Partial differential equations I, 2011/2012

VI: Harmonic functions

All the functions in this problem set are assumed to be at least C^2 smooth.

1. Show that the following functions are harmonic

$$\begin{aligned} &a)\, f(x) = \langle v, x\rangle : \mathbb{R}^n \to \mathbb{R} \quad \text{(here } v \in \mathbb{R}^n \text{ is fixed)} \\ &b)\, f(x,y) = x^2 - y^2 : \mathbb{R}^2 \to \mathbb{R} \\ &c)\, f(x,y) = \exp(x)\cos(y) : \mathbb{R}^2 \to \mathbb{R} \\ &d)\, \mathrm{Re}(f), \mathrm{Im}(f) \quad \text{where } f : \mathbb{C} \to \mathbb{C} \text{ is holomorphic}\,. \end{aligned}$$

2. Let $p = (1,1) \in \mathbb{R}^2$. Calculate

$$\frac{1}{\pi}\int_{\mathbb{B}(p,1)} x^3 - 3xy^2 \; d\lambda_2(x,y)\,.$$

3. ["Invariance" under inverses] Let u be harmonic. Show that

$$v(x) = \frac{1}{|x|^{n-2}} u\big(\tfrac{x}{|x|^2}\big)$$

is also harmonic.

4. [Invariance under inverses for n-harmonic functions] Let $u : \mathbb{R}^n \to \mathbb{R}$ be n-*harmonic*, i.e.

$$\mathrm{div}(|\nabla u|^{n-2}\nabla u) = 0\,.$$

Show that the function

$$v(x) = u\big(\tfrac{x}{|x|^2}\big)$$

is also n-harmonic.

5. Let u, v be harmonic in $\Omega \subseteq \mathbb{R}^n$ (a bounded domain) and such that $u \geqslant v$ on $\partial\Omega$. Show that $u \geqslant v$ on all of Ω.

6. Let $n \geqslant 3$ and let u be a solution to

$$-\Delta u = f \quad \text{in } \mathbb{B}(0,r)\,, \qquad u = g \quad \text{on } \partial\mathbb{B}(0,r)\,.$$

Show that

$$u(0) = \fint_{\partial\mathbb{B}(0,r)} g(x)\; dS(x) + \tfrac{1}{n(n-2)\alpha(n)} \int_{\mathbb{B}(0,r)} \Big(\tfrac{1}{|x|^{n-2}} - \tfrac{1}{r^{n-2}}\Big) f(x)\; dx\,.$$

Hint: One formula worth remembering

$$\fint_{\partial\mathbb{B}(x,r)} u(x)\; dS(x) = \fint_{\partial\mathbb{B}(0,1)} u(x+rz)\; dS(z)\,.$$

7. [Invariance under orthogonal transformations of the domain] The function u satisfies $\Delta u = 0$ in $\Omega \subseteq \mathbb{R}^n$. Let A be an orthogonal matrix, i.e. $AA^T = I = A^TA$. Show that $v(x) = u(Ax)$ satisfies $\Delta v = 0$ in $A^{-1}\Omega$.

8. Let u be harmonic in R^n and let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex. Show that the following functions are subharmonic

$$a)\, w = |\nabla u|^2 \qquad\qquad b)\, v = \varphi(u)\,.$$

9. Let $x = r\cos\varphi$ and $y = r\sin\varphi$ be polar coordinates in $\mathbb{R}^2$. Show that

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\,.$$

10. For which $a \in \mathbb{R}$ the following functions are subharmonic

$$a)\; u(x) = \exp(a|x|^2) : \mathbb{R}^n \to \mathbb{R} \qquad\qquad b)\; v(x) = \big(\ln|x|\big)^a : \mathbb{R}^n \to \mathbb{R}\,.$$

11. Let $u : \mathbb{R}^n \to \mathbb{R}$ be harmonic. Knowing that $u(x) = v(|x|)$ for some $v : \mathbb{R} \to \mathbb{R}$, find the ODE satisfied by v and solve it.

12. Let $u \in C^2(\mathbb{B}) \cap C(\overline{\mathbb{B}})$, where $\mathbb{B} \subseteq \mathbb{R}^3$ is a unit ball. Assume that $\Delta u = u - 1$ and that $u = 1$ on the set $S^2_+ = \{(x_1, x_2, x_3) \in \partial\mathbb{B} : x_3 > 0\}$ and that $\frac{\partial u}{\partial \vec{n}} = 0$ on the set $S^2_- = \partial\mathbb{B} \setminus S^2_+$. Show that $u = 1$ in $\mathbb{B}$.

13. Let $u : \mathbb{B} \to \mathbb{R}$, where $\mathbb{B} \subseteq \mathbb{R}^n$ is a unit ball. Assume that $\Delta u = u^3$ in $\mathbb{B}$ and $u = 0$ on $S_+ = \{x \in \partial\mathbb{B} : x_1 > 0\}$ and $\frac{\partial u}{\partial \vec{n}} = 0$ on the set $S_- = \partial\mathbb{B} \setminus S_+$. Show that $u = 0$ in $\mathbb{B}$.

14. Let $u, h : \mathbb{B} \to \mathbb{R}$, where $\mathbb{B} \subseteq \mathbb{R}^n$ is a unit ball. Assume that h is harmonic in $\mathbb{B}$ and u satisfies

$$\Delta u = u - h \text{ in } \mathbb{B}\,, \qquad u = h \text{ on } \partial\mathbb{B}\,.$$

Is $u = h$ the only possibility?

15. Let u be harmonic in $\mathbb{R}^n$. Assume that there exists a constant $C > 0$ such that $|u(x)| \leqslant C(1 + |x|)^\varepsilon$ for some $\varepsilon \in (0, 1)$. Show that u is constant.

16. Let u be harmonic in $\mathbb{R}^n$ which does not change the sign. Show that u is constant.

17. Let $u \in K := \{w \in C^2(\Omega) : w = g \text{ on } \partial\Omega\}$. Show that the following conditions are equivalent

$$a)\; \Delta u = 0 \text{ in } \Omega\,,$$

$$b)\; \int_\Omega \nabla u \nabla \varphi \; dx = 0 \quad \text{for each } \varphi \in C_0^\infty(\Omega)\,,$$

$$c)\; \int_\Omega |\nabla u|^2 \; dx \leqslant \int_\Omega |\nabla w|^2 \; dx \quad \text{for each } w \in K\,.$$

Remark. The functional $I : K \to \mathbb{R}$ defined by $I[u] = \int_\Omega |\nabla u|^2$ is called the *Dirichlet energy* and harmonic functions are minimizers of this energy.

18. Using the above characterization of harmonic maps, solve problems 3 and 7.