# Partial differential equations I, 2011/2012

## VII: The Green representation formula

### Reminder

Recall that the *fundamental solution of the Laplace equation* $\Phi : \mathbb{R}^n \to \mathbb{R}$ is defined as follows

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & \text{for } n = 2\,, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & \text{for } n \geqslant 3\,. \end{cases}$$

This function satisfies

$$-\Delta\Phi = \delta_0 \quad \equiv \quad \forall f \in C_0^2(\mathbb{R}^n) \quad \int_{\mathbb{R}^n} (-\Delta f)(x)\Phi(x)\, dx = f(0)\,. \tag{1}$$

Let $\Omega$ be an open subset of $\mathbb{R}^n$. The *Green function* $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ for $\Omega$ is defined as

$$G(x, y) = \Phi(y - x) - \varphi_x(y)\,,$$

$$\text{where} \quad -\Delta\varphi_x = 0 \text{ in } \Omega \quad \text{and} \quad \varphi = \Phi(y - x) \text{ on } \partial\Omega\,.$$

The function

$$u(x) = -\int_{\partial\Omega} g(y) \frac{\partial G}{\partial n}(x, y)\, dS(y) - \int_{\Omega} f(y)G(x, y)\, dy$$

$$\text{solves} \quad -\Delta u = f \text{ in } \Omega\,, \qquad u = g \text{ on } \partial\Omega\,.$$

The Green function satisfies (in distributional sense)

$$-\Delta_y G = \delta_x \text{ in } \Omega\,, \qquad G = 0 \text{ on } \partial\Omega\,. \tag{2}$$

For some domains $\Omega$ the Green function is given with an explicit formula.

- If $\Omega = \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$ then

  $$G(x, y) = \Phi(y - x) - \Phi(y - \tilde{x})\,, \quad \text{where} \quad \tilde{x} = (x_1, \ldots, x_{n-1}, -x_n)\,.$$

  For $y \in \partial\mathbb{R}^n_+$ (i.e. $y = (y_1, \ldots, y_{n-1}, 0)$) we have

  $$K(x, y) = -\frac{\partial G}{\partial n}(x, y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x - y|^n}\,.$$

  Hence the function

  $$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x - y|^n}\, dS(y) \quad \text{solves} \quad -\Delta u = 0 \text{ in } \Omega\,, \quad u = g \text{ on } \partial\Omega\,. \tag{3}$$

- If $\Omega = \mathbb{B}(0, R)$ then

  $$G(x, y) = \Phi(y - x) - \Phi\big(\tfrac{|x|}{R}(y - \tilde{x})\big)\,, \quad \text{where} \quad \tilde{x} = \frac{x}{|x|^2}\,.$$

  For $y \in \partial\Omega$ we have

  $$K(x, y) = -\frac{\partial G}{\partial n}(x, y) = \frac{1}{n\alpha(n)R} \frac{R^2 - |x|^2}{|x - y|^n}\,.$$

  Hence the function

  $$u(x) = \frac{R^2 - |x|^2}{n\alpha(n)R} \int_{\partial\Omega} \frac{g(y)}{|x - y|^n}\, dS(y) \quad \text{solves} \quad -\Delta u = 0 \text{ in } \Omega\,, \quad u = g \text{ on } \partial\Omega\,. \tag{4}$$

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## Problems

1. [Harnack inequality for balls] Let $u : \mathbb{B}(0,R) \to \mathbb{R}$ be positive and harmonic in $\mathbb{B}(0,R)$. Show that
$$R^{n-2}\frac{R-|x|}{(R+|x|)^{n-1}}u(0) \leqslant u(x) \leqslant R^{n-2}\frac{R+|x|}{(R-|x|)^{n-1}}u(0)\,.$$

2. Let $u$ be the solution to
$$-\Delta u = 0 \text{ in } \mathbb{R}^n_+\,, \quad u = g \text{ on } \partial\mathbb{R}^n_+$$
given by (3). Assume that $g$ is bounded and $g(x) = |x|$ for $x \in \mathbb{B}(0,R) \cap \partial\mathbb{R}^n_+$. Show that $Du$ is not bounded in a neighborhood of $0 \in \mathbb{R}^n$.

   *Remark:* Note that $u$ is harmonic in $\mathbb{R}^n_+$, hence $u \in C^\infty(\mathbb{R}^n_+)$.

3. Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be given by (3). Assume that $-g(x) = g(-x)$. Show that $-u(x) = u(-x)$.

4. Find explicit formulas for the solution to
$$-\Delta u = 0 \text{ in } \mathbb{R}_+ \times \mathbb{R}_+\,, \quad u(x,0) = g(x) \text{ for } x \in \mathbb{R}_+\,, \quad u(0,y) = h(y) \text{ for } y \in \mathbb{R}_+\,.$$
Describe the Green function for $\Omega = \mathbb{R}_+ \times \mathbb{R}_+$.

5. Let $G$ be the Green function for $\Omega$. Let $0 < a < b < \infty$ and $x \in \Omega$. Set
$$W := \{y \in \Omega : a \leqslant G_x(y) \leqslant b\}$$
(we assume that $\overline{W} \subseteq \Omega$). Show that
$$\int_W |\nabla G_x(y)|^2 \, dy = b - a\,.$$

6. Let $u \in C^2(\mathbb{R}^n)$ be harmonic in $\mathbb{R}^n$. Let $\varphi \in C^\infty(\mathbb{R}^n)$ be any nonnegative function such that $\varphi(x) = \psi(|x|)$ for some $\psi : \mathbb{R} \to \mathbb{R}$ and such that $\int \varphi = 1$. For $\varepsilon > 0$ we set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Let $u_\varepsilon$ be the convolution $u_\varepsilon = u * \varphi_\varepsilon$, i.e.
$$u_\varepsilon(x) = \int u(y)\varphi_\varepsilon(x-y) \, dy\,.$$
Show that $u_\varepsilon(x) = u(x)$ for all $\varepsilon$ and all $x \in \mathbb{R}^n$. Deduce that harmonic functions are $C^\infty$ smooth.

7. Prove that $\Phi$ and $G$ satisfy (1) and (2) respectively.