1. Niech $\mathbf{i} = \sqrt{-1}$. Dla $z \in \mathbb{C}$ mamy

$$\sin(z) = \frac{1}{2\mathbf{i}}\big(\exp(\mathbf{i}z) - \exp(-\mathbf{i}z)\big) \quad \text{oraz} \quad \cos(z) = \frac{1}{2}\big(\exp(\mathbf{i}z) + \exp(-\mathbf{i}z)\big)\,.$$

   (a) Dla $x \in \mathbb{R} \setminus \{2\pi l : l \in \mathbb{Z}\}$ udowodnić wzory

$$\sum_{k=1}^{n} \sin(kx) = \frac{\sin(nx/2)\sin((n+1)x/2)}{\sin(x/2)}\,,$$
$$\sum_{k=1}^{n} \cos(kx) = \frac{\sin(nx/2)\cos((n+1)x/2)}{\sin(x/2)}\,.$$

   (b) Pokazać, że dla $w, z \in \mathbb{C}$ mamy

$$\cos(z) - \cos(w) = -2\sin((z-w)/2)\sin((z+w)/2)\,,$$
$$\sin(z) - \sin(w) = 2\sin((z-w)/2)\cos((z+w)/2)\,.$$

2. Obliczyć granice

$$\lim_{n\to\infty} n\sin(2\pi e n!) \quad \text{oraz} \quad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sin^2(k)\,.$$

3. Zbadać zbieżność szeregów

(1) $$\sum_{n=1}^{\infty} \frac{1}{n}\big(\sin(na)\sin(n^2 a)\big) \quad \text{gdzie } a \in \mathbb{R}\,,$$

(2) $$\sum_{n=1}^{\infty} \frac{1}{n}\big(\sin(na)\cos(n^2 a)\big) \quad \text{gdzie } a \in \mathbb{R}\,,$$

(3) $$\sum_{n=1}^{\infty} \frac{1}{n}\big(\cos(n)\cos(na)\big) \quad \text{gdzie } a \in \mathbb{R}\,,$$

(4) $$\sum_{n=1}^{\infty} 1 - \cos(\sin(1/n))\,,$$

(5) $$\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n}\,,$$

(6) $$\sum_{n=1}^{\infty} \cos(1/n) - \cos(\sin(1/n))\,,$$

(7) $$\sum_{n=1}^{\infty} \cos(1/(n+1)) + \sin(1/n)\,,$$

(8) $$\sum_{n=1}^{\infty} \frac{\cos(\pi n^2)}{\ln(n)}\,,$$

(9) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\big(\cos(\sqrt{n+1}) - \cos(\sqrt{n})\big)\,,$$

(10) $$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \quad \text{gdzie } x \in \mathbb{R} \text{ takie, że } \sin(x/2) \neq 0\,,$$

(11) $$\sum_{n=1}^{\infty} \frac{\cos(n+1/n)}{n}\,.$$