:10.1007/s00205-011-0468-1`.

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