Partial differential equations I, 2011/2012

VIII: Weak derivatives and the Sobolev spaces

Reminder

Let $u, v \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index. We say that v is the α**-weak derivative** of u (and we write "$D^\alpha u = v$") if

$$\forall \varphi \in C_c^\infty(\Omega) \qquad \int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v \varphi \, dx \, .$$

A function $u \in L^1_{loc}(\Omega)$ belongs to the **Sobolev space** $W^{k,p}(\Omega)$ if for all multi-indices α such that $|\alpha| \leqslant k$ there exists $D^\alpha u$ in the weak sense and $D^\alpha u \in L^p(\Omega)$.

Problems

1. Let $u : [a, b] \to \mathbb{R}$ be piecewise C^1. Show that the weak derivative u' exists and that it equals the classical derivative at all the points where u' exists in the classical sense.
2. Let $p \geqslant 1$. Give example of a function $u \in C^\infty(\mathbb{B}^n) \setminus W^{1,p}(\mathbb{B}^n)$.
3. Let $u : [0, 1] \to [0, 1]$ be the Cantor function (so called "Devil's staircase"). Does the weak derivative u' exists? What if u is the Cantor function on a thick Cantor set (i.e. a Cantor set of positive measure)?
4. Let α, β be some multi-indices and let $u \in L^1_{loc}(\mathbb{R}^n)$ be such that two of $D^\alpha D^\beta u$, $D^\beta D^\alpha u$, $D^{\alpha+\beta} u$ exist in the weak sense. Show that the third one also exists and all of them are equal.
5. Let $u \in W^{k,p}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$ for some $k \in \mathbb{N}$ and $p \in [1, \infty)$. Show that $\varphi u \in W^{k,p}(\Omega)$ and that the Leibniz rule holds

$$D^\alpha(\varphi u) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha - \beta} u \, ,$$

   where $\binom{\alpha}{\beta} = \alpha!/(\beta!(\alpha - \beta)!)$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.

6. Let $n > 1$, $s \in (0, n)$ $u : \mathbb{B}^n \to \mathbb{R}$ be given by $u(x) = |x|^{-s}$. For which $p \geqslant 1$ does u belong to: $W^{1,p}(\mathbb{B}^n)$? $W^{2,p}(\mathbb{B}^n)$?
7. Let $u(x) = \log\log(1 + |x|^{-1})$. For which $p \geqslant 1$ does u belong to $W^{1,p}(\mathbb{B}^n)$?
8. Let $u \in W^{1,p}([0, 1]) \cap C^\infty([0, 1])$. Show that

$$|u(x) - u(y)| \leqslant |x - y|^{1 - 1/p} \|u\|_{W^{1,p}} \, .$$

9. Let $\Omega \subseteq \mathbb{R}^n$ be connected and let $u \in W^{1,p}(\Omega)$ be such that the weak derivative $u_{x_i} = 0$ a.e. for each $i = 1, \ldots, n$. Show that u is constant a.e.
10. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$u(x) = \begin{cases} 1 - |x|^2 & \text{for } |x| \leqslant 1 \\ 0 & \text{for } |x| > 1 \, . \end{cases}$$

   Does u belong to: $W^{1,2}(\mathbb{B}^2(0, 2))$? $W^{2,2}(\mathbb{B}^2(0, 2))$? $W_0^{1,2}(\mathbb{B}^2(0, 2))$?

11. For which $s > 0$ there exists a weak derivative of the function $u(x) = |x|^{-s} \log |x|$?