Partial differential equations I, 2011/2012

Example solutions to some problems using the separation of variables

Example 1. Solve the following problem

$$u_t - u_{xx} = 0 \tag{1}$$
$$u(0,t) = u(\pi,t) = 0 \tag{2}$$
$$u(x,0) = 2\sin(x) + 3\sin(2x) + 7\sin(5x)\,. \tag{3}$$

Solution.
We know that the solution has to be of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t}\sin(nx)\,. \tag{4}$$

Comparing this with (3) we obtain

$$u(x,t) = 2e^{-t}\sin(x) + 3e^{-4t}\sin(2x) + 7e^{-25t}\sin(5x)\,. \tag{5}$$

This obviously satisfies (3). Since $\sin(n\pi) = \sin(n\cdot 0) = 0$ for all $n \in \mathbb{N}$, our function (5) satisfies (2). Of course any function of the form (4) satisfies (1). Hence, (5) is a solution to our problem.

Example 2. Solve the non-homogeneous problem

$$u_t - u_{xx} = e^{-t}\sin(x) \tag{6}$$
$$u(0,t) = u(\pi,t) = 0 \tag{7}$$
$$u(x,0) = 2\sin(x) + 3\sin(2x) + 7\sin(5x)\,. \tag{8}$$

Solution.
We know that the solution has to be of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin(nx)\,. \tag{9}$$

Differentiating and comparing with (6) we get the following identity

$$\sum_{n=1}^{\infty}(A_n'(t) + n^2 A_n(t))\sin(nx) = e^{-t}\sin(x)\,.$$

Comparing the coefficients we obtain the following system of ODEs

$$A_1' = -A_1 + e^{-t} \quad \text{and} \quad \forall n \neq 1 \quad A_n' = -n^2 A_n\,.$$

The solutions are

$$A_1(t) = (t + a_1)e^{-t} \quad \text{and} \quad \forall n \neq 1 \quad A_n(t) = a_n e^{-n^2 t}\,.$$

Therefore a general solution of (6) has to be of the form

$$u(x,t) = te^{-t}\sin(x) + \sum_{n=1}^{\infty} a_n e^{-n^2 t}\sin(nx)\,. \tag{10}$$

Comparing this with (8) we obtain

$$u(x,t) = 2e^{-t}\sin(x) + 3e^{-4t}\sin(2x) + 7e^{-25t}\sin(5x) + te^{-t}\sin(x)\,. \tag{11}$$

One can easily check that (11) satisfies (6), (7) and (8).

Example 3. Solve the problem

$$u_{tt} - u_{xx} = 0 \tag{12}$$
$$u(0,t) = u(1,t) = 0 \tag{13}$$
$$u(x,0) = 2\sin(3\pi x) + 3\sin(2\pi x) \tag{14}$$
$$u_t(x,0) = 4\sin(\pi x) + 5\sin(3\pi x)\,. \tag{15}$$

Solution.
We know that the solution has to be of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\sin(n\pi t) + b_n \sin(n\pi x)\cos(n\pi t)\,. \tag{16}$$

Any such function certainly satisfies (12) and (13). Comparing coefficients in (16) with the coefficients in (14) we see that

$$b_3 = 2\,, \quad b_2 = 3 \quad \text{and} \quad \forall n \notin \{2,3\} \quad b_n = 0\,.$$

Differentiating with respect to t and comparing with (15) we get

$$\pi a_1 = 4\,, \quad 3\pi a_3 = 5 \quad \text{and} \quad \forall n \notin \{1,3\} \quad n\pi a_n = 0\,.$$

Hence, the solution is

$$u(x,t) = \frac{4}{\pi}\sin(\pi x)\sin(\pi t) + \frac{5}{3\pi}\sin(3\pi x)\sin(3\pi t) + 3\sin(2\pi x)\cos(2\pi t) + 2\sin(3\pi x)\cos(3\pi t)\,.$$

Example 4. Solve the non-homogeneous problem

$$u_{tt} - u_{xx} = \sin(2\pi x)(1 - \cos(4\pi t)) \tag{17}$$
$$u(0,t) = u(1,t) = 0 \tag{18}$$
$$u(x,0) = 2\sin(3\pi x) + 3\sin(2\pi x) \tag{19}$$
$$u_t(x,0) = 4\sin(\pi x) + 5\sin(3\pi x)\,. \tag{20}$$

Solution.
We know that the solution has to be of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin(n\pi x)\,. \tag{21}$$

Any such function certainly satisfies (18). We differentiate and compare with (17).

$$\sum_{n=1}^{\infty} (A_n''(t) + n^2\pi^2 A_n(t))\sin(n\pi x) = \sin(2\pi x)(1 - \cos(4\pi t))\,,$$

$$\text{hence} \quad A_2'' + 4\pi^2 A_2 = 1 - \cos(4\pi t) \quad \text{and} \quad \forall n \neq 2 \quad A_n'' = -n^2\pi^2 A_n \,.$$

The solutions are

$$A_2(t) = a_2 \sin(2\pi t) + b_2 \cos(2\pi t) + \frac{1}{12\pi^2}\cos(4\pi t) + \frac{1}{4\pi^2}$$

$$\text{and} \quad \forall n \neq 2 \quad A_n(t) = a_n \sin(n\pi t) + b_n \cos(n\pi t) \,.$$

Therefore, the general solution to our problem has to be of the form

$$u(x,t) = \frac{1}{12\pi^2}\cos(4\pi t)\sin(2\pi x) + \frac{1}{4\pi^2}\sin(2\pi x) + \sum_{n=1}^{\infty} a_n \sin(n\pi x)\sin(n\pi t) + b_n \sin(n\pi x)\cos(n\pi t) \,. \quad (22)$$

Comparing coefficients in (22) with the coefficients in (19) we see that

$$b_3 = 2\,, \quad b_2 + \frac{1}{3\pi^2} = 3 \quad \text{and} \quad \forall n \notin \{2,3\} \quad b_n = 0\,.$$

Differentiating with respect to t and comparing with (20) we also get

$$\pi a_1 = 4\,, \quad 3\pi a_3 = 5\,, \quad 2\pi a_2 + \frac{1}{4\pi^2} = 0 \quad \text{and} \quad \forall n \notin \{1,2,3\} \quad n\pi a_n = 0\,.$$

Hence, the solution is

$$u(x,t) = \frac{4}{\pi}\sin(\pi x)\sin(\pi t) + \frac{-1}{8\pi^3}\sin(2\pi x)\sin(2\pi t) + \frac{5}{3\pi}\sin(3\pi x)\sin(3\pi t) + \frac{9\pi^2-1}{3\pi^2}\sin(2\pi x)\cos(2\pi t) + 2\sin(3\pi x)\cos(3\pi t)\,.$$

Example 5. Solve the problem

$$u_t - u_{xx} = 0 \quad (23)$$
$$u(0,t) = u(\pi,t) = 0 \quad (24)$$
$$u(x,0) = 1 - \cos(2x)\,. \quad (25)$$

Solution.
In this case the boundary conditions are compatible. To solve this problem we need to express $1 - \cos(2x)$ as a sum of sines. To do that we define the function $f : [-\pi,\pi] \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} 1 - \cos(2x) & \text{for } x \geqslant 0 \\ \cos(2x) - 1 & \text{for } x \leqslant 0 \end{cases}$$

Now f is antisymmetric (i.e. $f(x) = -f(-x)$), so the Fourier series of f will contain only sines. We calculate the Fourier series of f on the interval $[-\pi,\pi]$.

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx) \quad \text{where} \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\,.$$

Simple calculation gives

$$a_n = \frac{8(\cos(n\pi) - 1)}{n\pi(n^2 - 4)} = \frac{8((-1)^n - 1)}{n\pi(n^2 - 4)}\,.$$

Therefore the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8((-1)^n - 1)}{n\pi(n^2 - 4)} e^{-n^2 t} \sin(nx)\,. \tag{26}$$

Remark. If $t > 0$ then (29) describes a convergent series. Actually it converges uniformly on compact subsets of $(0,\pi) \times (0,\infty)$. We can differentiate this series component-wise as many times as we want and we always get an almost uniformly convergent series. This shows that $u(x,t)$ is of class C^∞ on $(0,\pi) \times (0,\infty)$.