Setup: $0 < k \le m \le n$, $U \subseteq \mathbf{R}^n$ is open, M is a smooth m dimensional submanifold of U such that the inclusion map $i : M \hookrightarrow \mathbf{R}^n$ is proper, X, Y are vectorspaces, ϕ, ψ are measures, $E \subseteq X$, $g \in \mathscr{C}_c^\infty(M, \mathbf{R}^n)$, $V \in \mathbf{V}_k(M)$, $1 \le p \le \infty$, Z a locally compact Hausdorff space, $\alpha \in \mathscr{K}(\mathbf{G}_k(M))$, $\beta \in \mathscr{K}(\mathbf{G}(n,k))$.

Definitions

1. $\mathbf{O}^*(n,k) = \left\{ p \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^k) : p \circ p^* = \operatorname{id}_{\mathbf{R}^k} \right\}$
2. $\mathbf{G}(n,k)$ - Grassmannian of k-dim. subspaces of $\mathbf{R}^n$;
3. $T \in \mathbf{G}(n,k) \quad \Rightarrow \quad T_\natural \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$;
4. $f, g \in \operatorname{Hom}(X,Y) \;\Rightarrow\; f \bullet g = \operatorname{tr}(f^* \circ g)$,
   $|f| = (f \bullet f)^{1/2}$, $\|f\| = \sup\{ f(x) : x \in X, |x| \le 1 \}$;
5. $\boldsymbol{\Theta}^k(\phi, x) = \lim_{r \downarrow 0} \frac{\phi\, \mathbf{B}(x,r)}{\boldsymbol{\alpha}(k) r^k}$;
6. $\operatorname{Tan}(E, a) = \{ v \in X : \forall \varepsilon > 0\ \exists x \in E\ \exists r > 0$
   $|x - a| < \varepsilon$ and $|r(x-a) - v| < \varepsilon \}$;
7. $\operatorname{Tan}^m(\phi, a) =$
   $\bigcap \{ \operatorname{Tan}(E, a) : E \subseteq X,\ \boldsymbol{\Theta}^m(\phi \llcorner X \sim E, a) = 0 \}$;
8. $\mathscr{X}(M) = \{ g \in \mathscr{C}_c^\infty(M, \mathbf{R}^n) : g(x) \in \operatorname{Tan}(M, x) \}$;
9. $\mathscr{X}^\perp(M) = \{ g \in \mathscr{C}_c^\infty(M, \mathbf{R}^n) : g(x) \in \operatorname{Nor}(M, x) \}$;
10. $\operatorname{Tan}(M, g) \in \mathscr{X}(M)$ and $\operatorname{Nor}(M, g) \in \mathscr{X}^\perp(M)$;
11. $\bigodot^m(X,Y)$ symmetric Y-valued m-forms on X;
    $\bigodot^m(X,Y) \simeq \operatorname{Hom}(\bigodot_m X, Y)$;
12. $\bigwedge^m(X,Y)$ antisymmetric Y-valued m-forms on X;
    $\bigwedge^m(X,Y) \simeq \operatorname{Hom}(\bigwedge_m X, Y)$;
13. $\mathbf{b}(M,a) \in \bigodot^2(\operatorname{Tan}(M,a), \operatorname{Nor}(M,a))$ s.t.
    $\mathrm{D}g(a)w \bullet v = -\mathbf{b}(M,a)(v,w) \bullet g(a)$ for $g \in \mathscr{X}^\perp(M)$;
14. $\mathbf{h}(M,a) \in \operatorname{Nor}(M,a)$ s.t. for $g \in \mathscr{X}^\perp(M)$
    $(\mathrm{D}g(a) \circ \operatorname{Tan}(M,a)_\natural) \bullet \operatorname{Tan}(M,a)_\natural = -g(a) \bullet \mathbf{h}(M,a)$;
15. $\mathbf{G}_k(M) =$
    $\{ (x,S) : x \in M,\ S \in \mathbf{G}(n,k),\ S \subseteq \operatorname{Tan}(M,x) \}$;
16. $\mathscr{K}(Z)$ space of continuous compactly supported functions $Z \to \mathbf{R}$ endowed with locally convex topology;
17. $\mathbf{V}_k(M)$ space of Radon measures over $\mathbf{G}_k(M)$;
18. $\mathbf{V}_k(M) \subseteq \mathscr{K}(\mathbf{G}_k(M))^* \subseteq \mathbf{R}^{\mathscr{K}(\mathbf{G}_k(M))}$;
19. $\|V\|(B) = V(\{ (x,S) \in \mathbf{G}_k(M) : x \in B \})$ for $B \subseteq M$;
20. $f \in \operatorname{Hom}(X,Y) \Rightarrow \bigwedge_m f \in \operatorname{Hom}(\bigwedge_m X, \bigwedge_m Y)$;
21. $h \in \mathscr{C}^1(M, M') \;\Rightarrow\; h_\# V \in \mathbf{V}_k(M')$ s.t.
    $h_\# V(\alpha) = \int \alpha(h(x), \mathrm{D}h(x)[S]) |\bigwedge_k \mathrm{D}h(x) \circ S_\natural| \,\mathrm{d}V(x,S)$;
22. $V^{(x)}(\beta) = \lim_{r \downarrow 0} ⨍_{\mathbf{B}(x,r) \times \mathbf{G}(n,k)} \beta(S) \,\mathrm{d}(i_\# V)(y,S)$;
23. $\boldsymbol{\mu}_r(x) = rx$; $\quad \boldsymbol{\tau}_a(x) = x + a$;
24. $a \in M$, $j : \operatorname{Tan}(M,a) \hookrightarrow \mathbf{R}^n$,
    $C \in \operatorname{VarTan}(V,a) \iff$
    $j_\# C = \lim_{j \to \infty} (\boldsymbol{\mu}_{r_j} \circ \boldsymbol{\tau}_{-a} \circ i)_\# V$ for some $r_j \uparrow \infty$;
25. $\mathbf{v}_k(E)(\alpha) = \int_E \alpha(x, \operatorname{Tan}^k(\mathscr{H}^k \llcorner E, x)) \,\mathrm{d}\mathscr{H}^k(x)$
    given $E \subseteq \mathbf{R}^n$ is a countably $(\mathscr{H}^k, k)$ rectifiable and $\mathscr{H}^k(E \cap K) < \infty$ for $K \subseteq U$ compact;
26. $V \in \mathbf{RV}_k(M) \iff \mathbf{V}_k(M) \ni V = \sum_{i=1}^\infty c_i \mathbf{v}_k(E_i)$
    where $c_i \in (0, \infty)$ and $E_i \subseteq M$;
27. $V \in \mathbf{IV}_k(M) \iff \mathbf{V}_k(M) \ni V = \sum_{i=1}^\infty c_i \mathbf{v}_k(E_i)$
    where $c_i \in \mathbb{N}$ and $E_i \subseteq M$;
28. $\delta V(g) = \int (\mathrm{D}g(x) \circ S_\natural) \bullet S_\natural \,\mathrm{d}V(x,S)$;
29. $\|\delta V\|(G) = \sup\{ \delta V(g) : g \in \mathscr{X}(M), \operatorname{spt} g \subseteq G, |g| \le 1 \}$
    for $G \subseteq M$ open,
    $\|\delta V\|(A) = \inf\{ \|\delta V\|(G) : A \subseteq G,\ G \subseteq M \text{ open} \}$
    for arbitrary $A \subseteq M$.
30. $\mathbf{D}(\phi, \psi, x) = \lim_{r \downarrow 0} \frac{\phi\, \mathbf{B}(x,r)}{\psi\, \mathbf{B}(x,r)}$;
31. $\|\delta V\|$ is a Radon measure $\Rightarrow$
    - $\delta V(g) = \int g(x) \bullet \boldsymbol{\eta}(V,x) \,\mathrm{d}\|\delta V\|(x)$,
    - $\mathbf{h}(V,x) = -\mathbf{D}(\|\delta V\|, \|V\|, x) \boldsymbol{\eta}(V,x)$,
    - $\delta V(g) = -\int g(x) \bullet \mathbf{h}(V,x) \,\mathrm{d}\|V\|(x)$
      $+ \int g(x) \bullet \boldsymbol{\eta}(V,x) \,\mathrm{d}\|\delta V\|_{\mathrm{sing}}(x)$;
32. V is *stationary* $\iff \delta V = 0$
    V is *stationary in an open set* $G \iff \|\delta V\|(G) = 0$;
33. $x \in \operatorname{reg} V \iff x \in \operatorname{spt} \|V\|$ and there exists G open in M with $x \in G$ s.t. $\operatorname{spt} \|V\| \cap G$ is a $\mathscr{C}^1$ submanifold of M of dim. k and $V \llcorner \mathbf{G}_k(G) \in \mathbf{RV}_k(M)$;
    $\operatorname{sing} V = \operatorname{spt} \|V\| \sim \operatorname{reg} V$;
34. V *satisfies* $H(p)$ if
    - in case $p = 1$, $\|\delta V\|$ is a Radon measure;
    - in case $p > 1$, $\|\delta V\|$ is a Radon measure, the mean curvature vector $\mathbf{h}(V, \cdot)$ belongs to $L^p_{\mathrm{loc}}(\|V\|)$, and $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$;
35. $\mathbf{C}(T, a, r, h) =$
    $\{ x \in \mathbf{R}^n : |T_\natural(x - a)| < r,\ |T_\natural^\perp(x-a)| < h \}$
    $\mathbf{C}(T, a, r) = \mathbf{C}(T, a, r, \infty)$;
36. $\text{H-dist}_A(B, C) = \sup\{ |\operatorname{dist}(x, B) - \operatorname{dist}(x, C)| : x \in A \}$, whenever $A, B, C \subseteq X$.

Exercises

1. $\omega \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R})$, $v \in \mathbf{R}^n$, $S \in \mathbf{G}(n,k) \Rightarrow$
   $(\omega \cdot v) \bullet S_\natural = \langle S_\natural v, \omega \rangle = \omega(S_\natural v)$;
2. $S, T \in \mathbf{G}(n,k), R \in \mathbf{G}(n,m), \eta, \eta_1, \eta_2 \in \operatorname{Hom}(S, S^\perp) \Rightarrow$
   - $\|S_\natural - T_\natural\| = \|S_\natural^\perp \circ T_\natural\| = \|S_\natural \circ T_\natural^\perp\|$,
   - $|S_\natural - T_\natural|^2 = 2 S_\natural \bullet T_\natural^\perp = 2 S_\natural^\perp \bullet T_\natural$,
   - $2 |T_\natural \bullet (\eta \circ S_\natural)|^2 \le |S_\natural - T_\natural|^2 |\eta|^2$,
   - $|S_\natural \circ R_\natural|^2 = S_\natural \bullet R_\natural$,
   - $S_i = \{ x + \eta_i(x) : x \in S \}$ for $i = 1, 2 \Rightarrow$
     - ⋆ $\|S_{1\natural} - S_{2\natural}\| \le \|\eta_1 - \eta_2\|$,
     - ⋆ $(1 - \|S_{1\natural} - S_\natural\|^2) \|\eta_1 - \eta_2\|^2 \le (1 + \|\eta_2\|^2) \|S_{1\natural} - S_{2\natural}\|^2$;
3. assume $T \in \mathbf{G}(n,k)$, $\mu \in (1, \infty)$, $A \subseteq \mathbf{R}^n$, $|a - b| \ge \mu |T_\natural^\perp(a - b)|$ for $a, b \in A$. Then there exists $f : T \to T^\perp$ such that $\operatorname{Lip} f < (\mu^2 - 1)^{-1/2}$ and $A = \{ T_\natural x + f(T_\natural x) : x \in T_\natural[A] \}$;
4. $f \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n), t \in (0, \infty) \Rightarrow$
   $\det(\operatorname{id}_{\mathbf{R}^n} + t f) = \sum_{k=0}^n t^k \operatorname{tr}(\bigwedge_k f)$;
5. $g \in \mathscr{C}^1(\mathbf{R}^n, \mathbf{R}^n)$, $S \in \mathbf{G}(n,k)$, $x \in \mathbf{R}^n$, $p, q \in \mathbf{O}^*(n,k)$,
   $\operatorname{im} p^* = S$, $\operatorname{im} q^* = \operatorname{im} \mathrm{D}g(x) \circ S_\natural \Rightarrow$
   $\| \bigwedge_k \mathrm{D}g(x) \circ S_\natural \| = | \bigwedge_k \mathrm{D}g(x) \circ S_\natural | = \det(q \circ \mathrm{D}g(x) \circ S_\natural \circ p^*)$;

6. $\|\boldsymbol{\mu}_{r\#}V\| = r^k \boldsymbol{\mu}_{r\#}\|V\|$,
$\|\delta(\boldsymbol{\mu}_{r\#}V)\| = r^{k-1}\boldsymbol{\mu}_{r\#}\|\delta V\|$;
7. $p \in [1,\infty)$, $r > 0$, V satisfies $H(p)$, $\psi_V = |\mathbf{h}(V,\cdot)|^p\|V\| \Rightarrow$
$\psi_{\boldsymbol{\mu}_{r\#}V} = r^{k-p}(\boldsymbol{\mu}_{r\#}\psi_V)$;
8. $f : Z \to \mathbf{R}$ is bounded,
$\lambda(x) = \limsup_{y\to x} f(y)$, $\mu(x) = \liminf_{y\to x} f(y) \Rightarrow$
λ is upper semi-continuous, μ is lower semi-continuous;
9. $f : Z \to (0,\infty)$ is upper semi-continuous $\Rightarrow$
$\{x : f(x) > \liminf_{y\to x} f(y)\}$ is of first Baire category;
10. $\{f_i : i \in I\}$ a family of real valued upper semi-continuous functions on $Z \Rightarrow$
$\lambda(x) = \inf\{f_i(x) : i \in I\}$ is upper semi-continuous;
11. μ is a Borel regular measure over U, $t \in (0,\infty)$, $A \subseteq U \Rightarrow$
   - $\boldsymbol{\Theta}^{*k}(\mu,x) < t$ for $x \in A \Rightarrow \mu(A) \le 2^k t\mathscr{H}^k(A)$,
   - $\mu(U) < \infty, \boldsymbol{\Theta}^{*k}(\mu,x) > t$ for $x \in A \Rightarrow$
   $\mu(A) \ge t\mathscr{H}^k(A)$,
   - $\mu(A) < \infty$, A is μ-measurable $\Rightarrow \boldsymbol{\Theta}^k(\mu \llcorner A, x) = 0$ for μ almost all $x \in U \sim A$.
12. $S,T \in \mathbf{G}(n,k)$, $a \in \mathbf{R}^n$, $f(x) = |T_\natural^\perp(x-a)|$ for $x \in \mathbf{R}^n \Rightarrow$
$|S_\natural(\operatorname{grad} f(x))| \le \|S_\natural - T_\natural\|$;
13. $L \in (0,\infty)$, $S,T \in \mathbf{G}(n,k)$, $\varphi \in \mathscr{D}(\mathbf{R},\mathbf{R})$, $\operatorname{Lip}\varphi < L$, $g \in \mathscr{X}(\mathbf{R}^n)$, $g(x) = \varphi(x)^2 T_\natural^\perp x$ for $x \in \mathbf{R}^n \Rightarrow$
$\mathrm{D}g(x) \bullet S = 2\varphi(x)(S_\natural \circ T_\natural^\perp x) \bullet \operatorname{grad}\varphi(x) + \varphi(x)^2 T_\natural^\perp \bullet S_\natural$,
$\varphi(x)^2\|S_\natural - T_\natural\|^2 \le \mathrm{D}g(x) \bullet S_\natural + 2L\varphi(x)\|S_\natural - T_\natural\| \cdot |T_\natural^\perp x|$;
14. μ, μ_i are Radon measures over $\mathbf{R}^n$, $\mu_i \to \mu$ as $i \to \infty \Rightarrow$
$\lim_{i\to\infty} \sup\{\operatorname{dist}(x, \operatorname{spt}\mu_i) : x \in K \cap \operatorname{spt}\mu\} = 0$
for any compact set $K \subseteq \mathbf{R}^n$.
15. μ, μ_i are Radon measures over $\mathbf{R}^n$, $\mu_i \to \mu$ as $i \to \infty$, $\omega : \mathbf{R}^n \times (0,\infty) \to (0,\infty)$ is continuous in the first variable, $\mu_i \mathbf{B}(x,r) > \omega(x,r)$ for all $i \in \mathbb{N}$, $r \in (0,1)$, $x \in \operatorname{spt}\mu_i \Rightarrow$
$\lim_{i\to\infty} \sup\{\operatorname{dist}(x, \operatorname{spt}\mu) : x \in K \cap \operatorname{spt}\mu_i\} = 0$
for any compact set $K \subseteq \mathbf{R}^n$.
16. μ is a Radon measure over $\mathbf{R}^n$, $a \in \mathbf{R}^n \Rightarrow$
$\operatorname{Tan}^k(\mu,a) \subseteq \operatorname{Tan}(\operatorname{spt}\mu, a)$;
17. $C > 0$, μ is a Radon measure over $\mathbf{R}^n$, $\mu\mathbf{B}(x,r) \ge Cr^k$ for $x \in \operatorname{spt}\mu$, $r \in (0,1) \Rightarrow$
$\operatorname{Tan}(\operatorname{spt}\mu, x) = \operatorname{Tan}^k(\mu,x)$ for μ almost all x.

Facts

1. $V \in \mathbf{V}_k(\mathbf{R}^n)$, $M \in [0,\infty)$, $a \in \operatorname{spt}\|V\|$, $R \in \mathbf{R}$, $0 < R < \operatorname{dist}(a, \mathbf{R}^n \sim U)$, $\|\delta V\|\mathbf{B}(a,r) \le M\|V\|\mathbf{B}(a,r)$ for $r \in (0,R) \Rightarrow$
$\varphi(r) = r^{-k}\|V\|\mathbf{B}(a,r)\exp(Mr)$
is non-decreasing on $(0,R)$.
(see [All72, 5.1(3)])
2. $V \in \mathbf{V}_k(\mathbf{R}^n)$, $p > k$, $a \in \operatorname{spt}\|V\|$, $R \in \mathbf{R}$,
$0 < R < \operatorname{dist}(a, \mathbf{R}^n \sim U)$, V satisfies $H(p)$,
$\int_{\mathbf{B}(a,R)} |\mathbf{h}(V,\cdot)|^p \,\mathrm{d}\|V\| = \Gamma^p \Rightarrow$
$\varphi(r) = r^{-k}\|V\|\mathbf{B}(a,r) + \frac{\Gamma}{p-k} r^{1-k/p}$
is non-decreasing on $(0,R)$.
(see [Sim83, 17.7])
3. $V \in \mathbf{V}_k(\mathbf{R}^n)$, $d \in (0,\infty)$, $\boldsymbol{\Theta}^k(\|V\|,x) \ge d$ for $\|V\|$ almost all $x \Rightarrow$
$\|V\|(\mathbf{R}^n)^{(k-1)/k} \le \boldsymbol{\gamma}(k) d^{-1/k}\|\delta V\|(\mathbf{R}^n)$.
(see [Sch16, 6.11])
4. M is connected, $V \in \mathbf{V}_m(U)$, $\operatorname{spt}\|V\| \subseteq M$, $\|\delta V\|$ is a Radon measure, $\delta V(g) = 0$ for $g \in \mathscr{X}(U)$ such that $\operatorname{Nor}(M, g|_M) = 0 \Rightarrow$
$V = c\mathbf{v}_m(M)$ where $\mathbf{R} \ni c = \|V\|(A)/\mathscr{H}^m(A)$ for any $A \subseteq M$ with $\mathscr{H}^m(A) > 0$.
(see [All72, 4.6(3)])
5. $r \in \mathbf{R}$, $V \in \mathbf{V}_k(U)$, $\|\delta V\|$ is a Radon measure, $f : U \to \mathbf{R}$ is continuous, $g \in \mathscr{X}(U)$, f is smooth in a neighborhood of $\operatorname{spt}\|V\| \cap f^{-1}\{r\} \cap \operatorname{spt} g \Rightarrow$
$(\delta V \llcorner \{x : f(x) > r\})(g)$
$= \delta\big(V \llcorner \{(x,S) : f(x) > r\}(g)\big)(g)$
$+ \lim_{h\downarrow 0} \frac{1}{h} \int_{\{x : r < f(x) \le r+h\}} S_\natural(g(x)) \bullet \operatorname{grad} f(x) \,\mathrm{d}V(x,S)$.
(see [All72, 4.10(1)])
6. $V \in \mathbf{V}_k(U)$, $\|\delta V\|$ is a Radon measure, $-\infty \le a < b \le \infty$, $f : U \to \mathbf{R}$ is continuous and smooth in a neighborhood of $\operatorname{spt}\|V\| \cap f^{-1}\{(a,b)\} \Rightarrow$
$\int_a^b \|\delta(V \llcorner \{(x,S) : f(x) > r\})\|(B) \,\mathrm{d}\mathscr{L}^1(r)$
$\le \int_{B \cap f^{-1}\{(a,b)\} \times \mathbf{G}(n,k)} |S_\natural(\operatorname{grad} f(x))| \,\mathrm{d}V(x,S)$
$+ \int_a^b \|\delta V\|(B \cap \{x : f(x) > r\}) \,\mathrm{d}\mathscr{L}^1(r)$
for any Borel set $B \subseteq U$.
(see [All72, 4.10(2)])
7. $W \subseteq \mathbf{R}^n$ is $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable, $m \le \nu$, $f : W \to \mathbf{R}^\nu$, $\operatorname{Lip}(f) < \infty$. $\Rightarrow$
$\int_W (g \circ f) J_m f \,\mathrm{d}\mathscr{H}^m = \int_{R^\nu} g(z) N(f,z) \,\mathrm{d}\mathscr{H}^m(z)$
for any $g : \mathbf{R}^\nu \to \bar{\mathbf{R}}$.
(see [Fed69, 3.2.20])
8. $m \ge \mu$, $W \subseteq \mathbf{R}^n$ is $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable, $Z \subseteq \mathbf{R}^\nu$ is $(\mathscr{H}^\mu, \mu)$ rectifiable and $\mathscr{H}^\mu$ measurable, $f : W \to Z$, $\operatorname{Lip}(f) < \infty$ (we write "ap" for "$(\mathscr{H}^m \llcorner W, m)$ ap") $\Rightarrow$
   - for $\mathscr{H}^m$ almost all $w \in W$, either $\operatorname{ap} J_\mu f(w) = 0$ or $\operatorname{im}\operatorname{ap}\mathrm{D}f(w) = \operatorname{Tan}^\mu(\mathscr{H}^\mu \llcorner Z, f(w)) \in \mathbf{G}(\nu,\mu)$;
   - $f^{-1}\{z\}$ is $(\mathscr{H}^{m-\mu}, m-\mu)$ rectifiable and $\mathscr{H}^{m-\mu}$ measurable for $\mathscr{H}^\mu$ almost all $z \in Z$;
   - for any $(\mathscr{H}^m \llcorner W)$ integrable function $g : W \to \bar{\mathbf{R}}$
   $\int_W g \cdot \operatorname{ap} J_\mu f \,\mathrm{d}\mathscr{H}^m = \int_Z \int_{f^{-1}\{z\}} g \,\mathrm{d}\mathscr{H}^{m-\mu} \,\mathrm{d}\mathscr{H}^\mu(z)$.

   (see [Fed69, 3.2.22])
9. $R \in (0,\infty)$, $C \in \mathbf{V}_k(\mathbf{U}(0,R))$, C is stationary, $0 \in \operatorname{spt}\|C\|$, $\|C\|\mathbf{U}(0,R) \le \boldsymbol{\Theta}^k(\|C\|,x)\boldsymbol{\alpha}(k)R^k$ for $\|C\|$ almost all $x \in \mathbf{U}(0,R) \Rightarrow$
there exists $T \in \mathbf{G}(n,k)$ such that
$C = \boldsymbol{\Theta}^k(\|C\|,0)\mathbf{v}_k(T \cap \mathbf{U}(0,R))$.
(see [All72, 5.3])
10. $d \in (0,\infty)$, $\lim_{i\to\infty} V_i = V \in \mathbf{V}_k(U)$, $W \subseteq U$ is open, $\liminf_{i\to\infty} \|\delta V_i\|(W) < \infty$,
$\lim_{i\to\infty} \|V_i\|(\{x \in W : \boldsymbol{\Theta}^k(\|V_i\|,x) < d\}) = 0 \Rightarrow$
$\boldsymbol{\Theta}^k(\|V\|,x) \ge d$ for $\|V\|$ almost all $x \in W$.
(see [All72, 5.4])

Theorem ([All72, §8])**.** *Suppose $1 \le k < p < \infty$ and $p \ge 2$. For each $\varepsilon \in (0,1)$ there exists $\eta \in (0,\infty)$ such that if*

$$R \in (0,\infty), \quad d \in (0,\infty), \quad V \in \mathbf{V}_k(\mathbf{R}^n), \quad a \in \operatorname{spt}\|V\|,$$

(1) $$\boldsymbol{\Theta}^k(\|V\|, x) \ge d \quad \text{for } \|V\| \text{ almost all } x \in \mathbf{U}(a,R),$$

(2) $$\|V\|\,\mathbf{U}(a,R) \le (1+\eta) d \boldsymbol{\alpha}(k) R^k,$$

(3) $$V \text{ satisfies } H(p) \quad \text{and} \quad \left(\int_{\mathbf{U}(a,R)} |\mathbf{h}(V,\cdot)|^p \,\mathrm{d}\|V\| \right)^{1/p} \le \frac{\eta d^{1/p}}{R^{1-k/p}},$$

then there exist $T \in \mathbf{G}(n,k)$ and $f : T \to T^\perp$ of class $\mathscr{C}^{1,1-k/p}$ such that

$$\|\mathrm{D}f(y) - \mathrm{D}f(z)\| \le \varepsilon \left(\frac{|y-z|}{R} \right)^{1-k/p} \quad \text{and} \quad \mathbf{U}(a,(1-\varepsilon)R) \cap \operatorname{spt}\|V\| = \mathbf{U}(a,(1-\varepsilon)R) \cap \operatorname{graph} f.$$

Remark. Setting $\bar{V} = \frac{1}{d}(\boldsymbol{\mu}_{1/R} \circ \boldsymbol{\tau}_{-a})_{\#} V$ we obtain

$$\boldsymbol{\Theta}^k(\|\bar{V}\|, x) \ge 1 \quad \text{for } \|\bar{V}\| \text{ almost all } x \in \mathbf{U}(0,1),$$
$$\|\bar{V}\|\,\mathbf{U}(0,1) \le (1+\eta)\boldsymbol{\alpha}(k),$$
$$\bar{V} \text{ satisfies H(p)} \quad \text{and} \quad \left(\int_{\mathbf{U}(0,1)} |\mathbf{h}(\bar{V},\cdot)|^p \,\mathrm{d}\|\bar{V}\| \right)^{1/p} \le \eta.$$

Remark. Assume $T = \mathbf{R}^k$, $a = 0$, $R = 1$, and V is stationary. The function f from the theorem is of class $\mathscr{C}^{1,1-k/p}$ and its graph corresponds to a stationary varifold. In particular, $f : \mathbf{R}^k \to \mathbf{R}^{n-k}$ must be a critical point of the functional

$$W^{1,k}_f(\mathbf{R}^k, \mathbf{R}^{n-k}) \ni g \mapsto \int_{\mathbf{U}(0,1-\varepsilon)} \Phi^{\S}(z, g(z), \mathrm{D}g(z)) \,\mathrm{d}\mathscr{L}^k(z),$$

where $\Phi^{\S}$ is the non-parametric integrand corresponding to the area integrand $\Phi : \mathbf{R}^n \times \bigwedge_k \mathbf{R}^n \to \mathbf{R}$ given by $\Phi(x,\xi) = |\xi|$; see [Fed69, 5.1.1, 5.1.9]. Therefore, one may apply PDE methods (see [Fed69, 5.2.15-18]) to verify that f is in fact real analytic.

Remark. Assume V satisfies $H(p)$ with $p > k$ and $\|V\|(\{x : \boldsymbol{\Theta}^k(\|V\|,x) = 0\}) = 0$. Then for $\|V\|$ almost all a there exists $R > 0$ such that conditions (2) and (3) hold. Only the condition (1) is problematic; cf. [All72, 8.1(2)]. From fact 2 and exercise 10 it follows that $\boldsymbol{\Theta}^k(\|V\|,\cdot)$ is upper semi-continuous and from exercise 9 that the set of points $a \in \operatorname{spt}\|V\|$ where (1) is satisfied is dense in $\operatorname{spt}\|V\|$. Hence, the Allard Regularity Theorem implies that the regular set $\operatorname{reg} V$ of V is open and dense in $\operatorname{spt}\|V\|$.

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