 $1 \le k < p < \infty$, $q = \frac{p}{p-1} = p'$, $p \geq 2$ 

Regularity theorem

Theorem

$$\forall \varepsilon \in (0,1) \ \exists \eta \in (0,\infty)$$
$$\forall R, d, V, a$$
$$R \in (0,\infty), \quad d \in (0,\infty), \quad V \in \mathbb{V}_k(\mathbb{R}^n), \quad a \in \operatorname{spt}\|V\|$$

if $\circ$ $\Theta^k(\|V\|, x) \geq d$ for $\|V\|$ almost all $x \in \mathbb{U}(a,R)$

$\circ\circ$ $\|V\| \, \mathbb{U}(a,R) \leq (1+\eta)\, d\, \alpha(k) R^k$

$\circ\circ\circ$ $V$ satisfies $\mathbb{H}(p)$ with $\left( \int_{\mathbb{U}(a,R)} |h(V,\cdot)|^p \, d\|V\| \right)^{1/p} \leq \frac{\eta d^{1/p}}{R^{1-k/p}}$

then $\exists T \in G(n,k) \ \exists f : T \to T^\perp \in \mathcal{C}^{1,1-\frac{k}{p}}$

$$\|Df(y) - Df(z)\| \leq \varepsilon \left( \frac{|y-z|}{R} \right)^{1-\frac{k}{p}} \qquad \forall y,z \in T$$

$$\mathbb{U}(a,(1-\varepsilon)R) \cap \operatorname{spt}\|V\| = \mathbb{U}(a,(1-\varepsilon)R) \cap \operatorname{graph} f .$$

Remark: Setting $\bar{V} = \frac{1}{d}(\mu_{1/R} \circ \tau_{-a})_{\#} V$ we obtain

$\circ$ $\Theta^k(\|\bar{V}\|, x) \geq 1$ for $\|\bar{V}\|$ almost all $x \in \mathbb{U}(0,1)$

$\circ\circ$ $\|\bar{V}\| \, \mathbb{U}(0,1) \leq (1+\eta)\, \alpha(k)$

$\circ\circ\circ$ $\bar{V}$ satisfies $\mathbb{H}(p)$ with $\int_{\mathbb{U}(0,1)} |h(\bar{V},\cdot)|^p \, d\|\bar{V}\| \leq \eta$ .

Remark: If $V \in \mathbb{V}_k(\mathbb{R}^n)$, $a \in \operatorname{spt}\|V\|$, and $\Theta^k(\|V\|, a) = d \in (0,\infty)$, then for all $R > 0$ small enough $V$ satisfies $\circ\circ$ and $\circ\circ\circ$.

[8.1(3)]: In particular, if $V = \mathbf{v}_k(E)$ for some countably $(\mathcal{H}^k, k)$ rectifiable set $E$ such that $\mathcal{H}^k(E \cap K) < \infty$ whenever $K \subseteq \mathbb{R}^n$ is compact, then for $\|V\|$ almost all $a \in \operatorname{spt}\|V\|$ there exists $R > 0$ such that $V$ satisfies $\circ, \circ\circ, \circ\circ\circ$.

Remark: The regular set of $V$ is always open by definition.

[8.2] $V \in \mathbb{V}_k(\mathbb{R}^n)$, $\alpha \in [0,\infty)$, $r \in (0,\infty)$, $G \subseteq \mathbb{R}^n$ open

$V$ satisfies $\mathbb{H}(p)$ $\left(\text{i.e. } \delta V(g) = \int g \bullet \mathbf{h}(V,\cdot)\, d\|V\| \text{ where } \mathbf{h}(V,\cdot) \in L^p_{loc}(\|V\|,\mathbb{R}^n)\right)$

$$\left(\int_G |\mathbf{h}(V,\cdot)|^p \, d\|V\|\right)^{1/p} = \alpha$$

$$\Updownarrow$$

$$\left(\int_{\mu_r[G]} |\mathbf{h}(\mu_{r\#}V,\cdot)|^p \, d\|\mu_{r\#}V\|\right)^{1/p} = \alpha\, r^{\left(\frac{k}{p}-1\right)}$$

$$\boxed{\begin{aligned} \psi_{V,p} &= |\mathbf{h}(V,\cdot)|^p \|V\| \qquad p > 1 \\ \psi_{\mu_{r\#}V,p} &= r^{k-p}(\mu_{r\#}\psi_{V,p}) \end{aligned}}$$

<u>Proof</u>: $\alpha = \sup \left\{ \delta V(g) : g \in L^q(\|V\|,\mathbb{R}^n) \cap \mathscr{X}(\mathbb{R}^n),\ \operatorname{spt} g \subseteq G,\ \|g\|_{L^q(\|V\|)} \le 1 \right\}$

$$\boxed{q = \frac{p}{p-1}} \qquad \boxed{\frac{k}{q} = k\left(1-\frac{1}{p}\right) = k - \frac{k}{p}} \qquad \boxed{k-1-\frac{k}{q} = \frac{k}{p}-1}$$

$$\begin{aligned}
\delta(\mu_{r\#}V)(g) &= \int Dg(x) \bullet S \, d(\mu_{r\#}V)(x,S) = \\
&= r^k \int Dg(rx) \bullet S \, dV(x,S) \\
&= r^{k-1} \int D(g \circ \mu_r)(x) \bullet S \, dV(x,S) \\
&= r^{k-1}\, \delta V(g \circ \mu_r) \le r^{k-1} \cdot \alpha \left(\int |g \circ \mu_r|^q \, d\|V\|\right)^{1/q} \\
&= r^{k-1} \cdot \alpha \cdot \left(\int |g|^q \, d\, \mu_{r\#}\|V\|\right)^{1/q} \\
&= r^{k-1} \alpha \cdot \left(r^{-k} \int |g|^q \, d\|\mu_{r\#}V\|\right)^{1/q} \\
&= r^{k-1-\frac{k}{q}} \cdot \alpha \left(\int |g|^q \, d\|\mu_{r\#}V\|\right)^{1/q} \\
&= r^{\left(\frac{k}{p}-1\right)} \cdot \alpha \left(\int |g|^q \, d\|\mu_{r\#}V\|\right)^{1/q}
\end{aligned}$$

$\square$

[8.3] **If** $d \in (0,\infty)$, $\alpha \in [0,\alpha)$, $\Delta = d^{1/k} \cdot \gamma(k)^{-1} - \alpha > 0$

$a \in \operatorname{spt}\|V\|$, $s \in (0,\infty)$, $V \in \mathbf{V}_k(\mathbb{R}^m)$

$V$ satisfies $\mathbb{H}(k)$, $\Theta^k(\|V\|, x) \geq d$ for $\|V\|$ almost all $x \in \mathbf{U}(a,s)$

$\left(\int_{\mathbf{U}(a,s)} |h(V,\cdot)|^k \, d\|V\|\right)^{1/k} \leq \alpha$

**then**

$$\frac{\|V\| \mathbf{B}(a,r)}{r^k} \geq \left(\frac{\Delta}{k}\right)^k \quad \text{whenever } 0 < r < s.$$

**Proof.** Assume $k > 1$.

Recall the isoperimetric inequality:

$$(*) \quad d^{1/k} \|V\| \mathbf{U}(a,r)^{\frac{k-1}{k}} \leq \gamma(k) \|\delta(V \llcorner \mathbf{U}(a,r) \times G(m,k))\|(\mathbb{R}^m).$$

for $r \in (0,s)$

Recall 4.10(1) ($\leftarrow$ cutting varifolds with smooth functions)

$$(**) \quad \|\delta(V \llcorner \mathbf{U}(a,r) \times G(m,k))\|(\mathbb{R}^m) \leq \|\delta V\| \mathbf{U}(a,r) + \lim_{h \downarrow 0} \frac{1}{h} \|V\|(\mathbf{U}(a,r) \sim \mathbf{U}(a,r-h))$$

$$\leq \alpha \|V\| \mathbf{U}(a,r)^{\frac{k-1}{k}} + \lim_{h \downarrow 0} \frac{\mu(r) - \mu(r-h)}{h},$$

where $\mu(r) = \|V\| \mathbf{U}(a,r)$.

Combining $(*)$ and $(**)$ we get

$$d^{1/k} \mu(r)^{\frac{k-1}{k}} \leq \gamma(k) \, \alpha \, \mu(r)^{\frac{k-1}{k}} + \gamma(k) \mu'(r) \quad \text{for } \mathcal{L}^1 \text{ almost all } r \text{ with } 0 < r < s$$

$$\Rightarrow \Delta \mu(r)^{\frac{k-1}{k}} \leq \mu'(r)$$

Note $(\mu^{1/k})'(r) = \frac{1}{k} \cdot \mu^{\frac{1}{k}-1}(r) \cdot \mu'(r) \geq \frac{\Delta}{k}$ for $\mathcal{L}^1$ almost all $r \in (0,s)$

$$\Rightarrow \mu^{1/k}(r) = \mu^{1/k}(r) - \mu^{1/k}(0) \geq \int_0^r \frac{\Delta}{k} \, d\mathcal{L}^1 = \frac{\Delta}{k} \cdot r$$

$$\Rightarrow \mu(r) \geq \left(\frac{\Delta}{k}\right)^k \cdot r^k \quad \text{for } \mathcal{L}^1 \text{ almost all } r \in (0,s).$$

□

[8.4] $\forall \varepsilon \in (0,1)$ $\exists \eta \in (0,\infty)$ $\forall V \in \mathbf{V}_k(\mathbf{U}(0,1))$

<u>if</u> $0 \in \operatorname{spt}\|V\|$, $\|\delta V\|\,\mathbf{U}(0,1) \leq \eta$

$\Theta^k(\|V\|, x) \geq 1$ for $\|V\|$ almost all $x \in \mathbf{U}(0,1)$

$\|V\|\,\mathbf{U}(0,1) \leq (1+\eta)\,\alpha(k)$

$V$ satisfies $\mathbb{H}(k)$ and $\displaystyle\int_{\mathbf{U}(0,1)} |\mathbf{h}(V,\cdot)|^k \, d\|V\| \leq \frac{1}{\gamma(k)+\varepsilon}$

<u>then</u> $\exists\, T \in G(n,k)$

$$\sup\left\{ \left| \operatorname{dist}(x, \operatorname{spt}\|V\|) - \operatorname{dist}(x,T) \right| : x \in \mathbf{U}(0,1-\varepsilon) \right\} \leq \varepsilon$$

<u>Proof</u>. Assume [8.4] is not true. Then there exists $\varepsilon \in (0,1)$ such that for each $i \in \mathbb{N}$ there exists $V_i \in \mathbf{V}_k(\mathbf{U}(0,1))$ s.t.

$0 \in \operatorname{spt}\|V_i\|$, $\|\delta V_i\|\,\mathbf{U}(0,1) \leq \frac{1}{i}$, $\Theta^k(\|V_i\|, x) \geq 1$ for $\|V\|$ a.a. $x \in \mathbf{U}(0,1)$.

$\|V_i\|\,\mathbf{U}(0,1) \leq \left(1+\frac{1}{i}\right)\alpha(k)$, $V_i$ satisfies $\mathbb{H}(k)$

and $\displaystyle\int |\mathbf{h}(V_i,\cdot)|^k \, d\|V_i\| \leq \frac{1}{\gamma(k)+\varepsilon}$

but for each $T \in G(n,k)$ there holds

$$\sup\left\{ \left| \operatorname{dist}(x, \operatorname{spt}\|V_i\|) - \operatorname{dist}(x,T) \right| : x \in \mathbf{U}(0,1-\varepsilon) \right\} > \varepsilon .$$

W.l.o.g. $V_i \longrightarrow C \in \mathbf{V}_k(\mathbf{U}(0,1))$

$$\boxed{\Delta = \frac{1}{\gamma(k)} - \frac{1}{\gamma(k)+\varepsilon} > 0}$$

[4.11] $\Rightarrow$ $\|\delta C\|\,\mathbf{U}(0,1) = 0$

⊛ $\|C\|\,\mathbf{U}(0,1) \leq \alpha(k)$

[5.4] $\Rightarrow$ $\Theta^k(\|C\|, x) \geq 1$ for $\|C\|$ almost all $x \in \mathbf{U}(0,1)$

☺ [8.3] with $\alpha = \frac{1}{\gamma(k)+\varepsilon}$ $\Rightarrow$ $\dfrac{\|V_i\|\,\mathbf{B}(a,r)}{r^k} \geq \left(\dfrac{\Delta}{k}\right)^k$ for $r \in (0, 1-|a|)$, $a \in \operatorname{spt}\|V_i\|$

Hence, $0 \in \operatorname{spt}\|C\|$

[5.3] $\Rightarrow$ $\exists\, T \in G(n,k)$ $\quad C = \overbrace{\Theta^k(\|C\|, 0)}^{=1 \text{ by monotonicity } + ⊛}\, \mathbf{V}_k(T \cap \mathbf{U}(0,1))$.

Therefore, ☺ implies that

$$\sup\left\{ \left| \operatorname{dist}(x, \operatorname{spt}\|V_i\|) - \operatorname{dist}(x, \underset{\operatorname{spt}\|C\|}{\underset{\parallel}{T}}) \right| : x \in \mathbf{U}(0,1-\varepsilon) \right\} \xrightarrow[i \to \infty]{} 0$$

$\square$.

$V \in \mathbb{V}_k(\mathbb{R}^n)$, $a \in \operatorname{spt}\|V\|$



Recall:

- if $\|\delta V\| \mathbb{B}(a,r) \le M \|V\| \mathbb{B}(a,r)$ for $0 < r < R$

  then $r \longmapsto \frac{1}{r^k} \|V\| \mathbb{B}(a,r) \cdot \exp(Mr)$ is non-decreasing for $r \in (0,R)$.

  $\Downarrow$ $\Theta^k(\|V\|, a)$ exists

- if $V$ satisfies $\mathbb{H}(p)$ for some $p > k$,

  $$\left( \int_{\mathbb{B}(a,R)} |\mathbf{h}(V,\cdot)|^p \, d\|V\| \right)^{1/p} = \Gamma \in (0,\infty)$$

  then

  $$r \longmapsto \left( \frac{\|V\| \mathbb{B}(a,r)}{r^k} \right)^{1/p} + \frac{\Gamma}{p-k} r^{1-\frac{k}{p}}$$ is non-decreasing for $r \in (0,R)$.

[8.6] [Sim83, 17.8] if $V \in \mathbb{V}_k(\mathbb{R}^n)$, $G \subseteq \mathbb{R}^n$ open, $p > k$, $V$ satisfies $\mathbb{H}(p)$ in $G$

then $\Theta^k(\|V\|, \cdot)$ is u.s.c. on $G$.

**8.8** $\forall \varepsilon \in (0,1)\ \exists \eta \in (0,\infty)\ \forall V \in \mathbf{V}_k(\mathbb{R}^n)\ \forall a \in \operatorname{spt}\|V\|$
$\forall R \in (0,\infty)\ \forall d \in (0,\infty)$

if
- $V$ satisfies $\mathbb{H}(p)$ with $\left(\int_{\mathbb{U}(a,R)} |\mathbf{h}(V,\cdot)|^p \, d\|V\|\right)^{\frac{1}{p}} \le \eta\, d^{\frac{1}{p}} R^{\left(\frac{k}{p}-1\right)}$
- $\|V\|\,\mathbb{U}(a,R) \le (1+\eta)\,\alpha(k)\, d\, R^k$
- $\Theta^k(\|V\|,x) \ge d$ for $\|V\|$ almost all $x \in \mathbb{U}(a,R)$

then
$$(1-\varepsilon)d \le \frac{\|V\|\,\mathbb{B}(b,r)}{r^k \alpha(k)} \le (1+\varepsilon)d$$
for $b \in \mathbb{U}(a,(1-\varepsilon)R) \cap \operatorname{spt}\|V\|$
$0 < r < (1-\varepsilon)(R-|a-b|)$.

Moreover
$\forall b \in \mathbb{U}(a,(1-\varepsilon)R) \cap \operatorname{spt}\|V\|$
$\forall r \in (0,(1-\varepsilon)(R-|a-b|))$
$\exists X(b,r)$
$$\sup\left\{ \left| \operatorname{dist}(x, X(b,r)) - \operatorname{dist}(x, \operatorname{spt}\|V\|) \right| : x \in \mathbb{U}(b,r) \right\} \le \varepsilon r.$$

---

Proof. Wlog $a = 0$. Note $\mathbb{U}(0,R) = \mu_R[\mathbb{U}(0,1)]$ so from [8.2]

$$\left(\int_{\mathbb{U}(0,1)} |\mathbf{h}(\bar V,\cdot)|^p \, d\|\bar V\|\right)^{1/p} \le \eta, \quad \text{where } \bar V = \frac{1}{d}\,\mu_{1/R\,\#} V$$

and $\Theta^k(\bar V, x) \ge 1$ for $\|\bar V\|$ almost all $x \in \mathbb{U}(0,1)$.

and $\|\bar V\|\,\mathbb{U}(0,1) \le (1+\eta)\,\alpha(k)$

and $$\|\delta \bar V\|\,\mathbb{U}(0,1) = \int_{\mathbb{U}(0,1)} |\mathbf{h}(\bar V,\cdot)|\, d\|\bar V\| \le \eta\, \|\bar V\|\,\mathbb{U}(0,1)^{1-\frac{1}{p}} \le (1+\eta)^{1-\frac{1}{p}} \alpha(k)^{1-\frac{1}{p}} \cdot \eta$$

and $$\int_{\mathbb{U}(0,1)} |\mathbf{h}(\bar V,\cdot)|^k \, d\|\bar V\| \le \eta^k \cdot \|\bar V\|\,\mathbb{U}(0,1)^{1-\frac{k}{p}} \le \eta^k \left(\alpha(k)(1+\eta)\right)^{1-\frac{k}{p}}$$

Choose $\Delta_i \to 0$.



**Proof of 8.8 cont.** Assume 8.8. is not true.

Then there is $\varepsilon > 0$ and a sequence $V_i \in \mathbb{V}_k(\mathbb{R}^n)$, $a_i \in \operatorname{spt}\|V_i\|$, $R_i \in (0,\infty)$, $d_i \in (0,\infty)$ s.t.

- $V_i$ satisfies $\mathbb{H}(p)$ with $\left(\int_{\mathbb{U}(a_i,R_i)} |\mathbf{h}(V_i,\cdot)|^p \, d\|V_i\|\right)^{\frac{1}{p}} \leq \Delta_i d^{1/p} R_i^{\frac{k}{p}-1}$
- $\|V_i\| \, \mathbb{U}(a_i, R_i) \leq (1+\Delta_i)\alpha(k) d_i R_i^k$
- $\Theta^k(\|V_i\|, x) \geq d_i$ for $\|V_i\|$ almost all $x \in \mathbb{U}(a_i, R_i)$

and there exist $b_i \in \mathbb{U}(a_i, (1-\varepsilon)R_i) \cap \operatorname{spt}\|V_i\|$
$r_i \in (0, (1-\varepsilon)(R_i - |a_i - b_i|))$

s.t. either $\dfrac{\|V_i\| \, \mathbb{B}(b_i, r_i)}{r_i^k \, \alpha(k)} > (1+\varepsilon) d_i$ or $\dfrac{\|V_i\| \, \mathbb{B}(b_i, r_i)}{r_i^k \alpha(k)} < (1-\varepsilon) d_i$

We set $\bar{V}_i = \frac{1}{d_i} (\mu_{1/R_i} \circ \tau_{-a_i})_{\#} V_i \in \mathbb{V}_k(\mathbb{R}^n)$

Then

- $\bullet$ $\bar{V}_i$ sat. $\mathbb{H}(p)$ with $\left(\int_{\mathbb{U}(0,1)} |\mathbf{h}(V_i,\cdot)|^p d\|V_i\|\right)^{\frac{1}{p}} \leq \Delta_i$
- $\bullet\bullet$ $\|\bar{V}_i\| \, \mathbb{U}(0,1) \leq (1+\Delta_i)\alpha(k)$
- $\because$ $\Theta^k(\|\bar{V}_i\|, x) \geq 1$ for $\|\bar{V}_i\|$ almost all $x \in \mathbb{U}(0,1)$
- $::$ $\|\bar{V}_i\| \, \mathbb{B}\left(\frac{b_i - a_i}{R_i}, \frac{r_i}{R_i}\right) > (1+\varepsilon)\alpha(k)\left(\frac{r_i}{R_i}\right)^k$ or $\|\bar{V}_i\| \, \mathbb{B}\left(\frac{b_i - a_i}{R_i}, \frac{r_i}{R_i}\right) < (1-\varepsilon) \alpha(k)\left(\frac{r_i}{R_i}\right)^k$

Wlog $\bar{V}_i \llcorner \mathbb{U}(0,1) \times G(n,k) \to C \in \mathbb{V}_k(\mathbb{R}^n)$ and $\|\delta C\| \, \mathbb{U}(0,1) = 0$. by $\bullet$

We know $0 \in \operatorname{spt}\|\bar{V}_i\|$ and, by [Sim 83, 17.8],

$$(*) \quad \frac{\|\bar{V}_i\| \, \mathbb{B}(0,\varrho)}{\varrho^k} = \underbrace{\left(\frac{\|\bar{V}_i\| \, \mathbb{B}(0,\varrho)}{\varrho^k} + \frac{\Delta_i}{p-k} \varrho^{1-\frac{k}{p}}\right)}_{\substack{\text{non-decreasing} \\ \geq \alpha(k)}} - \underbrace{\frac{\Delta_i}{p-k} \varrho^{1-\frac{k}{p}}}_{\xrightarrow[i\to\infty]{} 0}$$

Therefore, $0 \in \operatorname{spt}\|C\|$.

In particular, the second alternative in $::$ cannot hold for $i \in \mathbb{N}$ large enough

Moreover, $\|C\| \, \mathbb{U}(0,1) \leq \alpha(k)$ by $\bullet\bullet$ and, by [5.4] and $\because$, $\Theta^k(\|C\|, x) \geq 1$ for $\|C\|$ almost all $x \in \mathbb{U}(0,1)$

Thus, [5.3] yields $T \in G(n,k)$ s.t. $C = \mathbb{V}_k(T \cap \mathbb{U}(0,1))$.

However, this contradicts $::$

Proof of [8.8] cnt.

Indeed, wlog $\frac{b_i - a_i}{R_i} \longrightarrow c \in \mathbb{U}(0,1)$ and $\frac{r_i}{R_i} \longrightarrow s \in [0, 1-\varepsilon]$

If $s = 0$, then we use ⊛ to replace $r_i$ with $\underbrace{\tfrac{1}{2}(1-\varepsilon)(R_i - |a_i - b_i|)}_{\tilde{r}_i}$ so that for large enough $i \in \mathbb{N}$ we still get

$$\|\bar{V}_i\| \mathbb{B}\left(\frac{b_i - a_i}{R_i}, \tilde{r}_i\right) > (1+\varepsilon)\,\alpha(k)\left(\frac{\tilde{r}_i}{R_i}\right)^k$$

and then $s = \lim_{i\to\infty} \frac{\tilde{r}_i}{R_i} > 0$.

Since $\frac{b_i - a_i}{R_i} \in \operatorname{spt}\|\bar{V}_i\|$ we argue using ⊛ to see that $c \in \operatorname{spt}\|C\|$.

Now, for any $\delta > 0$ we have

$$\|C\| \mathbb{B}(c, (1+\delta)s) \overset{\text{u.s.c.}}{\geq} \lim_{i\to\infty} \|\bar{V}_i\| \mathbb{B}(c, (1+\delta)s)$$

since $\frac{b_i - a_i}{R_i} \to c$, $\frac{\tilde{r}_i}{R_i} \to s$ we have $\mathbb{B}(c, (1+\delta)) \supseteq \mathbb{B}\left(\frac{b_i - a_i}{R_i}, \frac{\tilde{r}_i}{R_i}\right)$ for large enough $i \in \mathbb{N}$

$$\geq \lim_{i\to\infty} \|\bar{V}_i\| \mathbb{B}\left(\frac{b_i - a_i}{R_i}, \frac{\tilde{r}_i}{R_i}\right)$$

$$\geq \lim_{i\to\infty} (1+\varepsilon)\,\alpha(k)\left(\frac{\tilde{r}_i}{R_i}\right)^k = (1+\varepsilon)\,\alpha(k)\, s^k$$

But $\|C\| = \mathcal{H}^k \llcorner T$

so $\|C\| \mathbb{B}(c, (1+\delta)s) = (1+\delta)^k s^k$

If $(1+\delta)^k < (1+\varepsilon)$, we get a contradiction.

The second part of [8.8] is a direct consequence of [8.4] applied to $W = \frac{1}{d_i}(\mu_{1/r} \circ \tau_{-b})_* V$

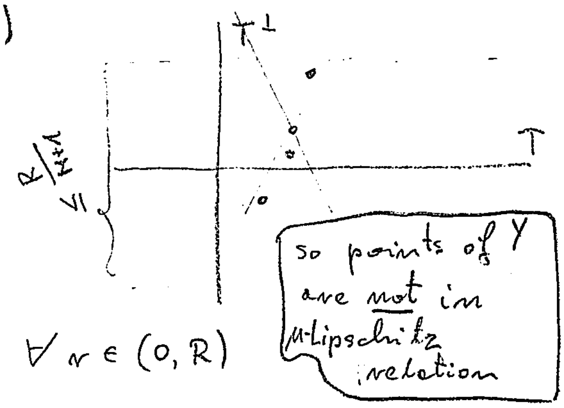
where $b \in \mathbb{U}(a, (1-\varepsilon)R) \cap \operatorname{spt}\|V\|$

and $r \in (0, (1-\varepsilon)(R - |a-b|))$. □

Corollary. If $\varepsilon > 0$ is small ~~enough~~, then [8.8] shows that one can apply the Reifenberg topological disc theorem, which asserts that there exists a bi-hölder map $\varphi : T \cap \mathbb{B}(0,1) \longrightarrow \mathbb{R}^n \in \mathcal{C}^{0,\alpha}$ $[\alpha = \alpha(\varepsilon)]$ such that $\operatorname{spt}\|V\| \cap \mathbb{B}(a, \rho) = \operatorname{im}\varphi \cap \mathbb{B}(a, \rho)$ for some $\rho > 0$.

**6.1 Lemma:** Suppose

(1) $\nu \in \mathbb{N}_+$, $\mu \in [1,\infty)$, $\xi \in (0,1)$, $M \in (1,\infty)$, $R \in (0,\infty)$
$T \in G(n,k)$, $Y \subseteq \mathbb{R}^n$, $V \in \mathbb{V}_k(\mathbb{R}^n)$




(2) $\# Y \le \nu + 1$

(3) $\forall y, z \in Y \quad |y-z| \le \mu \, |T_\natural^\perp(y-z)|$

(4) $\operatorname{diam} Y \le \dfrac{R}{M+1}$

(5) $R \, \|\delta V\| \, \mathbb{B}(y,r) \le \xi \, \|V\| \, \mathbb{B}(y,r) \quad \forall y \in Y \quad \forall r \in (0,R)$

(6) $\displaystyle\int_{\mathbb{B}(y,r)\times G(n,k)} \|S_\natural - T_\natural\| \, dV(x,S) \le \xi \, \|V\| \, \mathbb{B}(y,r) \quad \forall y \in Y \quad \forall r \in (0,R)$

Then $\exists V_1, V_2 \in \mathbb{V}_k(\mathbb{R}^n) \quad \exists Y_0, Y_1, Y_2 \subseteq Y$
$Y_0 \cup Y_1 \cup Y_2 = Y, \quad Y_i \cap Y_j = \emptyset \quad \forall i \ne j$

(7) $V \ge V_1 + V_2$

(8) $\# Y_1 \le \nu$ and $\# Y_2 \le \nu$

(9) $(M \cdot \operatorname{diam} Y) \cdot \|\delta V_j\| \, \mathbb{B}(y,r) \le 2M(\nu+1)(\exp \xi)(3\nu M\mu)^{k+1} \xi \, \|V\| \, \mathbb{B}(y,r)$
$\forall j \in \{1,2\} \quad \forall y \in Y_j \quad \forall r \in (0, M\cdot\operatorname{diam} Y)$

(10) $\displaystyle\int_{\mathbb{B}(y,r)\times G(n,k)} \|S_\natural - T_\natural\| \, dV_j(x,S) \le M \, (\underbrace{3\nu M\mu}_{\ge 1})^k \xi \, \|V_j\| \, \mathbb{B}(y,r)$
$\forall j \in \{1,2\} \quad \forall y \in Y_j \quad \forall r \in (0, M\cdot\operatorname{diam} Y)$

(11) $\Theta^k(\|V\|, y) = \Theta^k(\|V_j\|, y) \quad \forall j \in \{1,2\} \; \forall y \in Y_j$

(12) $$\left(\left(1+\frac{1}{M}\right)^k + \frac{\nu+1}{M}\right)(\exp \xi) \, \frac{\|V\|\left(\bigcup\{\mathbb{B}(y,R) : y \in Y\}\right)}{R^k}$$
$$\ge \sum_{y \in Y_0} \Theta^k(\|V\|, y) + \sum_{j=1}^{2} \frac{\|V_j\|\left(\bigcup\{\mathbb{B}(y, M\cdot\operatorname{diam} Y) : y \in Y_j\}\right)}{(M\cdot\operatorname{diam} Y)^k}$$

## Proof of 6.1

From 5.1(3) and (5) we have

(13) $$\frac{\|V\|\mathbb{B}(y,r)}{r^k}\exp(3/R\cdot r) \le \frac{\|V\|\mathbb{B}(y,s)}{s^k}\exp(3/R\cdot s) \quad \text{for } y\in Y \text{ and } 0<r<s<R.$$

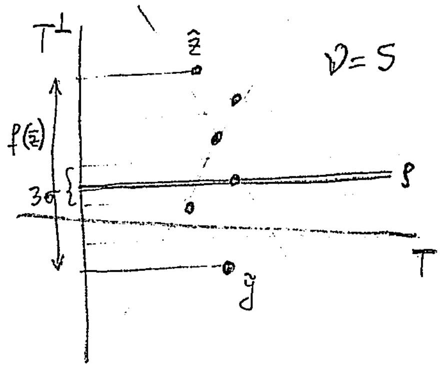
Choose $\tilde{y},\hat{z}\in Y$ s.t. $|\tilde{y}-\hat{z}| = \operatorname{diam} Y$

Set $\eta = M\cdot\operatorname{diam} Y$, $\boxed{f(x) = |T_h^\perp(x-\tilde{y})|}$, $\sigma = \frac{f(\hat{z})}{3\nu}$, $f:\mathbb{R}^n\to\mathbb{R}$

Choose $\rho\in(0,f(\hat{z}))$ s.t. $\forall y\in Y$ $f(y)\le\rho$ or $f(y)\ge\rho+3\sigma$ $= \frac{f(\hat{z})}{\nu}$

Note that by (3) we have

(14) $$\frac{\eta}{\sigma} = 3\nu\frac{M|\tilde{y}-\hat{z}|}{|T^\perp(\tilde{y}-\hat{z})|} \le 3\nu M\mu$$




Set

$$Y_0 = Y\cap\{y : \|V\|\mathbb{B}(y,\eta) > M\,\Theta^k(\|V\|,y)\,\alpha(k)\,\eta^k\}$$
$$Y_1 = Y\cap\{y : f(y)\le\rho\}\sim Y_0$$
$$Y_2 = Y\cap\{y: f(y)\ge\rho+3\sigma\}\sim Y_0$$

Note $\tilde{y}\notin Y_2$ and $\hat{z}\notin Y_1$ so (8) holds by (2).

For $\tau>0$ define $V_{1,\tau} = V\llcorner\{(x,S): f(x)<\tau\}$
$V_{2,\tau} = V\llcorner\{(x,S): f(x)>\tau\}$

For $y\in Y$ estimate

$$\int_{\rho+\sigma}^{\rho+2\sigma}\|\delta V_{1,\tau}\|\mathbb{B}(y,\eta)\,d\mathcal{L}^1(\tau) = \int_{-\rho-\sigma}^{-\rho-2\sigma}\|\delta V_{1,-\tau}\|\mathbb{B}(y,\eta)\,d\mathcal{L}^1(\tau)$$

$$\overset{4.10(2)}{\le} \int_{\mathbb{B}(y,\eta)\cap\{x:\rho+\sigma<f(x)<\rho+2\sigma\}} |S_h(\operatorname{grad} f(x))|\,dV(x,S) + \int_{\rho+\sigma}^{\rho+2\sigma}\|\delta V\|(\mathbb{B}(y,\eta)\cap\{f<\tau\})\,d\mathcal{L}^1(\tau) \quad \circledast$$

$$Df(x)u = \frac{T_h^\perp(x-\tilde{y})}{|T_h^\perp(x-\tilde{y})|}\bullet(T_h^\perp u) = \frac{T_h^\perp(x-\tilde{y})}{|T_h^\perp(x-\tilde{y})|}\bullet u \Rightarrow \operatorname{grad} f(x) = \frac{T_h^\perp(x-\tilde{y})}{|T_h^\perp(x-\tilde{y})|}$$

$$|S_h\operatorname{grad} f(x)| = \sup\{S_h\circ Df(x)^*1\bullet u : u\in\mathbb{R}^n, |u|=1\} = \|Df(x)\circ S_h\|$$

$$= \sup\left\{\frac{T_h^\perp(x-\tilde{y})}{|T_h^\perp(x-\tilde{y})|}\bullet(T_h^\perp\circ S_h)u : |u|=1\right\} = \frac{|S_h\circ T_h^\perp(x-\tilde{y})|}{|T_h^\perp(x-\tilde{y})|}$$

$$= \left|S_h\frac{T_h^\perp(x-\tilde{y})}{|T_h^\perp(x-\tilde{y})|}\right| \le \|S_h\circ T^\perp\| = \|S_h - T_h\|$$

$$\circledast \leq \int_{B(y,\eta)\cap\{x:\, \rho+\sigma < f(x) < \rho+2\sigma\}} \|S_h - T_h\| \, dV(x,S) + \sigma \cdot \|\delta V\| B(y,\eta) \qquad \boxed{\eta < R}$$

$$\overset{(5)\,\&\,(6)}{\leq} \zeta \|V\| B(y,\eta) + \sigma \frac{\zeta}{R} \|V\| B(y,\eta) = \zeta\left(1 + \frac{\sigma}{R}\right)\|V\| B(y,\eta).$$

$$\leq 2\zeta \|V\| B(y,\eta)$$

Similarly:
$$\int_{\sigma+\rho}^{\rho+2\sigma} \|\delta V_{2,\tau}\| B(y,\eta) \, d\mathcal{L}^1(\tau) \leq 2\zeta \|V\| B(y,\eta)$$

By Чебышев inequality there exists $t \in (\rho+\sigma, \rho+2\sigma)$ s.t.

$$(15) \qquad \|\delta V_{j,t}\| B(y,\eta) \leq (\nu+1) \frac{2\zeta}{\sigma} \|V\| B(y,\eta) \quad \forall y \in Y_j \quad \forall j \in \{1,2\}.$$

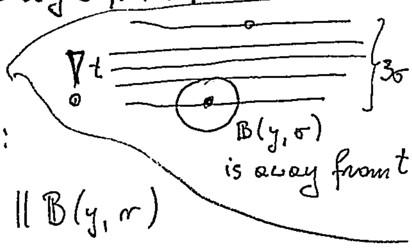
Define $V_1 = V_{1,t}$ and $V_2 = V_{2,t}$. Clearly (7) holds.

$$\boxed{\begin{aligned} &\rho+\sigma < t < \rho+2\sigma \\ \Rightarrow &\begin{cases} \rho < t-\sigma \\ t+\sigma < \rho+3\sigma \end{cases} \end{aligned}}$$

Since for $y \in Y_1$: $f(y) \leq \rho < t - \sigma$

and for $y \in Y_2$: $f(y) \geq \rho + 3\sigma > t + \sigma$

we have $\boxed{V_j \llcorner B(y,\sigma) \times G(n,k) = V \llcorner B(y,\sigma) \times G(n,k) \text{ for } y \in Y_j}$ and $j \in \{1,2\}$




so (11) holds.

It follows also that for $0 < r < \sigma$ and $y \in Y_j$:

(9) & (10) for small $r$.
$$\begin{cases} \eta \|\delta V_j\| B(y,r) \overset{(4)}{<} R \|\delta V_j\| B(y,r) \overset{(5)}{\leq} \zeta \|V_j\| B(y,r) \\ \displaystyle\int_{B(y,r)} \|S_h - T_h\| \, dV_j(x,S) \overset{(6)}{\leq} \zeta \|V_j\| B(y,r) \end{cases}$$

We employ (13) with $r \downarrow 0$ and $s = \sigma$ to get for $y \in Y \sim Y_0$

$$\boxed{\odot \quad \|V\| B(y,\eta) \overset{y \notin Y_0}{\leq} M \, \Theta^k(\|V\|, y)\, \alpha(k) \eta^k \overset{(13)}{\leq} M \left( \frac{\|V\| B(y,\sigma)}{\sigma^k} \exp\Big(\underbrace{\tfrac{\zeta}{R}\sigma}_{\leq \zeta}\Big) \right) \eta^k}$$

Hence, for $r \in [\sigma, \eta)$ we get

$$\eta \|\delta V_j\| B(y,r) \leq \eta \|\delta V_j\| B(y,\eta) \overset{(15)}{\leq} \eta(\nu+1)\frac{2\zeta}{\sigma} \|V\| B(y,\eta)$$

$$\overset{\odot}{\leq} \eta(\nu+1)\frac{2\zeta}{\sigma} M \Big( \sigma^{-k} \overbrace{\|V\| B(y,\sigma)}^{=\|V_j\| B(y,\sigma)} \exp(\zeta) \Big) \eta^k$$

$$\overset{(14)}{\leq} 6\nu M_\mu \cdot \zeta \cdot (\nu+1) \cdot M \cdot (3\nu M_\mu)^k \cdot \|V_j\| B(y,r) \qquad (r \geq \sigma)$$

$$= 2\,(3\nu M_\mu)^{k+1} \cdot M \cdot (\nu+1) \cdot \zeta \, \|V_j\| B(y,r)$$

$$\Longrightarrow (9).$$

We compute still for $\sigma \le r < \eta$

$$\int_{B(y,r)} \|S_\natural - T_\natural\| \, dV_j(x,S) \overset{(6)}{\le} \xi \|V_j\| B(y,\eta) \le \xi M \Big(\sigma^{-k} \overbrace{\|V\| B(y,\sigma)}^{=\|V_j\| B(y,\sigma)} \exp(\xi)\Big) \eta^k$$

$$\overset{(14)}{\le} \xi \cdot M \cdot (3\nu M_\mu)^k \|V_j\| B(y,r) \leftarrow \boxed{r \ge \sigma}$$

$$\Rightarrow (10).$$

To prove (12) we write

$$\eta^{-k} \sum_{j=1}^{2} \|V_j\| \big(\cup\{B(y,\eta) : y \in Y_j\}\big) \overset{(7)}{\le} \eta^{-k} \|V\| \; B(\tilde{y}, (M+1)\cdot \operatorname{diam} Y)$$

$$= \Big(1+\frac{1}{M}\Big)^k \big((M+1) \operatorname{diam} Y\big)^{-k} \|V\| B(\tilde{y}, (M+1) \operatorname{diam} Y)$$

$$\overset{(4),(13)}{\le} \Big(1+\frac{1}{M}\Big)^k R^{-k} \|V\| B(\tilde{y}, R) \cdot \exp\Big(\underbrace{\frac{\xi}{R}\cdot (R - (M+1)\operatorname{diam} Y)}_{\le \xi}\Big)$$

and

$$\alpha(k) \sum_{y \in Y_0} \Theta^k(\|V\|, y) \le \frac{1}{M} \sum_{y\in Y_0} \eta^{-k} \|V\| B(y,\eta) \le$$

$$\overset{(4),(13)}{\le} \frac{1}{M} \sum_{y \in Y_0} R^{-k} \|V\| B(y,R) \exp\Big(\overbrace{\tfrac{\xi}{R}(R-\eta)}^{\le \xi}\Big)$$

$$\le \frac{\nu+1}{M} \exp(\xi) \; R^{-k} \|V\| \big(\cup\{B(y,R) : y \in Y\}\big)$$

$$\Rightarrow (12) \qquad \square$$

---

## 6.2 Theorem

$\forall \nu \in \mathbb{N}_+ \quad \forall \lambda \in (1,\infty) \quad \forall \mu \in [1,\infty) \qquad \exists \gamma \in (0,\infty)$

$\forall V \in \mathbb{V}_k(\mathbb{R}^n) \quad \forall Y \subseteq \mathbb{R}^n \quad \forall R \in (0,\infty) \quad \forall T \in G(n,k)$

$$\# Y \le \nu, \quad \operatorname{diam} Y \le \gamma R, \quad |y-z| \le \mu |T^\perp(y-z)| \;\; \forall y,z \in Y$$

$$\left.\begin{aligned} &R \|\delta V\| B(y,r) \le \gamma \|V\| B(y,r) \\ &\int_{B(y,r)} \|S_\natural - T_\natural\| \, dV(x,S) \le \gamma \|V\| B(y,r) \end{aligned}\right\} \quad \forall y \in Y \quad \forall r \in (0,R)$$

$$\Longrightarrow \quad \alpha(k) \sum_{y \in Y} \Theta^k(\|V\|, y) \le \lambda R^{-k} \|V\| \big(\cup\{B(y,R) : y \in Y\}\big)$$

Proof of 6.2

We shall prove the claim by induction w.r.t. $\nu \in \mathbb{N}_+$.

Case $\nu = 1$: Let $\lambda \in (1,\infty)$ and $\mu \in [1,\infty)$ be given.

Set $\boxed{\gamma = \ln \lambda \in (0,\infty)}$. From 5.1(3) we deduce:

if $y \in \mathbb{R}^n$, $V \in \mathbb{V}_k(\mathbb{R}^n)$, $R \in (0,\infty)$, $\|\delta V\| \mathbb{B}(y,r) \le \frac{\gamma}{R} \|V\| \mathbb{B}(y,r)$ $\forall r \in (0,R)$

then $\alpha(k)\, \Theta^k(\|V\|, y) \le \underbrace{\exp\left(\frac{\gamma}{R} R\right)}_{=\lambda} R^{-k} \|V\| \mathbb{B}(y,R)$

Inductive step:

Assume the claim holds for all $\nu \in \{1,2,\dots,\nu_0\}$ for some $\nu_0 \in \mathbb{N}_+$.

Let $\lambda \in (1,\infty)$ and $\mu \in [1,\infty)$ be given and set $\varepsilon = \lambda - 1$.

There exists $\gamma_0 = \gamma(\nu_0, 1+\frac{\varepsilon}{4}, \mu)$.

Let $M > \frac{2}{\gamma_0}$ and $\xi \in (0,1)$ be s.t.

$$2M(\nu_0+1)(3\nu_0 M \mu)^{k+1} \xi \le \gamma_0$$

and $$\left(\left(1+\frac{1}{M}\right)^k + \frac{\nu_0+1}{M}\right)\exp(\xi) \le 1 + \frac{\varepsilon}{4}$$

Choose any $\gamma \in (0,\infty)$ s.t. $\gamma < \min\{\xi, \frac{1}{M+1}\}$.

Let $V, Y, R, T$ be as in the theorem for $\nu = \nu_0 + 1$

$$(M+1)\operatorname{diam} Y \le (M+1)\gamma R \le R$$

Now (1)-(6) of $\boxed{6.1}_{\nu=\nu_0}$ are satisfied and we obtain

$Y = Y_0 \,\dot\cup\, Y_1 \,\dot\cup\, Y_2$ and $V_1, V_2$ satisfying (7)-(12).

We have

$$\left(1+\frac{\varepsilon}{4}\right) R^{-k} \|V\|\left(\bigcup\{\mathbb{B}(y,R) : y \in Y\}\right) \ge R^{-k}\left(\left(1+\frac{1}{M}\right)^k + \frac{\nu_0+1}{M}\right)\exp(\xi)\, \|V\|\left(\bigcup\{\mathbb{B}(y,R) : y \in Y\}\right)$$

$$\overset{6.1\,(12)}{\ge} \alpha(k) \sum_{y \in Y_0} \Theta^k(\|V\|, y) + \sum_{j=1}^{2} \underbrace{(M \cdot \operatorname{diam} Y)^{-k} \|V_j\|\left(\bigcup\{\mathbb{B}(y, M\cdot \operatorname{diam} Y) : y \in Y_j\}\right)}_{\ge \left(1+\frac{\varepsilon}{4}\right)^{-1} \alpha(k) \sum_{y \in Y_j} \Theta^k(\|V\|, y) \text{ by inductive hypothesis}}$$

To use inductive hypothesis with:

"$R$" $= M \cdot \operatorname{diam} Y$ and "$\gamma$" $= \gamma_0$ and "$\nu$" $= \nu_0$

we need to check: $*\ Y_j \le \nu_0$ ✓, $M \cdot \operatorname{diam} Y \le R$ ✓

$\operatorname{diam} Y_j \le \operatorname{diam} Y \le 2 \operatorname{diam} Y < \gamma_0 \cdot \underbrace{M}_{>2} \operatorname{diam} Y$ ✓

$$\overset{(11)}{\Longrightarrow} \left(1+\frac{\varepsilon}{4}\right)^2 R^{-k} \|V\|\left(\bigcup\{\mathbb{B}(y,R) : y \in Y\}\right) \ge \alpha(k) \sum_{y \in Y} \Theta^k(\|V\|, y) \qquad \square$$

**8.10 Definition:** $T \in G(n,k)$, $a \in \mathbb{R}^n$, $r \in (0,\infty)$

$$C(T,a,r) = \mathbb{R}^n \cap \{x : |T_\natural(x-a)| < r\}$$

**8.11 Lemma:** [size of holes] $V \in \mathbb{V}_k(\mathbb{R}^n)$, $r \in (0,\infty)$, $\zeta \in (0,1)$, $T \in G(n,k)$, $y \in T$
$T^\perp[\operatorname{spt}\|V\|]$ is compact and

$$\|V\|\, C(T,y,\zeta r) = 0$$

Then

$$(\zeta(k+1)-k)\,\|V\|\, C(T,y,r) \le \int_{C(T,y,r)\times G(n,k)} \|S_\natural - T_\natural\|^2 \, dV(x,S) + r\,\|\delta V\|\, C(T,y,r)$$

Proof. Wlog. $y = 0$. Suppose $\theta : \mathbb{R} \to \mathbb{R}$ is smooth and $\operatorname{spt}\theta \subseteq (-\infty, r)$, $0 \notin \operatorname{spt}\theta'$, $\theta \le 0$, and $\theta' \ge 0$. Let $g \in \mathcal{X}(\mathbb{R}^n)$ be given by $\boxed{g(x) = \theta(|T_\natural x|)\cdot T_\natural x}$ for $x$ near $\operatorname{spt}\|V\|$

For $V$ almost all $(x,S)$ we get $(u \in S)$

$$Dg(x)\,u = \theta'(|Tx|)\left(\frac{Tx}{|Tx|}\bullet Tu\right) Tx + \theta(|Tx|)\, Tu$$

$$\begin{aligned}
Dg(x)\bullet S &= \operatorname{tr}(Dg(x)\circ S) = \theta'(|Tx|)\,(S\,Tx)\bullet\frac{Tx}{|Tx|} + \theta(|Tx|)\, T\bullet S \\
&= \theta'(|Tx|)\,|Tx|\,\left|S\frac{Tx}{|Tx|}\right|^2 + \theta(|Tx|)\, T\bullet S \\
&= \theta'(|Tx|)|Tx| - \theta'(|Tx|)\left|S^\perp\frac{Tx}{|Tx|}\right|^2|Tx| + k\,\theta(|Tx|) \underbrace{- \theta(|Tx|)\overbrace{T^\perp\bullet S}^{0 \,\le\, \frac{1}{2}|S_\natural - T_\natural|^2 \text{ 8.9(1)}}}_{\ge 0 \text{ This we omit}} \\
&\ge \theta'(|Tx|)|Tx| + k\,\theta(|Tx|) - \theta'(|Tx|)\,\|S_\natural - T_\natural\|^2|Tx|
\end{aligned}$$

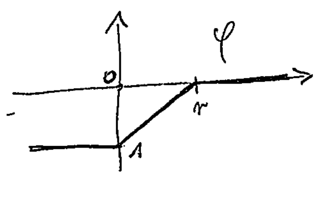
Consequently:

$$\int \theta'(|Tx|)\,|Tx| + k\,\theta(|Tx|)\, d\|V\|(x) \le \int \theta'(|Tx|)\,|Tx|\,\|S-T\|^2\, dV(x,S) + \delta V(g)$$

This integral is in fact over $C(T,0,r) \sim C(T,0,\zeta r)$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(s) = \begin{cases} -1 & \text{if } s < 0 \\ -1 + s/r & \text{if } 0 \le s < r \\ 0 & \text{if } r \le s \end{cases}$$




Observe that for $\zeta r < s < r$ we have: $\zeta < \frac{s}{r} < 1$ and

$$\varphi'(s)\cdot s + k\,\varphi(s) = \frac{s}{r} + k\left(-1 + s/r\right) > \zeta - k + k\zeta = \zeta(k+1) - k,$$

$$\varphi'(s)\cdot s = \frac{s}{r} < 1, \quad \text{and} \quad |\varphi(s)\cdot s| < r$$

Complete the proof by choosing $\theta$ approximating $\varphi$. □

8.12 Lemma:

$\forall \varepsilon \in (0,1)$ $\forall \mu \in (1,\infty)$ $\exists \xi \in (0,\infty)$ $\exists P \in (0,\infty)$

Lipschitz approximation

If (1) $V \in \mathbb{V}_k(\mathbb{R}^n)$ and $T \in G(n,k)$

(2) for $\|V\|$ almost all $x \in \mathbb{C}(T,0,5)$

Not more than one layer

$\Theta^k(\|V\|,x) \geq 1$, $\|V\|\mathbb{B}(x,1) \leq (2-\varepsilon)\alpha(k)$,

Some lower density ratio bound → $\|V\|\mathbb{B}(x,r) \geq \varepsilon r^k$ for $r \in (0,1)$

(3) $V$ satisfies $\mathbb{H}(2)$ and

$$\left(\int_{\mathbb{C}(T,0,5)} |\mathbf{h}(V,\cdot)|^2 \, d\|V\|\right)^{1/2} = \alpha \in [0,\infty)$$

(4) $\|V\|\,\mathbb{C}(T,0,1) > 0$

(5) $\operatorname{dist}(x,T) \leq \xi$ $\forall x \in \operatorname{spt}\|V\|$ ← A priori flatness

then $\exists f: T \to T^\perp$

(6) $F: T \to \mathbb{R}^n$, $F(x) = x + f(x)$

(7) $\sup \operatorname{im}|f| \leq \sup\{\operatorname{dist}(x,T) : x \in \operatorname{spt}\|V\|\}$

$\operatorname{Lip} f \leq \frac{1}{\sqrt{\mu^2-1}}$

(8) $\forall y,z \in T$ $|f(y)-f(z)| \leq \frac{1}{\sqrt{\mu^2-1}}|y-z|$

(9) $\|V\|\big(\mathbb{C}(T,0,1) \sim \operatorname{im} F\big) + \mathcal{H}^k\big(T \cap \mathbb{U}(0,1) \cap \{y : \Theta^k(\|V\|,F(y)) < 1\}\big)$

$$\leq P\left(\alpha^2 + \int_{\mathbb{C}(T,0,5)\times G(n,k)} \|S-T\|^2 \, dV(x,S)\right).$$

First variation + $L^2$-tilt-excess controll the holes

Proof. Let $\varepsilon \in (0,1)$ and $\mu \in (1,\infty)$ be given. We need to find $\xi$ and $P$.

Assume $V, T, \alpha$ satisfy (1)-(4).

Let $\beta^2 = \int_{\mathbb{C}(T,0,5)\times G(n,k)} \|S-T\|^2 \, dV(x,S)$.

Choose $\varrho \in (0,1)$ and $\lambda \in (1,\infty)$ s.t.

(☺) $\lambda \frac{2-\varepsilon}{(1-\varrho)^k} < 2$

Apply $\boxed{6.2}$ to find $\tilde{\gamma}$ for "$\nu$" $= 2$, | Set $\gamma = \min\{\frac{1}{12}, \tilde{\gamma}\}$

$\tilde{\gamma} = \gamma_{6.2}(\nu,\lambda,\mu) = \gamma_{6.2}(2,\lambda,\mu)$

Let $A \subseteq \mathbb{C}(T,0,4) \cap \operatorname{spt}\|V\|$ contain points $a$ s.t. $\Theta^k(\|V\|,a) \geq 1$

$$\left.\begin{aligned} &\text{and} \quad \|\delta V\|\,\mathbb{U}(a,r) \leq \gamma\, \|V\|\,\mathbb{B}(a,r) \\ &\text{and} \quad \int_{\mathbb{B}(a,r)\times G(n,k)} \|S-T\|^2 dV(x,S) \leq \gamma^2 \|V\|\,\mathbb{B}(a,r) \end{aligned}\right\} \quad \forall r \in (0,1)$$

Let $B = (\mathbb{C}(T,0,4) \cap \operatorname{spt}\|V\|) \sim A$ and $C = T_\natural[B]$ and

$\quad D = T \cap \mathbb{U}(0,1) \sim T_\natural\,[\operatorname{spt}\|V\|]$

Clearly $\displaystyle\int_{\mathbb{B}(a,r)\times G(n,k)} \|S-T\|\,dV(x,S) \leq \left(\int_{\mathbb{B}(a,r)\times G(n,k)} \|S-T\|^2 dV(x,S)\right)^{\frac{1}{2}} \|V\|\,\mathbb{B}(a,r)^{\frac{1}{2}} \leq \gamma\,\|V\|\,\mathbb{B}(a,r)$

whenever $a \in A$ and $r \in (0,1)$.

If $a, \tilde{a} \in A$ and $|a-\tilde{a}| \leq \min\{\rho, \gamma(1-\rho)\}$,

then if $|a-\tilde{a}| \leq \mu\,|T^\perp(a-\tilde{a})|$, then by 6.2 we have:

$$\begin{aligned} 2\,\alpha(k) &\leq \left(\Theta^k(\|V\|,a) + \Theta^k(\|V\|,\tilde{a})\right)\alpha(k) \\ &\leq \lambda\,(1-\rho)^{-k}\,\|V\|\left(\mathbb{B}(a,1-\rho)\cup\mathbb{B}(\tilde{a},1-\rho)\right) \\ &\leq \lambda\,(1-\rho)^{-k}\,\|V\|\,\mathbb{B}(a,1) \\ &\leq \frac{\lambda}{(1-\rho)^k}\,(2-\varepsilon)\,\alpha(k) \end{aligned}$$

which is impossible by (29); !

Not enough measure for 2 layers

hence, $|a-\tilde{a}| > \mu\,|T^\perp(a-\tilde{a})|$. ← $\forall a,\tilde{a} \in A$ $a$ and $\tilde{a}$ are in $\mu$-Lip. relation

Therefore, there exists $\xi > 0$ such that if (5) holds,

then $\forall a, \tilde{a} \in A \quad |a-\tilde{a}| > \mu\,|T^\perp(a-\tilde{a})|$

or equivalently:

$$\forall a,\tilde{a} \in A \quad |T^\perp(a-\tilde{a})| < (\mu^2-1)^{-\frac{1}{2}}\,|T(a-\tilde{a})|$$

Hence, by Kirszbraun's theorem, there exists $f: T \to T^\perp$ s.t. (6), (7), (8) hold and for $a \in A$ $f(Ta) = T^\perp a$ and

(10) $\quad A \subseteq \operatorname{im} F$ $\qquad$ ← clear from the definition of $F$

$\Rightarrow \operatorname{spt}\|V\| \cap \mathbb{C}(T,0,1) \sim \operatorname{im} F \subseteq B$

Also (2) implies

$$T \cap \mathbb{U}(0,1) \cap \{y : \Theta^k(\|V\|, F(y)) < 1\} \subseteq C \cup D$$

Either $(y+T^\perp)\cap\operatorname{spt}\|V\| = \emptyset$ and $y \in D$, or $y \in T_\natural[\operatorname{spt}\|V\|]$ and $F(y) \notin A$, so $y \in C$.

Hence, it suffices to estimate $\|V\|(B)$, $\mathcal{H}^k(C)$, and $\mathcal{H}^k(D)$.

$\boxed{\|V\|(B)}$

$$B = \mathbb{C}(T,0,4) \cap \operatorname{spt}\|V\| \sim A$$
$$= \Big\{ a \in \mathbb{C}(T,0,4) \cap \operatorname{spt}\|V\| : \exists r \in (0,1) \quad \|\delta V\|\,\mathbb{U}(a,r) > \gamma\,\|V\|\,\mathbb{B}(a,r) \text{ or } \int_{\mathbb{B}(a,r)\times G(n,k)} \|S-T\|^2 \, dV(x,S) > \gamma^2 \|V\|\,\mathbb{B}(a,r) \Big\}$$

Cover $B$ with balls $\mathbb{B}(a,r_a)$ for $a \in B$ s.t.

(FV) either: $\|V\|\,\mathbb{B}(a,r_a) < \frac{1}{\gamma}\|\delta V\|\,\mathbb{U}(a,r_a)$

(Tilt) or: $\|V\|\,\mathbb{B}(a,r_a) < \frac{1}{\gamma^2}\int_{\mathbb{B}(a,r_a)\times G(n,k)} \|S-T\|^2\, dV(x,S)$

Apply the Besicovitch covering theorem to obtain $\mathcal{B}(n)$ disjointed families of balls $\mathcal{B}_1, \dots, \mathcal{B}_{\mathcal{B}(n)}$ covering $B$.

For each $i \in \{1,2,\dots,\mathcal{B}(n)\}$ decompose $\mathcal{B}_i$ into $\mathcal{B}_i^{FV} \cup \mathcal{B}_i^{Tilt}$

Since $\mathcal{B}_i^{FV}$ is disjointed we have

$$\sum_{K\in\mathcal{B}_i^{FV}} \|V\|(K) < \frac{1}{\gamma}\sum_{K\in\mathcal{B}_i^{FV}} \|\delta V\|(K) = \frac{1}{\gamma}\|\delta V\|\left(\cup \mathcal{B}_i^{FV}\right) \le \frac{\alpha}{\gamma}\|V\|\left(\cup\mathcal{B}_i^{FV}\right)^{1/2}$$
$$= \frac{\alpha}{\gamma}\Big(\sum_{K\in\mathcal{B}_i^{FV}}\|V\|(K)\Big)^{1/2}$$

Hence,
$$\sum_{K\in\mathcal{B}_i^{FV}} \|V\|(K) \le \left(\frac{\alpha}{\gamma}\right)^2 \Rightarrow \|V\|(\cup\mathcal{B}_i) \le \frac{\alpha^2+\beta^2}{\gamma^2}$$

(11) $$\Rightarrow \|V\|(B) \le \sum_{i=1}^{\mathcal{B}(n)} \|V\|(\cup\mathcal{B}_i) \le \mathcal{B}(n)\frac{\alpha^2+\beta^2}{\gamma^2}$$

$\boxed{\mathcal{H}^k(C)}$ Recall that $C = T_\natural[B]$ so $\mathcal{H}^k(C) \le \mathcal{H}^k(B)$

Since we know $\Theta^k(\|V\|,x) \ge 1$ for $\|V\|$ almost all $x \in \mathbb{C}(T,0,5)$

it follows that

$$\|V\| \llcorner \mathbb{C}(T,0,5) \ge \mathcal{H}^k \llcorner \{x \in \mathbb{C}(T,0,5) : \Theta^k(\|V\|,x) > 0\}$$

(12) so $$\mathcal{H}^k(C) \le \mathcal{H}^k(B) \le \|V\|(B) \le \mathcal{B}(n)\frac{\alpha^2+\beta^2}{\gamma^2}.$$

$\boxed{\mathcal{H}^k(D)}$ Observe that $\|\delta V\| \llcorner A \le \gamma \|V\| \llcorner A$ ← simply from the def. of $A$.

(13)

Let $y \in D$ and set $s = \inf\{|T(x-y)| : x \in \operatorname{spt}\|V\|\}$

(4) $\Rightarrow s \in (0,2)$

Set $\zeta = \frac{2k+1}{2k+2} < 1$ and $r = \frac{s}{\zeta} > s$. Then $\zeta(k+1) - k = \frac{1}{2}$ and $r < 3$

Apply [8.11] to obtain

$$\frac{1}{2}\|V\|\,\mathbb{C}(T,y,r) \le \int_{\mathbb{C}(T,y,r)\times G(n,k)} \|S-T\|^2\,dV(x,S) + r\,\|\delta V\|\,\mathbb{C}(T,y,r)$$

By (13):

$$r\,\|\delta V\|\,\mathbb{C}(T,y,r) \overset{\boxed{r\le 3}}{\le} 3\gamma\,\|V\|\big(\mathbb{C}(T,y,r)\cap A\big) + 3\|\delta V\|\big(B\cap\mathbb{C}(T,y,r)\big)$$

Since $\gamma \le \frac{1}{12}$ we get $\boxed{3\gamma\le\frac{1}{4}}$ and

$$\frac{1}{4}\|V\|\,\mathbb{C}(T,y,r) \le \int_{\mathbb{C}(T,y,r)\times G(n,k)} \|S-T\|^2\,dV(x,S) + 3\|\delta V\|\big(B\cap\mathbb{C}(T,y,r)\big)$$

Let $x\in\operatorname{spt}\|V\|$ be such that $|T(x-y)| = s$.
Then (2) implies that

$$\frac{1}{4}\|V\|\,\mathbb{C}(T,y,r) \ge \frac{1}{4}\|V\|\,\mathbb{U}(x,r-s) \overset{(2)}{\ge} \frac{1}{4}\varepsilon\left(\frac{r}{2k+2}\right)^k$$

$\boxed{\text{noting } r-s = r-\zeta r = \frac{r}{2k+2} < \ell}$

In consequence

$$\mathcal{H}^k(D\cap\mathbb{B}(y,r)) \le \alpha(k)\,r^k \le \overbrace{\tfrac{1}{4}\alpha(k)\tfrac{1}{\varepsilon}(2k+2)^k}^{\Gamma}\,\|V\|\,\mathbb{C}(T,y,r)$$

$$\le \Gamma\left(\int_{\mathbb{C}(T,y,r)\times G(n,k)} \|S-T\|^2\,dV(x,S) + 3\|\delta V\|\big(B\cap\mathbb{C}(T,y,r)\big)\right)$$

Apply the Besicovitch covering theorem to get

(14) $$\mathcal{H}^k(D) \le \mathbf{B}(k)\cdot\Gamma\left(\int_{\mathbb{C}(T,0,4)\times G(n,k)} \|S-T\|^2\,dV(x,S) + 3\|\delta V\|\big(B\cap\mathbb{C}(T,0,4)\big)\right)$$

From (3) we know

$$\|\delta V\|(B) \overset{(3)}{\le} \alpha\,\|V\|(B)^{1/2} \overset{(11)}{\le} \alpha\left(\mathbf{B}(n)\frac{\alpha^2+\beta^2}{\gamma^2}\right)^{1/2}$$

$$\le \frac{\alpha}{\gamma}\mathbf{B}(n)^{1/2}(\alpha+\beta) = \frac{\mathbf{B}(n)^{1/2}}{\gamma}\left(\alpha^2+\alpha\beta\right) \overset{[C\text{-}S]}{\le} \frac{\mathbf{B}(n)^{1/2}}{\gamma}\left(\alpha^2+\frac{\alpha^2}{2}+\frac{\beta^2}{2}\right)$$

□

<u>Remark</u>: For <u>integral</u> varifolds $V\in\mathbf{IV}_k(\mathbb{R}^n)$ with $\|V\|\,\mathbb{C}(T,0,r,h)\approx Q\,\alpha(k)\,r^k$ one can find a Lipschitz function $f: T\to \mathcal{Q}_Q(T^\perp) = \{\mu : \mu = \sum_{i=1}^{Q}\operatorname{Dirac}(x_i) \text{ for some } x_1\ldots x_Q\in T^\perp\}$ with similar estimates.

<u>See</u>: U. Menne, Calc. Var. PDE (2010), Lemma 3.15
U. Menne, ARMA (2012), Lemma 5.7

Caccioppoli-type inequality

Tilt $\leq$ height + FirstVar

**8.13** $G, U \subseteq \mathbb{R}^n$ open, $0 < \delta < \infty$, $\mathbb{U}(x,\delta) \subseteq G \ \forall x \in U$

$T \in G(n,k)$, $V \in \mathbb{V}_k(G)$, $0 < \alpha < \infty$, $V \in \mathbb{H}(2)$

$$\int_G |\mathbf{h}(V,\cdot)|^2 \, d\|V\| = \alpha^2. \qquad \boxed{U + \mathbb{U}(0,\delta) \subseteq G}$$

Then

$$\int_{U \times G(n,k)} \|S - T\|^2 \, dV(x,S) \leq \max\left\{ \delta^2 \alpha^2, \ \frac{9}{\delta^2} \int_G |T^\perp x|^2 \, d\|V\|(x) \right\}$$

E.g. $U = \mathbb{U}(x,r)$, $G = \mathbb{U}(x,2r)$, $\delta = r$

$$\mathrm{Tilt}_2(x,r,T) \leq r^2 \Psi_2(\mathbb{U}(x,2r)) + \frac{9}{r^2} \int_{\mathbb{U}(x,2r)} |T^\perp y| \, d\|V\|(y)$$

Proof. Assume $\operatorname{Clos} U \subseteq \mathbb{R}^n$ is compact and $\int_{U \times G(n,k)} \|S - T\|^2 \, dV(x,S) > \delta^2 \alpha^2$

Let $\varphi \in \mathcal{D}(G,\mathbb{R})$ be s.t. $0 \leq \varphi \leq 1$, $U \subseteq \varphi^{-1}\{1\}$

Let $\Delta = \sup\{ |\operatorname{grad} \varphi(x)| : x \in G \}$

and $\boxed{g(x) = \varphi(x)^2 \cdot T^\perp x}$ ← height times cut-off

Recall **2.3(4)**: $\omega v \bullet S = \operatorname{tr}(\omega v \circ S) = \langle S(v), \omega \rangle$ for $\omega \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$, $v \in \mathbb{R}^n$, $S \in G(n,k)$

so $D\varphi(x) u = \operatorname{grad} \varphi(x) \bullet u$, $Dg(x) u = 2\varphi(x)(\operatorname{grad}\varphi(x) \bullet u) T^\perp x + \varphi^2(x) T^\perp u$

(*) $$Dg(x) \bullet S = 2\varphi(x) \ S \circ T^\perp(x) \bullet \operatorname{grad} \varphi(x) + \underbrace{\varphi(x)^2 T^\perp \bullet S}$$

Exercise **8.9(2)(3)**: $S \bullet T = |S \circ T|^2 = \operatorname{tr}(S \circ T \circ T^* \circ S^*) \quad \forall S \in G(n,k) \ \forall T \in G(n,m)$

$\|S - T\| = \|S^\perp \circ T\| = \|T^\perp \circ S\| = \|S \circ T^\perp\| = \|T \circ S^\perp\|$

Hence,

$$(\varphi(x) \|S - T\|)^2 = \varphi(x)^2 \|T^\perp \circ S\|^2 \leq \varphi(x)^2 |T^\perp \circ S|^2$$

(**) $$= \varphi(x)^2 \, T^\perp \bullet S \overset{(*)}{\leq} Dg(x) \bullet S + 2\varphi(x) \left| S \circ T^\perp(x) \bullet \operatorname{grad}\varphi(x) \right|$$

$$\leq Dg(x) \bullet S + 2\Delta \varphi(x) \|S - T\| \cdot |T^\perp x|$$

Thus,

(****) $$\delta V(g) = \int Dg(x) \bullet S \, dV(x,S) \overset{\text{(Hölder)}}{\leq} \alpha \left( \int |g|^2 d\|V\| \right)^{\frac{1}{2}} \leq \frac{1}{\delta} \left( \int \varphi(x)^2 \|S - T\|^2 dV(x,S) \right)^{\frac{1}{2}} \cdot \left( \int |T^\perp x|^2 d\|V\|(x) \right)^{\frac{1}{2}}$$

$$\delta V(g) = \underbrace{\int g(x) \bullet \mathbf{h}(V,x) \, d\|V\|(x)}$$

Moreover,

$$(***)\quad \int_G 2\Delta\, \varphi(x)\, \|S-T\| \cdot |T^\perp x|\, dV(x,S) \leq$$

$$\leq 2\Delta \left(\int_G \varphi(x)^2 \|S-T\|^2 dV(x,S)\right)^{1/2} \left(\int_G |T^\perp x|^2 d\|V\|(x)\right)^{1/2}$$

Combining $(**)$, $(***)$, and $(****)$ we obtain

$$\int (\varphi(x)\|S-T\|)^2 dV(x,S) \leq \left(\frac{1}{\delta} + 2\Delta\right) \left(\int_G \varphi(x)^2 \|S-T\|^2 dV(x,S)\right)^{1/2} \left(\int_G |T^\perp x|^2 d\|V\|(x)\right)^{1/2}$$

$$\Rightarrow \left(\int_G \varphi(x)^2 \|S-T\|^2 dV(x,S)\right)^{1/2} \leq \left(\frac{1}{\delta} + 2\Delta\right) \left(\int_G |T^\perp x|^2 d\|V\|(x)\right)^{1/2}$$

Since $U + \mathbb{U}(0,\delta) \subseteq G$, for each $\varepsilon > 0$ we can find $\varphi \in \mathcal{D}(G, \mathbb{R})$ with $U \subseteq \varphi^{-1}\{1\}$ such that $\underbrace{\sup\{|\operatorname{grad}\varphi(x)| : x \in G\}}_{=\Delta} \leq \frac{1}{\delta} + \varepsilon$.

Hence, $\left(\frac{1}{\delta} + 2\Delta\right)^2 \leq \left(\frac{3}{\delta} + 2\varepsilon\right)^2 = \frac{9}{\delta^2} + \left(\frac{12\varepsilon}{\delta} + 4\varepsilon^2\right) \xrightarrow[\varepsilon \downarrow 0]{} \frac{9}{\delta^2}$ $\square$

---

**8.14** **AUXILIARY**

$\exists\, C \in (0,\infty)$ $\forall T \in G(n,k)$ $\forall \ell \in \operatorname{Hom}(T, T^\perp)$ $\forall \Theta \in \operatorname{Hom}(T, T^\perp)$

Setting $L(x) = x + \ell(x)$, $L \in \operatorname{Hom}(T, \mathbb{R}^n)$

we get

$$(\Theta \circ T_\natural) \bullet (\operatorname{im} L)_\natural \cdot |\Lambda_k L| - \ell \bullet \Theta \leq \Gamma \|\Theta\| \max\{\|\ell\|^3, \|\ell\|^{4k-1}\}$$

where $\Gamma = \Gamma(k)$

*Proof.* Let $i : T \hookrightarrow \mathbb{R}^n$ be the inclusion map. $\left[\begin{array}{l} i^* \circ \mu \\ \| \end{array} \Rightarrow i \circ \mu = 0\right]$

Whenever $\mu \in \operatorname{Hom}(T, \mathbb{R}^n)$ is such that $T \circ \mu = 0$ we have

Exercise: $(*)$
$$|\Lambda_k(i+\mu)|^2 = 1 + \sum_{j=1}^{k} |\Lambda_j \mu|^2 \Longleftarrow \boxed{\text{Recall: } \det(id_X + tf) = \sum_{i=0}^{m} t^i \operatorname{trace}(\Lambda_i f)}$$

Observe that $\Lambda_j : \operatorname{Hom}(T, \mathbb{R}^n) \longrightarrow \operatorname{Hom}(\Lambda_j T, \Lambda_j \mathbb{R}^n)$

Let $\Psi_j = D\Lambda_j : \operatorname{Hom}(T, \mathbb{R}^n) \longrightarrow \operatorname{Hom}(\operatorname{Hom}(T, \mathbb{R}^n), \operatorname{Hom}(\Lambda_j T, \Lambda_j \mathbb{R}^n))$

We compute

$$(\Theta \circ T_\natural) \bullet (\operatorname{im} L)_\natural\, |\Lambda_k L| = \frac{d}{dt}\Big|_{t=0} \overbrace{\left|\Lambda_k\left(id_{\mathbb{R}^n} + t(\Theta \circ T) \circ (\operatorname{im} L)_\natural\right)\right|}^{= J_k(id_{\mathbb{R}^n} + t(\Theta\circ T))|_{\operatorname{im} L}} \cdot \overbrace{|\Lambda_k L|}^{= J_k L}$$

$$\overset{\substack{\det(A\circ B) \\ = \det(A)\det(B)}}{=} \frac{d}{dt}\Big|_{t=0} |\Lambda_k(\underbrace{i + \ell}_{=L} + t\Theta)| \overset{(*)}{=} \frac{d}{dt}\Big|_{t=0} \left(1 + \sum_{j=1}^{k} |\Lambda_j(\ell + t\Theta)|^2\right)^{1/2}$$

$$= \frac{1}{2}\left(1+\sum_{j=1}^{k}|\Lambda_j \ell|^2\right)^{-\frac{1}{2}} \cdot 2\sum_{j=1}^{k}(\Lambda_j \ell)\circ\langle\Theta, \Psi_j(\ell)\rangle$$

[ Note: $\Lambda_1 : \mathrm{Hom}(T, \mathbb{R}^n) \longrightarrow \mathrm{Hom}(T, \mathbb{R}^n)$, $\Lambda_1 \mu = \mu$ for $\mu \in \mathrm{Hom}(T, \mathbb{R}^n)$

Hence, $\Psi_1(\mu) = id_{\mathrm{Hom}(T, \mathbb{R}^n)}$ for $\mu \in \mathrm{Hom}(T, \mathbb{R}^n)$

$$= \ell\circ\Theta + \sum_{j=2}^{k}(\Lambda_j \ell)\circ\langle\Theta, \Psi_j(\ell)\rangle + \left(\left(1+\sum_{j=1}^{k}|\Lambda_j \ell|^2\right)^{-\frac{1}{2}} - 1\right)\sum_{j=1}^{k}(\Lambda_j \ell)\circ\langle\Theta, \Psi_j(\ell)\rangle$$

Observe $\Lambda_j(t\alpha) = t^j \Lambda_j(\alpha)$

$\Rightarrow$ $D\Lambda_j(t\alpha) = t^{j-1} D\Lambda_j(\alpha)$

[1.7.9] $f \in \mathrm{Hom}(\mathbb{R}^k, X)$
$|f| \le \sqrt{k}\,\|f\|$

$$(*)\quad \left|\sum_{j=2}^{k}(\Lambda_j \ell)\circ\langle\Theta, \Psi_j(\ell)\rangle\right| \le k\,\|\Theta\|\sum_{j=2}^{k}\|\ell\|^{2j-1}\left\|\Psi_j\left(\tfrac{\ell}{\|\ell\|}\right)\right\|$$

[1.7.6.] $\|\Lambda_j f\| \le \|f\|^j$

$$\le C(k)\,\|\Theta\| \begin{cases} \|\ell\|^3 & \text{if } \|\ell\| \le 1 \\ \|\ell\|^{2k-2} & \text{if } \|\ell\| > 1 \end{cases}$$

Note: $$\left|\frac{1}{\sqrt{1+x}} - 1\right| = 1 - \frac{1}{\sqrt{1+x}} \le 1 - \frac{1}{1+\frac{x}{2}} = \frac{\frac{x}{2}}{1+\frac{x}{2}} \le \begin{cases} \frac{x}{2} & \text{if } 0 \le x < 1 \\ \frac{1}{2} & \text{if } x \ge 1 \end{cases}$$

$f(x) = \frac{x}{1+x}$ $\nearrow$
$f'(x) = \frac{1}{(1+x)^2} > 0$ $\Uparrow$

$$(**)\quad \left(\sum_{j=1}^{k}|\Lambda_j \ell|^2\right)\left(\sum_{j=1}^{k}(\Lambda_j \ell)\circ\langle\Theta, \Psi_j(\ell)\rangle\right) \le$$

$$\le C(k)\,\|\Theta\|\left(\sum_{j=0}^{k-1}\|\ell\|^{2(j+1)}\right)\left(\sum_{j=0}^{k-1}\|\ell\|^{2j+1}\right)$$

$$= C(k)\,\|\Theta\|\sum_{j=0}^{2k-2}\sum_{\alpha+\beta=j}\|\ell\|^{2(\alpha+1)+2\beta+1}$$

$$= C(k)\,\|\Theta\|\sum_{j=0}^{2k-2}\|\ell\|^{2j+3} \le \tilde{C}(k)\,\|\Theta\| \begin{cases} \|\ell\|^3 & \text{if } \|\ell\| \le 1 \\ \|\ell\|^{4k-1} & \text{if } \|\ell\| > 1 \end{cases}$$

$\square$

Recall: $1 \le k < p < \infty$, $p \ge 2$, $q = \frac{p}{p-1} = p^* \in (1,2)$



**8.16** Pivotal lemma — height-excess decay

$\forall \varepsilon \in (0,1)$ $\exists \Delta \in (0,1)$ $\exists \Theta \in (0, \frac{1}{14})$ $\exists M \in (1,\infty)$

$\forall V, a, r, T, U, A, \mu, \alpha$

If (1) $V \in \mathbb{V}_k(\mathbb{R}^n)$, $a \in \operatorname{spt}\|V\|$, $r \in (0,\infty)$

(2) $T, U \in G(n,k)$, $\|T - U\| \le \Delta$,

$A \subseteq \mathbb{R}^n$ is a $k$-dim affine plane parallel to $T$

(3) $$\mu = \left( \frac{1}{r^{k+2}} \int_{\mathbf{C}(U,a,r)} \operatorname{dist}(x,A)^2 \, d\|V\|(x) \right)^{1/2} \le \Delta$$

(4) $\alpha \in [0,\infty)$, $V$ satisfies $\mathbb{H}(p)$

$$\left( \int_{\mathbf{C}(U,a,r)} |\mathbf{h}(V,\cdot)|^p \, d\|V\| \right)^{1/p} = \alpha$$

(5) $$\varepsilon \le \frac{1}{s^k} \|V\| \mathbf{B}(x,s) \le (2-\varepsilon)\alpha(k) \quad \text{for } s \in (0,r) \text{ and } \|V\| \text{ almost all } x \in \mathbf{C}(U,a,r)$$

(6) $1 \le \Theta^k(\|V\|, x) \le 1 + \Delta$ for $\|V\|$ almost all $x \in \mathbf{C}(U,a,r)$

then

$\exists \tilde{T} \in G(n,k)$ $\exists \tilde{A} \subseteq \mathbb{R}^n$

(7) $\tilde{A}$ is a $k$-dim. affine plane parallel to $\tilde{T}$

(8) $\|\tilde{T} - T\| \le M\mu$

(9) $$\left( \frac{1}{(\Theta r)^{k+2}} \int_{\mathbf{C}(U,a,\Theta r)} \operatorname{dist}(x,\tilde{A})^2 \, d\|V\|(x) \right)^{1/2} \le \Theta^{1-\frac{k}{p}} \max\left\{ \mu, M r^{1-\frac{k}{p}} \alpha \right\}$$

---

$\Downarrow$

Iterating [8.16]:

$$\operatorname{height}_2(a,A,r) = \left( \frac{1}{r^{k+2}} \int_{\mathbf{C}(U,a,r)} \operatorname{dist}(x,A)^2 \, d\|V\|(x) \right)^{1/2}$$

$A_0 = A$, $A_{j+1} = \tilde{A}_j$

$$\operatorname{height}_2(a, A_j, \Theta^j r) \le \cancel{(\Theta^{1-\frac{k}{p}})^j \operatorname{height}_2(a,A_0,r) + M\alpha r^{1-\frac{k}{p}} \sum_{l=1}^{j} \Theta^{l}}$$
$$\le \Theta^{j(1-\frac{k}{p})} \cdot \max\left\{ \operatorname{height}_2(a,A_0,r), M\alpha r^{1-\frac{k}{p}} \right\}$$

If $0 < s < r$, ~~then~~ and $\Theta^{j+1} \le \frac{s}{r} < \Theta^j$, then

$$\operatorname{height}_2(a, A_j, s) \lesssim \left(\frac{s}{r}\right)^{1-\frac{k}{p}} \cdot \max\left\{ \operatorname{height}_2(a,A_0,r), M\alpha r^{1-\frac{k}{p}} \right\}$$

**Proof of 8.16:** We proceed by contradiction.

Assume that 8.16 is false. Then

⊛ $\exists \varepsilon \quad \forall \Theta, \Delta, M \quad \exists V, a, r, T, U, A, \mu, \alpha$ satisfying (1)–(6) and $\neg[(7) \wedge (8) \wedge (9)]$.

$$M_i \geq \Gamma_{8.15} \cdot 7^{\frac{k+2}{2}}$$
$$\frac{((2-\varepsilon)\alpha(k)7^k)^{1/q - 1/2}}{\Theta^{(k+2)/2}\,\Delta_i}^{1-1/p} \leq M_i(7\Theta)$$

We choose:

(11) $\quad \Delta_i \xrightarrow[i\to\infty]{} 0, \quad M_i \xrightarrow[i\to\infty]{} \infty, \quad M_i \gtrsim \frac{1}{\Delta_i}, \quad \Theta$ small.

Applying ⊛ plus homotheties and scaling and rotations we get

(13) $\quad V_i \in \mathbb{V}_k(\mathbb{R}^n), \quad a_i \in \operatorname{spt}\|V_i\|, \quad (U_i)_{\natural} a_i = 0, \quad U_i \in G(n,k)$

(14), (15) $\quad \|T - U_i\| \leq \Delta, \quad \mu_i = \left(\frac{1}{7^{k+2}} \int_{\mathbb{C}(U_i,0,7)} \operatorname{dist}(x,T)^2 \, d\|V_i\|(x)\right)^{1/2} \leq \Delta_i$

(16) $\quad \left(\int_{\mathbb{C}(U_i,0,7)} |h(V_i,\cdot)|^p \, d\|V_i\|\right)^{1/p} = \alpha_i \in [0,\infty)$ and $V_i$ satisfies $H(p)$.

(17) $\quad \varepsilon \leq \frac{1}{s^k}\|V_i\| B(x,s) \leq (2-\varepsilon)\alpha(k)$ for $s \in (0,7)$ and $\|V_i\|$ almost all $x \in \mathbb{C}(U_i,0,7)$

(18) $\quad \Theta^k(\|V_i\|, x) \in [1, 1+\Delta_i]$

Moreover, for all $\tilde{T}_i$, $\tilde{A}_i$ at least one of the following three statements is false:

(19) $\quad \tilde{T}_i \in G(n,k)$, $\tilde{A}_i$ is parallel to $\tilde{T}_i$ k-dim. affine plane

(20) $\quad \|\tilde{T}_i - T\| \leq M_i \mu_i$

(21) $\quad \left(\frac{1}{(7\Theta)^{k+2}} \int_{\mathbb{C}(U_i,0,7\Theta)} \operatorname{dist}(x,\tilde{A}_i)^2 \, d\|V_i\|(x)\right)^{1/2} \leq \Theta^{1-\frac{k}{p}} \max\{\mu_i, M_i \cdot 7^{1-\frac{k}{p}}\alpha_i\}$

For $i \in \mathbb{N}$ we define

$$\beta_i = \left(\int_{\mathbb{C}(U_i,0,6)\times G(n,k)} \|S - T\|^2 \, dV(x,S)\right)^{1/2}$$

and note that $\mu_i > 0$. (otherwise (21) would be satisfied taking $\tilde{A}_i = \tilde{T}_i = T$ since then $\operatorname{spt}\|V\| \subseteq T$ and $h(V,x) = 0$ for $\|V\|$ a.e. $x$)

Define for $i \in \mathbb{N}$

$$\nu_i = \sup\left\{ \left(7 - |(U_i)_\natural(x)|\right)^k \operatorname{dist}(x,T) : x \in \operatorname{spt}\|V_i\| \cap C(U_i,0,7) \right\}$$

From (11), (14), (15) we see that

$$\mu_i \xrightarrow[i\to\infty]{} 0 \quad \text{and} \quad \|T - U_i\| \xrightarrow[i\to\infty]{} 0$$

Hence, by (17) we get

$$\nu_i \xrightarrow[i\to\infty]{} 0$$

In consequence, (13) gives $\quad a_i \xrightarrow[i\to\infty]{} 0$

Wlog assume:

$$\operatorname{spt}\|V_i\| \cap C(T,0,5) \subseteq U(a_i,7) \qquad \text{for } i \in \mathbb{N}.$$

<u>Claim</u>: For $i \in \mathbb{N}$ large enough there holds

(25) $$\left( \int_{U(a_i,7)} |h(V_i,\cdot)|^2 \, d\|V_i\| \right)^{1/2} \leq \Delta_i \mu_i$$

<u>Proof of claim</u>: Assume the countrary, i.e.,

(*) $$\left( \int_{U(a_i,7)} |h(V_i,\cdot)|^2 \, d\|V_i\| \right)^{1/2} > \Delta_i \mu_i$$

Then we check (21) with $\tilde{A}_i = \tilde{T}_i = T$:

(26)
$$\begin{aligned}
&\left( \frac{1}{(7\Theta)^{k+2}} \int_{C(U_i,0,7\Theta)} \operatorname{dist}(x,T)^2 \, d\|V_i\|(x) \right)^{1/2} \leq \frac{1}{\Theta^{\frac{k+2}{2}}} \mu_i \\
&\overset{(*)+\text{Hölder}}{<} \frac{1}{\Delta_i \Theta^{\frac{k+2}{2}}} \left( \int_{U(a_i,7)} |h(V_i,\cdot)|^p \, d\|V_i\| \right)^{1/p} \left( \|V_i\| \, U(a_i,7) \right)^{\frac{1}{2} - \frac{1}{p}} \\
&\overset{(16)+(17)}{\leq} \frac{\alpha_i}{\Delta_i \Theta^{(k+2)/2}} \left( (2-\varepsilon)\alpha(k) 7^k \right)^{\frac{1}{2} - 1/p} \\
&\leq \Theta^{1-k/p} M_i \cdot 7^{1-\frac{k}{p}} \cdot \alpha_i
\end{aligned}$$

given $M_i$ are chosen s.t. $M_i (7\Theta)^{1-\frac{k}{p}} \geq \dfrac{\left((2-\varepsilon)\alpha(k)7^k\right)^{\frac{1}{2}-\frac{1}{p}}}{\Delta_i \Theta^{(k+2)/2}}$

Thus, (19) ∧ (20) ∧ (21) holds for $V_i$ ☒

Using (17), (22) - (25) we may apply the Lipschitz-approximation $\boxed{8.12}$ to each $V_i$ with $\mu = \sqrt{2}$ to get a number $P > 0$ and mappings $f_i : T \to T^\perp$, $F_i : T \to \mathbb{R}^n$, $F_i(x) = x + f_i(x)$ s.t.

$$\sup\{|f_i(y)| : y \in T\} \leq \nu_i$$

$$|f_i(y) - f_i(z)| \leq |y - z| \quad \text{for } y, z \in T$$

and setting $C = \mathbb{C}(T, 0, 1)$ and $D = T \cap \mathbb{U}(0,1)$

$$X_i = C \cap \operatorname{im} F_i\,, \quad Y_i = \{y \in D : \Theta^k(\|V_i\|, F_i(y)) \geq 1\}$$

(30) $\quad \|V_i\|(C \sim X_i) + \mathcal{H}^k(D \sim Y_i) \leq P\left((\Delta_i \mu_i)^2 + \beta_i^2\right)$

The Rademacher theorem shows that $F_i$ is $(\mathcal{H}^k \llcorner T)$ almost everywhere differentiable and

(31) $\quad \|Df_i\| \leq 1$ and $\|DF_i\| \leq 2$.

From 8.13 and the claim (25) we see that

$$\beta_i^2 \lesssim \mu_i^2 + (\Delta_i \mu_i)^2 \quad \text{for large enough } i \in \mathbb{N}.$$

Hence,

(33) $\quad \max\{\beta_i^2, \|V_i\|(C \sim X_i) + \mathcal{H}^k(D \sim Y_i)\} \lesssim \mu_i^2$

We compute:

- $$\int_{Y_i} |f_i(y)|^2 \, d\mathcal{H}^k(y) \leq \int_{Y_i} |f_i(y)|^2 \overbrace{\Theta^k(\|V_i\|, F(y))}^{\geq 1 \text{ by } (18)} \cdot \overbrace{|\Lambda_k DF_i(y)|}^{\geq 1 \text{ by def.}} \, d\mathcal{H}^k(y)$$

$$\overset{\text{(area-formula)}}{=} \int_{X_i} \operatorname{dist}(x, T)^2 \, d\|V_i\|(x) \leq 7^{k+2} \mu_i^2$$

- (34) $$\int_{Y_i} \|Df_i(y)\|^2 \, d\mathcal{H}^k(y) \overset{8.9(5) \text{ with } \eta_1 = 0,\ \eta_2 = Df_i(y),\ S = T}{\leq} \int_{Y_i} \|DF_i(y)\|^2 \cdot \|(\operatorname{im} DF_i(y))_\natural - T_\natural\|^2 \, d\mathcal{H}^k(y)$$

$$\overset{(31)}{\leq} 4 \int_{Y_i} \|(\operatorname{im} DF_i(y))_\natural - T_\natural\|^2 \, \Theta^k(\|V_i\|, F_i(y)) \, |\Lambda_k DF_i(y)| \, d\mathcal{H}^k(y)$$

$$= 4 \int_{X_i \times G(n,k)} \|S - T\|^2 \, d\|V_i\|(x, S) \overset{(33)}{\lesssim} \mu_i^2 \leq N \mu_i^2$$

- $$\int_{D\sim Y_i} |f_i(y)|^2 \, d\mathcal{H}^k(y) \overset{(33)}{\lesssim} \mu_i^2 \cdot \nu_i^2 \leq N \, \nu_i^2 \mu_i^2$$
- $$\int_{D\sim Y_i} \| Df_i(y) \|^2 \, d\mathcal{H}^k(y) \overset{(33)}{\lesssim} \mu_i^2 \leq N \mu_i^2$$

for some $N \in \mathbb{R}$ big enough

Hence, Relich - Kondrachov + Banach - Alaoglu give us a function $\boxed{\text{Exercise}}$

$$h : T \to T^\perp \in W^{1,2}(D, T^\perp)$$

such that

(35) $$\liminf_{i\to\infty} \int_D \left| h - \frac{1}{\mu_i} f_i \right|^2 d\mathcal{H}^k = 0$$

(36) and $$\int_D |h|^2 \, d\mathcal{H}^k \leq 7^{k+2}$$

and $$\int_D \|Dh\|^2 d\mathcal{H}^k \leq N.$$

We will show that $h$ is harmonic. To this end we need to show that

Roughly $= \delta V_i(\varphi\circ T)$ + errors

(37) $$\lim_{i\to\infty} \frac{1}{\mu_i} \int_D Df_i(y) \bullet D\varphi(y) \, d\mathcal{H}^k(y) = 0 \qquad \forall \varphi \in \mathcal{D}(D, T^\perp)$$

Fix $\varphi \in \mathcal{D}(D, T^\perp)$. Set

$$B = \sup\{ |\varphi(y)| + |D\varphi(y)| : y \in D \}$$

Bad set $D\sim Y_i$ is small $\Rightarrow$ $$a_{1,i} = \int_{D\sim Y_i} Df_i(y) \bullet D\varphi(y) \, d\mathcal{H}^k(y)$$

By [8.14] with $\ell = Df_i(y)$ and $\overset{(33)}{\bullet\bullet}$ $\Rightarrow$ $$a_{2,i} = \int_{Y_i} Df_i(y) \bullet D\varphi(y) - (\operatorname{im} DF_i(y))_\natural \bullet (D\varphi(y)\circ T_\natural) \, |\Lambda_k DF_i(y)| \, d\mathcal{H}^k(y)$$

$|1 - \Theta^k(..)| \leq \Delta_i$ 8.9(4)(5) + $\overset{(33)}{\bullet\bullet}$ $\Rightarrow$ $$a_{3,i} = \int_{Y_i} (\operatorname{im} DF_i(y))_\natural \bullet (D\varphi(y)\circ T_\natural) \cdot \left(1 - \Theta^k(\|V_i\|, F_i(y))\right) |\Lambda_k DF_i(y)| \, d\mathcal{H}^k(y)$$

On good set this is 0, on bad set is bounded and the bad set is small $\Rightarrow$ $$a_{4,i} = \int_{Y_i} (\operatorname{im} DF_i(y))_\natural \bullet (D\varphi(y)\circ T_\natural) \, \Theta^k(\|V_i\|, F_i(y)) \, |\Lambda_k DF_i(y)| \, d\mathcal{H}^k(y) - \delta V_i(\varphi\circ T)$$

Use claim (25) $\Rightarrow$ $$\boxed{a_{5,i} = \delta V_i(\varphi\circ T)}.$$

Note that $\frac{1}{\mu_i}\int_D Df_i(y)\bullet D\varphi(y)\,d\mathcal{H}^k(y) = \frac{1}{\mu_i}\sum_{j=1}^{5} a_{j,i}$

so we only need to show that $\frac{1}{\mu_i}|a_{j,i}| \xrightarrow[i\to\infty]{} 0$ for $j\in\{1,2,3,4,5\}$.

$|a_{1,i}| \leq BN\mu_i^2$ by (31) and (33)

$$|a_{2,i}| \leq \Gamma_{8.14}\cdot B\int_{Y_i}\|Df_i\|^2 d\mathcal{H}^k$$ by 8.14 with $l = Df_i(y)$, $\Theta = D\varphi(y)$ and using (31)

$$\leq \Gamma_{8.14}\cdot B\cdot N\mu_i^2$$

$$|a_{3,i}| \leq \Delta_i\cdot 2^k\cdot\int_{Y_i}\left|(\operatorname{im}DF_i(y))_\natural\bullet(D\varphi(y)\circ T_\natural)\right| d\mathcal{H}^k(y)$$

$$\overset{[8.9(4)]}{\leq} 2^k\Delta_i\cdot\frac{1}{\sqrt{2}}\int_{Y_i}|(\operatorname{im}DF_i(y))_\natural - T_\natural|\cdot|D\varphi(y)|\,d\mathcal{H}^k(y)$$

$$\overset{[8.9(5)]}{\leq} 2^k\Delta_i\cdot B\cdot\sqrt{m}\int_{Y_i}\|Df_i\|\,d\mathcal{H}^k$$

$\Delta_i \to 0$

(Cauchy-Schwarz) + (34)
$$\leq 2^k\Delta_i B\sqrt{m}\sqrt{\alpha(k)}\cdot\sqrt{N}\,\mu_i$$

$$|a_{4,i}| = \left|\int_{X_i}(D\varphi(y)\circ T)\bullet S\,dV_i(y,S) - \int_C(D\varphi(y)\circ T)\bullet S\,dV_i(y,S)\right|$$

$$= \left|\int_{C\sim X_i}(D\varphi(y)\circ T)\bullet S\,dV_i(y,S)\right|$$

$$\leq B\,\|V_i\|(C\sim X_i) \leq BN\mu_i^2$$

$$|a_{5,i}| = \left|\int_C\varphi(T_\natural y)\bullet \mathbf{h}(V_i,y)\,d\|V_i\|(y)\right|$$

$$\overset{(25)}{\leq} B\cdot\Delta_i\mu_i\,\|V_i\|(C)^{1/2} \leq B\Delta_i\mu_i\left((2-\varepsilon)\alpha(k)7^k\right)^{1/2},$$

$\Delta_i \to 0$

Therefore, (37) holds,

so $h$ is harmonic.

$$h : \underset{\underset{T}{\cup}}{D} \longrightarrow T^{\perp} \quad \text{harmonic}$$

Define $\quad L_i(y) = y + \mu_i \langle y, Dh(0) \rangle$

$$K_i(x) = L_i(T(x)) + \mu_i h(0)$$

$$= \mu_i h(0) + T(x) + \mu_i \langle Tx, Dh(0) \rangle$$

[We will show that (21) holds]

$$\tilde{T}_i = \operatorname{im} L_i, \quad \tilde{A}_i = \operatorname{im} K_i$$

$K_i$ = <u>first Taylor polynomial of the function</u> $x \mapsto T(x) + h(T(x))$

Clearly $\tilde{T}_i$, $\tilde{A}_i$ satisfy (19)

Moreover,

$$(39) \quad \|\tilde{T}_i - T\| \overset{[8.9(5)]}{\leq} \mu_i \|Dh(0)\| \overset{\substack{[8.15] \\ \text{std. estimates} \\ \text{for harmonic funct.}}}{\leq} C\mu_i \left(\int_D |h|^2 \, d\mathcal{H}^k\right)^{\frac{1}{2}} \overset{\substack{\text{given } M_i \text{ is chosen} \\ \text{big enough}}}{\leq} M_i \mu_i$$

For $x \in C \cap \operatorname{spt}\|V_i\|$ we have

$$x - K_i(x) = T^{\perp}x - \mu_i h(0) - \mu_i \langle Tx, Dh(0) \rangle$$

$$(40) \quad \Rightarrow \operatorname{dist}(x, \tilde{A}_i) \overset{[8.15]}{\leq} |x - K_i(x)| \leq \operatorname{dist}(x, T) + C \cdot \mu_i$$

if additionally, $x \in X_i$ and $y = Tx$, then $F(y) = x$ and

$$x - K_i(x) = f_i(y) - \mu_i h(y) + \mu_i \big(\underbrace{h(y) - h(0) - \langle y, Dh(0) \rangle}_{\text{First Taylor polynomial of } h}\big)$$

hence,

$$(41) \quad \operatorname{dist}(F_i(y), \tilde{A}_i) \leq |x - K_i(x)| \overset{[8.15]}{\leq} |f_i(y) - \mu_i h(y)| + C\mu_i \cdot |y|^2$$

provided $|y| < \frac{1}{2}$.

Let $\gamma \in (7, \infty)$. We estimate

- $$\int_{C(T,0,\gamma\theta) \cap X_i} \operatorname{dist}(x, \tilde{A}_i)^2 \, d\|V_i\|(x) = \int_{U(0,\gamma\theta) \cap Y_i} \operatorname{dist}(F_i(y), \tilde{A}_i)^2 \underbrace{\Theta^k(\|V_i\|, F_i(y))}_{\leq 1+\Delta_i} \underbrace{|\Lambda_k DF_i(y)|}_{\leq 2^k} \, d\mathcal{H}^k(y)$$

$$\overset{(41)}{\leq} (1+\Delta_i) \cdot 2^k \cdot C\mu_i^2 \underbrace{\int_{U(0,\gamma\theta) \cap T} |y|^4 \, d\mathcal{H}^k(y)}_{= C \cdot (\gamma\theta)^{k+4}} + (1+\Delta_i) 2^k \underbrace{\int_D |f_i(y) - \mu_i h(y)|^2 \, d\mathcal{H}^k(y)}_{\frac{1}{\mu_i^2}(\ldots) \xrightarrow[i \to \infty]{} 0}$$

- $$\int_{C(T,0,\gamma\theta) \sim X_i} \operatorname{dist}(x, \tilde{A}_i)^2 \, d\|V_i\|(x) \underset{(40)}{\overset{(33)}{\leq}} \underbrace{(\nu_i^2 + (C\mu_i)^2)}_{\xrightarrow[i \to \infty]{} 0} \cdot N \mu_i^2$$

Thus, we obtain

$$\limsup_{i\to\infty} \frac{1}{(7\theta)^{k+2}\mu_i^2} \int_{\mathbf{C}(T,0,\gamma\theta)} \mathrm{dist}(x,\tilde{A}_i)^2 \, d\|V_i\|(x) \leq C\,\gamma^{k+4}\cdot\theta^2$$

Observe that (14) and (23) imply that

$$\mathbf{C}(U_i,0,7\theta)\cap \operatorname{spt}\|V_i\| \subseteq \mathbf{C}(T,0,\gamma\theta) \quad \text{for } i \text{ large enough.}$$

If $\theta\in(0,\frac{1}{14})$ is small enough, then we get

$$\limsup_{i\to\infty} \frac{1}{(7\theta)^{k+2}\mu_i^2} \int_{\mathbf{C}(U_i,0,7\theta)} \mathrm{dist}(x,\tilde{A}_i)^2 \, d\|V_i\|(x) < \left(\theta^{1-\frac{k}{p}}\right)^2$$

so (21) holds given $i\in\mathbb{N}$ is big enough.

$\square$.

**8.17** $\forall \varepsilon \in (0,1)\ \exists \gamma \in (0,1)\ \exists N \in (1,\infty)$

if $V, a, R, T, U, A, \mu, \alpha$ satisfy (1)–(6) of [8.16], with $\gamma$ in place of $\Delta$, and if

$$r^{1-\frac{k}{p}}\alpha \le \gamma$$

then

(1) $\operatorname{spt}\|V\| \cap \{x : Ux = Ua\} = a = \operatorname{spt}\|V\| \cap U_h^{-1}\{a\}$

(2) $\operatorname{Tan}(\operatorname{spt}\|V\|, a) \in G(n,k)$ and

$$\|T - \operatorname{Tan}(\operatorname{spt}\|V\|, a)\| \le N \max\{\mu, r^{1-\frac{k}{p}}\alpha\}$$

(3) $\forall s \in (0,r)\ \exists T_s \in G(n,k)\ \exists A_s$ - affine $k$-plane, $A_s \| T_s$

$$\max\left\{\|T_s - \operatorname{Tan}(\operatorname{spt}\|V\|, a)\|, \left(\frac{1}{s^{k+2}}\int_{C(U,a,s)} \operatorname{dist}(x, A_s)^2 \, d\|V\|(x)\right)^{1/2}\right\}$$
$$\le N \max\{\mu, r^{1-\frac{k}{p}}\alpha\}\left(\frac{s}{r}\right)^{1-\frac{k}{p}}$$

*Proof* Let $\Delta \in (0,1)$, $\Theta \in (0, \frac{1}{14})$, $M \in (1,\infty)$ be as in [8.16]

Choose $\gamma$ and $N$ so that $\boxed{\gamma \text{ small}, N \text{ big}}$

$\boxed{\begin{array}{l}\gamma = \gamma(\Delta, \Theta, M_{k,p}) \in (0,1)\\ N = N(\Delta, \Theta, M_{k,p}) \in (1,\infty)\end{array}}$

$$M\gamma < \Delta, \quad \gamma + \sum_{j=0}^{\infty} M(M\gamma)\Theta^{(j-1)(1-\frac{k}{p})} \le \Delta,$$
$$M^2 \sum_{j=0}^{\infty} \Theta^{(j-1)(1-k/p)} \le N, \quad \Theta^{-(k+2)/2-(1-\frac{k}{p})} \le N$$

Claim: $\forall j \in \mathbb{N}\ \exists T_j, A_j, \mu_j$

- $T_0 = T$, $A_0 = A$, $\mu_0 = \mu$
- $T_j \| A_j$
- $\mu_j = \left(\dfrac{1}{(\Theta^j r)^{k+2}} \displaystyle\int_{C(U,a,\Theta^j r)} \operatorname{dist}(x, A_j)^2 \, d\|V\|(x)\right)^{1/2}$
- $\|T_j - T_{j-1}\| \le M \max\{\mu, M r^{1-\frac{k}{p}}\alpha\}\, \Theta^{(j-1)(1-\frac{k}{p})}$
- $\mu_j \le \max\{\mu, M r^{1-\frac{k}{p}}\alpha\}\, \Theta^{j(1-\frac{k}{p})}$

Observe that for $\ell \in \mathbb{N}$

$$\mu_\ell \le M\gamma \le \Delta$$

$$\|T_\ell - U\| \le \|T_0 - U\| + \sum_{j=1}^{\ell} \|T_j - T_{j-1}\|$$

$$\le \underbrace{\|T - U\|}_{\le \gamma \text{ by (1)}} + \sum_{j=1}^{\ell} M \cdot \max\{\mu, M r^{1-\frac{k}{p}} \alpha\} \Theta^{(j-1)(1-\frac{k}{p})}$$

$$\le \Delta$$

Applying [8.16] with $\varepsilon$ fixed, $\Delta, \Theta, M$ depending only on $\varepsilon$ and $V, a, \Theta^\ell r, T_\ell, U, A_\ell, \mu_\ell, \alpha$ in place of $V, a, r, T, U, A, \mu, \alpha$ we get $T_{\ell+1}, A_{\ell+1}, \mu_{\ell+1}$ s.t.

- $\|T_{\ell+1} - T_\ell\| \le M\mu_j \le M \cdot \max\{\mu, M r^{1-\frac{k}{p}} \alpha\} \Theta^{\ell(1-\frac{k}{p})}$
- $$\mu_{\ell+1} = \left( \frac{1}{(\Theta^{\ell+1} r)^{k+2}} \int_{C(U,a,\Theta^{\ell+1} r)} \operatorname{dist}(x, A_{\ell+1})^2 \, d\|V\|(x) \right)^{1/2}$$

$$\le \Theta^{1-\frac{k}{p}} \max\{\mu_\ell, M(\Theta^\ell r)^{1-\frac{k}{p}} \alpha\}$$

$$\le \max\{\mu, M r^{1-\frac{k}{p}} \alpha\} \Theta^{(\ell+1)(1-\frac{k}{p})}$$

By assumption we have

$$\|V\| B(x,s) \ge \varepsilon \alpha(k) s^k \quad \text{for } s \in (0,r) \text{ and } \|V\| \text{ almost all } x \in C(U,a,r).$$

and $\mu_i \xrightarrow[i\to\infty]{} 0$ ; hence,

$$(\circledast) \quad \lim_{j\to\infty} \frac{2}{\Theta^j r} \sup\left\{ \operatorname{dist}(x, A_j) : x \in \operatorname{spt}\|V\| \cap C(U, a, \tfrac{1}{2}\Theta^j r) \right\} = 0$$

Note that • shows that $T_i \xrightarrow[i\to\infty]{} T_\infty$ in $G(n,k)$.
also that $(\ast\ast)$ $\|T_j - U\| \le \Delta < 1$ for all $j \in \mathbb{N}$.

From $\circledast$ it follows that $A_j \longrightarrow A_\infty$ and, since $a \in \operatorname{spt}\|V\|$, we have $a \in A_\infty$ so $(\ast\ast)$ gives

$$\operatorname{spt}\|V\| \cap \{x : Ux = Ua\} = \{a\}$$

and $\operatorname{Tan}(\operatorname{spt}\|V\|, a) \subseteq T_\infty$

If $C \in \operatorname{VarTan}(V,a)$, then $\operatorname{spt}\|C\| \subseteq \operatorname{Tan}(\operatorname{spt}\|V\|, a) \subseteq T_\infty$ so [6.5] and [4.6(3)] yield $C = v_k(T_\infty)$

$[4.6(3)]$ = stationary varifolds inside smooth manifolds (connected) $M$ of the same dimension must be equal to a positive multiple of $\mathbb{V}_k(M)$.

In particular, we obtain

$$\|T - \operatorname{Tan}(\operatorname{spt}\|V\|, a)\| \le M \cdot \max\{\mu, M r^{1-\frac{k}{p}} \alpha\} \sum_{j=1}^{\infty} \Theta^{(j-1)(1-\frac{k}{p})}$$

$$\le N \cdot \max\{\mu, r^{1-\frac{k}{p}} \alpha\}$$

For $s \in (0, r)$ we find $j \in \mathbb{N}$ s.t. $\underbrace{r\,\Theta^{j+1} \le s < r\,\Theta^{j}}_{\boxed{\Theta^j \approx \frac{s}{r}}}$.
Set $T_s = T_j$, $A_s = A_j$ and estimate:

$$\|T_s - \operatorname{Tan}(\operatorname{spt}\|V\|, a)\| \le \sum_{l=1}^{\infty} M \max\{\mu, M r^{1-\frac{k}{p}} \alpha\} \Theta^{(j+l-1)(1-\frac{k}{p})}$$

$$\le N \cdot \max\{\mu, r^{1-\frac{k}{p}} \alpha\} \left(\frac{s}{r}\right)^{1-\frac{k}{p}}$$

$$\left(\frac{1}{s^{k+2}} \int_{\mathbf{C}(U,a,s)} \operatorname{dist}(x, A_s)^2 \, d\|V\|(x)\right)^{1/2} \le \frac{1}{\Theta^{\frac{k+2}{2}}} \mu_j$$

$$\le \frac{1}{\Theta^{\frac{k+2}{2}}} \max\{\mu, M r^{1-\frac{k}{p}} \alpha\} \underbrace{\Theta^{j(1-\frac{k}{p})}}_{\approx \left(\frac{s}{r}\right)^{1-\frac{k}{p}}}$$

$$\le N \max\{\mu, r^{1-\frac{k}{p}} \alpha\} \left(\frac{s}{r}\right)^{1-\frac{k}{p}}.$$

$\square$

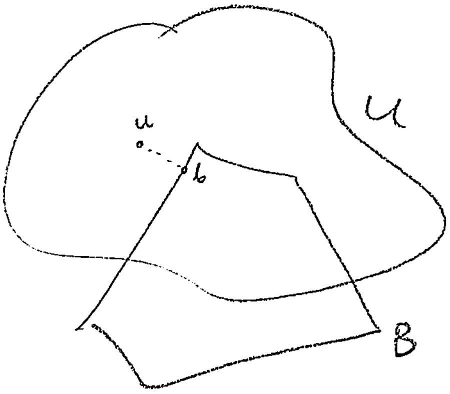
**8.18** $U \subseteq \mathbb{R}^k$ open, $B \subseteq \mathbb{R}^k$ closed, $\mathrm{Tan}(B, b) = \mathbb{R}^k$ $\forall b \in B \cap U$

Then $B \cap U$ is open.

Proof. Assume $B \cap U$ is not open.
Then $U \cap B \neq \emptyset$, $U \cap B \neq U$, and $U \sim B \neq \emptyset$.
Moreover $U \sim B$ is open.
$B \cap U \subseteq U = \bigcup \{ U(u, \operatorname{dist}(u, \mathbb{R}^k \sim U)) : u \in U \}$ ; one of these balls intersects B
hence, $\exists\, u \in U \sim B \quad \operatorname{dist}(u, B) < \operatorname{dist}(u, \mathbb{R}^k \sim U)$
Since B is closed, there exists $b \in B$ s.t.

$$\operatorname{dist}(u, B) = |u - b| \quad \text{and} \quad b \in B \cap U.$$

Then $\mathrm{Tan}(B, b) = \mathbb{R}^k$ but this is impossible because $U(u, |u-b|) \cap B = \emptyset$. □




Remark. Since $B \cap U$ is rel. closed in $U$ we see that $B \cap U = U$ or $B \cap U = \emptyset$.

**8.19** $\forall \varepsilon \in (0,1)\ \exists \eta \in (0,1)\ \exists C \in (1,\infty)$

(1) $\forall V \in \mathbb{V}_k(\mathbb{R}^n)$, $0 \in \operatorname{spt}\|V\|$
$\forall r \in (0,\infty)$, $\forall T \in G(n,k)$, $\forall \mu \in [0,\infty)$, $\forall \alpha \in [0,\infty)$

<u>if</u>

(2) $$\mu = \left( \frac{1}{r^{k+2}} \int_{\mathbb{C}(T,0,2r)} \operatorname{dist}(x,T)^2 \, d\|V\|(x) \right)^{\frac{1}{2}} \leq \eta$$

(3) $0 < r^{(p-k)} \cdot \alpha \leq \eta$
$V$ satisfies $\mathbb{H}(p)$ and $\left( \int_{\mathbb{C}(T,0,2r)} |h(V,\cdot)|^p \, d\|V\| \right)^{\frac{1}{p}} = \alpha$

(4) for $\|V\|$ almost all $x \in \mathbb{C}(T,0,2r)$
$$\varepsilon \leq \frac{\|V\| \mathbb{B}(x,s)}{s^k} \leq (2-\varepsilon)\,\alpha(k) \qquad \forall s \in (0,r)$$

(5) for $\|V\|$ almost all $x \in \mathbb{C}(T,0,2r)$
$$1 \leq \Theta^k(\|V\|, x) \leq 1+\eta$$

<u>then</u> $\exists f : T \cap \mathbb{U}(0,r) \longrightarrow T^\perp \in \mathcal{C}^{1,1-\frac{k}{p}}$

(6) $F : T \cap \mathbb{U}(0,r) \longrightarrow \mathbb{R}^n$, $F(x) = x + f(x)$

(7) $\operatorname{spt}\|V\| \cap \mathbb{C}(T,0,r) = \operatorname{im} F$

(8) $Df(x)$ exists for $x \in T \cap \mathbb{U}(0,r)$

(9) $$\|Df(y) - Df(z)\| \leq C \max\left\{ \mu, r^{1-\frac{k}{p}} \alpha \right\} \left( \frac{|y-z|}{r} \right)^{1-\frac{k}{p}}$$

Proof of [8.19]

Fix $\varepsilon \in (0,1)$ and choose $\gamma$, N as in [8.17]
Assume $V, r, T, \mu$ are as in (1) - (5).
Let $a \in \operatorname{spt}\|V\| \cap \mathbb{C}(T,0,r)$. From [8.17] it follows that

$$\operatorname{spt}\|V\| \cap \{x : Tx = Ta\} = \{a\}$$
$$\operatorname{Tan}(\operatorname{spt}\|V\|, a) \in G(n,k)$$

[given $\eta > 0$ is small enough]

$$\circledast \quad \|T - \operatorname{Tan}(\operatorname{spt}\|V\|, a)\| \leq N\eta < 1$$

In particular, $\operatorname{Tan}(T_\natural[\operatorname{spt}\|V\|], T_\natural a) = T$

Since $0 \in \operatorname{spt}\|V\|$ we see that [8.18] yields

$$T_\natural[\operatorname{spt}\|V\|] \supseteq T \cap \mathbb{U}(0,r).$$

Hence, (6), (7), and (8) hold true.

Let $y, z \in T \cap \mathbb{U}(0,r)$, $|y-z| < \frac{r}{2}$, $a = F(y)$, $b = F(z)$, $s = 2|y-z|$.
Apply [8.17] to obtain $T_s$, $A_s$ s.t. $T_s \parallel A_s$ and

$$\circledast\circledast \quad \max\left\{ \|T_s - \operatorname{Tan}(\operatorname{spt}\|V\|, a)\|, \left( \frac{1}{s^{k+2}} \int_{\mathbb{C}(T,a,s)} \operatorname{dist}(x, A_s)^2 \, d\|V\|(x) \right)^{\frac{1}{2}} \right\}$$

$$\leq N \max\left\{\mu, r^{1-\frac{k}{p}}\alpha\right\} \left(\frac{s}{r}\right)^{1-\frac{k}{p}}$$

Note that

$$\|T_s - T\| \leq \|T_s - \operatorname{Tan}(\operatorname{spt}\|V\|, a)\| + \|\operatorname{Tan}(\operatorname{spt}\|V\|, a) - T\|$$

$$\left( \frac{1}{|y-z|^{k+2}} \int_{\mathbb{C}(T,b,|y-z|)} \operatorname{dist}(x, A_s)^2 \, d\|V\|(x) \right)^{1/2} \leq 2^{\frac{k+2}{2}} \left( \frac{1}{s^{k+2}} \int_{\mathbb{C}(T,a,s)} \operatorname{dist}(x, A_s)^2 \, d\|V\|(x) \right)^{\frac{1}{2}}$$

so we may apply [8.17] once again at $b$ with $T_s$ in place of $T$ to get

$$\|\operatorname{Tan}(\operatorname{spt}\|V\|, a) - \operatorname{Tan}(\operatorname{spt}\|V\|, b)\| \leq \|\operatorname{Tan}(\operatorname{spt}\|V\|, a) - T_s\| + \|T_s - \operatorname{Tan}(\operatorname{spt}\|V\|, b)\|$$

$$\leq \Gamma \cdot \max\left\{\mu, r^{1-\frac{k}{p}}\alpha\right\} \left(\frac{|y-z|}{r}\right)^{1-\frac{k}{p}}.$$

$$\Rightarrow \quad DF \in \mathscr{C}^{0,1-\frac{k}{p}}.$$

$\square$