Partial differential equations I, 2011/2012

IX: The Lax-Milgram theorem

Reminder

• Lax-Milgram theorem: Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space equipped with the norm $|u| = \sqrt{\langle u, u\rangle}$, where $u \in H$. Let $B : H \times H \to \mathbb{R}$ be bilinear and such that there exist numbers $\alpha, \beta > 0$ such that for all $u, v \in H$

$$\begin{aligned} &\text{[boundedness/continuity]} \qquad |B(u,v)| \leqslant \alpha |u||v| \,, \\ &\text{[coercivity/ellipticity]} \qquad B(u,u) \geqslant \beta |u|^2 \,. \end{aligned}$$

Then for any bounded linear functional $L : H \to \mathbb{R}$ there exists a unique vector $v \in H$ such that for all $u \in H$

$$Lv = B(u,v)\,.$$

Remark: The functional B does not need to be symmetric, so it may not define a scalar product on H. This is the difference between the Lax-Milgram theorem and the Riesz representation theorem.

• Gagliardo-Nirenberg-Sobolev inequality: Let $1 \leqslant p < n$. There exists a constant $C = C(n,p)$ such that for all $u \in C_c^1(\mathbb{R}^n)$

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C \|Du\|_{L^p(\mathbb{R}^n)}\,,$$

where $p^* = \frac{np}{n-p}$ (in other words $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$).

• Poincaré inequality: Let $1 \leqslant p \leqslant \infty$ and $\Omega \subseteq \mathbb{R}^n$ be bounded, connected and open set with C^1-smooth boundary. There exists a constant $C = C(n,p,U)$ such that for all $u \in W^{1,p}(\Omega)$

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \leqslant C \|Du\|_{L^p(\Omega)}\,,$$

where $(u)_\Omega = |\Omega|^{-1} \int_\Omega u$.

Remark: If $\Omega = \mathbb{B}(x,r)$ is a ball then one can find a constant $C = C(n,p)$ such that

$$\|u - (u)_\Omega\|_{L^p(\mathbb{B}(x,r))} \leqslant Cr \|Du\|_{L^p(\mathbb{B}(x,r))}\,.$$

Problems

Unless otherwise stated, in the sequel $\Omega \subseteq \mathbb{R}^n$ is always a bounded domain with smooth boundary.

1. Let $n = 2$. Prove that there exists a weak solution of the problem

$$\begin{aligned} -u_{xx} - u_{yy} - u_{xy} &= f && \text{in } \Omega\,, \\ u &= 0 && \text{on } \partial\Omega\,. \end{aligned}$$

   What do we need to assume about f to be able to apply the Lax-Milgram theorem?

   Observe that in this case $n = 2 = p$. Can we use Gagliardo-Nirenberg-Sobolev inequality here?

2. Let $\Omega = \mathbb{B}^3(0,1) \subseteq \mathbb{R}^3$. For which $a, b \in \mathbb{R}$ there exists a weak solution of the problem

$$\begin{aligned} -4\Delta u(x) + aD_1 u(x) + u(x) &= \frac{b}{|x_1|} && \text{for } x = (x_1,x_2,x_3) \in \Omega\,, \\ u(x) &= x_1^2 && \text{for } x = (x_1,x_2,x_3) \in \partial\Omega\,. \end{aligned}$$

3. Let $\Omega = \mathbb{B}^2(0,1) \subseteq \mathbb{R}^2$. For which $a \in \mathbb{R}$ there exists a weak solution of the problem

$$\begin{aligned} -4\Delta u + aD_1 u + u = |x_1|^{-\frac{1}{4}} \qquad & \text{for } x = (x_1, x_2) \in \Omega\,, \\ u(x) = x_1 \qquad & \text{for } x = (x_1, x_2) \in \partial\Omega\,. \end{aligned}$$

4. Let $f \in L^2(\Omega)$. Show that there exists a solution of the problem

$$\begin{aligned} -\Delta u = f \qquad & \text{in } \Omega\,, \\ u = 0 \qquad & \text{on } \partial\Omega\,. \end{aligned} \tag{1}$$

5. Show that the solution of Problem 4 satisfies

$$\|u\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}\,.$$

6. Assume that $f \in L^{p+\varepsilon}(\Omega)$, where $\varepsilon > 0$ is some number. Show that the function $u \in W_0^{1,2}(\Omega)$ satisfying (1) (which exists and is determined uniquely by Lax-Milgram theorem) is Hölder continuous (i.e. there exists a representative which is Hölder continuous).

7. Show that there exists a weak solution of the problem

$$\begin{aligned} -k\Delta u + \mathbf{b}\cdot\nabla u + au = f \qquad & \text{in } \Omega\,, \\ u = 0 \qquad & \text{on } \partial\Omega\,, \end{aligned}$$

where let $f \in L^2(\Omega)$, $\mathbf{b} \in \mathbb{R}^n$, $a, k > 0$.

8. Show that there exists a unique solution to

$$\begin{aligned} -u_{xx} - 4u_{yy} + u_x + u_y + 5u = f \qquad & \text{in } \Omega\,, \\ u = 0 \qquad & \text{on } \partial\Omega\,, \end{aligned}$$

where let $f \in L^2(\Omega)$.

9. Show that there exists a unique solution to

$$\begin{aligned} -\operatorname{div}(\mathbf{k}\cdot\nabla u) + cu = f \qquad & \text{in } \Omega\,, \\ u = 0 \qquad & \text{on } \partial\Omega\,, \end{aligned}$$

where $f \in L^2(\Omega)$, $c \in C^0(\overline{\Omega})$, $c > 0$, $\mathbf{k} = (k_1, \dots, k_n) \in C^1(\overline{\Omega}, \mathbb{R}^n)$ and for all $i = 1, 2, \dots, n$ and all $x \in \Omega$ we have $k_i(x) \geqslant k_0 > 0$ for some k_0.

10. Let $f \in L^2(\Omega)$. For which $c \in \mathbb{R}$ there exists a unique solution to

$$\begin{aligned} 5u_{xx} + 4u_{yy} + u_x + u_y - cu = f \qquad & \text{in } \Omega\,, \\ u = 0 \qquad & \text{on } \partial\Omega \quad ? \end{aligned}$$